



Research article

The generalized Smarandache ruled surfaces generated by Legendre curves in the Euclidean 3-space

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Abstract: By Smarandache geometry, this paper introduces a class of ruled surfaces that are generated from spherical Legendre curves in the Euclidean 3-space. The developability and minimality of the generalized Smarandache ruled surface are investigated, as well as the geometric properties of the base curve. Moreover, since Legendre curves admit singularities, the generated ruled surfaces also contain singularities. Therefore, the singularities of the generalized Smarandache ruled surface are analyzed via the theory of framed curves and framed surfaces.

Keywords: generalized Smarandache ruled surfaces; Legendre curves; singularities

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1. Introduction

Ruled surfaces represent a classical topic in three-dimensional Euclidean space, with wide applications in surface machining, architectural design, and robot motion planning. The classical theory has fully described the developability, minimality, and curvature properties of ruled surfaces [1–3].

Recently, constructing curves and surfaces by Smarandache geometry has become a research direction. A Smarandache curve is defined as a linear combination of the Frenet frame vectors of a regular curve [4]. The ruled surface determined by a regular curve and its Smarandache curve is called a Smarandache ruled surface. In fact, the classical osculating-type, rectifying-type, and normal-type ruled surfaces are all special cases. Smarandache curves generated by null curves in Lorentz-Minkowski space were defined in [5]. This work presented geometric invariants when the base curve becomes a slant helix. Smarandache curves generated by a pedal curve were investigated in [6, 7], while [8] used a Bishop frame to examine adjoint curves generated by Smarandache curves. Moreover, [9–11] systematically discussed the developability, minimality, and geodesic properties of Smarandache ruled surfaces by different moving frames.

However, most of the above studies take a regular curve as the base curve to generate Smarandache curves and Smarandache ruled surfaces. Once the base curve has singular points, the classical Frenet frame is no longer available. To deal with singular cases, [12] classified the singularities by germs of maps, and [13–15] introduced the notions of front and frontal. [16] showed that such surfaces can still have a smooth normal vector field near singularities. [17] later developed the theory of framed surfaces. Moreover, [18] gave an efficient way to compute invariants and singularities by geometric algebra. [19] also investigated singularity properties of a non-developable ruled surface based on the Blaschke frame. [20] analyzed singular point types on generalized osculating-type ruled surfaces.

Based on the above work, this paper studies Smarandache ruled surfaces constructed by spherical curves from the viewpoint of singularity. Building on previous work, we investigate the geometric properties of Smarandache ruled surfaces generated by spherical curves, and derive explicit conditions for developability and minimality in terms of the base curve. Moreover, while most previous studies have focused on regular surfaces, we extend the construction to the singular case by taking a Legendre curve, which may have singularities, as the director curve. We then define a generalized Smarandache ruled surface. We show that it can be framed exactly when it has only the second class singularities and its curvature functions satisfy a suitable order condition near those singularities. Using framed surface theory, we give a complete classification of its singularities. This systematically extends the classical Smarandache ruled surface theory to the singular setting.

This paper is structured as follows. We first recall relevant theories of the Euclidean 3-space in Section 2. In Section 3, we define the generalized Smarandache ruled surfaces. We derive the conditions for developability and minimality by computing the fundamental forms and curvatures. If the base curve is geodesic, asymptotic, or principal, we give the relations between it and the Legendre curve. We also determine the singular set and classify the singular points into two disjoint classes. In Section 4, for surfaces that contain only the second class singularities under a suitable order condition, we obtain the framed surface by constructing a smooth normal vector field. In Section 5, we give conditions for the first class singularities to be cross caps or S_1^+ and for the second class to be locally cuspidal edges or cuspidal cross caps.

In this paper, we denote that U is a simply connected open set in $\mathbb{R}^2$ and I is an interval in $\mathbb{R}$. All maps and manifolds are smooth.

2. Preliminaries

We denote $\mathbb{R}^3$ as the three-dimensional Euclidean space with *inner product* $\cdot$, *vector product* $\times$, and the *norm* $\|\cdot\|$. The unit sphere in $\mathbb{R}^3$ is denoted by S^2 . We set $\Delta = \{(\mathbf{a}, \mathbf{b}) \in S^2 \times S^2 \mid \mathbf{a} \cdot \mathbf{b} = 0\}$, which is a three-dimensional smooth manifold.

Definition 2.1. [21] $(\gamma, \nu_1) : I \rightarrow \Delta \subset S^2 \times S^2$ is called a *Legendre curve* if $\dot{\gamma}(s) \cdot \nu_1(s) = 0$ for all $s \in I$. Then the Frenet-Serret-type formulas of (γ, ν_1) can be obtained:

$$\begin{pmatrix} \dot{\gamma}(s) \\ \dot{\nu}_1(s) \\ \dot{\nu}_2(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m(s) \\ 0 & 0 & n(s) \\ -m(s) & -n(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \nu_1(s) \\ \nu_2(s) \end{pmatrix},$$

where $\nu_2(s) = \gamma(s) \times \nu_1(s)$ and (m, n) is the curvature of the Legendre curve. We call s_0 a singular point of γ (resp. ν_1) iff $m(s_0) = 0$ (resp. $n(s_0) = 0$).

A ruled surface obtained by the Legendre curve is expressed as $\mathbf{x}(s, t) = \beta(s) + t\gamma(s)$, where $\beta(s)$ is not only a Smarandache curve of $\gamma(s)$ but also the base curve of $\mathbf{x}(s, t)$.

Definition 2.2. [22] Suppose that $\mathbf{x}(s, t)$ is a regular surface and $\beta(s)$ is a curve on $\mathbf{x}(s, t)$. We say $\mathbf{x}(s, t)$ is developable (resp. minimal) if its Gaussian curvature $K(s, t)$ (resp. mean curvature $H(s, t)$) vanishes identically, i.e., $K(s, t) \equiv 0$ (resp. $H(s, t) \equiv 0$). Moreover,

- (1) $\beta(s)$ is asymptotic on $\mathbf{x}(s, t)$ iff its normal curvature vanishes, i.e., $\kappa_n^\beta(s) = 0$.
- (2) $\beta(s)$ is geodesic on $\mathbf{x}(s, t)$ iff its geodesic curvature vanishes, i.e., $\kappa_g^\beta(s) = 0$.
- (3) $\beta(s)$ is a principal line on $\mathbf{x}(s, t)$ iff its geodesic torsion vanishes, i.e., $\tau_g^\beta(s) = 0$.

Definition 2.3. [17] We call $(\mathbf{x}, \phi_1, \phi_2) : U \rightarrow \mathbb{R}^3 \times \Delta$ a *framed surface* if $\mathbf{x}_s(s, t) \cdot \phi_1(s, t) = \mathbf{x}_t(s, t) \cdot \phi_1(s, t) = 0$ for all $(s, t) \in U$. Moreover, $(\mathbf{x}, \phi_1) : U \rightarrow \mathbb{R}^3 \times S^2$ is called a *Legendre surface*. If a Legendre surface $(\mathbf{x}, \phi_1)$ is an immersion, $(\mathbf{x}, \phi_1)$ is called a *Legendre immersion* and $\mathbf{x}$ is called a *front*.

Note that $\phi_3(s, t) = \phi_1(s, t) \times \phi_2(s, t)$. Then we can get a moving frame $\{\phi_1(s, t), \phi_2(s, t), \phi_3(s, t)\}$ along $\mathbf{x}(s, t)$. Consequently, we get the equations as follows:

$$\begin{pmatrix} \mathbf{x}_s(s, t) \\ \mathbf{x}_t(s, t) \end{pmatrix} = \begin{pmatrix} a_1(s, t) & b_1(s, t) \\ a_2(s, t) & b_2(s, t) \end{pmatrix} \begin{pmatrix} \phi_2(s, t) \\ \phi_3(s, t) \end{pmatrix},$$

$$\begin{pmatrix} \phi_{1s}(s, t) \\ \phi_{2s}(s, t) \\ \phi_{3s}(s, t) \end{pmatrix} = \begin{pmatrix} 0 & e_1(s, t) & f_1(s, t) \\ -e_1(s, t) & 0 & g_1(s, t) \\ -f_1(s, t) & -g_1(s, t) & 0 \end{pmatrix} \begin{pmatrix} \phi_1(s, t) \\ \phi_2(s, t) \\ \phi_3(s, t) \end{pmatrix},$$

$$\begin{pmatrix} \phi_{1t}(s, t) \\ \phi_{2t}(s, t) \\ \phi_{3t}(s, t) \end{pmatrix} = \begin{pmatrix} 0 & e_2(s, t) & f_2(s, t) \\ -e_2(s, t) & 0 & g_2(s, t) \\ -f_2(s, t) & -g_2(s, t) & 0 \end{pmatrix} \begin{pmatrix} \phi_1(s, t) \\ \phi_2(s, t) \\ \phi_3(s, t) \end{pmatrix}.$$

The basic invariants $a_i, b_i, e_i, f_i, g_i : U \rightarrow \mathbb{R}$, $i = 1, 2$ are smooth functions. From the integrability conditions of $(\mathbf{x}, \phi_1, \phi_2)$, these basic invariants have the equations given below:

$$\begin{cases} a_{1t} - b_1g_2 = a_{2s} - b_2g_1, \\ b_{1t} - a_2g_1 = b_{2s} - a_1g_2, \\ a_1e_2 + b_1f_2 = a_2e_1 + b_2f_1, \end{cases} \quad \begin{cases} e_{1t} - f_1g_2 = e_{2s} - f_2g_1, \\ f_{1t} - e_2g_1 = f_{2s} - e_1g_2, \\ g_{1t} - e_1f_2 = g_{2s} - e_2f_1. \end{cases} \quad (2.1)$$

The curvature of a framed surface is defined by $C^F = (J^F, K^F, H^F)$, where

$$J^F = \det \begin{pmatrix} a_1(s, t) & b_1(s, t) \\ a_2(s, t) & b_2(s, t) \end{pmatrix}, \quad K^F = \det \begin{pmatrix} e_1(s, t) & f_1(s, t) \\ e_2(s, t) & f_2(s, t) \end{pmatrix},$$

$$H^F = -\frac{1}{2} \left\{ \det \begin{pmatrix} a_1(s, t) & f_1(s, t) \\ a_2(s, t) & f_2(s, t) \end{pmatrix} - \det \begin{pmatrix} b_1(s, t) & e_1(s, t) \\ b_2(s, t) & e_2(s, t) \end{pmatrix} \right\}.$$

3. The generalized Smarandache ruled surfaces

Definition 3.1. Suppose that (γ, ν_1) is a Legendre curve. Note that $\{\gamma(s), \nu_1(s), \nu_2(s)\}$ is the Legendre frame of $\gamma(s)$. The surface $\mathbf{x}(s, t) : U \rightarrow \mathbb{R}^3$ defined by

$$\mathbf{x}(s, t) = \frac{\nu_1(s) + \nu_2(s)}{\sqrt{2}} + t\gamma(s), \quad (3.1)$$

is called a *generalized Smarandache ruled surface*, where $\beta(s) = (\nu_1(s) + \nu_2(s))/\sqrt{2}$ is the base curve and $\gamma(s)$ is the director curve.

Proposition 3.2. If $\gamma : I \rightarrow S^2$ is a regular unit-speed curve with the geodesic curvature $\kappa_g(s)$, then $\mathbf{x}(s, t) : U \rightarrow \mathbb{R}^3$ generated by $\gamma : I \rightarrow S^2$ is called a *regular Smarandache ruled surface*. Let (γ, ν_1) be a Legendre immersion and $(m(s), n(s))$ be the curvature. If $\gamma : I \rightarrow S^2$ is parameterized by arc length, $\kappa_g(s)$ and $(m(s), n(s))$ need to satisfy $\kappa_g(s) = -n(s)$ for all $s \in I$ with $m(s) = 1$.

Proposition 3.3. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface. The first and second fundamental forms of $\mathbf{x}(s, t)$, the Gaussian and mean curvature of $\mathbf{x}(s, t)$, are given as follows:

$$\begin{aligned} I(s, t) &= (t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s) + \frac{1}{2})ds^2 - \sqrt{2}dsdt + dt^2, \\ II(s, t) &= \frac{(2\kappa_g(s)(t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s)) + \kappa_g(s) + \sqrt{2}t\dot{\kappa}_g(s))ds^2 - 2\sqrt{2}\kappa_g(s)dsdt}{2\sqrt{t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s)}}, \\ K(s, t) &= -\frac{\kappa_g^2(s)}{2(t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s))^2}, \\ H(s, t) &= \frac{-\kappa_g(s) + 2\kappa_g(s)t^2 - 2\sqrt{2}\kappa_g^2(s)t + 2\kappa_g^3(s) + \sqrt{2}\dot{\kappa}_g(s)t}{4(t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s))^{\frac{3}{2}}}, \end{aligned}$$

respectively, where $\kappa_g(s)$ is the geodesic curvature of $\gamma(s)$ on S^2 .

Proof. By Proposition 3.2, let $\gamma(s)$ be parameterized by arc length. We obtain $\kappa_g(s) = -n(s)$ and $m(s) = 1$. Therefore, we can get

$$\dot{\gamma}(s) = \nu_2(s), \quad \dot{\nu}_1(s) = -\kappa_g(s)\nu_2(s), \quad \dot{\nu}_2(s) = -\gamma(s) + \kappa_g(s)\nu_1(s).$$

Then

$$\begin{aligned} \mathbf{x}_s(s, t) &= -\frac{\gamma(s)}{\sqrt{2}} + \frac{\kappa_g(s)}{\sqrt{2}}\nu_1(s) + (t - \frac{\kappa_g(s)}{\sqrt{2}})\nu_2(s), \\ \mathbf{x}_t(s, t) &= \gamma(s), \\ \mathbf{x}_{ss}(s, t) &= -(t - \frac{\kappa_g(s)}{\sqrt{2}})\gamma(s) + (-\frac{1}{\sqrt{2}} - \frac{\dot{\kappa}_g(s)}{\sqrt{2}} - \frac{\kappa_g^2(s)}{\sqrt{2}})\nu_2(s) \\ &\quad + (\frac{\dot{\kappa}_g(s)}{\sqrt{2}} + \kappa_g(s)(t - \frac{\kappa_g(s)}{\sqrt{2}}))\nu_1(s), \\ \mathbf{x}_{st}(s, t) &= \nu_2(s), \\ \mathbf{x}_{tt}(s, t) &= 0. \end{aligned}$$

Then the normal of $\mathbf{x}(s, t)$ is given as

$$\begin{aligned} \mathbf{n}(s, t) &= \frac{\mathbf{x}_s(s, t) \times \mathbf{x}_t(s, t)}{\|\mathbf{x}_s(s, t) \times \mathbf{x}_t(s, t)\|} \\ &= \frac{(\sqrt{2}t - \kappa_g(s))\nu_1(s) - \kappa_g(s)\nu_2(s)}{\sqrt{2(t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s))}}. \end{aligned}$$

By the orthogonality relations $\gamma(s) \cdot \nu_1(s) = 0$, $\gamma(s) \cdot \nu_2(s) = 0$, and $\nu_1(s) \cdot \nu_2(s) = 0$, we obtain the following components:

$$\begin{aligned} E(s, t) &= \mathbf{x}_s(s, t) \cdot \mathbf{x}_s(s, t) = t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s) + \frac{1}{2}, \\ F(s, t) &= \mathbf{x}_s(s, t) \cdot \mathbf{x}_t(s, t) = -\frac{1}{\sqrt{2}}, \\ G(s, t) &= \mathbf{x}_t(s, t) \cdot \mathbf{x}_t(s, t) = 1, \\ L(s, t) &= \mathbf{x}_{ss}(s, t) \cdot \mathbf{n}(s, t) = \frac{2\kappa_g(s)(t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s)) + \kappa_g(s) + \sqrt{2}\dot{\kappa}_g(s)t}{2\sqrt{t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s)}}, \\ M(s, t) &= \mathbf{x}_{st}(s, t) \cdot \mathbf{n}(s, t) = \frac{-\kappa_g(s)}{\sqrt{2(t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s))}}, \\ N(s, t) &= \mathbf{x}_{tt}(s, t) \cdot \mathbf{n}(s, t) = 0. \end{aligned}$$

Then, we get

$$\begin{aligned} I(s, t) &= E(s, t)ds^2 + 2F(s, t)dsdt + G(s, t)dt^2 \\ &= (t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s) + \frac{1}{2})ds^2 - \sqrt{2}dsdt + dt^2, \\ II(s, t) &= L(s, t)ds^2 + 2M(s, t)dsdt + N(s, t)dt^2 \\ &= \frac{(2\kappa_g(s)(t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s)) + \kappa_g(s) + \sqrt{2}t\dot{\kappa}_g(s))ds^2 - 2\sqrt{2}\kappa_g(s)dsdt}{2\sqrt{t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s)}}, \\ K(s, t) &= \frac{-M^2(s, t)}{E(s, t) - F^2(s, t)} = -\frac{\kappa_g^2(s)}{2(t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s))^2}, \\ H(s, t) &= \frac{L(s, t) - 2F(s, t)M(s, t)}{2(E(s, t) - F^2(s, t))} = \frac{-\kappa_g(s) + 2\kappa_g(s)t^2 - 2\sqrt{2}\kappa_g^2(s)t + 2\kappa_g^3(s) + \sqrt{2}\dot{\kappa}_g(s)t}{4(t^2 - \sqrt{2}t\kappa_g(s) + \kappa_g^2(s))^{\frac{3}{2}}}. \end{aligned}$$

□

Corollary 3.4. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface. If $\mathbf{x}(s, t)$ is non-developable, then all its regular points are hyperbolic.

Proposition 3.5. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface and $\gamma(s)$ is the director curve. Then $\mathbf{x}(s, t)$ is developable iff $\gamma(s)$ is part of a great circle.

Proposition 3.6. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface and $\gamma(s)$ is the director curve. If $\gamma(s)$ is part of a great circle, then $\mathbf{x}(s, t)$ is minimal.

Remark 3.7. If $\kappa_g(s) = 0$, that is, $\gamma(s)$ is part of a great circle, $\mathbf{x}(s, t)$ is not only developable but also minimal. Then $\mathbf{x}(s, t)$ can be regarded as a plane.

Proposition 3.8. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface with the regular unit-speed curve $\gamma(s)$, then we have the following:

- (1) The regular Smarandache ruled surface $\mathbf{x}(s, t)$ is not cylindrical.
 (2) The regular Smarandache ruled surface $\mathbf{x}(s, t)$ is tangent developable iff $\kappa_g(s) = 0$.

Proof. (1) By [23], $\mathbf{x}(s, t) = \boldsymbol{\beta}(s) + t\boldsymbol{\gamma}(s)$ is cylindrical exactly when its director curve $\boldsymbol{\gamma}(s)$ is parallel to a fixed vector, i.e., $\dot{\boldsymbol{\gamma}}(s) = 0$. But $\dot{\boldsymbol{\gamma}}(s) = \boldsymbol{\nu}_2(s) \in S^2$. Thus $\mathbf{x}(s, t)$ is not cylindrical.

(2) Let $\boldsymbol{\beta}(s) = (\boldsymbol{\nu}_1(s) + \boldsymbol{\nu}_2(s))/\sqrt{2}$ be the base curve of $\mathbf{x}(s, t)$. We can obtain

$$\begin{aligned}\dot{\boldsymbol{\beta}}(s) &= \frac{\dot{\boldsymbol{\nu}}_1(s) + \dot{\boldsymbol{\nu}}_2(s)}{\sqrt{2}} \\ &= \frac{-\boldsymbol{\gamma}(s) + \kappa_g(s)\boldsymbol{\nu}_1(s) - \kappa_g(s)\boldsymbol{\nu}_2(s)}{\sqrt{2}}.\end{aligned}$$

When $\kappa_g(s) = 0$, we get $\dot{\boldsymbol{\beta}}(s) = (-1/\sqrt{2})\boldsymbol{\gamma}(s)$, then $\dot{\boldsymbol{\beta}}(s)$ is parallel to $\boldsymbol{\gamma}(s)$. Hence the surface is a tangent developable. $\square$

Next we consider the characteristic of the base curve $\boldsymbol{\beta}(s)$ on $\mathbf{x}(s, t)$.

Theorem 3.9. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface and $\boldsymbol{\beta}(s)$ is the base curve. The curvature functions $\kappa_n^\beta(s)$, $\kappa_g^\beta(s)$, and $\tau_g^\beta(s)$ of $\boldsymbol{\beta}(s)$ are given as follows:

$$\begin{aligned}\kappa_n^\beta(s) &= -1, \\ \kappa_g^\beta(s) &= -\frac{2\kappa_g^2(s) + 2\dot{\kappa}_g(s) + 1}{(2\kappa_g^2(s) + 1)^{\frac{3}{2}}}, \\ \tau_g^\beta(s) &= 0,\end{aligned}$$

respectively, where $\kappa_g(s)$ is the geodesic curvature of $\boldsymbol{\gamma}(s)$ on S^2 .

Proof. For the base curve $\boldsymbol{\beta}(s) = (\boldsymbol{\nu}_1(s) + \boldsymbol{\nu}_2(s))/\sqrt{2}$, by differentiating, we obtain

$$\begin{aligned}\dot{\boldsymbol{\beta}}(s) &= \frac{\kappa_g(s)(\boldsymbol{\nu}_1(s) - \boldsymbol{\nu}_2(s)) - \boldsymbol{\gamma}(s)}{\sqrt{2}}, \\ \ddot{\boldsymbol{\beta}}(s) &= \frac{\kappa_g(s)\boldsymbol{\gamma}(s) + (\dot{\kappa}_g(s) - \kappa_g^2(s))\boldsymbol{\nu}_1(s) - (\dot{\kappa}_g(s) + \kappa_g^2(s) + 1)\boldsymbol{\nu}_2(s)}{\sqrt{2}}.\end{aligned}$$

From the proof of Proposition 3.3, the unit normal of $\mathbf{x}(s, t)$ along $\boldsymbol{\beta}(s)$ at $t = 0$ is taken as

$$\mathbf{n}_\beta(s) = \frac{\boldsymbol{\nu}_1(s) + \boldsymbol{\nu}_2(s)}{\sqrt{2}}.$$

And we also get

$$\dot{\mathbf{n}}_\beta(s) = \frac{\kappa_g(s)(\boldsymbol{\nu}_1(s) - \boldsymbol{\nu}_2(s)) - \boldsymbol{\gamma}(s)}{\sqrt{2}}.$$

Substituting the above results into the curvature formulas yields

$$\begin{aligned}\kappa_n^\beta(s) &= \frac{\ddot{\beta}(s) \cdot \mathbf{n}_\beta(s)}{\|\dot{\beta}(s)\|^2} = -1, \\ \kappa_g^\beta(s) &= \frac{\det(\dot{\beta}(s), \ddot{\beta}(s), \mathbf{n}_\beta(s))}{\|\dot{\beta}(s)\|^3} = -\frac{2\kappa_g^2(s) + 2\dot{\kappa}_g(s) + 1}{(2\kappa_g^2(s) + 1)^{\frac{3}{2}}}, \\ \tau_g^\beta(s) &= \frac{(\mathbf{n}_\beta(s) \times \dot{\mathbf{n}}_\beta(s)) \cdot \dot{\beta}(s)}{\|\dot{\beta}(s)\|^2} = 0.\end{aligned}$$

□

Proposition 3.10. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface. $\beta(s)$ is the base curve, and $\gamma(s)$ is the director curve with geodesic curvature $\kappa_g(s)$. Then $\beta(s)$ cannot be asymptotic.

Proposition 3.11. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface. $\beta(s)$ is the base curve, and $\gamma(s)$ is the director curve with geodesic curvature $\kappa_g(s)$. Then $\beta(s)$ is geodesic iff $\dot{\kappa}_g(s) = -(\kappa_g^2(s) + 1/2)$.

Proposition 3.12. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface. $\beta(s)$ is the base curve, and $\gamma(s)$ is the director curve with geodesic curvature $\kappa_g(s)$. Then $\beta(s)$ must be a principal line.

Proposition 3.13. Suppose that $\mathbf{x}(s, t)$ is a regular Smarandache ruled surface and $\beta(s)$ is the base curve. Then $\beta(s)$ is the striction curve of $\mathbf{x}(s, t)$ iff $\kappa_g(s) = 0$.

Proof. For $\mathbf{x}(s, t) = \beta(s) + t\gamma(s)$, the striction parameter can given by

$$\begin{aligned}\Gamma(s) &= \beta(s) - \frac{\dot{\beta}(s) \cdot \dot{\gamma}(s)}{\dot{\gamma}(s) \cdot \dot{\gamma}(s)} \gamma(s) \\ &= \beta(s) - \frac{(-\gamma(s) + \kappa_g(s)\mathbf{v}_1(s) - \kappa_g(s)\mathbf{v}_2(s)) \cdot \mathbf{v}_2(s)}{\sqrt{2}} \gamma(s) \\ &= \beta(s) + \frac{\kappa_g(s)}{\sqrt{2}} \gamma(s).\end{aligned}$$

Therefore, the base curve $\beta(s)$ is a striction curve when $\kappa_g(s) = 0$.

□

Now we consider the singular points of a generalized Smarandache ruled surface.

Theorem 3.14. Suppose that $\mathbf{x}(s, t)$ is a generalized Smarandache ruled surface with director curve $\gamma(s)$ and curvature functions $m(s)$ and $n(s)$. The singular set of $\mathbf{x}(s, t)$ is defined as

$$S = \{(s, t) \in I \times \mathbb{R} \mid n(s) = 0, tm(s) = 0\}.$$

Proof. Differentiating (3.1) partially, we obtain

$$\begin{aligned}\mathbf{x}_s(s, t) &= -\frac{m(s)}{\sqrt{2}} \gamma(s) + \left(\frac{n(s)}{\sqrt{2}} + tm(s)\right) \mathbf{v}_2(s) - \frac{n(s)}{\sqrt{2}} \mathbf{v}_1(s), \\ \mathbf{x}_t(s, t) &= \gamma(s).\end{aligned}\tag{3.2}$$

From Eq (3.2), we get

$$\mathbf{x}_s(s, t) \times \mathbf{x}_t(s, t) = \frac{n(s)}{\sqrt{2}} \mathbf{v}_2(s) + \left(\frac{n(s)}{\sqrt{2}} + tm(s)\right) \mathbf{v}_1(s).\tag{3.3}$$

The singularities of $\mathbf{x}(s, t)$ are constructed from the set S .

□

The singular set S naturally splits into two disjoint classes according to whether $m(s)$ is zero or not. If $m(s) \neq 0$, we have $tm(s) = 0$, yielding

$$S_1 = \{(s, 0) \in I \times \mathbb{R} \mid n(s) = 0, m(s) \neq 0\}.$$

If $m(s) = 0$, the condition $tm(s) = 0$ holds for all t , yielding

$$S_2 = \{(s, t) \in I \times \mathbb{R} \mid n(s) = 0, m(s) = 0\}.$$

4. Generalized Smarandache framed ruled surfaces

Definition 4.1. Let (γ, ν_1) be a Legendre curve with curvature functions $m(s)$ and $n(s)$, and let $\mathbf{x}(s, t)$ be the generalized Smarandache ruled surface generated by (γ, ν_1) . Suppose that $\mathbf{x}(s, t)$ admits only S_2 singularities and that $m(s) = o(n(s))$ near every S_2 singularity. We call $(\mathbf{x}, \phi_1, \phi_2) : U \rightarrow \mathbb{R}^3 \times \Delta$ a generalized Smarandache framed ruled surface if there exists a smooth function $\sigma(s, t) : U \rightarrow \mathbb{R}$ satisfying

$$(tm(s) + \frac{n(s)}{\sqrt{2}}) \sin \sigma(s, t) = \frac{n(s)}{\sqrt{2}} \cos \sigma(s, t). \quad (4.1)$$

The moving frame $\{\phi_1, \phi_2, \phi_3\}$ of $\mathbf{x}(s, t)$ is defined by

$$\begin{aligned} \phi_1(s, t) &= \nu_1(s) \cos \sigma(s, t) + \nu_2(s) \sin \sigma(s, t), \\ \phi_2(s, t) &= \nu_1(s) \sin \sigma(s, t) - \nu_2(s) \cos \sigma(s, t), \\ \phi_3(s, t) &= \gamma(s). \end{aligned}$$

Remark 4.2. The function $\sigma(s, t)$ represents the rotation angle of the moving frame $\{\phi_1, \phi_2\}$ in the plane spanned by $\nu_1(s)$ and $\nu_2(s)$, which is perpendicular to $\gamma(s)$. To construct a moving frame adapted to the surface, $\phi_1(s, t)$ must satisfy

$$\phi_1(s, t) \cdot \mathbf{x}_s(s, t) = \phi_1(s, t) \cdot \mathbf{x}_t(s, t) = 0.$$

By Eq (3.2), we get

$$(tm(s) + \frac{n(s)}{\sqrt{2}}) \sin \sigma(s, t) = \frac{n(s)}{\sqrt{2}} \cos \sigma(s, t).$$

When $(n(s), n(s) + \sqrt{2}tm(s)) \neq (0, 0)$, we obtain

$$\cos \sigma(s, t) = \frac{n(s) + \sqrt{2}tm(s)}{\sqrt{n^2(s) + (n(s) + \sqrt{2}tm(s))^2}}, \quad \sin \sigma(s, t) = \frac{n(s)}{\sqrt{n^2(s) + (n(s) + \sqrt{2}tm(s))^2}}. \quad (4.2)$$

Remark 4.3. The smoothness of $\sigma(s, t)$ is verified as follows.

- (1) At regular points, $(n(s), n(s) + \sqrt{2}tm(s)) \neq (0, 0)$ holds. By Eq (4.2), $\cos \sigma(s, t)$ and $\sin \sigma(s, t)$ are smooth, hence $\sigma(s, t)$ is smooth.

- (2) For any $(s_0, t_0) \in S_2$, we consider the limit of $\sigma(s, t)$ as regular points approach (s_0, t_0) . Since $m(s) = o(n(s))$ near S_2 , for any regular point with $n(s) \neq 0$, Eq (4.2) gives

$$\lim_{(s,t) \rightarrow (s_0,t_0)} \tan \sigma(s, t) = \lim_{(s,t) \rightarrow (s_0,t_0)} \frac{n(s)}{n(s) + \sqrt{2} t m(s)} = 1,$$

independent of the path. Since $\sigma(s, 0) = \pi/4$ by (4.1) and $\sigma(s, t)$ is continuous in the regular region, the branch of $\sigma(s, t)$ is fixed. We obtain a smooth extension by defining $\sigma(s, t) = \pi/4$ on S_2 .

- (3) At $(s_0, 0) \in S_1$, $n(s_0) = 0$ and $m(s_0) \neq 0$ hold. Consider the following two paths:

- Fix $s = s_0$ and let $t \rightarrow 0$. For $t \neq 0$, the point (s_0, t) is regular and $\sqrt{2}tm(s_0) \neq 0$.

$$\lim_{t \rightarrow 0} \tan \sigma(s_0, t) = \lim_{t \rightarrow 0} \frac{n(s_0)}{n(s_0) + \sqrt{2} t m(s_0)} = 0.$$

- Fix $t = 0$ and let $s \rightarrow s_0$. For $s \neq s_0$, the point $(s, 0)$ is regular and $n(s) \neq 0$.

$$\lim_{s \rightarrow s_0} \tan \sigma(s, 0) = \lim_{s \rightarrow s_0} \frac{n(s)}{n(s) + 0} = 1.$$

Since the limits along the two paths differ, $\sigma(s, t)$ cannot be smoothly extended. In conclusion, a globally smooth function $\sigma(s, t)$ exists exactly when the surface contains only S_2 singularities and $m(s) = o(n(s))$ near S_2 .

Proposition 4.4. Suppose that $(\mathbf{x}, \phi_1, \phi_2)$ is a generalized Smarandache framed ruled surface generated from the Legendre curve $\gamma(s)$. The basic invariants of $(\mathbf{x}, \phi_1, \phi_2)$ are given below.

$$\begin{aligned} a_1(s, t) &= -\frac{n(s)}{\sqrt{2}} \sin \sigma(s, t) - (tm(s) + \frac{n(s)}{\sqrt{2}}) \cos \sigma(s, t), & a_2(s, t) &= 0, \\ b_1(s, t) &= -\frac{m(s)}{\sqrt{2}}, & b_2(s, t) &= 1, \\ e_1(s, t) &= -(\sigma_s(s, t) + \sigma_t(s, t)), & e_2(s, t) &= -\sigma_t(s, t), \\ f_1(s, t) &= -m(s) \sin \sigma(s, t), & f_2(s, t) &= 0, \\ g_1(s, t) &= m(s) \cos \sigma(s, t), & g_2(s, t) &= 0. \end{aligned}$$

Corollary 4.5. The curvature $C^F = (J^F, K^F, H^F)$ of the generalized Smarandache framed ruled surface is given as follows.

$$\begin{aligned} J^F(s, t) &= -\frac{n(s)}{\sqrt{2}} \sin \sigma(s, t) - (tm(s) + \frac{n(s)}{\sqrt{2}}) \cos \sigma(s, t), \\ K^F(s, t) &= -\sigma_t(s, t)m(s) \sin \sigma(s, t), \\ H^F(s, t) &= \frac{1}{2} \left(\frac{\sigma_t(s, t)m(s)}{\sqrt{2}} + \sigma_s(s, t) + n(s) \right). \end{aligned}$$

Corollary 4.6. Suppose that $(\mathbf{x}, \phi_1, \phi_2)$ is a generalized Smarandache framed ruled surface obtained by the above construction.

(1) If $(n(s_0), n(s_0) + \sqrt{2}tm(s_0)) \neq (0, 0)$, then $\mathbf{x}(s, t)$ is regular at (s_0, t_0) . Especially, $\mathbf{x}(s, t)$ is a front at (s_0, t_0) .

(2) If $n(s_0) = 0, m(s_0) = 0$, then $(\mathbf{x}, \phi_1)$ is a Legendre immersion at (s_0, t_0) iff $\sigma_s(s_0, t_0) \neq 0$.

Proof. (1) From Eq (3.3), $\mathbf{x}(s, t)$ is regular at (s_0, t_0) iff $(n(s_0), n(s_0) + \sqrt{2}tm(s_0)) \neq (0, 0)$.

(2) By Corollary 4.5 and the assumption $n(s_0) = 0, m(s_0) = 0$, we obtain $J^F(s_0, t_0) = K^F(s_0, t_0) = 0$ and $H^F(s_0, t_0) = \sigma_s(s_0, t_0)$. Since $(\mathbf{x}, \phi_1)$ is a Legendre immersion at (s_0, t_0) exactly when $C^F(s_0, t_0) \neq 0$ (see [17]), therefore, $(\mathbf{x}, \phi_1)$ is a Legendre immersion at (s_0, t_0) exactly when $H^F(s_0, t_0) \neq 0$, that is, $\sigma_s(s_0, t_0) \neq 0$. $\square$

5. Singular points of generalized Smarandache ruled surfaces

This section is devoted to establishing a criterion for the singularities of generalized Smarandache ruled surfaces in $\mathbb{R}^3$.

Theorem 5.1. Suppose that $\mathbf{x}(s, t)$ is a generalized smarandache ruled surface. Then $p = (s_0, t_0) \in S$ is a corank 1 singular point of $\mathbf{x}(s, t)$.

Proof. From Eq (3.2), we get

$$\begin{pmatrix} \mathbf{x}_s(s_0, t_0) \\ \mathbf{x}_t(s_0, t_0) \end{pmatrix} = \begin{pmatrix} -m(s_0)/\sqrt{2} & -n(s_0)/\sqrt{2} & vm(s_0) + n(s_0)/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma(s_0) \\ \nu_1(s_0) \\ \nu_2(s_0) \end{pmatrix}.$$

Therefore, $\mathbf{x}(s, t)$ only has a corank 1 singular point. $\square$

5.1. Non-degenerate and degenerate singular points

This subsection classifies the singular points of the generalized Smarandache framed ruled surfaces defined in Section 4. Since these surfaces have only S_2 singularities, the analysis below is limited to this class.

Suppose that $(\mathbf{x}, \phi_1, \phi_2)$ is a generalized Smarandache framed ruled surface obtained from a Legendre curve and $p = (s_0, t_0) \in S_2$ is a corank 1 singularity. Then we have $n(s) = 0, m(s) = 0$. The discriminant function $\lambda : I \times J \rightarrow \mathbb{R}$ is given as follows:

$$\begin{aligned} \lambda(p) &= \det(\mathbf{x}_s, \mathbf{x}_t, \phi_1)(p) \\ &= (tm(s_0) + \frac{n(s_0)}{\sqrt{2}}) \cos \sigma + \frac{n(s_0)}{\sqrt{2}} \sin \sigma(s_0, t_0), \\ \lambda_s(p) &= (t\dot{m}(s_0) + \frac{\dot{n}(s_0)}{\sqrt{2}}) \cos \sigma(s_0, t_0) + \frac{\dot{n}(s_0)}{\sqrt{2}} \sin \sigma(s_0, t_0) \\ &= \dot{n}(s_0) \sin(\sigma(s_0, t_0) + \frac{\pi}{4}) + t\dot{m}(s_0) \cos \sigma(s_0, t_0), \\ \lambda_t(p) &= m(s_0) \cos \sigma(s_0, t_0) = 0. \end{aligned}$$

Consequently, p is a degenerate singularity of $\mathbf{x}(s, t)$ exactly when $\dot{n}(s_0) = 0$ and $t\dot{m}(s_0) = 0$.

Suppose that $p \in S_2$ is a non-degenerate singularity of $\mathbf{x}(s, t)$. We denote that $\delta(u) = (s_0, u)$ is the singular curve and $\eta(u) = \partial/\partial s$ is the vector field. Let $\widehat{\phi}_1(s, t)$ be the normal vector field which does not vanish around p .

Theorem 5.2. Let $\mathbf{x}(s, t)$ be a generalized Smarandache framed ruled surface. Suppose that $p \in S_2$ is a non-degenerate singularity of $\mathbf{x}(s, t)$. Then $\mathbf{x}(s, t)$ has only first-kind singularities.

Proof. Since the singular point $p = (s_0, t_0) \in S_2$, it satisfies $n(s) = 0$ and $m(s) = 0$. Then we obtain

$$\begin{aligned} d\mathbf{x}(p) &= \left(-\frac{m(s_0)}{\sqrt{2}} \boldsymbol{\gamma}(s_0) + \left(\frac{n(s_0)}{\sqrt{2}} + t_0 m(s_0) \right) \boldsymbol{\nu}_2(s_0) - \frac{n(s_0)}{\sqrt{2}} \boldsymbol{\nu}_1(s_0) \right) ds + \boldsymbol{\gamma}(s_0) dt \\ &= \boldsymbol{\gamma}(s_0) dt. \end{aligned}$$

Thus, the null vector field of $\mathbf{x}(s, t)$ is given by

$$\boldsymbol{\eta}(u) = \frac{\partial}{\partial s}.$$

Consequently, $\boldsymbol{\eta}\lambda(s, t) = \lambda_s(s, t)$. Since p is a non-degenerate singularity iff $\lambda_s(p) \neq 0$, it follows that $\boldsymbol{\eta}\lambda(p) \neq 0$. By the definition of singularities of the k -th type, the non-degenerate singularity p of the surface $\mathbf{x}(s, t)$ is of the first type. $\square$

Cuspidal edges and cuspidal cross caps are typical singularities in $\mathbb{R}^3$. In particular, since $\mathbf{x}(s, t)$ has only first-kind singularity, it does not possess any swallowtail singularities.

Theorem 5.3. Let $\mathbf{x}(s, t)$ be a generalized Smarandache framed ruled surface. If $p = (s_0, t_0) \in S_2$ is a non-degenerate singularity of $\mathbf{x}(s, t)$, the following results hold.

- (1) p is a cuspidal edge singularity iff $\sigma_s(p) \neq 0$.
- (2) p is a cuspidal cross cap singularity iff $\sigma_s(p) = 0$ and $\sigma_{st}(p) \neq 0$.

Proof. According to calculation, we can obtain

$$\begin{aligned} \phi_{\mathbf{x}}(u) &= \det((\mathbf{x} \circ \delta)'(u), \widehat{\boldsymbol{\phi}}_1 \circ \delta(u), d\widehat{\boldsymbol{\phi}}_1(\boldsymbol{\eta})(u)) \\ &= \det(\mathbf{x}_t(s_0, u), \widehat{\boldsymbol{\phi}}_1(s_0, u), \widehat{\boldsymbol{\phi}}_{1s}(s_0, u)) \\ &= \sigma_s(s_0, u) \cos^2 \sigma(s_0, u) + \sigma_s(s_0, u) \sin^2 \sigma(s_0, u) \\ &= \sigma_s(s_0, u). \end{aligned}$$

(1) By the definition of cuspidal edge (see [15]), the singularity p is a cuspidal edge exactly when $\mathbf{x}(s, t)$ is a front near p and $\boldsymbol{\eta}\lambda(p) \neq 0$. Since p is a non-degenerate singularity, it satisfies $\lambda_s(p) = \boldsymbol{\eta}\lambda(p) \neq 0$. Moreover, since $\boldsymbol{\eta}\lambda(p) \neq 0$, $\mathbf{x}(s, t)$ is a front near p iff $\phi_{\mathbf{x}}(u) \neq 0$, that is, $\sigma_s(p) \neq 0$. Therefore, p is a cuspidal edge singularity iff $\sigma_s(p) \neq 0$.

(2) Since $\phi_{\mathbf{x}}(u) = \sigma_s(s_0, u)$, we obtain $\phi'_{\mathbf{x}}(u) = \sigma_{st}(s_0, u)$ under the assumption $\phi_{\mathbf{x}}(t_0) = 0$. It follows that $\phi'_{\mathbf{x}}(t_0) \neq 0$ exactly when $\sigma_{st}(p) \neq 0$. By the definition of cuspidal cross cap singularity, $\mathbf{x}(s, t)$ is a cuspidal cross cap at p iff $\boldsymbol{\eta}\lambda(p) \neq 0$, $\phi_{\mathbf{x}}(p) = 0$, and $\phi'_{\mathbf{x}}(p) \neq 0$. Therefore, p is a cuspidal cross cap singularity iff $\sigma_s(p) = 0$ and $\sigma_{st}(p) \neq 0$. $\square$

Example 1. Suppose that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1) : (-0.5, 0.5) \rightarrow \Delta \subset S^2 \times S^2$ is a Legendre curve defined by

$$\boldsymbol{\gamma}(s) = (\cos \sigma_1(s) \cos \sigma_2(s), \sin \sigma_1(s) \cos \sigma_2(s), \sin \sigma_2(s)), \quad \sigma_1(s) = s^3, \quad \sigma_2(s) = s^3 + s^5.$$

By differentiating, we obtain

$$\dot{\boldsymbol{\gamma}}(s) = 3s^2 \cos \sigma_2(s) \mathbf{e}_1 + (3s^2 + 5s^4) \mathbf{e}_2,$$

where

$$\begin{aligned} \mathbf{e}_1 &= (-\sin \sigma_1(s), \cos \sigma_1(s), 0), \\ \mathbf{e}_2 &= (-\cos \sigma_1(s) \sin \sigma_2(s), -\sin \sigma_1(s) \sin \sigma_2(s), \cos \sigma_2(s)). \end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ can be obtained by rotating $\{\mathbf{e}_1, \mathbf{e}_2\}$ so that $\mathbf{v}_1(s) \cdot \dot{\boldsymbol{\gamma}}(s) = 0$. It follows that

$$\begin{aligned} \mathbf{v}_1(s) &= \frac{(3 + 5s^2)\mathbf{e}_1 - 3 \cos \sigma_2(s)\mathbf{e}_2}{\sqrt{(3 + 5s^2)^2 + (3 \cos \sigma_2(s))^2}}, \\ \mathbf{v}_2(s) &= \frac{3 \cos \sigma_2(s)\mathbf{e}_1 + (3 + 5s^2)\mathbf{e}_2}{\sqrt{(3 + 5s^2)^2 + (3 \cos \sigma_2(s))^2}}. \end{aligned}$$

Since $m(s) = \dot{\boldsymbol{\gamma}}(s) \cdot \mathbf{v}_2(s)$, $n(s) = \dot{\mathbf{v}}_1(s) \cdot \mathbf{v}_2(s)$, we obtain

$$\begin{aligned} m(s) &= s^2 \sqrt{(3 + 5s^2)^2 + (3 \cos \sigma_2(s))^2}, \\ n(s) &= 3s^2 \sin \sigma_2(s) + \frac{3s^2(3 + 5s^2)^2 \sin \sigma_2(s) + 30s \cos \sigma_2(s)}{(3 + 5s^2)^2 + (3 \cos \sigma_2(s))^2}. \end{aligned}$$

Differentiating $m(s)$ and $n(s)$ with respect to s gives

$$\begin{aligned} \dot{m}(s) &= 2s(\sqrt{(3 + 5s^2)^2 + (3 \cos \sigma_2(s))^2}) + s^2 \dot{D}(s), \\ \dot{n}(s) &= 6s \sin \sigma_2 + 3s^2(3s^2 + 5s^4) \cos \sigma_2 + \frac{\dot{B}(s)A(s) - B(s)\dot{A}(s)}{A^2(s)}, \end{aligned}$$

where

$$\begin{aligned} A(s) &= (3 + 5s^2)^2 + (3 \cos \sigma_2)^2, \quad \dot{A}(s) = 20s(3 + 5s^2) - 18(3s^2 + 5s^4) \cos \sigma_2 \sin \sigma_2, \\ B(s) &= 3s^2(3 + 5s^2)^2 \sin \sigma_2 + 30s \cos \sigma_2, \\ \dot{B}(s) &= 6s(3 + 5s^2)^2 \sin \sigma_2 + 3s^2 \cdot 2(3 + 5s^2) \cdot 10s \sin \sigma_2 \\ &\quad + 3s^2(3 + 5s^2)^2(3s^2 + 5s^4) \cos \sigma_2 + 30 \cos \sigma_2 - 30s(3s^2 + 5s^4) \sin \sigma_2. \end{aligned}$$

Next we consider the generalized Smarandache framed ruled surface

$$\mathbf{x}_1(s, t) = \frac{\mathbf{v}_1(s) + \mathbf{v}_2(s)}{\sqrt{2}} + t\boldsymbol{\gamma}(s).$$

Assuming $s = 0$, it follows that $m(0) = n(0) = 0$, that is, $(0, 0) \in S_2$. Since $\dot{n}(0) = 5/3$ and $\dot{m}(0) = 0$, we have $\lambda_s(0, 0) \neq 0$. Therefore, $(0, 0)$ is a non-degenerate singularity of $\mathbf{x}_1(s, t)$. Moreover, $(0, 0)$ satisfies $\boldsymbol{\eta}\lambda(0, 0) \neq 0$. For a generalized Smarandache framed ruled surface having only S_2 singularities, $\sigma(s, t)$ must satisfy Eq (4.1). Then we have $\sigma(s, 0) = \pi/4$ under the assumption $t = 0$. By continuity, it follows that $\sigma(0, 0) = \pi/4$. Denote Eq (4.1) as $F(s, t) = 0$, where

$$F(s, t) = (tm(s) + \frac{n(s)}{\sqrt{2}}) \sin \sigma(s, t) - \frac{n(s)}{\sqrt{2}} \cos \sigma(s, t).$$

Differentiate $F(s, t)$ implicitly and evaluate at $(0, 0)$. Since

$$m(s) = n(s) = 0, \quad \dot{m}(0) = 0, \quad \dot{n}(0) = 5/3, \quad \ddot{m}(0) = 6\sqrt{2}, \quad \sigma(0, 0) = \pi/4,$$

we have

$$\begin{aligned}
 F_s(0, 0) &= \frac{\dot{n}(0)}{\sqrt{2}}(\sin \sigma(0, 0) - \cos \sigma(0, 0)) \\
 &= \frac{5}{3\sqrt{2}}(\sin \frac{\pi}{4} - \cos \frac{\pi}{4}), \\
 F_{ss}(0, 0) &= \frac{2\dot{n}(0)}{\sqrt{2}}(\cos \sigma(0, 0) + \sin \sigma(0, 0))\sigma_s(0, 0) \\
 &= \frac{10}{3\sqrt{2}}(\cos \frac{\pi}{4} + \sin \frac{\pi}{4})\sigma_s(0, 0), \\
 F_{sst}(0, 0) &= \ddot{m}(0) \sin \sigma(0, 0) + \frac{10}{3}\sigma_{st}(0, 0) \\
 &= 6\sqrt{2} \sin \frac{\pi}{4} + \frac{10}{3}\sigma_{st}(0, 0).
 \end{aligned}$$

Moreover, since $F_{ss}(0, 0) = F_{sst}(0, 0) = 0$, we get $\sigma_s(0, 0) = 0$ and $\sigma_{st}(0, 0) = -9/5$. Substitution into $F_s(0, 0)$ gives an identity and does not determine the derivatives. By Theorem 5.3, p is a cuspidal cross cap singularity iff $\sigma_s(p) = 0$ and $\sigma_{st}(p) \neq 0$. Therefore $(0, 0)$ is a cuspidal cross cap. The graph of $\mathbf{x}_1(s, t)$ is given by Figure 1.

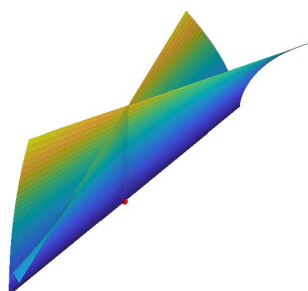


Figure 1. A generalized Smarandache framed ruled surface with a cuspidal cross cap singularity (red point).

The following discussion focuses on singularities of the cuspidal lip and cuspidal beak type.

Proposition 5.4. Suppose that $p = (s_0, t_0) \in S_2$ is a corank 1 degenerate singularity of the generalized Smarandache framed ruled surface $\mathbf{x}(s, t)$. Then $\mathbf{x}(s, t)$ has neither cuspidal lips nor cuspidal beaks.

Proof. We obtain $n(s_0) = m(s_0) = 0$ and $\dot{n}(s_0) = \dot{m}(s_0) = 0$ because $p = (s_0, t_0)$ is a degenerate singularity. And we have

$$\begin{aligned}
 \lambda_{ss}(p) &= \ddot{n}(s_0) \sin(\sigma(s_0, t_0) + \frac{\pi}{4}) + \ddot{m}(s_0) \cos \sigma(s_0, t_0), \\
 \lambda_{st}(p) &= \dot{m}(s_0) \cos \sigma(s_0, t_0), \quad \lambda_{tt}(p) = 0, \\
 \det \text{Hess} \lambda(p) &= \lambda_{ss}(p)\lambda_{tt}(p) - \lambda_{st}^2(p) \\
 &= -(\dot{m}(s_0) \cos \sigma(s_0, t_0))^2.
 \end{aligned}$$

According to the integrability conditions of the framed surfaces, the fundamental quantities of the

generalized Smarandache framed ruled surface $\mathbf{x}(s, t)$ must satisfy Eq (2.1). Hence, we obtain

$$\sigma_t(s, t) \left((tm(s) + \frac{n(s)}{\sqrt{2}}) \cos \sigma(s, t) + \frac{n(s) \sin \sigma(s, t)}{\sqrt{2}} \right) = -m(s) \sin \sigma(s, t).$$

Therefore

$$\sigma_t(s, t) \lambda(s, t) = -m(s) \sin \sigma(s, t). \quad (5.1)$$

Differentiation of (5.1) with respect to t yields

$$\sigma_{st}(s, t) \lambda(s, t) + \sigma_t(s, t) \lambda_s(s, t) = -\dot{m}(s) \sin \sigma(s, t).$$

Moreover, the smooth function $\sigma(s, t)$ needs to satisfy Eq (4.1). Differentiating (4.1) twice partially with respect to t gives

$$t\ddot{m}(s) \sin \sigma(s, t) + t\dot{m}(s) \sin \sigma(s, t) = \ddot{n}(s) \sin \sigma(s, t).$$

Since p is a degenerate singularity of $\mathbf{x}(s, t)$, $\lambda(p) = 0$ and $\lambda_s(p) = 0$ hold. Then we can obtain $-\dot{m}(s_0) \sin \sigma(s_0, t_0) = 0$.

(1) Assuming $\dot{m}(s_0) = 0$, we have $\det \text{Hess} \lambda(p) = 0$.

(2) Assuming $\dot{m}(s_0) \neq 0$, we have $t = 0$, $\sin \sigma(s_0, t_0) = 0$, and $\cos \sigma(s_0, t_0) = \pm 1$. Since $t = 0$ and $-\dot{m}(s_0) \sin \sigma(s_0, t_0) = 0$, we obtain $\ddot{n}(s_0) \sin(\pi/4 - \sigma(s_0, t_0)) = 0$ and $\det \text{Hess} \lambda(p) < 0$. Since $\sin \sigma(s_0, t_0) = 0$ and $\cos \sigma(s_0, t_0) = \pm 1$, it follows that $\ddot{n}(s_0) = 0$. Consequently,

$$\eta \eta \lambda(p) = \ddot{n}(s_0) \sin(\frac{\pi}{4} + \sigma(s_0, t_0)) + t\ddot{m}(s_0) \cos \sigma(s_0, t_0) = 0.$$

Therefore, by [24], $\mathbf{x}(s, t)$ has neither cuspidal lips nor cuspidal beaks. $\square$

5.2. Chen-Matsumoto-Mond $\pm$ and cross cap singular points

In contrast to the generalized Smarandache framed ruled surfaces considered in the previous subsection, we now consider generalized Smarandache ruled surfaces with singularities of both classes:

$$S_1 = \{(s, 0) \mid n(s) = 0, m(s) \neq 0\}, \quad S_2 = \{(s, t) \mid n(s) = 0, m(s) = 0\}.$$

Theorem 5.5. *Let $\mathbf{x}(s, t)$ be a generalized Smarandache ruled surface. Suppose that $S = S_1 \cup S_2$ and that $p = (s_0, t_0) \in S$ is a singularity of $\mathbf{x}(s, t)$ of corank 1. Then the classification is given as follows.*

- (1) *If $p \in S_1$ and $\dot{n}(s_0) \neq 0$, then p is a cross cap S_0 singularity.*
- (2) *If $p \in S_1$ and $\dot{n}(s_0) = 0$, $\ddot{n}(s_0) \neq 0$, then p is an S_1^+ singularity.*
- (3) *If $p \in S_2$, p is neither an $S_1^\pm$ (Chen-Matsumoto-Mond $\pm$) nor a cross cap S_0 singularity.*

Proof. We define the function $\varphi(s, t) = \det(\xi \mathbf{x}, \eta \mathbf{x}, \eta \eta \mathbf{x})(s, t)$, where

$$\xi = \frac{\partial}{\partial t}, \quad \eta = \frac{\partial}{\partial s} + \frac{m(s)}{\sqrt{2}} \frac{\partial}{\partial t}.$$

Then, by Eq (3.2), we obtain

$$\begin{aligned}
\mathbf{x}_{ss}(s, t) &= \left(-\frac{\dot{m}(s)}{\sqrt{2}} - \frac{m(s)n(s)}{\sqrt{2}} - t m^2(s)\right) \boldsymbol{\gamma}(s) + \left(-\frac{\dot{n}(s)}{\sqrt{2}} - \frac{n^2(s)}{\sqrt{2}} - t m(s)n(s)\right) \mathbf{v}_1(s) \\
&\quad + \left(\frac{\dot{n}(s)}{\sqrt{2}} + t \dot{m}(s) - \frac{m^2(s) + n^2(s)}{\sqrt{2}}\right) \mathbf{v}_2(s), \\
\varphi(s, t) &= \det(\xi \mathbf{x}, \eta \mathbf{x}, \eta \eta \mathbf{x})(s, t) \\
&= \det(\mathbf{x}_t, \mathbf{x}_s, \mathbf{x}_{ss})(s, t) \\
&= (-n^3(s) + \frac{n(s)m^2(s)}{2} - \sqrt{2}tm(s)n^2(s) - t^2m^2(s)n(s) \\
&\quad + t \frac{n(s)\dot{m}(s) - m(s)\dot{n}(s)}{\sqrt{2}}) \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)), \\
\varphi_t(s, t) &= (-\sqrt{2}m(s)n^2(s) - 2tm^2(s)n(s) + \frac{n(s)\dot{m}(s) - m(s)\dot{n}(s)}{\sqrt{2}}) \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)), \\
\varphi_{tt}(s, t) &= -2m^2(s)n(s) \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)), \\
\varphi_s(s, t) &= (-3n^2(s)\dot{n}(s) + \frac{1}{2}m^2(s)\dot{n}(s) + n(s)m(s)\dot{m}(s) \\
&\quad + t(-\sqrt{2}\dot{m}(s)n^2(s) - 2\sqrt{2}m(s)n(s)\dot{n}(s) + \frac{n(s)\ddot{m}(s) - m(s)\ddot{n}(s)}{\sqrt{2}}) \\
&\quad + t^2(-2m(s)n(s)\dot{m}(s) - m^2(s)\dot{n}(s))) \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)), \\
\varphi_{ss}(s, t) &= ((-6n(s) - 2\sqrt{2}tm(s))\dot{n}^2(s) + n(s)(1 - 2t^2)\dot{m}^2(s) \\
&\quad + (2m(s) - 4\sqrt{2}n(s)t - 4m(s)t^2)\dot{n}(s)\dot{m}(s) \\
&\quad + (-3n^2(s) + \frac{1}{2}m^2(s) - 2\sqrt{2}m(s)n(s)t - m^2(s)t^2)\ddot{n}(s) \\
&\quad + n(s)(m(s) - \sqrt{2}n(s)t - 2m(s)t^2)\ddot{m}(s) \\
&\quad + \frac{t}{\sqrt{2}}(\dot{n}(s)\ddot{m}(s) + n(s)\ddot{m}(s) - \dot{m}(s)\ddot{n}(s) - m(s)\ddot{n}(s))) \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)), \\
\varphi_{st}(s, t) &= (-\sqrt{2}(n^2(s)\dot{m}(s) + 2m(s)n(s)\dot{n}(s)) \\
&\quad - 2t(2m(s)n(s)\dot{m}(s) + m^2(s)\dot{n}(s)) + \frac{n(s)\ddot{m}(s) - m(s)\ddot{n}(s)}{\sqrt{2}}) \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)).
\end{aligned}$$

If $p \in S_1$, we have $n(s_0) = 0$, $m(s_0) \neq 0$, and $t = 0$. It follows that

$$\begin{aligned}
\varphi_s(p) &= \frac{m^2(s_0)\dot{n}(s_0)}{2} \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)), \\
\varphi_t(p) &= -\frac{m(s_0)\dot{n}(s_0)}{2} \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)), \\
\varphi_{ss}(p) &= (2m(s_0)\dot{n}(s_0)\dot{m}(s_0) + \frac{m^2(s_0)\ddot{n}(s_0)}{2}) \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)), \\
\varphi_{tt}(p) &= 0, \\
\varphi_{st}(p) &= -\frac{m^2(s_0)\ddot{n}(s_0)}{2} \det(\boldsymbol{\gamma}(s), \mathbf{v}_2(s), \mathbf{v}_1(s)).
\end{aligned}$$

Hence, by [25], p is a cross cap S_0 singularity exactly when $\dot{n}(s_0) \neq 0$. Moreover, assuming $\dot{n}(s_0) = 0$, then we get

$$\begin{aligned}\det(\text{Hess}\varphi(p)) &= \varphi_{ss}(p)\varphi_{tt}(p) - \varphi_{st}^2(p) = -\frac{m^2(s_0)\ddot{n}(s_0)}{2}, \\ \xi\mathbf{x}(p) &= \boldsymbol{\gamma}(s_0), \\ \boldsymbol{\eta}\boldsymbol{\eta}\mathbf{x}(p) &= -(tm(s_0) + \frac{n(s_0)}{\sqrt{2}})m(s_0)\boldsymbol{\gamma}(s_0) - (\frac{\dot{n}(s_0)}{\sqrt{2}} + tm(s_0)n(s_0) + \frac{n^2(s_0)}{\sqrt{2}})\boldsymbol{\nu}_1(s_0) \\ &\quad + (\frac{m^2(s_0)}{\sqrt{2}} - \frac{n^2(s_0)}{\sqrt{2}} + tm(s_0) + \frac{\dot{n}(s_0)}{\sqrt{2}})\boldsymbol{\nu}_2(s_0), \\ &= \frac{m^2(s_0)\boldsymbol{\nu}_2(s_0)}{\sqrt{2}}.\end{aligned}$$

Since $m(s_0) \neq 0$, we have that $\xi\mathbf{x}(p)$ and $\boldsymbol{\eta}\boldsymbol{\eta}\mathbf{x}(p)$ are linearly independent. Hence, p is an S_1^+ singularity exactly when $\dot{n}(s_0) = 0$ and $\ddot{n}(s_0) \neq 0$.

If $p \in S_2$, we have $n(s_0) = 0$ and $m(s_0) = 0$. It follows that $\varphi_s(p) = \varphi_t(p) = 0$ and $\det(\text{Hess}\varphi(p)) = 0$. Therefore, If $p \in U_2$, then p is not an $S_1^\pm$ or a cross cap S_0 singularity. $\square$

Example 2. Suppose that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1) : I \rightarrow \Delta \subset S^2 \times S^2$ is a Legendre curve defined by

$$\boldsymbol{\gamma}(s) = (\cos f_1(s), \sin f_1(s) \cos f_2(s), \sin f_1(s) \sin f_2(s)), \quad f_1(s) = \frac{s^2}{2} - s + \frac{\pi}{2}, \quad f_2(s) = \frac{s^5}{5} - \frac{s^4}{4}.$$

By differentiating and normalizing $\boldsymbol{\gamma}(s)$, we obtain

$$\boldsymbol{\nu}_2(s) = \frac{\dot{f}_1(s)}{\|\dot{\boldsymbol{\gamma}}(s)\|} \mathbf{e}_1 + \frac{\dot{f}_2(s) \sin f_1(s)}{\|\dot{\boldsymbol{\gamma}}(s)\|} \mathbf{e}_2,$$

where

$$\begin{aligned}\mathbf{e}_1 &= (0, -\sin f_1(s) \sin f_2(s), \sin f_1(s) \cos f_2(s)), \\ \mathbf{e}_2 &= (-\sin f_1(s), \cos f_1(s) \cos f_2(s), \cos f_1(s) \sin f_2(s)).\end{aligned}$$

Then a smooth function $\sigma(s) : I \rightarrow \mathbb{R}$ exists that satisfies

$$\cos \sigma(s) = \frac{s-1}{\|\dot{\boldsymbol{\gamma}}(s)\|}, \quad \sin \sigma(s) = \frac{(s^4 - s^3) \sin f_1(s)}{\|\dot{\boldsymbol{\gamma}}(s)\|}. \quad (5.2)$$

Therefore, the orthonormal frame $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2\}$ is given by

$$\begin{aligned}\boldsymbol{\nu}_1(s) &= \sin \sigma(s) \mathbf{e}_1 - \cos \sigma(s) \mathbf{e}_2, \\ \boldsymbol{\nu}_2(s) &= \cos \sigma(s) \mathbf{e}_1 + \sin \sigma(s) \mathbf{e}_2.\end{aligned}$$

From $m(s) = \dot{\boldsymbol{\gamma}}(s) \cdot \boldsymbol{\nu}_2(s)$, $n(s) = \dot{\boldsymbol{\nu}}_1(s) \cdot \boldsymbol{\nu}_2(s)$, we obtain

$$\begin{aligned}m(s) &= |s-1| \sqrt{1 + s^6 \sin^2 f_1(s)}, \\ n(s) &= \dot{\sigma}(s) - (s^4 - s^3) \sin f_1(s) \cos f_1(s).\end{aligned}$$

From the expressions for $m(s)$, we have

$$\dot{m}(s) = -\sqrt{1 + s^6 \sin^2 f_1(s)} + (1 - s) \frac{6s^5 \sin^2 f_1(s) + 2s^6(s - 1) \sin f_1(s) \cos f_1(s)}{2\sqrt{1 + s^6 \sin^2 f_1(s)}}.$$

According to Eq (5.2), we obtain

$$\tan \sigma(s) = \frac{(s^4 - s^3) \sin f_1(s)}{s - 1}.$$

By the expansions near $s = 0$, we get

$$\tan \sigma(s) = \frac{(s^4 - s^3)(1 - \frac{1}{2}s^2 + O(s^4))}{s - 1} = s^3 - \frac{1}{2}s^5 + O(s^6).$$

It follows that

$$\begin{aligned}\sigma(s) &= s^3 - \frac{1}{2}s^5 + O(s^7), \\ \dot{\sigma}(s) &= 3s^2 - \frac{5}{2}s^4 + O(s^6).\end{aligned}$$

Since near $s = 0$,

$$(s^4 - s^3) \sin f_1(s) \cos f_1(s) = (s^4 - s^3)(1 - \frac{1}{2}s^2)(s - \frac{1}{2}s^2) + O(s^6) = -s^4 + \frac{3}{2}s^5 + O(s^6).$$

Therefore

$$n(s) = (3s^2 - \frac{5}{2}s^4 + O(s^6)) - (-s^4 + \frac{3}{2}s^5 + O(s^6)) = 3s^2 - \frac{3}{2}s^4 - \frac{3}{2}s^5 + O(s^6).$$

Differentiating gives

$$\dot{n}(s) = 6s - 6s^3 - \frac{15}{2}s^4 + O(s^5), \quad \ddot{n}(s) = 6 - 18s^2 - 30s^3 + O(s^4).$$

Next we consider the generalized Smarandache ruled surface

$$\mathbf{x}_2(s, t) = \frac{\mathbf{v}_1(s) + \mathbf{v}_2(s)}{\sqrt{2}} + t\boldsymbol{\gamma}(s).$$

We show that $(s, t) = (0, 0)$ is an S_1^+ singularity of $\mathbf{x}_2(s, t)$. First, the origin $(0, 0)$ is a corank 1 singularity. Moreover, we obtain $m(0) = 1$, $n(0) = 0$, $\dot{n}(0) = 0$, and $\ddot{n}(0) = 6$. By the above calculation, we have

$$\det(\text{Hess}(\varphi))(0, 0) = -\frac{1}{2}m(s)^2\ddot{n}(s) = -18 < 0.$$

By [25] and Theorem 5.5, p is an S_1^+ singularity exactly when $\dot{n}(s_0) = 0$, $\ddot{n}(s_0) \neq 0$. Therefore, $(0, 0)$ is an S_1^+ singularity of $\mathbf{x}_2(s, t)$. Figure 2 shows the graph of $\mathbf{x}_2(s, t)$.

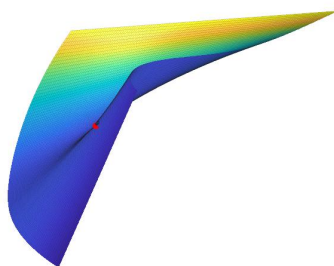


Figure 2. A generalized Smarandache ruled surface with an S_1^+ singular point (red point).

6. Conclusions

This paper studies a class of generalized Smarandache ruled surfaces in $\mathbb{R}^3$ from the theory of singularity. We first explore the relation between developability, minimality, and the Legendre curve. The surface is developable exactly when the Legendre curve is part of a great circle. If it is also minimal, it degenerates to a plane. A difficulty is that when both singularity classes S_1 and S_2 coexist, no global smooth unit normal vector field exists, so classical tools cannot be applied. Therefore, we restrict to surfaces with only S_2 singularities that satisfy the required order condition on the curvature functions. Then we construct a unit normal vector field via a smooth function. Within this framework, we give conditions for the first class singularities to be cross caps or S_1^+ and for the second class to be locally cuspidal edges or cuspidal cross caps.

The above results reveal how the curvature functions of the Legendre curve are intrinsically linked to the developability, minimality, and singularities of the surface. They successfully extend the classical Smarandache theory to the singular case. Although this paper focuses on the geometric properties of the Smarandache surface family with one representative coefficient, choosing more general coefficients may lead to richer geometric structures. In fact, using the analytical framework established in this paper, we can further study the singularity classification and developability of surfaces with general coefficients, as well as the deeper properties of the base curve. This is undoubtedly a theoretically meaningful and interesting issue. Extending these results to more general surfaces and higher-dimensional spaces will be a promising direction for future work.

Author contributions

Rong Zhao was responsible for conceptualization, methodology, original draft writing, and review and editing; Donghe Pei was responsible for review, editing, and funding acquisition.

All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors state that this article was created without the use of Artificial Intelligence tools.

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Conflict of interest

All authors state that there is no conflict of interest associated with this work.

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