



Research article**Quasi-multiplier homotopes of ternary rings of operators****Hong-Wei Wang¹, Selim Çetin², Mehmet Gürdal³, Ömer Kişi⁴ and Qing-Bo Cai^{1,*}**

¹ Fujian Provincial Key Laboratory of Data-Intensive Computing, Key Laboratory of Intelligent Computing and Information Processing, School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China

² Department of Mathematics, Burdur Mehmet Akif Ersoy University, Burdur, Turkey

³ Department of Mathematics, Süleyman Demirel University, Isparta 32260, Turkey

⁴ Department of Mathematics, Bartın University, Bartın, Turkey

* **Correspondence:** Email: qbcai@126.com.

Abstract: We study associative products $x \star_z y = xzy$ on a ternary ring of operators X , where $z \in X^*$ is contractive. Completely isometric isomorphisms are characterized by the orbit of the parameter under ternary isomorphisms. For every such parameter, we compute the Jacobson radical and show that it has cube zero. For closed-range parameters, the semisimple quotient is completely boundedly isomorphic to a C^* -corner. For nonzero closed-range parameters on $\mathcal{B}(H, K)$, the existence of a bounded algebra isomorphism is equivalent to the existence of a completely bounded one. These isomorphism classes are determined by the dimensions of the range, kernel, and cokernel. We also obtain an explicit divided-difference functional calculus and extend the closed-range results to adjointable operators on Hilbert C^* -modules.

Keywords: ternary ring of operators; quasi-multiplier; operator algebra; Jacobson radical; completely bounded isomorphism

Mathematics Subject Classification: Primary 47L25, 46L07, Secondary 46H10, 47A60, 46L08

1. Introduction

The Blecher–Ruan–Sinclair theorem characterizes abstract operator algebras [1]. For a fixed operator space, the quasi-multiplier theory of Kaneda and Paulsen then identifies the associative products that admit completely isometric operator-algebra representations [2–4]. In a concrete realization, such a product has the form

$$x \star_z y = xzy. \quad (1.1)$$

We study the isomorphism classes of the algebras obtained from these products, compute their Jacobson radicals, and derive formulas for their holomorphic functional calculus. Structural rigidity of maps on operator algebras under zero-product constraints has recently been obtained in [5]. In a different setting, continuous orbit equivalence of groupoid actions can be recovered from Cartan-preserving isomorphisms of the associated C^* -algebras [6].

The ternary product of a ternary ring of operators (TRO) is determined by its operator-space structure; see [7–9]. Zettl [10] gives a characterization of ternary rings of operators. For a closed-range parameter, the polar part and its support projections belong to the linking algebra. This question concerns a different radical from that used in ternary radical theory. The latter tests quasi-invertibility over all homotopes, whereas we ask for the ordinary Jacobson radical of one fixed associative product. Thus the semisimplicity result for the ambient ternary system in [11] does not preclude a nonzero radical for X_z . Finite-dimensional homotopes and their radicals also occur in algebraic representation theory [12]. Here we work with norm-closed TROs and study the bounded and completely bounded structure of a fixed homotope.

Let $X \subseteq \mathcal{B}(H, K)$ be a TRO and let $z \in X^*$ with $\|z\| \leq 1$. We write X_z for X equipped with (1.1). For nonzero TROs X, Y , we prove that

$$X_z \cong_{\text{c.i.}} Y_w \iff \theta^\#(z) = w \text{ for some ternary isomorphism } \theta : X \rightarrow Y. \quad (1.2)$$

Thus, complete-isometric classification is an orbit problem for quasi-multipliers. When $X = \mathcal{B}(H, K)$, every ternary isomorphism is spatial, so (1.2) reduces to $w = VzU^*$ for unitaries U, V . In finite dimensions the invariant is the full singular-value list. This differs from the algebraic classification of finite sandwich semigroups in [13]; our results also apply to infinite-dimensional Hilbert spaces and retain the operator-space structure.

The radical formula does not require a closed-range assumption. For every $z \in X^*$, we prove

$$\text{Rad}(X_z) = \{x \in X : zxz = 0\}, \quad \text{Rad}(X_z)^3 = 0. \quad (1.3)$$

The nilpotency bound is sharp already for M_2 . Closed range is needed for the subsequent identification of the quotient with a C^* -algebra. It makes the Moore–Penrose inverse bounded and allows the positive part of the parameter to be removed. Closed-range Moore–Penrose techniques also occur in operator estimates [14], whereas here they produce an algebra isomorphism. If $z = u|z|$, $p = u^*u$ and $q = uu^*$, then left multiplication by $|z| + (1 - p)$ gives a completely bounded isomorphism $X_z \simeq X_u$. The Peirce decomposition identifies its semisimple quotient with the C^* -corner pXq . A compact diagonal example shows that without closed range, a semisimple homotope need not be boundedly isomorphic to any C^* -algebra.

For nonzero closed-range parameters on full operator TROs, the quotient and its two annihilator modules are isomorphic, respectively, to

$$\mathcal{B}(\text{Ran } z), \quad \mathcal{B}(\ker z^*, \text{Ran } z), \quad \mathcal{B}(\text{Ran } z, \ker z).$$

For nonzero closed-range parameters z and w , the induced module isomorphisms give

$$\mathcal{B}(H, K)_z \cong \mathcal{B}(H', K')_w \text{ boundedly} \iff \begin{cases} \dim \text{Ran } z = \dim \text{Ran } w, \\ \dim \ker z = \dim \ker w, \\ \dim \ker z^* = \dim \ker w^*. \end{cases} \quad (1.4)$$

These conditions also give a completely bounded isomorphism with an explicit bound for the product of its completely bounded norm and that of its inverse. On a fixed rectangular matrix TRO they reduce to equality of rank, whereas complete isometry requires equality of all singular values.

For a partial isometry $u \in X^*$ and $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in X_u$ in Peirce form, the spectrum in the external unitization is $\sigma_{C_u}(a) \cup \{0\}$. We express the resolvent and the holomorphic functional calculus in terms of the corner variable a and the first two divided differences at zero and obtain estimates at every matrix level.

Finally, the closed-range results extend to $\mathcal{L}_A(F, E)$ for Hilbert C^* -modules and closed-range adjointable parameters. We also show that the crossed inner product of [15] exists precisely when the underlying contraction is an isometry, in which case it is uniquely determined.

The paper has six sections. Section 2 constructs representations using defect operators and characterizes completely isometric isomorphisms. Section 3 computes the radical for arbitrary parameters, develops the closed-range classification, and gives a nonclosed-range example. Section 4 constructs the resolvent, the divided-difference functional calculus, and the complete spectral estimates. Section 5 treats Hilbert C^* -modules, crossed inner products, and two further questions. Section 6 summarizes the main results.

2. TRO homotopes and complete-isometric rigidity

Let H and K be complex Hilbert spaces. Inner products are linear in the first variable. A norm-closed subspace $X \subseteq \mathcal{B}(H, K)$ is a TRO if $xy^*w \in X$ ($x, y, w \in X$). We write

$$\mathcal{A}_X = \overline{\text{span}(XX^*)} \subseteq \mathcal{B}(K) \quad \text{and} \quad \mathcal{B}_X = \overline{\text{span}(X^*X)} \subseteq \mathcal{B}(H)$$

for its left and right C^* -algebras. Their standard linking algebra is

$$\mathfrak{L}(X) = \begin{bmatrix} \mathcal{A}_X & X \\ X^* & \mathcal{B}_X \end{bmatrix}.$$

Throughout,

$$X^* = \{x^* : x \in X\} \subseteq \mathcal{B}(K, H)$$

denotes the adjoint TRO; it should not be confused with the Banach dual of X .

We use standard facts about TROs and their canonical operator-space structure; see [9, 10, 16]. The intrinsic quasi-multiplier space $QM(X)$ is a corner of the injective or ternary envelope of X [17–19]. In particular, the completely contractive products on X that admit completely isometric operator-algebra representations are implemented by contractive elements of $QM(X)$ [2–4]. These background results will be used without proof.

For a fixed concrete representation of X , set

$$QM_c(X) = \{z \in \mathcal{B}(K, H) : XzX \subseteq X\}.$$

For $z \in QM_c(X)$ define

$$X_z = (X, \star_z), \quad x \star_z y = xzy.$$

Associativity is immediate. If $\|z\| \leq 1$, the product is completely contractive on the given operator-space structure of X . The representations below apply to every contractive $z \in QM_c(X)$. Starting with

the classification theorem we restrict to $z \in X^*$; then $XzX \subseteq X$ is automatic from the TRO identity, and the parameter lies in the opposite corner of the linking algebra. If z has closed range, its polar part and support projections also belong to the linking algebra.

The following representations are expressed directly in terms of the left and right defect operators of z .

Theorem 2.1. *Let $X \subseteq \mathcal{B}(H, K)$ be an operator space, and let $z \in \mathcal{B}(K, H)$ be a contraction satisfying $XzX \subseteq X$. Set $d_K = (I_K - z^*z)^{1/2}$ and $d_H = (I_H - zz^*)^{1/2}$. The maps $\rho_r, \rho_\ell : X_z \rightarrow \mathcal{B}(K \oplus H)$ given by*

$$\rho_r(x) = \begin{bmatrix} 0 & d_K x \\ 0 & zx \end{bmatrix}, \quad \rho_\ell(x) = \begin{bmatrix} xz & xd_H \\ 0 & 0 \end{bmatrix}$$

are completely isometric algebra homomorphisms.

Proof. For $x, y \in X$, block multiplication gives

$$\rho_r(x)\rho_r(y) = \begin{bmatrix} 0 & d_K xzy \\ 0 & zxzy \end{bmatrix} = \rho_r(x \star_z y).$$

Likewise,

$$\rho_\ell(x)\rho_\ell(y) = \begin{bmatrix} xzyz & xzyd_H \\ 0 & 0 \end{bmatrix} = \rho_\ell(x \star_z y).$$

Let

$$V_z = \begin{bmatrix} d_K \\ z \end{bmatrix} : K \rightarrow K \oplus H, \quad W_z = \begin{bmatrix} z & d_H \end{bmatrix} : K \oplus H \rightarrow H.$$

Then, $V_z^* V_z = d_K^2 + z^*z = I_K$ and $W_z W_z^* = zz^* + d_H^2 = I_H$. Thus, V_z is an isometry and W_z is a coisometry. If $P_H : K \oplus H \rightarrow H$ is the coordinate projection and $J_K : K \rightarrow K \oplus H$ is the coordinate inclusion, then

$$\rho_r(x) = V_z x P_H, \quad \rho_\ell(x) = J_K x W_z.$$

It follows that $\|\rho_r(x)\| = \|x\| = \|\rho_\ell(x)\|$. The same factorizations after amplification by I_n prove the equality at every matrix level. $\square$

Remark 2.2. For $A = [x_{ij}] \in M_n(X)$, amplification of either representation means the entrywise map $\rho^{(n)}(A) = [\rho(x_{ij})]$. After the canonical reordering of direct sums,

$$\rho_r^{(n)}(A) = \begin{bmatrix} 0 & (I_n \otimes d_K)A \\ 0 & (I_n \otimes z)A \end{bmatrix}, \quad \rho_\ell^{(n)}(A) = \begin{bmatrix} A(I_n \otimes z) & A(I_n \otimes d_H) \\ 0 & 0 \end{bmatrix}.$$

Thus, the parameter is amplified diagonally as $I_n \otimes z$, while A is an arbitrary operator matrix. The factorizations in the proof give $\|\rho_r^{(n)}(A)\| = \|A\| = \|\rho_\ell^{(n)}(A)\|$. Theorem 2.1 realizes the given matrix norms and product xzy in explicit operator algebras without an identity assumption. We use this realization for the unitization and matrix estimates in Section 4 and again for the defect representations in Section 5.

2.1. The orbit theorem

We next determine when two parameters give completely isometrically isomorphic algebras. A surjective complete isometry between TROs preserves the ternary product [7]. Multiplicativity then determines the parameter itself.

If $\theta : X \rightarrow Y$ is a ternary isomorphism, define the associated linear map $\theta^\# : X^* \rightarrow Y^*$, $\theta^\#(t^*) = \theta(t)^*$ ($t \in X$). It satisfies

$$\theta(xt^*y) = \theta(x)\theta^\#(t^*)\theta(y) \quad (x, t, y \in X).$$

Theorem 2.3. *Let X and Y be nonzero TROs, and let $z \in X^*$ and $w \in Y^*$ satisfy $\|z\|, \|w\| \leq 1$. The following are equivalent:*

- (i) X_z and Y_w are completely isometrically isomorphic as operator algebras.
- (ii) There is a ternary isomorphism $\theta : X \rightarrow Y$ such that $\theta^\#(z) = w$.

Thus, the completely isometric isomorphism classes of the products implemented by contractive parameters in X^* are exactly their orbits under the ternary automorphism group of X .

Proof. Let $\theta : X_z \rightarrow Y_w$ be a surjective complete isometric algebra homomorphism. Since the underlying operator spaces are TROs, uniqueness of the TRO product implies

$$\theta(xy^*v) = \theta(x)\theta(y)^*\theta(v) \quad (x, y, v \in X).$$

Hence, θ is a ternary isomorphism. Multiplicativity now gives

$$\theta(x)\theta^\#(z)\theta(y) = \theta(xzy) = \theta(x)w\theta(y).$$

By surjectivity,

$$Y(\theta^\#(z) - w)Y = \{0\}.$$

Write $\theta^\#(z) - w = t^*$ with $t \in Y$. Taking the two outer variables equal to t yields $tt^*t = 0$, and therefore $\|t\|^3 = \|tt^*t\| = 0$. Thus, $\theta^\#(z) = w$.

Conversely, if θ is a ternary isomorphism and $\theta^\#(z) = w$, then

$$\theta(x \star_z y) = \theta(x)w\theta(y) = \theta(x) \star_w \theta(y).$$

A ternary isomorphism is completely isometric, so θ has all the required properties. $\square$

For full operator TROs the preceding orbit statement becomes a two-sided unitary classification. Since finite-rank operators need not be norm dense in $\mathcal{B}(H, K)$, we include the matrix-coefficient argument. We use $k \boxtimes h$ for the rank-one operator $\xi \mapsto \langle \xi, h \rangle k$.

Lemma 2.4. *Let H, K, H', K' be nonzero Hilbert spaces. Every ternary isomorphism $\theta : \mathcal{B}(H, K) \rightarrow \mathcal{B}(H', K')$ has the form $\theta(x) = UxV^*$, where $U : K \rightarrow K'$ and $V : H \rightarrow H'$ are unitaries.*

Proof. Fix unit vectors $k_0 \in K$ and $h_0 \in H$ and put $e = k_0 \boxtimes h_0$. Minimal tripotents are characterized intrinsically by $ee^*\mathcal{B}(H, K)e^*e = \mathbb{C}e$, so $\theta(e)$ is a rank-one tripotent. Choose unit vectors $k'_0 \in K'$ and $h'_0 \in H'$ so that $\theta(e) = k'_0 \boxtimes h'_0$.

The ternary identities

$$(k \boxtimes h_0)e^*e = k \boxtimes h_0, \quad ee^*(k_0 \boxtimes h) = k_0 \boxtimes h$$

show that there are linear isometries $U : K \rightarrow K'$ and $V : H \rightarrow H'$ such that

$$\theta(k \boxtimes h_0) = Uk \boxtimes h'_0, \quad \theta(k_0 \boxtimes h) = k'_0 \boxtimes Vh.$$

Surjectivity on the corresponding column and row corners makes U and V onto. Since

$$k \boxtimes h = (k \boxtimes h_0)e^*(k_0 \boxtimes h),$$

we obtain $\theta(k \boxtimes h) = Uk \boxtimes Vh$ for every k, h .

To treat an arbitrary operator $x \in \mathcal{B}(H, K)$, we compare its matrix coefficients. For unit k, h , let $a = k \boxtimes h_0$ and $c = k_0 \boxtimes h$. Applying θ twice to the ternary products in $(aa^*x)c^*c = P_kxP_h$ shows that

$$P_{Uk}\theta(x)P_{Vh} = \theta(P_kxP_h) = U(P_kxP_h)V^*.$$

All matrix coefficients of $\theta(x)$ and UxV^* therefore agree. Hence, $\theta(x) = UxV^*$ for every $x \in \mathcal{B}(H, K)$. $\square$

Theorem 2.5. *Let H, K, H' , and K' be nonzero Hilbert spaces, and let $z \in \mathcal{B}(K, H)$ and $w \in \mathcal{B}(K', H')$ be contractions. Then,*

$$\mathcal{B}(H, K)_z \cong_{\text{c.i.}} \mathcal{B}(H', K')_w,$$

if and only if there are unitaries $U : K \rightarrow K'$ and $V : H \rightarrow H'$ such that $w = VzU^$. Every completely isometric algebra isomorphism is then $x \mapsto UxV^*$ for such a pair (U, V) .*

Proof. Suppose first that the two homotopes are completely isometrically isomorphic. Theorem 2.3 supplies a ternary isomorphism θ with $\theta^\sharp(z) = w$, and Lemma 2.4 gives $\theta(x) = UxV^*$ for unitaries U, V . Its adjoint-corner map is

$$\theta^\sharp(z) = VzU^*.$$

Thus, $w = VzU^*$. Conversely, if this identity holds, then $\theta(x) = UxV^*$ is a ternary isomorphism with $\theta^\sharp(z) = w$. Theorem 2.3 shows that θ is a completely isometric algebra isomorphism. The same argument also proves the final assertion. $\square$

Remark 2.6. There is no additional transpose case in Theorem 2.5. Matrix transposition reverses the outer variables in the ternary product and is a ternary anti-isomorphism; except in the scalar case, it is not a complete isometry between the relevant rectangular operator spaces. The scalar case is already spatial.

Corollary 2.7. *In the setting of Theorem 2.5, complete isometric isomorphism is equivalent to two-sided unitary equivalence of z and w . Equivalently, there is a unitary $U : K \rightarrow K'$ such that*

$$U(z^*z)U^* = w^*w, \quad \dim \ker z^* = \dim \ker w^*.$$

If the spaces are finite-dimensional, this says precisely that the domain and codomain dimensions agree and z, w have the same singular values, including multiplicities.

Proof. The forward implication follows from $w = VzU^*$. Conversely, suppose $U(z^*z)U^* = w^*w$. If $z\xi = z\eta$, then $wU(\xi - \eta) = 0$, so the rule $z\xi \mapsto wU\xi$ is well defined. Moreover,

$$\langle wU\xi, wU\eta \rangle = \langle z^*z\xi, \eta \rangle = \langle z\xi, z\eta \rangle.$$

It therefore extends to a unitary from $\overline{\text{Ran } z}$ onto $\overline{\text{Ran } w}$. The equality of cokernel dimensions extends this unitary to $V : H \rightarrow H'$ with $Vz = wU$. The finite-dimensional assertion is the singular-value decomposition. $\square$

Corollary 2.8. *For nonzero H, K ,*

$$\text{Aut}_{\text{c.i.}}(\mathcal{B}(H, K)_z) \cong \frac{\{(U, V) : U \in \mathcal{U}(K), V \in \mathcal{U}(H), VzU^* = z\}}{\{(\lambda I_K, \lambda I_H) : |\lambda| = 1\}}.$$

Proof. Theorem 2.5 gives the numerator. The pairs inducing the identity map on $\mathcal{B}(H, K)$ are exactly the simultaneous scalar unitaries in the denominator. $\square$

3. The radical and closed-range isomorphism types

3.1. Polar normalization

We now assume that $X \subseteq \mathcal{B}(H, K)$ is a TRO and that $z \in X^*$ has closed range. The restriction $z \in X^*$ guarantees that $z^*z \in \mathcal{A}_X$ and $zz^* \in \mathcal{B}_X$.

Write

$$z = u|z|, \quad p = u^*u \in \mathcal{A}_X, \quad q = uu^* \in \mathcal{B}_X$$

for the polar decomposition and support projections. Since z has closed range, zero either lies outside the spectrum of $|z|$ or is an isolated spectral point. Hence, p, q , the Moore–Penrose inverse $|z|^\dagger$, and $u = z|z|^\dagger$ belong to the appropriate corners of the linking algebra. In particular, $u \in X^*$ and $XuX \subseteq X$.

Closed range is equivalent to bounded invertibility of $|z|$ on its support. This allows us to replace z by its polar part through a bounded algebra isomorphism. Without closed range, the inverse of $|z|$ on its range is unbounded, so the following construction does not give a Banach-algebra isomorphism.

Theorem 3.1. *Let $z \in X^*$ have closed range and let $z = u|z|$ be as above. Define $s = |z| + (I_K - p) \in \mathcal{B}(K)$. Then, s is positive and invertible, $s^{-1} = |z|^\dagger + (I_K - p)$, and $sX = X$. The map $\Phi_s : X_z \rightarrow X_u$, $\Phi_s(x) = sx$, is a completely bounded algebra isomorphism. Moreover,*

$$\|\Phi_s\|_{\text{cb}} \leq \|s\|, \quad \|\Phi_s^{-1}\|_{\text{cb}} \leq \|s^{-1}\|.$$

Proof. The assertions concerning s follow from functional calculus on the orthogonal decomposition $K = pK \oplus (1 - p)K$. Since $|z|, |z|^\dagger, p \in \mathcal{A}_X$, the operators $p, |z|$, and $|z|^\dagger$ act on X by left multiplication. Hence, so do $s = I_K + (|z| - p)$ and $s^{-1} = I_K + (|z|^\dagger - p)$, and these two actions are inverse to one another. Thus, $sX = X$.

The identities $u(I_K - p) = 0$ and $up = u$ give $us = u|z| = z$. Consequently, for $x, y \in X$,

$$\Phi_s(x \star_z y) = sxzy = sxusy = \Phi_s(x) \star_u \Phi_s(y).$$

The inverse is left multiplication by s^{-1} . The completely bounded norm estimates are the standard amplification estimates for left multiplication by an operator on K . $\square$

Remark 3.2. For $z \neq 0$, let $\gamma(z) = \inf\{\|z\xi\| : \xi \in (\ker z)^\perp, \|\xi\| = 1\}$ be the reduced minimum modulus of z . Then,

$$\|s^{-1}\| \leq \max\{1, \gamma(z)^{-1}\}.$$

The estimate is an equality for nonzero contractions. For $z = 0$, it is also an equality under the convention $\gamma(0) = \infty$. The bound therefore depends on the reciprocal of the reduced minimum modulus.

The next corollary gives a completely bounded isomorphism whenever the polar parts are two-sided unitarily equivalent.

Corollary 3.3. Let $z \in \mathcal{B}(K, H)$ and $w \in \mathcal{B}(K', H')$ be closed-range contractions, with polar decompositions $z = u|z|$ and $w = v|w|$. Suppose there are unitaries $U : K \rightarrow K'$ and $V : H \rightarrow H'$ such that $v = VuU^*$. Set $s_z = |z| + (I_K - u^*u)$ and $s_w = |w| + (I_{K'} - v^*v)$. Then,

$$\Theta(x) = s_w^{-1} U s_z x V^*$$

is a completely bounded algebra isomorphism $\mathcal{B}(H, K)_z \rightarrow \mathcal{B}(H', K')_w$, and

$$\|\Theta\|_{\text{cb}} \leq \|s_w^{-1}\| \|s_z\|, \quad \|\Theta^{-1}\|_{\text{cb}} \leq \|s_z^{-1}\| \|s_w\|.$$

Proof. The map is the composition

$$\mathcal{B}(H, K)_z \xrightarrow{\Phi_{s_z}} \mathcal{B}(H, K)_u \xrightarrow{x \mapsto UxV^*} \mathcal{B}(H', K')_v \xrightarrow{\Phi_{s_w}^{-1}} \mathcal{B}(H', K')_w.$$

Theorem 2.5 and 3.1, applied to u and v , prove the assertion and the norm bounds. $\square$

3.2. The radical and the C^* -corner quotient

We first work with the partial-isometry homotope X_u . Relative to the support projections $p = u^*u$ and $q = uu^*$, every $x \in X$ has the Peirce decomposition

$$x = \begin{bmatrix} pxq & px(1-q) \\ (1-p)xq & (1-p)x(1-q) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The notation is relative to the domain decomposition $H = qH \oplus (1-q)H$ and range decomposition $K = pK \oplus (1-p)K$.

Set $C_u = pXq$. On C_u define $a \circ b = aub$ and $a^\sharp = u^*a^*u^*$.

Lemma 3.4. The space C_u , with the preceding product and involution, is a C^* -algebra with identity u^* . The map $\Psi : C_u \rightarrow q\mathcal{B}_Xq$, $\Psi(a) = ua$, is a completely isometric $*$ -isomorphism onto $q\mathcal{B}_Xq$.

Proof. If $a, b \in C_u$, then $aub \in pXq$, and the partial-isometry relations show that u^* is a two-sided identity. Direct computation gives

$$\Psi(a \circ b) = uaub = \Psi(a)\Psi(b), \quad \Psi(a^\sharp) = uu^*a^*u^* = (ua)^*.$$

Left multiplication by u preserves the matrix norms on pXq , since $a = pa$. If $t \in q\mathcal{B}_Xq$, then $u^*t \in pXq$ because $X\mathcal{B}_X \subseteq X$, and $\Psi(u^*t) = t$. Thus, Ψ is onto, with inverse $t \mapsto u^*t$. $\square$

If X is a W^* -TRO, meaning that it is weak* closed in $\mathcal{B}(H, K)$, then C_u is a von Neumann algebra in the preceding representation. Indeed,

$$uC_u = \{T \in q\mathcal{B}(H)q : u^*T \in X\}$$

is weak* closed, since multiplication by a fixed operator is weak* continuous. The maps $a \mapsto ua$ and $T \mapsto u^*T$ are normal inverse *-isomorphisms. This observation applies whenever the partial isometry u belongs to X^* , in particular to the polar part of a closed-range parameter.

For compact formulas below, multiplication in C_u and its natural module actions will be written without the symbol $\circ$. Thus,

$$aa' = aud', \quad ab = aub, \quad ca = cua, \quad cb = cub,$$

whenever the Peirce corners make the expressions meaningful.

The Peirce multiplication formula gives the quotient map and the nilpotent ideal used below. It is also used to derive the functional-calculus formula in Section 4.

Lemma 3.5. *For*

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad y = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \quad \text{in } X_u,$$

one has

$$x \star_u y = \begin{bmatrix} aa' & ab' \\ ca' & cb' \end{bmatrix}. \quad (3.1)$$

Consequently, the compression $\pi_u(x) = a$ is a surjective algebra homomorphism onto C_u , and its kernel

$$J_u = \begin{bmatrix} 0 & pX(1-q) \\ (1-p)Xq & (1-p)X(1-q) \end{bmatrix}$$

is an ideal satisfying

$$J_u^2 \subseteq (1-p)X(1-q), \quad J_u^3 = 0. \quad (3.2)$$

Proof. Relative to the support decompositions, the partial isometry has the block form $u = \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}$. Multiplying xuy gives (3.1). The assertions concerning π_u and J_u follow immediately. If both x and y lie in J_u , their product is $\begin{bmatrix} 0 & 0 \\ 0 & cb' \end{bmatrix}$; this corner annihilates X_u on both sides, which proves (3.2). $\square$

The radical can in fact be computed without closed range. In the next theorem, we drop that assumption and use the support projections only as operators on the ambient Hilbert spaces.

Theorem 3.6. *Let $X \subseteq \mathcal{B}(H, K)$ be a TRO and let $z \in X^*$. Then,*

$$\text{Rad}(X_z) = J_z := \{x \in X : zxz = 0\}, \quad J_z^3 = \{0\}.$$

If p and q are the orthogonal projections onto $(\ker z)^\perp$ and $\overline{\text{Ran } z}$, respectively, then $J_z = \{x \in X : pxq = 0\}$. If, in addition, z has closed range and $z = u|z|$, the quotient X_z/J_z is completely boundedly isomorphic to the C^ -algebra $C_u = pXq$.*

Proof. The space J_z is a closed two-sided ideal. For example, if $x \in J_z$ and $v \in X$, then $z(v \star_z x)z = zv(zxz) = 0$ and $z(x \star_z v)z = (zxz)vz = 0$. If $x_1, x_2, x_3 \in J_z$, then

$$x_1 \star_z x_2 \star_z x_3 = x_1(zx_2z)x_3 = 0.$$

Thus, $J_z^3 = 0$, and the nilpotent ideal J_z is contained in $\text{Rad}(X_z)$.

Conversely, let $x \in \text{Rad}(X_z)$ and put $v = zxz$. The TRO identity gives $y = v^* = z^*x^*z^* \in X$. The map $\rho : X_z \rightarrow \mathcal{B}(H)$, $\rho(t) = zt$, is a bounded algebra homomorphism. The element $x \star_z y$ belongs to the radical and hence has spectral radius zero. Since $\rho(t^{*n}) = \rho(t)^n$, the spectral-radius formula gives

$$r_{\mathcal{B}(H)}(\rho(t)) \leq r_{X_z}(t).$$

But $\rho(x \star_z y) = zxzy = vv^*$ is positive. It follows that $\|vv^*\| = 0$, so $zxz = 0$. Therefore, $\text{Rad}(X_z) = J_z$.

Since $z = zp = qz$, the equality $pxq = 0$ implies $zxz = 0$. Conversely, $zxz = 0$ implies that px vanishes on $\text{Ran } z$, and therefore on qH by continuity. This proves $J_z = \{x : pxq = 0\}$ without a closed-range assumption. These projections need not act as multipliers of X in this generality.

When z has closed range, Theorem 3.1 sends J_z onto J_u . Lemma 3.5 identifies X_u/J_u completely isometrically with C_u . The quotient assertion follows. $\square$

Thus, every closed-range homotope fits, up to completely bounded isomorphism, into the canonical exact sequence

$$0 \longrightarrow J_u \longrightarrow X_u \xrightarrow{\pi_u} C_u \longrightarrow 0, \quad J_u^3 = 0. \quad (3.3)$$

The radical formula holds for each fixed product, including parameters with nonclosed range. The quotient description in the displayed exact sequence uses closed range.

Corollary 3.7. *Let H, K be nonzero Hilbert spaces and let $z \in \mathcal{B}(K, H)$ have closed range. For $\mathcal{B}(H, K)_z = (\mathcal{B}(H, K), \star_z)$ one has*

$$\text{Rad}(\mathcal{B}(H, K)_z) = \{x : zxz = 0\}, \quad \text{Rad}(\mathcal{B}(H, K)_z)^3 = 0.$$

The algebra is semisimple if and only if z is boundedly invertible from K onto H .

Proof. Only the last assertion needs proof. If z is invertible, then $p = I_K$ and $q = I_H$, so the radical is zero. Conversely, suppose the radical is zero. If $p \neq I_K$ and $q \neq 0$, a nonzero operator from qH into $(1 - p)K$ belongs to the radical. If $q \neq I_H$ and $p \neq 0$, a nonzero operator from $(1 - q)H$ into pK belongs to the radical. The remaining case $p = q = 0$ means $z = 0$ and the whole nonzero algebra is radical. Hence, $p = I_K$ and $q = I_H$. Closed range then makes z boundedly invertible. $\square$

For full operator TROs, the left and right annihilator ideals determine the two defect spaces as modules over the semisimple quotient. The following lemma identifies the bounded isomorphisms between these operator modules.

Lemma 3.8. *Let E, E' be nonzero Hilbert spaces.*

- (i) *Every bounded algebra isomorphism $\alpha : \mathcal{B}(E) \rightarrow \mathcal{B}(E')$ has the form $\alpha(a) = SaS^{-1}$ ($a \in \mathcal{B}(E)$) for some boundedly invertible $S : E \rightarrow E'$.*

- (ii) Let F, F' be Hilbert spaces and let $\Gamma : \mathcal{B}(F, E) \rightarrow \mathcal{B}(F', E')$ be a bounded linear bijection satisfying $\Gamma(ab) = \alpha(a)\Gamma(b)$. Then, there is a boundedly invertible $R : F' \rightarrow F$ such that $\Gamma(b) = SbR$.
- (iii) Let G, G' be Hilbert spaces and let $\Delta : \mathcal{B}(E, G) \rightarrow \mathcal{B}(E', G')$ be a bounded linear bijection satisfying $\Delta(ca) = \Delta(c)\alpha(a)$. Then, there is a boundedly invertible $T : G \rightarrow G'$ such that $\Delta(c) = TcS^{-1}$.

The assertions also cover zero modules. For example, an isomorphism in (ii) with $F = 0$ forces $F' = 0$.

Proof. For (i), choose $e \in E$ and $f \in E^*$ with $f(e) = 1$, and set $P = e \otimes f$. The algebraic condition $P\mathcal{B}(E)P = \mathbb{C}P$ characterizes rank-one idempotents. Hence, $Q = \alpha(P)$ is a rank-one idempotent. Choose a nonzero vector $e' \in E'$ with $Qe' = e'$ and define

$$S\xi = \alpha(\xi \otimes f)e' \quad (\xi \in E).$$

The map S is bounded and

$$\alpha(a)S\xi = Sa\xi. \quad (3.4)$$

If $S\xi = 0$, then $\alpha(\xi \otimes f)Q = \alpha((\xi \otimes f)P) = 0$, so $\xi = 0$. Moreover, $\text{Ran } S$ is a nonzero invariant linear subspace for $\alpha(\mathcal{B}(E)) = \mathcal{B}(E')$ and therefore equals E' . Thus, S is boundedly invertible, and (3.4) proves (i).

For (ii), replace Γ by $\widetilde{\Gamma}(b) = S^{-1}\Gamma(b)$. It is enough to treat the case $E' = E$ and α is the identity. For $y \in F'$, the evaluation

$$L_y : \mathcal{B}(F, E) \rightarrow E, \quad L_y(b) = \widetilde{\Gamma}(b)y,$$

is a bounded left $\mathcal{B}(E)$ -module map. We claim that there is a unique $r_y \in F$ such that $L_y(b) = br_y$ for every b .

Indeed, with e, f, P as above and $h \in F^*$, write

$$L_y(e \otimes h) = \lambda_y(h)e.$$

The functional λ_y belongs to F^{**} , so Hilbert-space reflexivity gives a unique $r_y \in F$ with $\lambda_y(h) = h(r_y)$. For $g \in E^*$ and $b \in \mathcal{B}(F, E)$, the module identity gives

$$g(L_y(b))e = (e \otimes g)L_y(b) = L_y((e \otimes g)b) = L_y(e \otimes (g \circ b)) = g(br_y)e.$$

Hence, $L_y(b) = br_y$. The assignment $Ry = r_y$ is linear, and testing on contractions $b : F \rightarrow E$ gives $\|Ry\| \leq \|\Gamma\| \|y\|$. Thus, $R : F' \rightarrow F$ is bounded and $\widetilde{\Gamma}(b) = bR$. Applying the same argument to $\widetilde{\Gamma}^{-1}$ produces $R' : F \rightarrow F'$; the two composition identities imply $RR' = I_F$ and $R'R = I_{F'}$. This proves (ii).

For (iii), put $\widetilde{\Delta}(c) = \Delta(c)S$; then $\widetilde{\Delta}(ca) = \widetilde{\Delta}(c)a$. For $x \in G$ and $\xi \in E$, write $x \otimes f \in \mathcal{B}(E, G)$ for the rank-one operator $\zeta \mapsto f(\zeta)x$. Since

$$(x \otimes f)(\xi \otimes h) = f(\xi)x \otimes h \quad (f, h \in E^*),$$

the right-module identity implies that, for every $x \in G$, there is a unique vector $Tx \in G'$ such that

$$\widetilde{\Delta}(x \otimes f) = Tx \otimes f \quad (f \in E^*).$$

Indeed, first take $f_0 \in E^*$ and $e_0 \in E$ with $f_0(e_0) = 1$; the module identity with $e_0 \otimes f_0$ gives $\widetilde{\Delta}(x \otimes f_0) = \widetilde{\Delta}(x \otimes f_0)(e_0 \otimes f_0)$. Thus, with $Tx = \widetilde{\Delta}(x \otimes f_0)e_0$, one has $\widetilde{\Delta}(x \otimes f_0) = Tx \otimes f_0$. Right multiplication by $e_0 \otimes f$ then gives the displayed identity for every f . The assignment $T : G \rightarrow G'$ is linear and bounded, since $\|Tx\| = \|\widetilde{\Delta}(x \otimes f_0)\|$ after normalizing $\|f_0\| = 1$. Every $c \in \mathcal{B}(E, G)$ satisfies

$$\widetilde{\Delta}(c)(\xi \otimes f) = \widetilde{\Delta}(c(\xi \otimes f)) = \widetilde{\Delta}(c\xi \otimes f) = Tc\xi \otimes f,$$

so $\widetilde{\Delta}(c) = Tc$. Applying the same argument to the inverse module map gives $T' : G' \rightarrow G$ with $T'T = I_G$ and $TT' = I_{G'}$. Hence, T is boundedly invertible and $\Delta(c) = TcS^{-1}$, proving (iii). $\square$

The following theorem classifies bounded isomorphism in arbitrary Hilbert-space dimension. We exclude the zero parameter, for which the product vanishes identically and the question reduces to Banach-space isomorphism of rectangular operator spaces.

Theorem 3.9. *Let H, K, H', K' be nonzero Hilbert spaces, and let $z \in \mathcal{B}(K, H)$ and $w \in \mathcal{B}(K', H')$ be nonzero closed-range contractions. The following are equivalent:*

- (i) $\mathcal{B}(H, K)_z$ and $\mathcal{B}(H', K')_w$ are isomorphic as Banach algebras.
- (ii) $\mathcal{B}(H, K)_z$ and $\mathcal{B}(H', K')_w$ are completely boundedly isomorphic.
- (iii)

$$\begin{aligned} \dim \operatorname{Ran} z &= \dim \operatorname{Ran} w, \\ \dim \ker z &= \dim \ker w, \\ \dim \ker z^* &= \dim \ker w^*. \end{aligned} \tag{3.5}$$

- (iv) *If u and v are the polar parts of z and w , respectively, then $v = VuU^*$ for some unitaries $U : K \rightarrow K'$ and $V : H \rightarrow H'$.*

Here $\dim$ denotes Hilbert-space dimension, possibly an infinite cardinal. When these conditions hold, the isomorphism in (ii) may be chosen as in Corollary 3.3, and its cb distortion satisfies

$$\|\Theta\|_{\text{cb}}\|\Theta^{-1}\|_{\text{cb}} \leq (\|s_z\| \|s_z^{-1}\|)(\|s_w\| \|s_w^{-1}\|). \tag{3.6}$$

In particular, since z, w are contractions, the right-hand side is at most

$$\max\{1, \gamma(z)^{-1}\} \max\{1, \gamma(w)^{-1}\}, \tag{3.7}$$

where γ denotes the reduced minimum modulus.

Proof. The implication (ii) $\Rightarrow$ (i) is immediate. Assume (i). Polar normalization allows us to replace z, w by their polar parts u, v . Put $p = u^*u$, $q = uu^*$ and $p' = v^*v$, $q' = vv^*$. For $A = \mathcal{B}(H, K)_u$, let $J = \operatorname{Rad}(A)$ and define the intrinsic annihilator ideals

$$\mathcal{L} = \{x \in A : xA = 0\}, \quad \mathcal{R} = \{x \in A : Ax = 0\}, \quad \mathcal{D} = \mathcal{L} \cap \mathcal{R}.$$

The Peirce table in Lemma 3.5 gives

$$\begin{aligned} \mathcal{L} &= \begin{bmatrix} 0 & p\mathcal{B}(H, K)(1-q) \\ 0 & (1-p)\mathcal{B}(H, K)(1-q) \end{bmatrix}, \\ \mathcal{R} &= \begin{bmatrix} 0 & 0 \\ (1-p)\mathcal{B}(H, K)q & (1-p)\mathcal{B}(H, K)(1-q) \end{bmatrix}, \\ \mathcal{D} &= (1-p)\mathcal{B}(H, K)(1-q). \end{aligned} \tag{3.8}$$

All four spaces $J, \mathcal{L}, \mathcal{R}, \mathcal{D}$ are preserved by algebra isomorphisms. Set

$$E = qH, \quad F = (1 - q)H, \quad G = (1 - p)K.$$

The corner identification $a \mapsto ua$ and the off-diagonal identifications $b \mapsto ub, c \mapsto c$ yield

$$A/J \cong \mathcal{B}(E), \quad \mathcal{L}/\mathcal{D} \cong \mathcal{B}(F, E), \quad \mathcal{R}/\mathcal{D} \cong \mathcal{B}(E, G), \quad (3.9)$$

where the last two are, respectively, a left and a right module over A/J . The quotient norms in (3.9) are the usual operator norms. Indeed, compression to the indicated corner is contractive, while a representative having only that corner nonzero has exactly the same norm. The Peirce formula also shows directly that the induced products are the usual left and right compositions.

A bounded algebra isomorphism $A \rightarrow A' = \mathcal{B}(H', K')_v$ therefore induces a bounded algebra isomorphism between $\mathcal{B}(E)$ and $\mathcal{B}(E')$, together with bounded bijective intertwining maps between the two modules in (3.9). Lemma 3.8 gives

$$E \cong E', \quad F \cong F', \quad G \cong G'.$$

Since $E = \text{Ran } z, F = \ker z^*$ and $G = \ker z$, and similarly for w , condition (3.5) follows. This proves (i) $\Rightarrow$ (iii).

If (iii) holds, choose a unitary $U_0 : pK \rightarrow p'K'$ and unitaries between the two kernel spaces. Extend U_0 to a unitary $U : K \rightarrow K'$ and define a unitary $V_0 : qH \rightarrow q'H'$ by $V_0 = vU_0u^*$. Extending V_0 by a unitary $\ker u^* \rightarrow \ker v^*$ gives $V : H \rightarrow H'$ and $v = VuU^*$. Thus, (iii) $\Rightarrow$ (iv). Corollary 3.3 proves (iv) $\Rightarrow$ (ii), and multiplying its two norm bounds gives (3.6). $\square$

Theorem 3.9 applies to arbitrary Hilbert-space dimensions. The three dimensions determine the two-sided unitary orbit of the polar part, whereas complete isometry also depends on the positive part of the parameter. On ℓ^2 , let S be the unilateral shift. The parameters I, S , and S^* have respective triples

$$(\aleph_0, 0, 0), \quad (\aleph_0, 0, 1), \quad (\aleph_0, 1, 0).$$

Their homotopes are therefore pairwise nonisomorphic as Banach algebras, although all three ranges have infinite dimension. In contrast, I and tI , with $0 < t < 1$, give boundedly isomorphic homotopes, but no completely isometric isomorphism exists because the parameter norms differ. The three dimensions are therefore complete invariants for bounded algebra isomorphism under the closed-range hypothesis. The classification of nonclosed-range parameters remains open.

Rectangular matrices with the product $x \star_z y = xzy$ also occur as finite sandwich semigroups. Dolinka–East classified their semigroup isomorphisms over arbitrary fields [13]. On a fixed complex rectangular matrix space, the bounded part of the next corollary is consistent with that classification; the operator-space statement adds the strictly finer complete-isometric invariant.

Corollary 3.10. *Fix $m, n \geq 1$, let $X = M_{m,n} = \mathcal{B}(\mathbb{C}^n, \mathbb{C}^m)$, and let $z, w \in M_{n,m}$ be contractions.*

- (i) $X_z \cong_{\text{c.i.}} X_w$ if and only if z and w have the same singular values, including multiplicities.
- (ii) *The following are equivalent:*

$$X_z \cong X_w \text{ as abstract algebras}, \quad X_z \cong_{\text{cb}} X_w, \quad \text{rank } z = \text{rank } w.$$

Thus the complete-isometric classes are determined by the singular values, whereas the algebraic isomorphism classes are determined by the rank.

Proof. Part (i) is Theorem 2.5 and the singular-value decomposition. If z, w are nonzero, equality of rank makes all three dimensions in (3.5) equal, so Theorem 3.9 gives a completely bounded algebra isomorphism. If both parameters vanish, the identity map is such an isomorphism.

Conversely, an algebra isomorphism preserves the Jacobson radical. If $r = \text{rank } z$, Theorem 3.6 identifies the semisimple quotient of X_z with M_r , and hence

$$\dim(X_z / \text{Rad}(X_z)) = r^2.$$

The corresponding quotient dimension for w is $(\text{rank } w)^2$, so the two ranks agree. In finite dimensions every linear algebra isomorphism is automatically completely bounded, which completes the equivalence. $\square$

We next show that the nilpotency bound in Theorem 3.6 is sharp and characterize the square-zero case for full operator TROs.

Example 3.11. Let $H = K = \mathbb{C}^2$, let $z = \text{diag}(1, 0)$, and put

$$x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, $x \in \text{Rad}((M_2)_z)$, while

$$x \star_z x = E_{22} \neq 0, \quad x \star_z x \star_z x = 0.$$

Thus the exponent three in Theorem 3.6 cannot be improved.

Proposition 3.12. For $X = \mathcal{B}(H, K)$ and a nonzero closed-range operator z , the equality $J_z^2 = 0$ holds if and only if $\ker z = 0$ or $\ker z^* = 0$. If both defect spaces are nonzero, then $J_z^2 \neq 0$ and $J_z^3 = 0$.

Proof. After polar normalization, apply the multiplication formula in Lemma 3.5. Products of two radical elements have only a lower-right corner, of the form cub with

$$b : (1 - q)H \rightarrow pK, \quad c : qH \rightarrow (1 - p)K.$$

If either $1 - p = 0$ or $1 - q = 0$, every such product vanishes. If both defect spaces are nonzero, choose rank-one b and c whose intermediate vectors are paired nontrivially through the unitary $u : pK \rightarrow qH$. Then, $cub \neq 0$. $\square$

3.3. Nonclosed-range parameters

The following semisimple homotope has a nonclosed-range parameter and is not boundedly isomorphic to any C^* -algebra.

Example 3.13. Let $H = \ell^2(\mathbb{N})$ and define $De_n = n^{-1}e_n$. Then, D is a positive compact contraction with dense nonclosed range. Every finitely supported vector lies in $\text{Ran } D$, whereas $(n^{-1})_{n \geq 1} \in H$ does not. Both support projections equal I_H , and Theorem 3.6 gives

$$\text{Rad}(\mathcal{B}(H)_D) = \{0\} = \{x \in \mathcal{B}(H) : Dx D = 0\}.$$

Nevertheless, this homotope has no left approximate identity. Such an approximate identity (a_i) would satisfy $a_i \star_D I_H = a_i D \rightarrow I_H$ in norm. Every $a_i D$ is compact, which is impossible on the infinite-dimensional space H . The same argument with $D a_i$ excludes a right approximate identity. Since every C^* -algebra has a bounded approximate identity, $\mathcal{B}(H)_D$ is not boundedly isomorphic to any C^* -algebra.

For the finite-rank truncations $D_N = \text{diag}(1, 1/2, \dots, 1/N, 0, \dots)$, one has $\|D - D_N\| = (N + 1)^{-1}$. If p_N projects onto $\text{span}\{e_1, \dots, e_N\}$, then $s_N = D_N + (I_H - p_N)$ satisfies $\|s_N^{-1}\| = N$. Thus, the inverse norms in polar normalization diverge although the parameters converge in norm.

More generally, if $\varepsilon > 0$ lies outside $\sigma(|z|)$, the projection $p_\varepsilon = \mathbf{1}_{[\varepsilon, \infty)}(|z|)$ belongs to $\mathcal{A}_X$ by continuous functional calculus. Hence, $z_\varepsilon = z p_\varepsilon \in X^*$ has closed range and $\|z - z_\varepsilon\| \leq \varepsilon$. This provides closed-range approximations when the positive spectrum is discrete and accumulates only at zero. Example 3.13 shows that their normalization constants need not remain bounded. A gap in the essential spectrum does not suffice, since $\sigma_{\text{ess}}(D) = \{0\}$ while the positive eigenvalues approach zero. Bounded polar normalization requires a gap between zero and the positive part of the full spectrum.

4. Functional calculus and divided differences

4.1. Resolvents and divided differences

Let $u \in X^*$ be a partial isometry, and use the Peirce notation and module conventions of Section 3.2. For $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in X_u$, Lemma 3.5 shows that all products of length at least two factor through the C^* -corner. Iteration gives the following formula for the powers of x , which we use below in the functional calculus.

Lemma 4.1. *For every integer $n \geq 2$,*

$$x^{\star n} = \begin{bmatrix} a^n & a^{n-1}b \\ ca^{n-1} & ca^{n-2}b \end{bmatrix},$$

where powers and module actions are taken in the u -corner conventions.

Proof. The case $n = 2$ follows directly from Peirce multiplication:

$$x \star_u x = \begin{bmatrix} a^2 & ab \\ ca & cb \end{bmatrix}.$$

Multiplication of the displayed formula on the right by x proves the induction step. □

Let $\widetilde{X}_u = \mathbb{C}1 \oplus X_u$ denote the external unitization, also when X_u already has an identity. We equip it with the operator-algebra matrix norms given by

$$\mu 1 + y \mapsto (\mu I_{K \oplus H} + \rho_r(y)) \oplus \mu,$$

where ρ_r is the representation from Theorem 2.1. This map is injective, its restriction to X_u is completely isometric, and the external identity has norm one at every matrix level. Let $e = u^*$ be the identity of C_u . In general X_u itself is not a C^* -algebra.

Theorem 4.2. Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in X_u$. If $\lambda \neq 0$ and $\lambda e - a$ is invertible in C_w set $R_\lambda = (\lambda e - a)^{-1}$. Then,

$$(\lambda 1 - x)^{-1} = \lambda^{-1} 1 + \begin{bmatrix} R_\lambda - \lambda^{-1} e & \lambda^{-1} R_\lambda b \\ \lambda^{-1} c R_\lambda & \lambda^{-2} d + \lambda^{-2} c R_\lambda b \end{bmatrix}.$$

Consequently,

$$\sigma_{\widetilde{X}_u}(x) = \sigma_{C_u}(a) \cup \{0\}.$$

Proof. To obtain the inverse, write $y = \begin{bmatrix} \alpha & \beta \\ \gamma & \eta \end{bmatrix}$ and solve $(\lambda 1 - x)(\lambda^{-1} 1 + y) = 1$. The scalar part is already correct. The four Peirce equations are

$$\begin{aligned} (\lambda e - a)\alpha &= \lambda^{-1} a, & (\lambda e - a)\beta &= \lambda^{-1} b, \\ \lambda\gamma - c\alpha &= \lambda^{-1} c, & \lambda\eta - c\beta &= \lambda^{-1} d. \end{aligned}$$

They can be solved in this order, giving

$$\begin{aligned} \alpha &= R_\lambda - \lambda^{-1} e, & \beta &= \lambda^{-1} R_\lambda b, \\ \gamma &= \lambda^{-1} c R_\lambda, & \eta &= \lambda^{-2} d + \lambda^{-2} c R_\lambda b. \end{aligned}$$

The identity $R_\lambda(\lambda e - a) = e$ then gives, by the same Peirce multiplication, $(\lambda^{-1} 1 + y)(\lambda 1 - x) = 1$. Thus, the expression is a two-sided inverse. The form of the resolvent follows from these triangular equations; the term involving d occurs only in the last equation.

Conversely, the scalar homomorphism $\varepsilon : \widetilde{X}_u \rightarrow \mathbb{C}$, $\varepsilon(\mu 1 + y) = \mu$, shows that 0 belongs to the spectrum of x in $\widetilde{X}_u$. The map

$$\widetilde{\pi}_u : \widetilde{X}_u \rightarrow C_u, \quad \widetilde{\pi}_u(\mu 1 + y) = \mu e + pyq,$$

is a unital homomorphism. Hence, invertibility of $\lambda 1 - x$ would imply invertibility of $\lambda e - a$. The spectral formula follows. $\square$

For a function f holomorphic on a neighborhood of $\sigma_{C_u}(a) \cup \{0\}$, define the removable divided differences

$$g_f(t) = \begin{cases} \frac{f(t) - f(0)}{t}, & t \neq 0, \\ f'(0), & t = 0, \end{cases}$$

and

$$h_f(t) = \begin{cases} \frac{f(t) - f(0) - f'(0)t}{t^2}, & t \neq 0, \\ \frac{1}{2}f''(0), & t = 0. \end{cases}$$

Theorem 4.3. With the preceding notation, the holomorphic functional calculus in $\widetilde{X}_u$ satisfies

$$f(x) = f(0)1 + \begin{bmatrix} f(a) - f(0)e & g_f(a)b \\ cg_f(a) & f'(0)d + ch_f(a)b \end{bmatrix}.$$

The same formula holds entrywise for matrix-valued holomorphic functions and for all matrix amplifications of X_u .

Proof. Choose a positively oriented contour Γ in the domain of f that surrounds $\sigma_{C_u}(a) \cup \{0\}$. By the holomorphic functional calculus and Theorem 4.2,

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda 1 - x)^{-1} d\lambda.$$

The scalar part is $f(0)1$, and the upper-left corner is $f(a) - f(0)e$. To justify the remaining terms without introducing singular integrands, note first that

$$\frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1}(\lambda e - a)^{-1} d\lambda = 0, \quad \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-2}(\lambda e - a)^{-1} d\lambda = 0. \quad (4.1)$$

Indeed, after enlarging Γ to a sufficiently large circle, the two integrands are respectively $O(|\lambda|^{-2})$ and $O(|\lambda|^{-3})$ at infinity. Since $f(\lambda) = f(0) + \lambda g_f(\lambda)$ and $f(\lambda) = f(0) + f'(0)\lambda + \lambda^2 h_f(\lambda)$, (4.1) and the C_u functional calculus give

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\lambda)}{\lambda} (\lambda e - a)^{-1} d\lambda = g_f(a), \quad \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\lambda)}{\lambda^2} (\lambda e - a)^{-1} d\lambda = h_f(a).$$

These formulas give the off-diagonal and $ch_f(a)b$ terms, while

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\lambda)}{\lambda^2} d\lambda = f'(0)$$

gives the coefficient of d . This proves the formula. Every step is stable under matrix amplification, which proves the last assertion. $\square$

Remark 4.4. In the formula of Theorem 4.3, the term $ch_f(a)b$ comes from multiplication of the two off-diagonal radical corners. If $J_u^2 = 0$, this term vanishes, and only the first divided difference remains.

4.2. Completely bounded estimates

We now formulate the quantitative consequence at all matrix levels. Let Ω be a compact set containing zero. For a matrix-valued function F holomorphic on a neighborhood of Ω , let G_F and H_F denote the removable continuations

$$G_F(t) = \frac{F(t) - F(0)}{t}, \quad H_F(t) = \frac{F(t) - F(0) - F'(0)t}{t^2}.$$

Define the second divided-difference seminorm by

$$\|F\|_{\mathcal{D}^2(\Omega)} = \max\{\|F\|_{\Omega}, \|G_F\|_{\Omega}, \|H_F\|_{\Omega}, \|F'(0)\|\}. \quad (4.2)$$

The four quantities in this seminorm are those appearing in Theorem 4.3. The formula involves no divided difference of order greater than two, since $J_u^3 = 0$.

Recall that $\Omega \supseteq \sigma_{C_u}(a)$ is a complete K -spectral set for a if

$$\|F(a)\| \leq K \sup_{t \in \Omega} \|F(t)\|, \quad (4.3)$$

for every matrix size and every matrix-valued rational function with poles off Ω (equivalently here, for functions holomorphic on a neighborhood of Ω). This formulation makes the following estimate independent of the faithful representation chosen for the C^* -corner.

Theorem 4.5. Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in X_w$ and let Ω be a compact set containing $\sigma_{C_u}(a) \cup \{0\}$. If Ω is a complete K -spectral set for a , then

$$\|F(x)\| \leq C_x(K)\|F\|_{\mathcal{D}^2(\Omega)} \quad (4.4)$$

for every matrix-valued F holomorphic near Ω , uniformly over all matrix sizes, where

$$C_x(K) = 1 + \left((K+1)^2 + K^2\|b\|^2 + K^2\|c\|^2 + (\|d\| + K\|c\|\|b\|)^2 \right)^{1/2}. \quad (4.5)$$

Consequently, the functional-calculus homomorphism is completely bounded with respect to the matrix seminorms in (4.2) and therefore descends to a completely bounded homomorphism on the corresponding quotient by their common null space.

Proof. Apply Theorem 4.3 at an arbitrary matrix level and abbreviate $M = \|F\|_{\mathcal{D}^2(\Omega)}$. Complete K -spectrality and complete contractivity of the corner-module actions give

$$\begin{aligned} \|F(a) - F(0)e\| &\leq (K+1)M, & \|G_F(a)b\| &\leq K\|b\|M, \\ \|cG_F(a)\| &\leq K\|c\|M, & \|F'(0)d + cH_F(a)b\| &\leq (\|d\| + K\|c\|\|b\|)M. \end{aligned} \quad (4.6)$$

For a 2×2 operator matrix, its norm is bounded by the Frobenius norm of the scalar matrix of block norms. Applying this to the Peirce block in Theorem 4.3, and then adding the external scalar term whose norm is at most M , yields (4.4)-(4.5). Every estimate in (4.6) is independent of the matrix size, proving complete boundedness. $\square$

The constant in (4.5) is an explicit upper bound, not an optimal numerical coefficient. For $x = 0$ and $K = 1$, the functional-calculus map has completely bounded norm one, whereas (4.5) gives three. The orders of dependence on its parameters can, however, be attained.

For this purpose, let $\kappa \geq 1$, $X = M_3$, $u = \text{diag}(I_2, 0)$ and $\Omega = \overline{\mathbb{D}}$. In the Peirce decomposition take

$$a_\kappa = \kappa E_{12}, \quad b = \beta e_2, \quad c = \gamma e_1^*, \quad d = \delta, \quad \beta, \gamma, \delta \geq 0,$$

where e_1, e_2 are the standard basis vectors of $\mathbb{C}^2$. The disk is a complete κ -spectral set for a_κ . Indeed, $a_\kappa = D_\kappa E_{12} D_\kappa^{-1}$ with $D_\kappa = \text{diag}(\kappa, 1)$. At every matrix size, $F(E_{12})$ is unitarily equivalent to the compression of multiplication by F on vector-valued Hardy space to the first two monomials. Its norm is therefore at most $\sup_{t \in \overline{\mathbb{D}}} \|F(t)\|$. Similarity by D_κ gives the constant κ , and the function $t \mapsto t$ shows that it is minimal.

For $f_j(t) = t^j$, with $j = 1, 2, 3$, each seminorm $\|f_j\|_{\mathcal{D}^2(\overline{\mathbb{D}})}$ equals one. Lemma 4.1 gives

$$\begin{aligned} \|f_1(x)\| &\geq \max\{\kappa, \beta, \gamma, \delta\}, \\ \|f_2(x)\| &= \kappa \max\{\beta, \gamma\}, \\ \|f_3(x)\| &= \kappa \beta \gamma. \end{aligned}$$

Consequently, uniform estimates require terms of orders K , $K\|b\|$, $K\|c\|$, $\|d\|$, and $K\|b\|\|c\|$. These examples concern the orders of growth and do not assert optimality of the coefficient in (4.5).

Related numerical and symbol-based methods have been developed for concrete operators on Lebesgue and Sobolev spaces [20], for Toeplitz operators on Hardy space [21], and for operators on

Fock and reproducing-kernel Hilbert spaces [22, 23]. More recently, multiparametric operator products on reproducing-kernel Hilbert spaces were investigated through the $\mathbb{A}$ -Berezin norm in [24]. The estimates in Theorem 4.5 use the radical filtration and the Peirce decomposition, and hold uniformly at all matrix levels. The completely bounded form of Crouzeix-type estimates is also studied in [25]. The general numerical-range theorem gives the following immediate choice of spectral set.

Corollary 4.6. *Let $\pi : C_u \rightarrow \mathcal{B}(\mathcal{H})$ be a faithful $*$ -representation and put*

$$\Omega = \text{conv}(\overline{W(\pi(a))} \cup \{0\}).$$

Then, Theorem 4.5 holds with $K = 1 + \sqrt{2}$.

Proof. Crouzeix–Palencia proved that the closure of the numerical range is a complete $(1 + \sqrt{2})$ -spectral set [26]. Enlarging a spectral set preserves the same inequality, and faithfulness of π preserves the norms of the matrix functional calculus. $\square$

5. Hilbert-module extensions and further questions

5.1. Hilbert C^* -module homotopes

Let A be a C^* -algebra and let E, F be right Hilbert A -modules. We write $\mathcal{L}_A(E, F)$ for the adjointable operators from E to F . The space $X = \mathcal{L}_A(F, E)$ is a TRO under ST^*R . Let $\delta \in \mathcal{L}_A(E, F)$ be a contraction and define

$$S \star_\delta T = S\delta T \quad (S, T \in X).$$

We recall that an adjointable operator with closed range has orthogonally complemented kernel and range and admits an adjointable polar decomposition; see [27]. Thus, if δ has closed range and $\delta = u|\delta|$, then

$$E = pE \oplus (1 - p)E, \quad F = qF \oplus (1 - q)F, \quad p = u^*u, \quad q = uu^*.$$

Theorem 5.1. *Let E, F be Hilbert A -modules and let $\delta \in \mathcal{L}_A(E, F)$ be a closed-range contraction. Then, $\mathcal{L}_A(F, E)_\delta$ has two canonical completely isometric defect representations. Moreover,*

$$\text{Rad}(\mathcal{L}_A(F, E)_\delta) = \{S \in \mathcal{L}_A(F, E) : \delta S \delta = 0\},$$

and the radical has cube zero. The semisimple quotient is completely boundedly isomorphic to the C^ -algebra $\mathcal{L}_A(qF)$.*

Proof. Choose a faithful representation of the unital C^* -algebra $\mathcal{L}_A(E \oplus F)$. Its (E, F) -corner identifies $\mathcal{L}_A(F, E)$ completely isometrically with a concrete TRO, while the opposite corner contains δ . The defect representations and the radical formula therefore follow from Theorems 2.1, 3.1, and 3.6; faithfulness makes these conclusions independent of the chosen representation.

It remains to identify the quotient. The corner $p\mathcal{L}_A(F, E)q$ is a C^* -algebra under $S \circ T = SuT$, and

$$\Xi : p\mathcal{L}_A(F, E)q \rightarrow \mathcal{L}_A(qF), \quad \Xi(S) = uS|_{qF},$$

is multiplicative because $\Xi(S \circ T) = uSuT = \Xi(S)\Xi(T)$, and it preserves the corner involution. It is injective since $S = u^*\Xi(S)$ on qF . Given $R \in \mathcal{L}_A(qF)$, extend R by zero on $(1 - q)F$ and put $S = u^*R$; then $S \in p\mathcal{L}_A(F, E)q$ and $\Xi(S) = R$. Thus, Ξ is onto, and the quotient assertion follows. $\square$

Remark 5.2. Neither a unit in A nor self-duality of E and F is required. The adjointable operator algebras $\mathcal{L}_A(E)$ and $\mathcal{L}_A(F)$ contain their identity operators even when A is nonunital. Closed range gives adjointable orthogonal projections p and q and hence the stated direct sums. Extension by zero from qF is adjointable, because the inclusion $qF \hookrightarrow F$ is adjointable. Thus, all four Peirce corners and their products are available. After polar normalization, Theorem 4.3 and the estimates of Theorem 4.5 apply with the unital corner $\mathcal{L}_A(qF)$ and its adjointable off-diagonal modules. For the original parameter δ , they are transported through Φ_s , with the corresponding normalization bounds.

Lemma 3.8 concerns full operator spaces on Hilbert spaces. Its proof uses rank-one operators and Hilbert-space reflexivity; it is not used in Theorem 5.1 or in the module functional calculus. We make no corresponding dimension classification for general Hilbert A -modules. The radical formula itself remains valid for arbitrary adjointable δ , without closed range, by Theorem 3.6; the complemented-corner description still requires the closed-range hypothesis.

5.2. Rigidity of crossed inner products

We now return to the crossed-inner-product setting of [15]. Let $\delta : K \rightarrow H$ be a contraction. A crossed inner product in the sense used there is a sesquilinear map $\beta : K \times H \rightarrow \mathbb{C}$, linear in the first variable, satisfying

$$\beta(k_1, \delta k_2) = \langle k_1, k_2 \rangle, \quad |\beta(k, h)| \leq \|k\| \|h\|.$$

We use inner products linear in the first variable.

Theorem 5.3. *A contraction $\delta : K \rightarrow H$ admits a crossed inner product if and only if δ is an isometry. In that case the crossed inner product is unique and is given by $\beta(k, h) = \langle \delta k, h \rangle$.*

Proof. Suppose β exists. For every $k \in K$,

$$\|k\|^2 = |\beta(k, \delta k)| \leq \|k\| \|\delta k\| \leq \|k\|^2.$$

Thus, $\|\delta k\| = \|k\|$, so δ is an isometry.

For fixed k , the Riesz representation theorem gives a unique vector $Vk \in H$ such that $\beta(k, h) = \langle Vk, h \rangle$. The generalized Cauchy–Schwarz bound implies $\|V\| \leq 1$. The extension identity gives $\delta^*V = I_K$. Since δ is an isometry, write $V = \delta + W$, where $\text{Ran } W \perp \text{Ran } \delta$. Then,

$$\|Vk\|^2 = \|k\|^2 + \|Wk\|^2.$$

The contractivity of V forces $W = 0$, and hence $V = \delta$.

Conversely, if δ is an isometry, the displayed formula for β has all the required properties. $\square$

For $D \in \mathcal{B}(H, K)$, define as in [15]

$$w_{\times}(D) = \sup_{\|h\|=1} |\beta(Dh, h)|.$$

Corollary 5.4. *Under the crossed-Hilbert hypotheses, $w_{\times}(D) = w(\delta D)$ and $\|\delta D\| = \|D\|$. Thus, the crossed numerical-radius inequalities are exactly the classical numerical-radius inequalities applied to δD .*

Proof. The first equality follows from Theorem 5.3; the second follows because left composition by the isometry δ preserves the operator norm. Both statements remain true at every matrix level. $\square$

We next identify the entire crossed module. Put $p = \delta\delta^*$ and

$$\mathcal{T}_p = \{T \in \mathcal{B}(H) : (1 - p)Tp = 0\}.$$

This is the triangular operator algebra associated with the two-step nest $\{0, pH, H\}$. We also write

$$\mathcal{B}(K, H, \delta) = \{(S, T) \in \mathcal{B}(K) \times \mathcal{B}(H) : T\delta = \delta S\},$$

equipped with componentwise operations and the maximum norm.

Theorem 5.5. *Let $\delta : K \rightarrow H$ be an isometry. The maps $\Theta : \mathcal{B}(H, K)_\delta \rightarrow p\mathcal{B}(H)$, $\Theta(D) = \delta D$, and $\Omega : \mathcal{B}(K, H, \delta) \rightarrow \mathcal{T}_p$, $\Omega(S, T) = T$, are completely isometric algebra isomorphisms. Under these maps, the boundary*

$$D \mapsto (D\delta, \delta D)$$

becomes the inclusion $p\mathcal{B}(H) \hookrightarrow \mathcal{T}_p$, and the left and right crossed actions become ordinary multiplication.

Proof. The map Θ is a complete isometry because δ is an isometry, and

$$\Theta(D \star_\delta E) = \delta D \delta E = \Theta(D)\Theta(E).$$

Its range is $p\mathcal{B}(H)$: if $A = pA$, then $A = \delta(\delta^*A)$.

If $(S, T) \in \mathcal{B}(K, H, \delta)$, then $T\delta = \delta S$, so T leaves $pH = \text{Ran } \delta$ invariant and $T \in \mathcal{T}_p$. Conversely, for $T \in \mathcal{T}_p$, set $S = \delta^*T\delta$. Invariance gives $T\delta = \delta S$. The pair is unique, and $\|S\| \leq \|T\|$; hence, the max norm of (S, T) equals $\|T\|$. The same argument at matrix levels proves complete isometry.

Finally,

$$\Omega(D\delta, \delta D) = \delta D = \Theta(D).$$

Moreover, $\Theta(SD) = \delta SD = T\delta D = \Omega(S, T)\Theta(D)$ and $\Theta(DT) = \Theta(D)\Omega(S, T)$. This identifies the boundary and actions. $\square$

Corollary 5.6. *If δ is an isometry, then*

$$\text{Rad}(\mathcal{B}(H, K)_\delta) = \{D \in \mathcal{B}(H, K) : D\delta = 0\} \quad \text{and} \quad \text{Rad}(\mathcal{B}(H, K)_\delta)^2 = 0.$$

The semisimple quotient is $$ -isomorphic to $\mathcal{B}(K)$.*

Proof. Under Θ , the radical is $p\mathcal{B}(H)(1 - p)$, whose square is zero. The compression $A \mapsto A|_{pH}$ gives the quotient $\mathcal{B}(pH) \cong \mathcal{B}(K)$. $\square$

Since δ is an isometry, one defect space is zero and the radical in this application has square zero.

5.3. Further questions

Theorem 3.6 determines the radical without a closed-range assumption. What remains is to understand the Banach-algebra structure of the quotient and the isomorphism classes when the positive part of the parameter cannot be removed by a bounded map.

Problem 5.7. Let $D_\alpha = \text{diag}(\alpha_n)$ and $D_\beta = \text{diag}(\beta_n)$ on ℓ^2 , where $\alpha_n, \beta_n > 0$ and both sequences decrease to zero. Characterize when $\mathcal{B}(\ell^2)_{D_\alpha}$ and $\mathcal{B}(\ell^2)_{D_\beta}$ are boundedly or completely boundedly isomorphic. In particular, determine which information about the decay of the two sequences is preserved by such an isomorphism.

For full operator TROs with nonzero closed-range parameters, the proof of Theorem 3.9 determines the dimensions of the range, kernel and cokernel from the quotient and its annihilator modules. For a general TRO, these modules need not be full operator spaces.

Problem 5.8. Let X and Y be TROs and let $z \in X^*$ and $w \in Y^*$ be nonzero closed-range parameters. Decide whether the C^* -algebras $X_z/\text{Rad}(X_z)$ and $Y_w/\text{Rad}(Y_w)$, together with their left and right annihilator modules and the multiplication pairing between them, form a complete invariant for bounded algebra isomorphism. Determine the answer first for W^* -TROs.

6. Conclusions

We have studied the associative homotopes X_z defined by $x \star_z y = xzy$ on a TRO X , with a contractive parameter $z \in X^*$. Completely isometric isomorphism is determined by the orbit of the parameter under ternary isomorphisms. For every such parameter, the Jacobson radical is

$$\text{Rad}(X_z) = \{x \in X : zxz = 0\}, \quad \text{Rad}(X_z)^3 = \{0\}.$$

This formula requires no closed-range assumption, and the nilpotency bound is sharp.

For closed-range parameters, polar normalization gives a completely bounded isomorphism to the partial-isometry homotope and identifies the semisimple quotient with a C^* -corner. For full operator TROs with nonzero closed-range parameters, the bounded and completely bounded isomorphism classes coincide and are determined by the dimensions of the range, kernel, and cokernel. On a fixed rectangular matrix space, this classification depends only on rank, whereas complete isometry depends on the full singular-value list.

The Peirce decomposition also yields explicit resolvent and holomorphic functional-calculus formulas involving the first two divided differences at zero, together with estimates valid at every matrix level. The closed-range construction extends to adjointable operators on Hilbert C^* -modules. The crossed-inner-product application reduces to the isometric case, where the inner product is uniquely determined.

The nonclosed-range example shows that semisimplicity alone does not ensure bounded isomorphism to a C^* -algebra. The classification of nonclosed-range homotopes and the reconstruction problem for general TROs remain open, as formulated in Section 5.

Author contributions

Hong-Wei Wang: Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing. **Selim Çetin:** Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing. **Mehmet Gürdal:** Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing. **Ömer Kişi:** Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing. **Qing-Bo Cai:** Conceptualization, methodology, formal analysis, writing-original draft, writing-review and editing, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

During the preparation of this work, the authors used OpenAI's ChatGPT and Codex to improve clarity, language, and formatting. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

Acknowledgments

This work is supported by Fujian Provincial Natural Science Foundation of China (Grant Nos. 2024J01792, 2022J011108).

Conflict of interest

The authors declare no conflicts of interest.

References

1. D. P. Blecher, Z. J. Ruan, A. M. Sinclair, A characterization of operator algebras, *J. Funct. Anal.*, **89** (1990), 188–201. [https://doi.org/10.1016/0022-1236\(90\)90010-I](https://doi.org/10.1016/0022-1236(90)90010-I)
2. M. Kaneda, Quasi-multipliers and algebrizations of an operator space, *J. Funct. Anal.*, **251** (2007), 346–359. <https://doi.org/10.1016/j.jfa.2006.12.014>
3. M. Kaneda, Quasi-multipliers and algebrizations of an operator space. II. Extreme points and quasi-identities, *J. Operator Theory*, **76** (2016), 219–248. <https://doi.org/10.7900/jot.2015jan31.2107>
4. M. Kaneda, V. I. Paulsen, Quasi-multipliers of operator spaces, *J. Funct. Anal.*, **217** (2004), 347–365. <https://doi.org/10.1016/j.jfa.2004.06.002>
5. W. Huang, J. Li, S. Pan, Some zero product preserving additive mappings of operator algebras, *AIMS Math.*, **9** (2024), 22213–22224. <https://doi.org/10.3934/math.20241080>
6. X. Qiang, Continuous orbit equivalence of groupoid actions, *AIMS Math.*, **11** (2026), 13287–13303. <https://doi.org/10.3934/math.2026548>
7. M. Neal, B. Russo, Operator space characterizations of C^* -algebras and ternary rings, *Pac. J. Math.*, **209** (2003), 339–364. <https://doi.org/10.2140/pjm.2003.209.339>

8. R. Pluta, B. Russo, Ternary operator categories, *J. Math. Anal. Appl.*, **505** (2022), 125590. <https://doi.org/10.1016/j.jmaa.2021.125590>
9. M. A. Youngson, Completely contractive projections on C^* -algebras, *Q. J. Math.*, **34** (1983), 507–511. <https://doi.org/10.1093/qmath/34.4.507>
10. H. Zettl, A characterization of ternary rings of operators, *Adv. Math.*, **48** (1983), 117–143. [https://doi.org/10.1016/0001-8708\(83\)90083-X](https://doi.org/10.1016/0001-8708(83)90083-X)
11. R. Pluta, B. Russo, Anti- C^* -algebras, *Banach J. Math. Anal.*, **18** (2024), 63. <https://doi.org/10.1007/s43037-024-00367-5>
12. I. Zhdanovskiy, Homotopes of finite-dimensional algebras, *Commun. Algebra*, **49** (2021), 43–57. <https://doi.org/10.1080/00927872.2020.1792918>
13. I. Dolinka, J. East, Semigroups of rectangular matrices under a sandwich operation, *Semigroup Forum*, **96** (2018), 253–300. <https://doi.org/10.1007/s00233-017-9873-6>
14. N. Altwaijry, S. S. Dragomir, Three operators' product norm and numerical radius inequalities in terms of the Moore–Penrose inverse, *AIMS Math.*, **11** (2026), 21355–21384. <https://doi.org/10.3934/math.2026866>
15. S. Çetin, M. Gürdal, U. Gürdal, F. Kittaneh, Analytic theory of crossed C^* -modules and generalized numerical radius inequalities, *J. Anal.*, **34** (2026), 2363–2385. <https://doi.org/10.1007/s41478-026-01089-7>
16. Z. J. Ruan, Subspaces of C^* -algebras, *J. Funct. Anal.*, **76** (1988), 217–230. [https://doi.org/10.1016/0022-1236\(88\)90057-2](https://doi.org/10.1016/0022-1236(88)90057-2)
17. D. P. Blecher, The Shilov boundary of an operator space and the characterization theorems, *J. Funct. Anal.*, **182** (2001), 280–343. <https://doi.org/10.1006/jfan.2000.3734>
18. D. P. Blecher, V. I. Paulsen, Multipliers of operator spaces, and the injective envelope, *Pac. J. Math.*, **200** (2001), 1–17. <https://doi.org/10.2140/pjm.2001.200.1>
19. M. Hamana, Triple envelopes and Šilov boundaries of operator spaces, *Math. J. Toyama Univ.*, **22** (1999), 77–93.
20. M. Gürdal, M. T. Garayev, S. Saltan, Some concrete operators and their properties, *Turk. J. Math.*, **39** (2015), 970–989. <https://doi.org/10.3906/mat-1502-48>
21. M. Gürdal, M. T. Garayev, S. Saltan, U. Yamancı, On some numerical characteristics of operators, *Arab J. Math. Sci.*, **21** (2015), 118–126. <https://doi.org/10.1016/j.ajmsc.2014.05.001>
22. U. Yamancı, M. Gürdal, Statistical convergence and operators on Fock space, *N. Y. J. Math.*, **22** (2016), 199–207. Available from: <https://api.semanticscholar.org/CorpusID:2968134>.
23. M. T. Garayev, H. Guediri, M. Gürdal, G. M. Alsahli, On some problems for operators on the reproducing kernel Hilbert space, *Linear Multilinear Algebra*, **69** (2021), 2059–2077. <https://doi.org/10.1080/03081087.2019.1659220>
24. X. L. Qiu, M. Gürdal, S. Çetin, Ö. Kişi, Q. B. Cai, Spectral estimates for multiparametric operator products via the $\mathbb{A}$ -Berezin norm in RKHS, *AIMS Math.*, **11** (2026), 6217–6230. <https://doi.org/10.3934/math.2026256>

-
25. R. Clouâtre, M. Ostermann, T. Ransford, An abstract approach to the Crouzeix conjecture, *J. Operator Theory*, **90** (2023), 209–221. <https://doi.org/10.7900/jot.2021nov15.2364>
26. M. Crouzeix, C. Palencia, The numerical range is a $(1 + \sqrt{2})$ -spectral set, *SIAM J. Matrix Anal. Appl.*, **38** (2017), 649–655. <https://doi.org/10.1137/17M1116672>
27. E. C. Lance, *Hilbert C^* -modules: a toolkit for operator algebraists*, *London mathematical society lecture note series*, Cambridge: Cambridge University Press, 1995. <https://doi.org/10.1017/CBO9780511526206>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)