
Research article

t-Semi-Lipschitz functions and t-best approximation in fuzzy quasi-metric spaces

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Abstract: In this paper, we introduced the concept of a t-semi-Lipschitz function defined on a fuzzy quasi-metric space with values in a fuzzy quasi-normed space. We showed that the set of such functions vanishing at a fixed point forms a quasi-normed cone. By means of t-semi-Lipschitz functions, we characterized the t-nearest points to a point from a subset and provided a formula for the corresponding nearest distance, specifically within an important subclass of fuzzy quasi-metric spaces known as fuzzy quasi-ultrametric spaces. The obtained results provide powerful tools for the qualitative description and quantitative analysis of best approximation problems in fuzzy environments.

Keywords: semi-Lipschitz function; fuzzy quasi-metric space; fuzzy quasi-normed space; best approximation

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1. Introduction

The Lipschitz condition can be viewed as a weakened form of differentiability; therefore, Lipschitz functions are useful substitutes for smooth functions in the framework of metric spaces. They appear in many branches of mathematics, such as differential equations, measure theory, nonlinear functional analysis, topology, metric geometry, and fractal theory, e.g., see [1]. The applications of Lipschitz functions, particularly in computer science, have also undergone rapid development, for example, in machine learning [2,3], algorithm design and analysis [4], and large margin classification [5]. Recently, Backusa and Ze-An [6] studied the Lipschitz extension problem with prescribed local Lipschitz constants and eikonal mappings on a complete Riemannian manifold.

In 2000, by extending the domain of Lipschitz functions to quasi-metric spaces, the concept of a semi-Lipschitz function was introduced in [7,8]. Romaguera and Sanchis [7] endowed the set of real-valued semi-Lipschitz functions with a structure of normed cone and characterized both the points of the best approximation and the semi-Chebyshevian subsets of a quasi-metric space. Meanwhile, Mustata [8] studied the uniqueness of the extension of semi-Lipschitz functions on quasi-metric spaces. Subsequent research on semi-Lipschitz functions has yielded a series of results. For example, Mustata proved an extension theorem and presented a procedure to obtain extremal elements of the unit ball of the quasi-normed semilinear space in [9,10], respectively. Romaguera and Sanchis obtained several properties of the normed cone of semi-Lipschitz functions in [11] and showed that semi-Lipschitz utility functions provide an efficient tool for interpreting the processes that appear in some relevant examples of theoretical computer science in [12]. Using the theory of semi-Lipschitz functions, Falciani and Sanchez-Perez [13] proposed an asymmetric topological approach for reinforcement learning on graph-based structures. For applications of semi-Lipschitz functions in best approximation, see [14] and references therein.

In 1975, Kramosil and Michalek [15] introduced the concept of fuzzy metric, now known as the KM-fuzzy metric. Later, to generate Hausdorff topology, George and Veeramani introduced another form of fuzzy metric [16] (see also [17,18]), now called the GV-fuzzy metric. In fact, under certain conditions, the GV-fuzzy metric can be regarded as a special case of the KM-fuzzy metric (e.g., [19]). In 2004, asymmetric generalizations of fuzzy metrics—namely, fuzzy quasi-metrics—were provided in [20] (see also [21,22]). Subsequent research in this area led to significant progress. For example, Gao et al. [23,24] investigated the connections between fuzzy quasi-metric spaces and the associated order structures of formal balls. Gregori et al. [25] presented a characterization of bicompletable fuzzy quasi-metric spaces. Nadaban and Dzitac [26] showed that a fuzzy quasi-pseudo-metric can be considered on the domain of words and algorithm complexity analysis. Recently, Tang and Wu [27] offered an order relation in fuzzy quasi-metric space and its applications in optimization. Jiménez-Fernandez et al. [28] presented an extended version of McShane–Whitney extensions for fuzzy Lipschitz maps.

The purpose of this paper is to extend the theory of Lipschitz functions to the framework of fuzzy quasi-metrics. We first introduce the definition of a t -semi-Lipschitz function from a fuzzy quasi-metric space to a fuzzy quasi-normed space; to the best of our knowledge, this definition is new. We then prove that the collection of all such functions vanishing at a fixed point constitutes a quasi-normed cone. Finally, as an application of the obtained results, we employ t -semi-Lipschitz functions to provide a characterization of t -nearest points, specifically within an important subclass of fuzzy quasi-metric spaces, known as fuzzy quasi-ultrametric spaces. In this way, our results generalize the corresponding findings for quasi-metric spaces in [7,14] to the fuzzy quasi-metric setting.

The paper is organized as follows: After this introductory section, Section 2 recalls some foundations on fuzzy quasi-metric spaces and introduces the concept of a fuzzy quasi-ultrametric space, together with two examples. Section 3 introduces t -semi-Lipschitz functions on fuzzy quasi-metric spaces and establishes their fundamental properties; in particular, it proves that the set of t -semi-Lipschitz functions vanishing at a fixed point forms a quasi-normed cone. Section 4 addresses best approximation problems, utilizing the existence of t -semi-Lipschitz functions to characterize nearest points. Section 5 concludes the paper.

In this paper, $\mathbb{R}$ and $\mathbb{N}$ denote the set of real numbers and positive integers, respectively. $\emptyset$ denotes the empty set, X is a nonempty set. A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t -norm [29]. Two paradigmatic examples of continuous t -norms are the minimum operator “ $\wedge$ ” and

the product operator “ $\cdot$ ”, which are defined by $a \wedge b = \min\{a, b\}$ and $a \cdot b = ab$, respectively.

2. Preliminaries

This section recalls some concepts and results needed for the sequel and gives several examples to illustrate the concepts.

Definition 2.1. [20] A GV-fuzzy quasi-metric on a set X is a pair $(M, *)$, where $*$ is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0, +\infty)$, satisfying for all $x, y, z \in X$ and $t, s > 0$:

$$(GV1) \quad M(x, y, t) > 0;$$

$$(GV2) \quad M(x, y, t) = M(y, x, t) = 1 \text{ for all } t > 0 \text{ if and only if } x = y;$$

$$(GV3) \quad M(x, y, t + s) \geq M(x, z, t) * M(z, y, s);$$

$$(GV4) \quad M(x, y, _): (0, +\infty) \rightarrow (0, 1] \text{ is continuous.}$$

In this case, a triple $(X, M, *)$ is called a GV-fuzzy quasi-metric space.

A GV-fuzzy quasi-metric $(M, *)$, satisfying the symmetry condition $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$, is called a fuzzy metric in the sense of George and Veeramani [16].

In this paper, a GV-fuzzy quasi-metric $(M, *)$ is simply called a fuzzy quasi-metric and simplified as M unless the continuous t-norm $*$ is specified.

If M is a fuzzy quasi-metric, then the function $M(x, y, _)$ is nondecreasing for each $x, y \in X$. Moreover, $\overline{M}$ and M^s are fuzzy quasi-metric and fuzzy metric, respectively, where $\overline{M}$ and M^s are defined by $\overline{M}(x, y, t) = M(y, x, t)$, $M^s(x, y, t) = M(x, y, t) \wedge \overline{M}(x, y, t)$ for all $x, y \in X$, $t > 0$. We refer to $\overline{M}$ and M^s as the conjugate of M and the fuzzy metric induced by M , respectively.

Each fuzzy quasi-metric M on X induces a topology τ_M , which has as a base the family of open balls

$$\mathcal{B}(x) = \{B_M(x, r, t) : r \in (0, 1), t > 0\}$$

at $x \in X$, where

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A sequence $\{x_n\}$ converges to x with respect to τ_M if and only if $\lim_{n \rightarrow \infty} M(x, x_n, t) = 1$ for any $t > 0$.

In [30], Gregori et al. introduced the concepts of fuzzy strong metrics and fuzzy ultrametrics (in the sense of Kramosil and Michalek [15]). In [31], fuzzy ultrametrics were also called non-Archimedean fuzzy metrics. We now present the asymmetric generalizations of these concepts as follows.

Definition 2.2. A fuzzy quasi-metric $(M, *)$ on X is said to be strong if it satisfies

$$(GV5) \quad M(x, y, t) \geq M(x, z, t) * M(z, y, t)$$

for each $x, y, z \in X$, and each $t > 0$.

If $(M, \wedge)$ is strong, then it is called a fuzzy quasi-ultrametric.

Remark 2.1. In general, (GV5) does not imply (GV3). However, it is easy to see that if $M(x, y, _)$ is

nondecreasing for each $x, y \in X$, then (GV5) is equivalent to $M(x, y, t \vee s) \geq M(x, z, t) * M(z, y, s)$, $\forall x, y, z \in X$, $\forall s, t > 0$. In this case, (GV5) implies (GV3). Thus, to show that $(X, M, *)$ is a fuzzy quasi-ultrametric on X , it is sufficient to verify that $(M, *)$ satisfies (GV1), (GV2), (GV4), and (GV5), and that $M(x, y, _)$ is nondecreasing for each $x, y \in X$.

We now provide an example of a fuzzy quasi-ultrametric space.

Example 2.1. Let $\{x_n\}$ and $\{y_n\}$ be two strictly increasing sequences of positive real numbers converging to 1 with respect to the Euclidean metric, with $A \cap B = \emptyset$, where $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$. Put $X = A \cup B$ and define a fuzzy set M on $X \times X \times (0, \infty)$ by

$$\begin{aligned} M(x_n, x_n, t) &= M(y_n, y_n, t) = 1 \text{ for all } n \in \mathbb{N}, t > 0; \\ M(x_n, x_m, t) &= x_n \text{ for all } m, n \in \mathbb{N} \text{ with } m \neq n, t > 0; \\ M(y_n, y_m, t) &= y_n \text{ for all } m, n \in \mathbb{N} \text{ with } m \neq n, t > 0; \\ M(x_n, y_m, t) &= x_n \text{ and } M(y_m, x_n, t) = y_m \text{ for all } m, n \in \mathbb{N}, t \geq 1; \\ M(x_n, y_m, t) &= x_n \wedge t \text{ and } M(y_m, x_n, t) = y_m \wedge t \text{ for all } m, n \in \mathbb{N}, 0 < t < 1. \end{aligned}$$

Then, $(X, M, \wedge)$ is a fuzzy quasi-ultrametric space.

Proof. The proofs of (GV1), (GV2), and (GV4) are straightforward and thus omitted. Additionally, it is clear that $M(x, y, _)$ is nondecreasing for each $x, y \in X$. By Remark 2.1, to end the proof, it is sufficient to verify that M satisfies (GV5). Let $x, y, z \in X$ and $t > 0$. We distinguish eight cases.

Case 1. $x, y, z \in A$. Then,

$$\begin{aligned} M(x, y, t) &= M(x_n, x_m, t) = \begin{cases} 1, & n = m \\ x_n, & n \neq m \end{cases} \\ M(x, z, t) \wedge M(z, y, s) &= M(x_n, x_k, t) \wedge M(x_k, x_m, s) = \begin{cases} 1 & n = m = k \\ x_n \wedge x_k & n = m \neq k \\ x_n & n \neq k = m \\ x_k & n = k \neq m \\ x_n \wedge x_k & n \neq m, m \neq k, n \neq k \end{cases} \\ &\leq M(x, y, t). \end{aligned}$$

Case 2. $x, y \in A$, $z \in B$. Then,

$$\begin{aligned} M(x, y, t) &= M(x_n, x_m, t) = \begin{cases} 1, & n = m \\ x_n, & n \neq m \end{cases} \\ M(x, z, t) \wedge M(z, y, s) &= M(x_n, y_k, t) \wedge M(y_k, x_m, s) = \begin{cases} x_n \wedge y_k \wedge t, & 0 < t < 1 \\ x_n \wedge y_k, & t \geq 1 \end{cases} \\ &\leq M(x, y, t). \end{aligned}$$

Case 3. $x, z \in A$, $y \in B$.

If $0 < t < 1$, then $M(x, y, t) = M(x_n, y_m, t) = x_n \wedge t$, and

$$M(x, z, t) \wedge M(z, y, s) = M(x_n, x_k, t) \wedge M(x_k, y_m, t) = \begin{cases} x_k \wedge t, & n = k \\ x_n \wedge x_k, \wedge t & n \neq k \end{cases} \leq M(x, y, t).$$

If $t \geq 1$, then $M(x, y, t) = M(x_n, y_m, t) = x_n$, and

$$M(x, z, t) \wedge M(z, y, s) = M(x_n, x_k, t) \wedge M(x_k, y_m, t) = \begin{cases} x_k, & n = k \\ x_n \wedge x_k, & n \neq k \end{cases} \leq M(x, y, t).$$

Case 4. $x \in A$, $y, z \in B$.

If $0 < t < 1$, then $M(x, y, t) = M(x_n, y_m, t) = x_n \wedge t$, and

$$M(x, z, t) \wedge M(z, y, s) = M(x_n, y_k, t) \wedge M(y_k, y_m, t) = \begin{cases} x_n \wedge t, & m = k \\ x_n \wedge t \wedge y_k, & m \neq k \end{cases} \leq M(x, y, t).$$

If $t \geq 1$, then $M(x, y, t) = M(x_n, y_m, t) = x_n$, and

$$M(x, z, t) \wedge M(z, y, s) = M(x_n, y_k, t) \wedge M(y_k, y_m, t) = \begin{cases} x_n, & m = k \\ x_n \wedge y_k, & m \neq k \end{cases} \leq M(x, y, t).$$

By swapping A and B , the remaining four cases—Case 5: $x, y, z \in B$; Case 6: $x, y \in B$, $z \in A$; Case 7: $x, z \in B$, $y \in A$; and Case 8: $x \in B$, $y, z \in A$ —follow directly from Cases 1–4 established above. Thus, M satisfies (GV5); consequently, M is a fuzzy quasi-ultrametric. $\square$

Example 2.2. Let $\{x_n\}$ be a strictly increasing sequence in $(0, 1)$ converging to 1 with respect to the Euclidean metric. Put $X = \{x_n : n \in \mathbb{N}\} \cup \{1\}$ and define a fuzzy set M on $X \times X \times (0, \infty)$ by

$$\begin{aligned} M(x, x, t) &= 1 \text{ for all } x \in X, t > 0; \\ M(x_n, x_m, t) &= x_n \wedge \frac{t}{2} \text{ for all } m, n \in \mathbb{N}, t > 0; \\ M(1, x_n, t) &= 1 \wedge t, \quad M(x_n, 1, t) = x_n \wedge t \text{ for all } n \in \mathbb{N}, t > 0. \end{aligned}$$

Then, $(M, \wedge)$ is a fuzzy quasi-metric, but not a fuzzy quasi-ultrametric on X .

Proof. Clearly, M satisfies conditions (GV1), (GV2), and (GV4). We now verify (GV3).

Let $x, y, z \in X$ and $t, s > 0$. In case $x = y$, $y = z$, or $x = z$, it is easy to see that (GV3) holds. Suppose that $x \neq y$, $y \neq z$, and $x \neq z$. We distinguish the following four cases.

Case 1. $x = x_m$, $y = x_n$, $z = x_k$.

$$M(x, y, t+s) = x_m \wedge \frac{t+s}{2} \geq x_m \wedge t \wedge s \geq x_m \wedge \frac{t}{2} \wedge x_k \wedge \frac{s}{2} = M(x, z, t) \wedge M(z, y, s).$$

Case 2. $x = 1$, $y = x_n$, $z = x_k$.

$$M(x, y, t+s) = 1 \wedge (t+s) \geq 1 \wedge t \wedge x_k \wedge \frac{s}{2} = M(x, z, t) \wedge M(z, y, s).$$

Case 3. $x = x_m, y = 1, z = x_k$.

$$M(x, y, t+s) = x_m \wedge (t+s) \geq x_m \wedge \frac{t}{2} \wedge x_k \wedge s = M(x, z, t) \wedge M(z, y, s).$$

Case 4. $x = x_m, y = x_n, z = 1$.

$$M(x, y, t+s) = x_m \wedge \frac{t+s}{2} \geq x_m \wedge t \wedge 1 \wedge s = M(x, z, t) \wedge M(z, y, s).$$

Thus, M satisfies (GV3), and hence, it is a fuzzy quasi-metric.

Since $\{x_n\}$ is a strictly increasing sequence in $(0,1)$ converging to 1, we can take $x_m, x_n \in \{x_n : n \in \mathbb{N}\}$ with $m \neq n$ such that both x_m and x_n are >0.5 . Then, for $0.5 < t < 1$, $M(x_m, x_n, t) = x_m \wedge \frac{t}{2} < 0.5$ and $M(x_m, 1, t) \wedge M(1, x_n, t) = x_m \wedge t \wedge 1 \wedge t = x_m \wedge t > 0.5$. That is, (GV5)

does not hold. Therefore, M is not a fuzzy quasi-ultrametric on X . $\square$

Remark 2.2. The above two examples can be seen as asymmetric generalizations of Example 2 in [32] and Example 3.2 in [33], respectively.

We now give the definition of a fuzzy quasi-norm, which is an asymmetric version of a fuzzy norm [34].

Definition 2.3. Let X be a real vector space. A fuzzy quasi-norm on X is a pair $(N, *)$, where N is a fuzzy set on $X \times (0, +\infty)$ satisfying the following conditions: for all $x, y \in X$, $t, s > 0$,

$$(FQN1) \quad N(x, t) > 0,$$

$$(FQN2) \quad N(x, t) = N(-x, t) = 1 \text{ if and only if } x = \theta, \text{ where } \theta \text{ is the null element in } X,$$

$$(FQN3) \quad N(\lambda x, t) = N(x, t/\lambda) \text{ for all } \lambda > 0,$$

$$(FQN4) \quad N(x+y, t+s) \geq N(x, t) * N(y, s),$$

$$(FQN5) \quad N(x, _): (0, +\infty) \rightarrow (0, 1] \text{ is continuous,}$$

$$(FQN6) \quad \lim_{t \rightarrow \infty} N(x, t) = 1.$$

The 3-tuple $(X, N, *)$ is called a fuzzy quasi-normed space. $(N, *)$ is often simplified as N .

A fuzzy quasi-norm N on X is called a fuzzy norm [34] if $N(\lambda x, t) = N(x, t/|\lambda|)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$.

Remark 2.3. Let $(X, N, *)$ be a fuzzy quasi-normed space. Define $M_N(x, y, t) = N(y-x, t)$ for each $x, y \in X$, $t > 0$, then $(M_N, *)$ is a GV-fuzzy quasi-metric satisfying the following condition:

$$(GV6) \quad \lim_{t \rightarrow \infty} M_N(x, y, t) = 1 \text{ for each } x, y \in X.$$

The topology induced by M_N is also called the topology induced by N , denoted by τ_N .

Remark 2.4. Since the induced fuzzy quasi-metric M_N is a GV-fuzzy quasi-metric, the fuzzy quasi-norm in Definition 2.3 is also referred to as a GV-fuzzy quasi-norm. In parallel, [34] and [35] defined the KM-fuzzy norm and KM-fuzzy quasi-norm, which can induce the KM-fuzzy metric and KM-fuzzy quasi-metric, respectively.

As with fuzzy quasi-metrics, we can define the conjugate fuzzy quasi-norm $\overline{N}$ of N and the fuzzy norm N^s induced by N , respectively, where

$$\overline{N}(x, t) = N(-x, t), \quad N^s(x, t) = N(x, t) \wedge \overline{N}(x, t)$$

for each $x, y \in X$ and $t > 0$.

Example 2.3. Define N_{st} and M_{st} by

$$N_{st}(x, t) = \frac{t}{t + x \vee 0}, \quad (2.1)$$

$$M_{st}(x, y, t) = N_{st}(y - x, t) = \frac{t}{t + (y - x) \vee 0} \quad (2.2)$$

for each $x, y \in \mathbb{R}$, $t > 0$. Then, it is easy to see that $(N_{st}, *)$ and $(M_{st}, *)$ are a fuzzy quasi-norm and a fuzzy quasi-metric on $\mathbb{R}$, respectively. They are called the standard fuzzy quasi-norm and standard fuzzy quasi-metric on $\mathbb{R}$, respectively.

3. t-Semi-Lipschitz functions

We now introduce the concept of a t-semi-Lipschitz function.

Definition 3.1. Let f be a function from a fuzzy quasi-metric space $(X, M, *_X)$ to a fuzzy quasi-normed space $(Y, N, *_Y)$, $t > 0$. If there exists an $L > 0$ such that

$$N(f(y) - f(x), Lt) \geq M(x, y, t) \quad (3.1)$$

for all $x, y \in X$, then f is called a t-semi-Lipschitz function, and L is called a t-semi-Lipschitz constant for f .

The set of all t-semi-Lipschitz functions from X to Y is denoted by $\text{Slip}_{M,N}^t(X, Y)$.

Theorem 3.1. Let $(X, M, *_X)$ be a fuzzy quasi-metric space, and let $(Y, N, *_Y)$ be a fuzzy quasi-normed space, $t > 0$. Then

(1) $\lambda \text{Slip}_{M,N}^t(X, Y) \subseteq \text{Slip}_{M,N}^t(X, Y)$ for all $\lambda > 0$, that is, $\text{Slip}_{M,N}^t(X, Y)$ is a cone.

(2) If $*_Y = \wedge$, then $\text{Slip}_{M,N}^t(X, Y) + \text{Slip}_{M,N}^t(X, Y) \subseteq \text{Slip}_{M,N}^t(X, Y)$; therefore, $\text{Slip}_{M,N}^t(X, Y)$ is a convex cone.

Proof. (1) Let $f \in \text{Slip}_{M,N}^t(X, Y)$. Then, there exists an $L > 0$ such that (3.1) holds. So,

$$N(\lambda f(y) - \lambda f(x), \lambda Lt) \geq M(x, y, t) \quad \text{for all } x, y \in X,$$

yielding that $\lambda f \in \text{Slip}_{M,N}^t(X, Y)$.

(2) Suppose $f_i \in \text{Slip}_{M,N}^t(X, Y)$ ($i=1, 2$). Then, there exists an $L_i > 0$ such that

$$N(f(y) - f(x), L_i t) \geq M(x, y, t)$$

for all $x, y \in X$ ($i=1, 2$). So,

$$\begin{aligned}
& N((f_1 + f_2)(y) - (f_1 + f_2)(x), (L_1 + L_2)t) \\
& \geq N(f_1(y) - f_1(x), L_1 t) \wedge N(f_2(y) - f_2(x), L_2 t) \\
& \geq M(x, y, t)
\end{aligned}$$

for all $x, y \in X$, yielding that $f_1 + f_2 \in \text{Slip}'_{M,N}(X, Y)$. $\square$

Definition 3.2. Let f be a function from a fuzzy quasi-metric space $(X, M, *_X)$ to a fuzzy quasi-normed space $(Y, N, *_Y)$, $t > 0$. Set

$$\|f\|_{t,M,N} = \inf \left\{ s > 0 : N(f(y) - f(x), st) \geq M(x, y, t), \forall x, y \in X \right\}, \quad (3.2)$$

and it is agreed that $\inf \emptyset = \infty$. Then, $\|f\|_{t,M,N}$ is called the t-norm of f .

Remark 3.1. It follows from (3.2) that $\|f\|_{t,M,N}$ is the infimum of the t-semi-Lipschitz constants of f . Moreover, Theorem 3.2 (2) below shows that, whenever f is a t-semi-Lipschitz function, $\|f\|_{t,M,N}$ is the smallest t-semi-Lipschitz constant of f . So, this t-norm admits an interpretation analogous to the classical case. For notational brevity, and whenever no confusion can arise, we shall abbreviate $\|f\|_{t,M,N}$ as $\|f\|_t$ in the sequel.

Theorem 3.2. Let f be a function from fuzzy quasi-metric space $(X, M, *_X)$ to fuzzy quasi-normed space $(Y, N, *_Y)$.

- (1) $f \in \text{Slip}'_{M,N}(X, Y)$ if and only if $\|f\|_t < \infty$;
- (2) If $f \in \text{Slip}'_{M,N}(X, Y)$, then $\|f\|_t$ is the smallest t-semi-Lipschitz constant of f .

Proof. (1) Follows directly from Definitions 3.1 and 3.2.

To prove (2), first suppose that $f \in \text{Slip}'_{M,N}(X, Y)$. For any $\varepsilon > 0$, by (3.2), there exists $s_0 > 0$ such that $s_0 < \|f\|_t + \varepsilon$ and $N(f(y) - f(x), s_0 t) \geq M(x, y, t)$ for all $x, y \in X$. So

$$N(f(y) - f(x), (\|f\|_t + \varepsilon)t) \geq M(x, y, t).$$

By the arbitrariness of ε and (FQN5), we know

$$N(f(y) - f(x), \|f\|_t t) \geq M(x, y, t)$$

for all $x, y \in X$. So, $\|f\|_t$ is a t-semi-Lipschitz constant of f .

Now, suppose that L is another t-semi-Lipschitz constant of f . It follows from (3.2) that $\|f\|_t \leq L$. Thus, $\|f\|_t$ is the smallest t-semi-Lipschitz constant of f . $\square$

The following theorem gives the specific expression of the t-norm for functions taking values in the real number space endowed with the standard fuzzy quasi-norm. The formula is a generalization of the classical argument for semi-Lipschitz functions in quasi-metric spaces (see, e.g., [7] and [14]).

Theorem 3.3. Let $(X, M, *)$ be a fuzzy quasi-metric space and $(N_{st}, \wedge)$ be the standard fuzzy quasi-norm on $\mathbb{R}$. If f is a function from $(X, M, *)$ to $(\mathbb{R}, N_{st}, \wedge)$, then

$$\|f\|_t = \sup \left\{ \frac{M(x, y, t)((f(y) - f(x)) \vee 0)}{t(1 - M(x, y, t))} : M(x, y, t) < 1, x, y \in X \right\}. \quad (3.3)$$

Proof. Let a be the right side of (3.3).

Case 1. $a = \infty$. Then, for any $s > 0$, there are $x_s, y_s \in X$ with $M(x_s, y_s, t) < 1$ such that

$$\frac{M(x_s, y_s, t)((f(y_s) - f(x_s)) \vee 0)}{t(1 - M(x_s, y_s, t))} > s,$$

equivalently,

$$\frac{st}{st + (f(y_s) - f(x_s)) \vee 0} < M(x_s, y_s, t),$$

or

$$N_{st}(f(y_s) - f(x_s), st) < M(x_s, y_s, t).$$

That is, there is no $s > 0$ such that $N_{st}(f(y) - f(x), st) \geq M(x, y, t)$ for all $x, y \in X$. This, together with Definition 3.2, implies that $\|f\|_t = \infty$. Thus $\|f\|_t = a$.

Case 2. $a \neq \infty$. Then, for every $x, y \in X$ with $M(x, y, t) < 1$,

$$a \geq \frac{M(x, y, t)((f(y) - f(x)) \vee 0)}{t(1 - M(x, y, t))},$$

that is,

$$\frac{at}{at + (f(y) - f(x)) \vee 0} \geq M(x, y, t),$$

implying $N_{at}(f(y) - f(x), at) \geq M(x, y, t)$. So, $\|f\|_t \leq a$.

Now, we suppose that $\|f\|_t < a$. Then, there exists $s_0 > 0$ such that $s_0 < a$ and

$$\frac{s_0 t}{s_0 t + (f(y) - f(x)) \vee 0} \geq M(x, y, t) \text{ for every } x, y \in X. \text{ So,}$$

$$s_0 \geq \frac{M(x, y, t)((f(y) - f(x)) \vee 0)}{t(1 - M(x, y, t))},$$

when $M(x, y, t) < 1$. Therefore, $s_0 \geq a$, which contradicts the fact that $s_0 < a$. Thus, $\|f\|_t = a$. $\square$

Corollary 3.1. Let $(X, M, \wedge)$ be a fuzzy quasi-ultrametric space, and let f be a function from $(X, M, \wedge)$ to $(\mathbb{R}, N_{st}, \wedge)$. Then, $f \in \text{Slip}_{M, N_{st}}^t(X, \mathbb{R})$ with t -semi-Lipschitz constant L if and only if

$$f(y) - f(x) \leq L \left(\frac{1}{M(x, y, t)} - 1 \right) t \quad (3.4)$$

for all $x, y \in X$.

Proof. Suppose $f \in \text{Slip}_{M, N_{st}}^t(X, \mathbb{R})$ with t -semi-Lipschitz constant L . Then, for each $x, y \in X$, we have the inequality

$$N_{st}(f(y) - f(x), Lt) \geq M(x, y, t),$$

that is

$$\frac{Lt}{Lt + (f(y) - f(x)) \vee 0} \geq M(x, y, t),$$

implying

$$f(y) - f(x) \leq (f(y) - f(x)) \vee 0 \leq L \left(\frac{1}{M(x, y, t)} - 1 \right) t.$$

Conversely, suppose that (3.4) holds, then

$$\begin{aligned} & \sup \left\{ \frac{M(x, y, t)((f(y) - f(x)) \vee 0)}{t(1 - M(x, y, t))} : M(x, y, t) < 1, x, y \in X \right\} \\ & \leq \sup \left\{ \frac{M(x, y, t)}{t(1 - M(x, y, t))} \cdot L \left(\frac{1}{M(x, y, t)} - 1 \right) t : M(x, y, t) < 1, x, y \in X \right\} \\ & = L. \end{aligned}$$

It follows from Theorem 3.3 that $f \in \text{Slip}_{M, N_{st}}^t(X, \mathbb{R})$ with t -semi-Lipschitz constant L . $\square$

It can be seen from formula (3.2) that $\|f\|_t = 0$ for any constant function f . Furthermore, we can obtain the following more general conclusion.

Theorem 3.4. Let $(X, M, *_X)$ be a fuzzy quasi-metric space and let $(Y, N, *_Y)$ be a fuzzy quasi-normed space, $t > 0$. If $f, g \in \text{Slip}_{M, N}^t(X, Y)$ with $f - g \equiv C$ (a constant), then $\|f\|_t = \|g\|_t$.

Proof. Since $f - g \equiv C$, we have $f(y) - f(x) = g(y) - g(x)$ for all $x, y \in X$. Therefore, it follows from (3.2) that

$$\begin{aligned} \|f\|_t &= \inf \left\{ s > 0 : N(f(y) - f(x), st) \geq M(x, y, t), \forall x, y \in X \right\} \\ &= \inf \left\{ s > 0 : N(g(y) - g(x), st) \geq M(x, y, t), \forall x, y \in X \right\} \\ &= \|g\|_t. \end{aligned}$$

This completes the proof. $\square$

The following result presents certain linear properties of the t -norm.

Theorem 3.5. Let $(X, M, *_X)$ be a fuzzy quasi-metric space, and let $(Y, N, *_Y)$ be a fuzzy quasi-

normed space, $t > 0$, f and $g \in \text{Slip}_{M,N}^t(X, Y)$.

- (1) If $\rho > 0$, then $\|\rho f\|_t = \rho \|f\|_t$.
- (2) If $*_Y = \wedge$, then $\|f + g\|_t \leq \|f\|_t + \|g\|_t$.
- (3) If $\|f\|_t = 0$, $(Y, N, *_Y)$ satisfies the following condition:

$$(\text{FQN7}) \quad \lim_{t \rightarrow 0^+} N(x, t) = 0 \quad \text{for any } x \in X \setminus \{\theta\},$$

then $f \equiv C$ (a constant).

Proof. (1) By using (3.2), we obtain

$$\begin{aligned} \|\rho f\|_t &= \inf \{s > 0 : N(\rho f(y) - \rho f(x), st) \geq M(x, y, t), \forall x, y \in X\} \\ &= \inf \{s > 0 : N(f(y) - f(x), st/\rho) \geq M(x, y, t), \forall x, y \in X\} \\ &= \inf \{\rho \cdot s' > 0 : N(f(y) - f(x), s't) \geq M(x, y, t), \forall x, y \in X\} \\ &= \rho \|f\|_t. \end{aligned}$$

(2) For any $\varepsilon > 0$, it follows from (3.2) that there exist $s_1, s_2 > 0$ with $s_1 < \|f\|_t + \varepsilon/2$ and $s_2 < \|g\|_t + \varepsilon/2$ such that

$$N(f(y) - f(x), s_1 t) \geq M(x, y, t) \quad \text{and} \quad N(g(y) - g(x), s_2 t) \geq M(x, y, t)$$

for all $x, y \in X$. So

$$\begin{aligned} &N(f(y) + g(y) - f(x) - g(x), (s_1 + s_2)t) \\ &\geq N(f(y) - f(x), s_1 t) \wedge N(g(y) - g(x), s_2 t) \\ &\geq M(x, y, t), \end{aligned}$$

implying that $\|f + g\|_t \leq s_1 + s_2 < \|f\|_t + \|g\|_t + \varepsilon$. By the arbitrariness of ε , we know $\|f + g\|_t \leq \|f\|_t + \|g\|_t$.

(3) If $\|f\|_t = 0$, then $\|f\|_t < 1/n$ for all $n \in \mathbb{N}$. By (3.2), there exists $1/n > s_n > 0$ such that $N(f(y) - f(x), s_n t) \geq M(x, y, t)$ for all $x, y \in X$. So,

$$N(f(y) - f(x), t/n) \geq M(x, y, t) > 0 \tag{3.5}$$

for all $n \in \mathbb{N}$, $x, y \in X$.

Suppose that there exist $x_0, y_0 \in X$ such that $f(x_0) \neq f(y_0)$. By (FQN7), we have $\lim_{n \rightarrow \infty} N(f(y_0) - f(x_0), t/n) = 0$, which contradicts (3.5). Thus, $f(x) = f(y)$ for all $x, y \in X$, that is $f \equiv C$ (a constant). $\square$

Remark 3.2. Suppose $(X, M, *_X)$ is a fuzzy quasi-metric space and $(Y, N, \wedge)$ is a fuzzy quasi-normed space satisfying (FQN7). It follows from Theorems 3.1, 3.4 and 3.5 that $(\text{Slip}_{M,N}^t(X, Y), \|\cdot\|_t)$ is a quasi-semi-normed convex cone. Now, consider the following subspaces:

$$\text{Slip}_{M,N}^{t,\theta}(X,Y) = \{f \in \text{Slip}_{M,N}'(X,Y) : f(\theta) = 0\}.$$

Then, it is easy to verify that $(\text{Slip}_{M,N}^{t,\theta}(X,Y), \|\cdot\|_t)$ is a quasi-normed convex cone. In fact, the above conclusion still holds if the zero vector θ is replaced by any fixed vector x_0 .

4. Applications in t-best approximation

The concept of t-best approximation was initiated by Veeramani [36] in 2001, later extended to fuzzy normed spaces by Vaezpour & Karimi [37], and further studied by Abbasi & Golshan [38]. In this section, we utilize the existence of t-semi-Lipschitz functions to characterize the features of t-nearest points, specifically within fuzzy quasi-ultrametric spaces. The obtained results generalize the classical argument for semi-Lipschitz functions in quasi-metric spaces (see, e.g., [7,14]).

Definition 4.1. Let A be a nonempty subset of a fuzzy quasi-metric space $(X, M, *)$, $t > 0$. An element $p \in A$ is said to be an M -t-nearest point to $x \in X$ from A if $d_M(x, A, t) = M(x, p, t)$, where

$$d_M(x, A, t) = \sup\{M(x, y, t) : y \in A\}. \quad (4.1)$$

We denote by $P_A^M(x, t)$ the set of all M -t-nearest points to x from A . For $t > 0$, a subset A of a fuzzy quasi-metric space $(X, M, *)$ is called M -t-proximinal if for every point $x \in X$, $P_A^M(x, t) \neq \emptyset$.

Remark 4.1. Note that if $M(x, p, t) = 1$ for some $p \in A$, then p is obviously an M -t-nearest point to $x \in X$ from A . Therefore, we focus on those points $x \in X$ satisfying $M(x, p, t) < 1$ for all $p \in A$. Let

$$W_M(A, t) = \{x \in X : M(x, p, t) < 1, \forall p \in A\},$$

then $\{x \in X : d_M(x, A, t) < 1\} \subseteq W_M(A, t) \subseteq A^c (= X \setminus A)$.

Remark 4.2. Since $d_M(A, x, t) = d_{\overline{M}}(x, A, t)$, $d_M(x, A, t)$ and $d_M(A, x, t)$ have similar properties, for simplicity, the following discussion focuses on only one of these two forms, from which the corresponding conclusions for the other form can be easily derived.

We now present some elementary properties of the distance function d_M .

Theorem 4.1. Let A be a nonempty subset of a fuzzy quasi-metric space $(X, M, *)$, $x \in X$. Then,

- (1) $d_M(x, A, s) \leq d_M(x, A, t)$ whenever $0 < s \leq t$.
- (2) $x \in \text{cl}_M A$ if and only if $d_M(x, A, t) = 1$, $\forall t > 0$.
- (3) The function $d_M(x, A, _): (0, +\infty) \rightarrow [0, 1]$ is left-continuous.

Proof. (1) is obvious.

(2) If $x \in \text{cl}_M A$ and $t > 0$, then there is $\{x_n\} \subseteq A$ such that $\{x_n\}$ is convergent to x with respect to the topology τ_M . Hence, $\lim_{n \rightarrow \infty} M(x, x_n, t) = 1$. Noting that

$$d_M(x, A, t) = \sup\{M(x, y, t) : y \in A\} \geq M(x, x_n, t),$$

we know $d_M(x, A, t) = 1$.

Conversely, if $d_M(x, A, t) = 1$ for all $t > 0$, then for any $n \in \mathbb{N}$, $d_M(x, A, 1/n) > 1 - 1/n$. It follows from (4.1) that there exists $x_n \in A$ such that $M(x, x_n, 1/n) > 1 - 1/n$, that is, $x_n \in B_M(x, 1/n, 1/n)$. Since $\{B_M(x, 1/n, 1/n) : n \in \mathbb{N}\}$ is a base of the neighborhood system of x with respect to τ_M , we have $x \in \text{cl}_M A$.

(3) Let $\{t_k\}$ be a strictly increasing sequence in $\mathbb{R}^+ = (0, \infty)$ converging to $t > 0$. Since for each $k \in \mathbb{N}$, $t_k < t$, it immediately follows from (1) that $d_M(x, A, t_k) \leq d_M(x, A, t)$ for all $k \in \mathbb{N}$.

Now, take $\varepsilon \in (0, 1)$ arbitrarily. Then, there is $y_{\varepsilon, t} \in A$ such that $M(x, y_{\varepsilon, t}, t) > d_M(x, A, t) - \varepsilon$. Since $M(x, y_{\varepsilon, t}, _)$ is left-continuous, we know $\lim_{k \rightarrow \infty} M(x, y_{\varepsilon, t}, t_k) = M(x, y_{\varepsilon, t}, t) > d_M(x, A, t) - \varepsilon$. So, there is $k_0 \in \mathbb{N}$ such that $d_M(x, A, t_k) \geq M(x, y_{\varepsilon, t}, t_k) > d_M(x, A, t) - \varepsilon$ for all $k \geq k_0$. Therefore,

$$d_M(x, A, t) - \varepsilon < d_M(x, A, t_k) \leq d_M(x, A, t) \text{ for all } k \geq k_0.$$

Thus, $d_M(x, A, _)$ is left-continuous in $(0, +\infty)$. $\square$

The following Lemma provides a useful t-semi-Lipschitz function.

Lemma 4.1. Let $(X, M, \wedge)$ be a fuzzy quasi-ultrametric space, $A \subseteq X$, $x \in X$, $t > 0$, and define

$$f_t(x) = \left(\frac{1}{d_M(A, x, t)} - 1 \right) t.$$

Then,

(1) f_t is a t-semi-Lipschitz function with $\|f_t\|_t \leq 1$, and $f_t(x) = 0$ for all $x \in \text{cl}_M(A)$.

(2) If $\text{cl}_M(A) \neq X$, then there exists $t_0 > 0$ such that $\|f_{t_0}\|_{t_0} = 1$.

Proof. (1) For any $x, y \in X$, $t > 0$, since $(X, M, \wedge)$ is a fuzzy quasi-ultrametric space, $M(a, y, t) \geq M(a, x, t) \wedge M(x, y, t)$ for each $a \in A$. Therefore,

$$\begin{aligned} \frac{1}{M(a, y, t)} &\leq \frac{1}{M(a, x, t) \wedge M(x, y, t)} = \frac{1}{M(a, x, t)} \vee \frac{1}{M(x, y, t)} \\ &< \frac{1}{M(a, x, t)} + \frac{1}{M(x, y, t)} - 1, \end{aligned}$$

yielding

$$\inf_{a \in A} \left(\frac{1}{M(a, y, t)} - 1 \right) t \leq \inf_{a \in A} \left(\frac{1}{M(a, x, t)} - 1 \right) t + \left(\frac{1}{M(x, y, t)} - 1 \right) t.$$

Hence,

$$f_t(y) - f_t(x) \leq \left(\frac{1}{M(x, y, t)} - 1 \right) t.$$

By Theorem 3.3, f_t is t -semi-Lipschitz, and $\|f_t\|_t \leq 1$.

Now, let $x \in \text{cl}_{\overline{M}}(A)$. By Theorem 4.1(2), we get $d_M(A, x, t) = d_{\overline{M}}(x, A, t) = 1$, implying $f_t(x) = 0$.

(2) Let $x_0 \in X \setminus \text{cl}_{\overline{M}}(A)$. By Theorem 4.1(2), there exists a point $t_0 > 0$ such that $d_{\overline{M}}(x_0, A, t_0) < 1$. Choose a sequence $\{y_n\}$ in A satisfying

$$\lim_{n \rightarrow \infty} M(y_n, x_0, t_0) = d_M(A, x_0, t_0) = d_{\overline{M}}(x_0, A, t_0).$$

It follows from (3.3) that

$$\begin{aligned} \|f_{t_0}\|_{t_0} &\geq \frac{M(y_n, x_0, t_0) \left((f_{t_0}(x_0) - f_{t_0}(y_n)) \vee 0 \right)}{t_0 (1 - M(y_n, x_0, t_0))} \\ &\geq \frac{M(y_n, x_0, t_0)}{t_0 (1 - M(y_n, x_0, t_0))} \left(\frac{1}{d_M(A, x_0, t_0)} - 1 \right) t_0 \rightarrow 1 \text{ as } n \rightarrow \infty, \end{aligned}$$

showing that $\|f_{t_0}\|_{t_0} \geq 1$, and so $\|f_{t_0}\|_{t_0} = 1$. $\square$

Remark 4.3. From the above proof, it can be seen that when $(X, M, \wedge)$ is merely a fuzzy quasi-metric space rather than a fuzzy quasi-ultrametric space, Lemma 4.1(2) still holds. However, Lemma 4.1(1) need not hold in general (see Example 4.1), although it can still hold in some cases (see Example 4.2).

Example 4.1. Let $(X, M, \wedge)$ be the fuzzy quasi-metric defined in Example 2.2, and let $A = \{x_1, x_2\}$, then

$$d_M(A, x, t) = M(x_1, x, t) \vee M(x_2, x, t) = \begin{cases} 1, & x = x_1 \text{ or } x_2 \\ x_2 \wedge \frac{t}{2}, & x = x_i (i \geq 3), \\ x_2 \wedge t, & x = 1 \end{cases}$$

so

$$f_t(x) = \left(\frac{1}{d_M(A, x, t)} - 1 \right) t = \begin{cases} 0, & x = x_1 \text{ or } x_2 \\ \left(\frac{1}{x_2 \vee \frac{2}{t}} - 1 \right) t, & x = x_i (i \geq 3) \\ \left(\frac{1}{x_2 \vee \frac{1}{t}} - 1 \right) t, & x = 1 \end{cases}.$$

For the case that $0 < t < x_2$ and $i \geq 3$,

$$f_t(x_i) - f_t(1) = \left(\frac{1}{x_2 \vee \frac{2}{t}} - 1 \right) t - \left(\frac{1}{x_2 \vee \frac{1}{t}} - 1 \right) t = \left(\frac{2}{t} - 1 \right) t - \left(\frac{1}{t} - 1 \right) t = 1,$$

and

$$\left(\frac{1}{M(1, x_i, t)} - 1\right)t = \left(\frac{1}{1 \wedge t} - 1\right)t = \left(\frac{1}{t} - 1\right)t = 1 - t.$$

Thus,

$$f_t(x_i) - f_t(1) \not\leq \left(\frac{1}{M(1, x_i, t)} - 1\right)t \quad \text{for all } x_i \quad (i \geq 3),$$

yielding that f_t is not a t-semi-Lipschitz function.

Example 4.2. Let $(\mathbb{R}, M_{st}, \wedge)$ be the standard fuzzy quasi-metric space. It is easy to see that $(M_{st}, \wedge)$ is not a fuzzy quasi-ultrametric. In fact, let $x=1, y=3, z=2, t=4$, it follows from (2.2) that

$$\begin{aligned} M_{st}(x, y, t) &= M_{st}(1, 3, 4) = \frac{4}{4+2} = \frac{4}{6}, \\ M_{st}(x, z, t) &= M_{st}(1, 2, 4) = \frac{4}{4+1} = \frac{4}{5}, \\ M_{st}(z, y, t) &= M_{st}(2, 3, 4) = \frac{4}{4+1} = \frac{4}{5}. \end{aligned}$$

Obviously, $M_{st}(x, y, t) \not\geq M_{st}(x, z, t) \wedge M_{st}(z, y, t)$.

Now, set $A = (0, 1]$. Then

$$d_{M_{st}}(A, x, t) = \sup_{a \leq 1} M_{st}(a, x, t) = \sup_{a \leq 1} \frac{t}{t + (x - a) \vee 0} = \begin{cases} 1, & x \leq 1 \\ \frac{t}{t + x - 1}, & x > 1 \end{cases}.$$

So,

$$f_t(x) = \left(\frac{1}{d_M(A, x, t)} - 1\right)t = \begin{cases} 0, & x \leq 1 \\ x - 1, & x > 1 \end{cases}.$$

For each $x, y \in \mathbb{R}$ with $M_{st}(x, y, t) < 1$, then $\frac{t}{t + (y - x) \vee 0} < 1$; hence, $y > x$. Therefore,

$$\begin{aligned} \frac{M_{st}(x, y, t)((f_t(y) - f_t(x)) \vee 0)}{t(1 - M_{st}(x, y, t))} &= \frac{\frac{t}{t + y - x}((f_t(y) - f_t(x)) \vee 0)}{t\left(1 - \frac{t}{t + y - x}\right)} \\ &= \frac{f_t(y) - f_t(x)}{y - x} = \begin{cases} 0, & x < y \leq 1 \\ \frac{y - 1}{y - x}, & x \leq 1 < y \\ 1, & 1 < x < y \end{cases} \end{aligned}$$

By (3.3), f_t is a t-semi-Lipschitz function, and $\|f_t\|_t = 1$.

The following theorem indicates that t-semi-Lipschitz functions can be employed to characterize

t-nearest points in certain fuzzy quasi-metric spaces—fuzzy quasi-ultrametric spaces.

Theorem 4.2. Let $(X, M, \wedge)$ be a fuzzy quasi-ultrametric space, $A \subseteq X$, $t > 0$, $x \in W_M(A, t)$. Then, $p \in P_A^M(x, t)$ if and only if there exists $f \in \text{Slip}_{M, N_{st}}^t(X, \mathbb{R}^+)$ satisfying

$$(a) \quad \|f\|_t = 1, (b) \quad f|_A = 0, (c) \quad M(p, x, t) = \frac{t}{t + f(x)},$$

where $\mathbb{R}^+ = [0, \infty)$.

Proof. Suppose $p \in P_A^M(x, t)$. Define $f: X \rightarrow \mathbb{R}^+$ by

$$f(y) = \left(\frac{1}{d_M(A, y, t)} - 1 \right) t. \quad (4.2)$$

It follows from Lemma 4.1 that f is a t-semi-Lipschitz function with $\|f\|_t \leq 1$.

(a) Since $p \in P_A^M(x, t)$, we have $p \in A$ and $d_M(A, x, t) = M(p, x, t) < 1$. Therefore,

$$f(p) = \left(\frac{1}{d_M(A, p, t)} - 1 \right) t \leq \left(\frac{1}{M(p, p, t)} - 1 \right) t = 0$$

and

$$f(x) = \left(\frac{1}{d_M(A, x, t)} - 1 \right) t = \left(\frac{1}{M(p, x, t)} - 1 \right) t, \quad (4.3)$$

implying that $f(p) = 0$ and

$$\|f\|_t \geq \frac{M(p, x, t)((f(x) - f(p)) \vee 0)}{t(1 - M(p, x, t))} = \frac{M(p, x, t)}{t(1 - M(p, x, t))} \cdot \left(\frac{1}{M(p, x, t)} - 1 \right) t = 1.$$

So, $\|f\|_t = 1$.

(b) and (c) follow from (4.2) and (4.3), respectively.

Conversely, suppose there is an $f \in \text{Slip}_{M, N_{st}}^t(X, \mathbb{R}^+)$ satisfying (a), (b), and (c). For each $y \in A$, by using (3.3), we get

$$1 = \|f\|_t \geq \frac{M(y, x, t)((f(x) - f(y)) \vee 0)}{t(1 - M(y, x, t))} = \frac{M(y, x, t)f(x)}{t(1 - M(y, x, t))}.$$

So that $M(y, x, t) \leq \frac{t}{t + f(x)} = M(p, x, t)$, implying that $p \in P_A^M(x, t)$. $\square$

Based on Theorem 4.2, we give another characterization of the t-nearest point in the fuzzy quasi-ultrametric spaces. To this end, we first present the following lemma.

Lemma 4.2. Let $(X, M, *)$ be a fuzzy quasi-metric space, $Y \subseteq X$, $t > 0$. Set $Y_0 = \{f: X \rightarrow \mathbb{R} \mid f \in \text{Slip}_{M, N_{st}}^t(X, \mathbb{R}), f|_Y = 0\}$. For each $x, y \in X$ with $M(x, y, t) < 1$, define

$$M_{Y_0}(x, y, t) = \inf \left\{ \frac{t \|f\|_t}{(f(y) - f(x)) \vee 0 + t \|f\|_t} \mid f \in Y_0, \|f\|_t \neq 0 \right\}. \quad (4.4)$$

Then, $M_{Y_0}(x, y, t) \geq M(x, y, t)$.

Proof. In fact, for each $f \in Y_0$ with $\|f\|_t \neq 0$, it follows from Theorem 3.3 that

$$\|f\|_t \geq \frac{M(x, y, t)((f(y) - f(x)) \vee 0)}{t(1 - M(x, y, t))},$$

for all $x, y \in X$ with $M(x, y, t) < 1$ $t > 0$. So,

$$\frac{t \|f\|_t}{(f(y) - f(x)) \vee 0 + t \|f\|_t} \geq M(x, y, t),$$

implying that $M_{Y_0}(x, y, t) \geq M(x, y, t)$. $\square$

Theorem 4.3. Let $(X, M, \wedge)$ be a fuzzy quasi-ultrametric space, $A \subseteq X$, $t > 0$, $x \in W_M(A, t)$. Then, $p \in P_A^M(x, t)$ if and only if $M_{Y_0}(p, x, t) = M(p, x, t)$.

Proof. Suppose $p \in P_A^M(x, t)$, then $M(p, x, t) = d_M(Y, x, t)$. By Theorem 4.2, there exists a $g \in \text{Slip}'_{M, N_{st}}(X, \mathbb{R}^+)$ such that

$$(a) \quad \|g\|_t = 1, (b) \quad g|_A = 0, (c) \quad M(p, x, t) = \frac{t}{t + g(x)}.$$

By (4.4), we obtain

$$M_{Y_0}(p, x, t) \leq \frac{t \|g\|_t}{(g(x) - g(p)) \vee 0 + t \|g\|_t} = \frac{t}{t + g(x)} = M(p, x, t) = d_M(Y, x, t).$$

This, together with Lemma 4.2, leads to $M_{Y_0}(p, x, t) = M(p, x, t)$.

Conversely, suppose that $M_{Y_0}(p, x, t) = M(p, x, t)$. Since $x \in W_M(A, t)$, $M(y, x, t) \leq d_M(Y, x, t) < 1$ for each $y \in Y$. By (4.4), we have

$$\begin{aligned} M(p, x, t) &= M_{Y_0}(p, x, t) = \inf \left\{ \frac{t \|f\|_t}{(f(x) - f(p)) \vee 0 + t \|f\|_t} \mid f \in Y_0, \|f\|_t \neq 0 \right\} \\ &= \inf \left\{ \frac{t \|f\|_t}{f(x) \vee 0 + t \|f\|_t} \mid f \in Y_0, \|f\|_t \neq 0 \right\} \\ &= \inf \left\{ \frac{t \|f\|_t}{(f(x) - f(y)) \vee 0 + t \|f\|_t} \mid f \in Y_0, \|f\|_t \neq 0 \right\} \\ &= M_{Y_0}(y, x, t) \geq M(y, x, t) \end{aligned}$$

for all $y \in Y$, implying that $p \in P_A^M(x, t)$. $\square$

5. Conclusions

This paper introduces the concept of a t -semi-Lipschitz function defined on a fuzzy quasi-metric space with values in a fuzzy quasi-normed space. We show that the set of all t -semi-Lipschitz functions vanishing at a fixed point forms a quasi-normed cone. In the final part of the paper, we show how this structure provides an appropriate setting to characterize the t -nearest points to a point from a subset. Unlike existing works that studied semi-Lipschitz functions solely in quasi-metric spaces, this paper is the first to extend the theory of semi-Lipschitz functions to fuzzy quasi-metric spaces and, more importantly, to employ this class of functions to characterize t -nearest points. This provides a new quantitative analysis tool for approximation problems in fuzzy settings. Our results generalize the related theory of Lipschitz functions and best approximation (e.g., [1, 39, 7]).

There is a well-known connection between the extension of Lipschitz functions and best approximation problems in metric spaces (e.g., [40]). Moreover, Lipschitz extensions have important applications in fields such as machine learning (e.g., [2]) and can be interpreted as foundations of reinforcement learning procedures. Therefore, it is of great significance to investigate both the extension of t -semi-Lipschitz functions in fuzzy quasi-metric spaces and the potential applications of this extension in machine learning. These topics will constitute the primary focus of our future work. Additionally, several key results in Section 4 are derived under the assumption that the relevant domain space is endowed with a fuzzy quasi-ultrametric, which inevitably restricts the generality of these conclusions. Hence, the following question deserves further investigation: Is it possible to obtain analogous results in the framework of general fuzzy quasi-metric spaces? Or what is the minimal condition under which these main results hold? Beyond the theoretical contributions, the t -semi-Lipschitz framework offers a less conservative alternative to global Lipschitz assumptions in fuzzy control, accommodating practical nonlinearities while preserving stability. This constitutes a key future direction.

Author contributions

Jian-Rong Wu: Conceptualization, methodology, investigation, writing—review and editing, supervision, funding acquisition; Lei Hua: Methodology, investigation, writing—original draft preparation; Fei Yang: Investigation, writing—original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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