



Research article

Dual G -orbital contractions: Common fixed points and applications to fractional differential equations

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Abstract: In this paper, we introduce the novel concept of dual G -orbital α - R -contractions for mapping pairs in b -metric spaces endowed with a directed graph. By employing the dual G -orbital transitive property alongside Property (E), we derive new existence and uniqueness theorems for common fixed points. To validate our theoretical results, we provide a concrete numerical example that demonstrates the computation of such points. Furthermore, we illustrate the applicability of our mathematical framework by proving the existence of a common solution to a system of nonlinear fractional differential equations involving Caputo fractional derivatives and nonlocal integral boundary conditions.

Keywords: common fixed point; b -metric space; directed graph; G -orbital ; dual connected-image contractions; fractional differential equations

Mathematics Subject Classification: 47H09, 47H10, 54H25

1. Introduction

Solving complex nonlinear equations is a central challenge in modern science and engineering, establishing fixed-point theory as an indispensable instrument in nonlinear analysis. This mathematical framework is particularly vital for investigating fractional differential equations (FDEs) [1–3], which are renowned for capturing nonlocal dynamics and memory phenomena. While the classical Caputo fractional derivative is extensively utilized due to its physical interpretability in

initial value problems, contemporary fractional-calculus literature exhibits a rich variety of alternative definitions that offer unique modeling advantages. Most notably, recent advancements have introduced the two-scale fractal derivative and fractal-based fractional calculus [4,5]. Unlike classical continuous fractional operators, these fractal approaches are explicitly designed to capture the discontinuous, multiscale behaviors inherent to highly porous or anomalous media. The integration of such fractal-fractional operators has become increasingly prominent, particularly when coupled with artificial intelligence (AI) and advanced computational tools to drive innovations in mechanical engineering and complex thermodynamic systems. By comparing these emerging fractal frameworks with the classical Caputo operators employed in this study, we gain a significantly broader theoretical perspective on how specialized mathematical operators can be tailored to capture distinct physical phenomena.

Furthermore, the landscape of modern fractional operator theory has rapidly evolved beyond classical definitions. Recent investigations have introduced highly generalized classes, such as (k, ψ) -Caputo multi-point systems and Θ -Caputo boundary value problems, which offer refined control over singular kernels and complex memory effects. Analyzing these intricate configurations often requires sophisticated methodologies, including the application of topological degree theory for solving tripled systems of multi-point boundary value problems [6]. Within this progressive context, our proposed Caputo-type system equipped with nonlocal integral conditions serves as a foundational bridge. The dual G -orbital contraction technique developed herein aligns seamlessly with these modern advancements, acting as a robust mathematical precursor that could be naturally adapted to analyze the solvability and well-posedness of these contemporary (k, ψ) -Caputo and Θ -Caputo generalized structures.

Beyond fractional dynamics, advanced fixed-point techniques remain fundamentally crucial for analyzing ordinary differential equations that model real-world scenarios, ranging from agricultural dynamics [7, 8] to climate fluctuations [9–11]. With the increasing complexity of mathematical models in economics and smart city infrastructures, the continuous development of generalized contraction mappings [12, 13] is more critical than ever to guarantee the existence and uniqueness of solutions in complicated dynamical systems.

Underpinning these advancements in nonlinear analysis is the topological structure of the spaces in which these operators act. To move beyond the constraints of standard metric topologies, Czerwik [14, 15] formulated the concept of b -metric spaces (see Definition 2.1). By softening the traditional triangle inequality with a scaling factor $b \geq 1$, this extension adeptly accommodates irregular geometries and unconventional contraction mappings.

To establish a broader context, it is imperative to align our proposed framework with recent advancements in generalized metric environments. Modern fixed-point research has successfully expanded into graph-structured controlled metric spaces and suprametric spaces, offering highly flexible topological structures. For instance, recent studies have explored analytical solutions for differential and nonlinear integral equations utilizing generalized contraction principles, such as F_{ϖ} -Suzuki contractions in modified ϖ -metric-like spaces [16], as well as Wardowski's contraction and fixed-point techniques [17]. While these advanced frameworks predominantly utilize generalized distance functions to relax the triangle inequality, our dual G -orbital approach distinguishes itself by directly coupling two distinct non-commuting mappings through the graph's edge connectivity in a b -metric space. This provides a dynamic, structure-preserving alternative to static controlled metrics,

particularly when dealing with coupled systems.

Parallel to this metric generalization, the integration of graph theory into analytical spaces has emerged as a powerful paradigm. Pioneered by Jachymski [18], this methodology evaluates contractive constraints exclusively along the edges of a directed graph. Consequently, it allows mathematicians to encode network topologies, dependencies, and partial orderings, providing a highly adaptable mechanism for examining fixed-point phenomena in topologically structured environments [19–21].

Building upon these foundations, contemporary scholars have sought to design novel contraction types that explicitly link mappings with underlying graphical structures. A notable example is the connected-image contraction introduced by Chaichana et al. [22], which enforces a contractive criterion while simultaneously preserving edge connectivity. Furthermore, the incorporation of auxiliary functions has proven advantageous in generalized spaces. For example, Suebcharoen et al. [23] explored fixed-point theorems in b -metric spaces equipped with graphs to manage intricate relational dependencies, whereas Karapınar et al. [1] applied auxiliary-based methods to derive solutions for FDEs.

Inspired by the aforementioned developments, the present study introduces the novel concept of a dual G -orbital α - R -contraction for a pair of non-commuting self-mappings operating within a graph-endowed b -metric space. By leveraging a specifically designed class of auxiliary functions, we formulate a highly versatile contractive condition that effectively models the synergistic interactions between the two mappings and the directed graph topology. Within this theoretical framework, we rigorously establish criteria guaranteeing both the existence and uniqueness of a common fixed point. To validate our theoretical contributions, we employ these newly derived results to resolve fractional differential equations, thereby confirming the framework's capability to tackle mathematical models characterized by complex dynamics and nonlocal operators.

The remainder of this paper is organized as follows. In Section 2, we establish the fundamental mathematical materials and methodological frameworks required for our analysis, beginning with the foundational concept of a b -metric space, which serves as the primary topological setting for our fixed-point theorems. Section 3 introduces the dual G -orbital property for an operator pair (f, g) alongside the novel concept of dual G -orbital α - R -contractions. Building upon these structures, we establish new existence and uniqueness theorems for common fixed points under appropriate conditions, supported by a concrete numerical example to validate our findings. Finally, Section 4 demonstrates the practical utility and broad applicability of our theoretical framework by employing it to solve a system of fractional differential equations.

2. Preliminaries

In this section, we establish the fundamental mathematical materials and methodological frameworks required for our analysis. We begin by recalling the foundational concept of a b -metric space, which serves as the primary topological setting for our fixed-point theorems.

Definition 2.1 ([14, 15]). *Let X be a nonempty set. A function $\beta : X \times X \rightarrow [0, \infty)$ is called a b -metric on X if, for all $x, y, z \in X$ and for some constant $b \geq 1$, it satisfies the following conditions:*

- $\beta(x, y) = 0$ if and only if $x = y$,

- $\beta(x, y) = \beta(y, x)$,
- $\beta(x, z) \leq b[\beta(x, y) + \beta(y, z)]$.

The pair (X, β) is known as a b -metric space.

As a result of this generalization, a wide array of fixed-point theorems has been developed within the context of b -metric spaces, offering greater adaptability for analyzing the existence and uniqueness of solutions. It is worth noting that standard metric spaces naturally emerge from this framework whenever the constant $b = 1$. To further solidify this mathematical environment, Czerwik [15] expanded the topological machinery of b -metric spaces by rigorously defining Cauchy sequences, convergence, and metric completeness, which are essential tools for our methodological approach.

Definition 2.2 ([14, 15]). Let (X, β) be a b -metric space and let $\{d_n\}$ be a sequence in X .

- The sequence $\{d_n\}$ is said to be **Cauchy** if, for every $\varepsilon > 0$, there exists $K_1 \in \mathbb{N}$ such that $\beta(d_n, d_m) < \varepsilon$ for all $n, m \geq K_1$.
- The sequence $\{d_n\}$ is said to **converge** to a point $d \in X$ if, for every $\varepsilon > 0$, there exists $K_2 \in \mathbb{N}$ such that $\beta(d_n, d) < \varepsilon$ for all $n \geq K_2$.
- The space (X, β) is said to be **complete** if every Cauchy sequence in X converges to a point in X .

3. Main results

In this section, we outline the fundamental concepts and basic definitions concerning dual G -orbital α - R -contractions, which form the theoretical backbone of our investigation. Relying on these prerequisites, we subsequently deduce the existence and uniqueness of common fixed points subject to our newly proposed contractive criteria. For the sake of clarity and to prevent the repetition of foundational conditions, the following notations and structural assumptions are uniformly adopted throughout the manuscript.

We consider the mathematical structure (X, β, f, g, G) satisfying the following criteria:

- X is a nonempty set;
- (X, β) denotes a b -metric space characterized by a parameter $b \geq 1$;
- $G = (V_G, E_G)$ represents a directed graph configured over X , possessing a vertex set $V_G = X$ and an edge set $E_G \subseteq X \times X$ that incorporates all self-loops (meaning $(x, x) \in E_G$ for every $x \in X$) without any parallel edges;
- $f, g : X \rightarrow X$ operate as self-mappings.

The concepts of the dual G -orbital property and dual G -orbital transitive property establish an intuitive mathematical environment for analyzing the interaction between two distinct self-mappings within a graph-equipped b -metric space. Specifically, these structures ensure the preservation of graph connectivity across successive applications of both operators, effectively generalizing the traditional connected-image property from a single mapping to a coupled pair.

Definition 3.1. Consider the configuration (X, β, f, g, G) . We say that the operator pair (f, g) possesses the **dual G -orbital property** if, for any element $a \in X$, the following conditions hold:

(O_1) if $(a, fa) \in E_G$, then $(fa, gfa) \in E_G$;

(O_2) if $(a, ga) \in E_G$, then $(ga, fga) \in E_G$.

Furthermore, the pair (f, g) is said to satisfy the **dual G -orbital transitive property** if it inherently possesses the dual G -orbital property and additionally satisfies the following two conditions for any $a, b \in X$:

(T_1) if $(a, b) \in E_G$ and $(b, fb) \in E_G$, then $(a, fb) \in E_G$;

(T_2) if $(a, b) \in E_G$ and $(b, gb) \in E_G$, then $(a, gb) \in E_G$.

Remark 3.2. Regarding the definitions presented above, we highlight the following key observations:

- 1) If the mappings f and g are traditionally edge-preserving (meaning $(x, y) \in E_G$ implies both $(fx, fy) \in E_G$ and $(gx, gy) \in E_G$ for all $x, y \in X$) and they commute, then the operator pair (f, g) automatically satisfies the dual G -orbital property.
- 2) In the specific scenario where the mappings are identical (i.e., $f = g$), the dual G -orbital property naturally reduces to the condition that the mapping f is orbitally edge-preserving. That is, $(x, fx) \in E_G \implies (fx, f^2x) \in E_G$ for all $x \in X$, a concept previously established in [13].

In the following, we define a fundamental class of auxiliary functions that serves as the theoretical core for our generalized contraction principle and the ensuing fixed-point results. We denote by $\mathbb{A}_b$ the collection of all mappings $\alpha : X^4 \rightarrow [0, \frac{1}{b}]$ that satisfy the following asymptotic property:

For any arbitrary sequences $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{d_n\}$ in X producing monotonically non-increasing distance sequences $\{\beta(a_n, b_n)\}$ and $\{\beta(c_n, d_n)\}$, the limiting behavior

$$\lim_{n \rightarrow \infty} \alpha(a_n, b_n, c_n, d_n) = \frac{1}{b}$$

guarantees that either

$$\lim_{n \rightarrow \infty} \beta(a_n, b_n) = 0 \quad \text{or} \quad \lim_{n \rightarrow \infty} \beta(c_n, d_n) = 0.$$

Remark 3.3. Intuitively, the asymptotic condition inherent in $\mathbb{A}_b$ prevents pseudo-contractive behavior by ensuring that iterative sequences do not stagnate at a positive distance when the multiplier approaches its supremum $\frac{1}{b}$.

Compared to standard simulation functions (e.g., $\mathcal{Z}$ -functions [24]), which typically employ bivariate conditions $(\zeta(t, s) < s - t)$ for a single mapping, our proposed class $\mathbb{A}_b$ utilizes a four-variable multiplier to capture the dynamic interplay between two distinct mappings across a graph structure. Furthermore, scaling the threshold specifically by $\frac{1}{b}$ directly counterbalances the relaxed triangle inequality in b -metric spaces an adjustment not required in classical settings.

Regarding topological regularity, $\alpha \in \mathbb{A}_b$ strictly does not require upper semi-continuity. While classical fixed-point theory frequently imposes this to prevent the contractive inequality from collapsing at the boundary, our framework encodes this safeguard directly into the asymptotic sequential condition (i.e., $\lim_{n \rightarrow \infty} \alpha = \frac{1}{b}$ rigidly forces the metric distances to vanish). Consequently, the control function α can be highly irregular or discontinuous while still rigorously guaranteeing convergence.

We now define a specific set that plays a crucial role in verifying the subsequent contraction condition, as follows:

$$E(f, g) := \{(u, v) \in X^2 : (fu, gv) \in E_G \text{ or } (gv, fu) \in E_G\}.$$

Guided by this setup, we present a novel class of contraction mappings specifically evaluated over the set $E(f, g)$. This generalized condition incorporates an auxiliary parameter $\alpha \in \mathbb{A}_b$ alongside a composite maximum function, purposely engineered to capture the metric displacements between points and their respective images under f and g .

Definition 3.4. *Given the structure (X, β, f, g, G) , the pair of mappings (f, g) is termed a **dual G -orbital α - R -contraction** if one can find an auxiliary function $\alpha \in \mathbb{A}_b$ ensuring that the condition*

$$b\beta(fx, gy) \leq \alpha(x, y, fx, gy)R(x, y) \quad (3.1)$$

is satisfied for any $x, y \in X$ with $(x, y) \in E(f, g)$.

Here, the constant $b \geq 1$ is the coefficient of the b -metric space, and the structural functional $R : X \times X \rightarrow [0, \infty)$ is defined as

$$R(x, y) = \max \left\{ \frac{\beta(x, fx)[1 + \beta(y, gy)]}{1 + \beta(x, y)}, \frac{\beta(y, gy)[1 + \beta(x, fx)]}{1 + \beta(x, y)}, \frac{\beta(x, gy) + \beta(y, fx)}{2b}, \beta(x, y) + |\beta(x, fx) - \beta(y, gy)| \right\}. \quad (3.2)$$

The subsequent lemma establishes a crucial behavior of the expression $R(x, y)$ when evaluated at the fixed points of the operators f and g . This specific characteristic is instrumental for demonstrating the validity of our principal theorems.

Lemma 3.5. *Consider the setting (X, β, f, g, G) . Suppose $x \in \text{Fix}(f) = \{u \in X : fu = u\}$ and $y \in \text{Fix}(g) = \{v \in X : gv = v\}$ are points satisfying the edge condition $(x, y) \in E_G$. Whenever the pair (f, g) constitutes a dual G -orbital α - R -contraction, or if $x = y$, it follows that*

$$R(x, y) = 0.$$

Proof. To begin, assume that the pair (f, g) forms a dual G -orbital α - R -contraction. Consider two points $x \in \text{Fix}(f)$ and $y \in \text{Fix}(g)$ sharing the edge relation $(x, y) \in E_G$. Because $fx = x$ and $gy = y$, we have $\beta(x, fx) = 0$ and $\beta(y, gy) = 0$. It is also evident that $(fx, gy) = (x, y) \in E_G$, ensuring that $(x, y) \in E(f, g)$.

By substituting these fixed points into the definition of $R(x, y)$ in (3.2), we evaluate the four components as follows:

$$\begin{aligned} R(x, y) &= \max \left\{ \frac{0 \cdot [1 + 0]}{1 + \beta(x, y)}, \frac{0 \cdot [1 + 0]}{1 + \beta(x, y)}, \frac{\beta(x, y) + \beta(y, x)}{2b}, \beta(x, y) + |0 - 0| \right\} \\ &= \max \left\{ 0, 0, \frac{\beta(x, y)}{b}, \beta(x, y) \right\}. \end{aligned}$$

Since $b \geq 1$, it is strictly true that $\frac{\beta(x, y)}{b} \leq \beta(x, y)$. Thus, the expression directly reduces to $R(x, y) = \beta(x, y)$. Invoking the contractive property (3.1) along with the fact that $fx = x$ and $gy = y$, we obtain:

$$b\beta(x, y) = b\beta(fx, gy) \leq \alpha(x, y, fx, gy)R(x, y).$$

Assume towards a contradiction that $R(x, y) > 0$, which inherently means $\beta(x, y) > 0$. Since the auxiliary function is bounded by $\alpha < \frac{1}{b}$ and $b \geq 1$, this assumption leads to the following sequence of strict inequalities:

$$\begin{aligned} \beta(x, y) &\leq b\beta(x, y) \leq \alpha(x, y, fx, gy)\beta(x, y) \\ &< \frac{1}{b}\beta(x, y) \\ &\leq \beta(x, y). \end{aligned}$$

This yields $\beta(x, y) < \beta(x, y)$, which is a clear mathematical contradiction. Thus, we must have $R(x, y) = 0$.

For the second case, suppose $x = y$ such that $x \in \text{Fix}(f)$ and $y \in \text{Fix}(g)$. In this scenario, we observe that

$$fx = x = y = gy,$$

which immediately implies

$$\beta(x, y) = \beta(x, fx) = \beta(y, gy) = 0.$$

As a result, each of the four components within the formulation of $R(x, y)$ evaluates strictly to zero:

$$\begin{aligned} \frac{\beta(x, fx)[1 + \beta(y, gy)]}{1 + \beta(x, y)} &= 0, \\ \frac{\beta(y, gy)[1 + \beta(x, fx)]}{1 + \beta(x, y)} &= 0, \\ \frac{\beta(x, gy) + \beta(y, fx)}{2b} &= 0, \\ \beta(x, y) + |\beta(x, fx) - \beta(y, gy)| &= 0. \end{aligned}$$

By taking the maximum of these expressions according to (3.2), we effortlessly deduce that $R(x, y) = 0$. This completes the proof. $\square$

It is well known that an element invariant under both operators f and g is referred to as a common fixed point. Investigating the existence and uniqueness of these points forms the core objective of our current research. Before presenting the subsequent lemma, which characterizes these elements via the expression $R(x, y)$, we formally define their collection.

Definition 3.6. Consider the mathematical framework (X, β, f, g, G) . We use $Cm(f, g)$ to represent the set of all **common fixed points** shared by f and g , which is explicitly given by

$$Cm(f, g) = \{x \in X : fx = gx = x\} = \text{Fix}(f) \cap \text{Fix}(g).$$

Lemma 3.7. *Under the framework (X, β, f, g, G) , the identity*

$$R(x, y) = 0 \quad \text{holds if and only if} \quad x = y \in Cm(f, g)$$

for any arbitrary elements $x, y \in X$.

Proof. Assume first that $R(x, y) = 0$. Keeping in mind that $R(x, y)$ is constructed as the maximum among four non-negative quantities as defined in (3.2), each individual component must inherently be bounded by zero. This allows us to deduce the following system of inequalities:

$$\begin{aligned} 0 = R(x, y) &\geq \frac{\beta(x, fx) [1 + \beta(y, gy)]}{1 + \beta(x, y)} \geq 0, \\ 0 = R(x, y) &\geq \frac{\beta(y, gy) [1 + \beta(x, fx)]}{1 + \beta(x, y)} \geq 0, \\ 0 = R(x, y) &\geq \frac{\beta(x, gy) + \beta(y, fx)}{2b} \geq 0, \\ 0 = R(x, y) &\geq \beta(x, y) + |\beta(x, fx) - \beta(y, gy)| \geq \beta(x, y) \geq 0. \end{aligned}$$

From the first two inequalities, observing that the bracketed terms and the denominators are strictly bounded below by 1 (e.g., $1 + \beta(y, gy) \geq 1$), it is mathematically forced that $\beta(x, fx) = 0$ and $\beta(y, gy) = 0$. Simultaneously, the fourth inequality strictly dictates that $\beta(x, y) = 0$. As a direct consequence, we obtain $fx = x = y = gy$, establishing that $x = y \in Cm(f, g)$.

To prove the reverse implication, let us assume $x = y \in Cm(f, g)$. This inherently means $x \in \text{Fix}(f)$ alongside $y \in \text{Fix}(g)$. Invoking lemma 3.5, we immediately conclude that $R(x, y) = 0$, which completes the proof. $\square$

To analyze the interplay between the operators f and g , we construct an iterative sequence via their alternating application. This approach enables us to track the connectivity of the iterates while ensuring that the underlying graph structure remains strictly preserved throughout the process. We formalize this concept in the following definition.

Definition 3.8. *Within the framework (X, β, f, g, G) , a sequence $\{e_n\}_{n=0}^{\infty}$ generated in X is designated as an **even-odd G -sequence** provided that it satisfies the following criteria:*

- $e_{2n+1} = ge_{2n}$ alongside $e_{2n+2} = fe_{2n+1}$ for every $n \in \mathbb{N} \cup \{0\}$;
- the directed edge (e_n, e_m) belongs to E_G whenever $m \geq n \geq 0$.

Building upon the preceding definition, the following lemma establishes the precise conditions under which an even-odd G -sequence is guaranteed to exist.

Lemma 3.9. *Consider the setup (X, β, f, g, G) . Assume that the pair of mappings (f, g) satisfies the dual G -orbital transitive property. If there exists $e_0 \in X$ such that $(e_0, ge_0) \in E_G$, then it is always possible to construct an even-odd G -sequence in X .*

Proof. Select an arbitrary initial element $e_0 \in X$ such that $(e_0, ge_0) \in E_G$. Defining $e_1 = ge_0$ ensures that the directed edge (e_0, e_1) belongs to E_G . Because (f, g) obeys the dual G -orbital transitive property, applying axiom (O_2) directly implies $(e_1, fe_1) \in E_G$. We then set $e_2 = fe_1$, which inherently yields

$(e_1, e_2) \in E_G$. Utilizing axiom (O_1) , we deduce that $(e_2, ge_2) \in E_G$, allowing us to define $e_3 = ge_2$ and ensuring $(e_2, e_3) \in E_G$.

By carrying out this iterative procedure, we construct a sequence $\{e_n\}$ characterized by the alternating relations:

$$e_{2n+1} = ge_{2n} \quad \text{and} \quad e_{2n+2} = fe_{2n+1}.$$

This construction naturally guarantees that $(e_n, e_{n+1}) \in E_G$ for all $n \geq 0$, while concurrently maintaining $(e_{2n}, ge_{2n}) \in E_G$ and $(e_{2n+1}, fe_{2n+1}) \in E_G$ for every non-negative integer n .

Next, we must demonstrate that $(e_n, e_m) \in E_G$ holds for any $m \geq n$. We accomplish this by invoking the transitivity axioms (T_1) and (T_2) . Starting with the known fact that $(e_{2k}, e_{2k+1}) \in E_G$ and $(e_{2k+1}, fe_{2k+1}) \in E_G$, applying (T_1) yields

$$(e_{2k}, fe_{2k+1}) = (e_{2k}, e_{2k+2}) \in E_G.$$

Following this, since $(e_{2k}, e_{2k+2}) \in E_G$ and $(e_{2k+2}, ge_{2k+2}) \in E_G$, axiom (T_2) dictates that

$$(e_{2k}, ge_{2k+2}) = (e_{2k}, e_{2k+3}) \in E_G.$$

Through standard mathematical induction, this alternating application of (T_1) and (T_2) confirms that $(e_{2k}, e_{2k+M}) \in E_G$ is valid for any integer $M \geq 1$. By adopting a strictly symmetrical inductive approach for the odd-indexed terms, we establish that $(e_{2k+1}, e_{2k+1+N}) \in E_G$ holds for all $N \geq 1$.

Therefore, the overarching relationship

$$(e_n, e_m) \in E_G$$

is preserved for all integers $m \geq n \geq 0$, which strictly fulfills the requirements of an even-odd G -sequence and concludes the proof. $\square$

The following lemma plays a crucial role in the proof of our main theorem. Specifically, it establishes that if the asymptotic distance between consecutive terms of an even-odd G -sequence vanishes, the sequence is necessarily a Cauchy sequence.

Lemma 3.10. *Consider the framework (X, β, f, g, G) . Suppose that the coupled mappings (f, g) possess the dual G -orbital transitive property and operate as a dual G -orbital α -R-contraction. If an even-odd G -sequence $\{e_n\}$ exists such that the distance between consecutive terms vanishes, meaning*

$$\lim_{n \rightarrow \infty} \beta(e_n, e_{n+1}) = 0,$$

then $\{e_n\}$ is necessarily a Cauchy sequence.

Proof. To establish that $\{e_n\}$ is a Cauchy sequence in X , it is mathematically sufficient, given the vanishing consecutive distance, to prove that its even-indexed subsequence $\{e_{2n}\}$ is Cauchy. We proceed by contradiction. Assume that $\{e_{2n}\}$ is not a Cauchy sequence. This implies the existence of some $\epsilon > 0$ and two subsequences of integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$, where $m(k)$ is chosen as the smallest integer satisfying:

$$\beta(e_{2m(k)}, e_{2n(k)}) \geq \epsilon \quad \text{and} \quad \beta(e_{2m(k)-2}, e_{2n(k)}) < \epsilon.$$

By employing the generalized b -metric triangle inequality (where $b \geq 1$), we obtain the bounds

$$\epsilon \leq \beta(e_{2m(k)}, e_{2n(k)}) \leq b\beta(e_{2m(k)}, e_{2n(k)+1}) + b\beta(e_{2n(k)+1}, e_{2n(k)}).$$

Since the consecutive distances approach zero as $n \rightarrow \infty$, evaluating the limit inferior of the above inequality as $k \rightarrow \infty$ yields

$$\frac{\epsilon}{b} \leq \liminf_{k \rightarrow \infty} \beta(e_{2m(k)}, e_{2n(k)+1}) \leq \limsup_{k \rightarrow \infty} \beta(e_{2m(k)}, e_{2n(k)+1}). \quad (3.3)$$

By the constructive definition of the even-odd G -sequence, the relation

$$(e_{2m(k)}, e_{2n(k)+1}) = (fe_{2m(k)-1}, ge_{2n(k)}) \in E_G$$

holds, which fundamentally means $(e_{2m(k)-1}, e_{2n(k)}) \in E(f, g)$. This allows us to utilize the contractive condition:

$$\begin{aligned} \beta(e_{2m(k)}, e_{2n(k)+1}) &\leq b\beta(fe_{2m(k)-1}, ge_{2n(k)}) \\ &\leq \alpha(e_{2m(k)-1}, e_{2n(k)}, fe_{2m(k)-1}, ge_{2n(k)})R(e_{2m(k)-1}, e_{2n(k)}). \end{aligned}$$

For notational simplicity, let $\Sigma_n = \beta(e_n, e_{n+1})$. Knowing that $fe_{2m(k)-1} = e_{2m(k)}$ and $ge_{2n(k)} = e_{2n(k)+1}$, we can expand the structural function R as follows:

$$\begin{aligned} R(e_{2m(k)-1}, e_{2n(k)}) &= \max \left\{ \frac{\Sigma_{2m(k)-1} [1 + \Sigma_{2n(k)}]}{1 + \beta(e_{2m(k)-1}, e_{2n(k)})}, \frac{\Sigma_{2n(k)} [1 + \Sigma_{2m(k)-1}]}{1 + \beta(e_{2m(k)-1}, e_{2n(k)})}, \right. \\ &\quad \frac{\beta(e_{2m(k)-1}, e_{2n(k)+1}) + \beta(e_{2n(k)}, e_{2m(k)})}{2b}, \\ &\quad \left. \beta(e_{2m(k)-1}, e_{2n(k)}) + |\Sigma_{2m(k)-1} - \Sigma_{2n(k)}| \right\}. \end{aligned}$$

Given the initial premise that

$$\lim_{k \rightarrow \infty} \Sigma_{2m(k)-1} = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \Sigma_{2n(k)} = 0,$$

it is clear that the first two fractional components vanish entirely as $k \rightarrow \infty$. Additionally, applying the triangle inequality to the third component demonstrates that it is ultimately bounded by terms dominated by $\beta(e_{2m(k)-1}, e_{2n(k)})$. Consequently, the asymptotic behavior of R is strictly governed by the distance $\beta(e_{2m(k)-1}, e_{2n(k)})$, providing the estimate:

$$\begin{aligned} \liminf_{k \rightarrow \infty} R(e_{2m(k)-1}, e_{2n(k)}) &= \liminf_{k \rightarrow \infty} \beta(e_{2m(k)-1}, e_{2n(k)}) \\ &\leq \liminf_{k \rightarrow \infty} [b\beta(e_{2m(k)-1}, e_{2m(k)-2}) + b\beta(e_{2m(k)-2}, e_{2n(k)})] \\ &\leq b\epsilon. \end{aligned}$$

By taking the limit inferior as $k \rightarrow \infty$ and combining this result with (3.3), we construct the following inequality chain:

$$\epsilon = b \left(\frac{\epsilon}{b} \right)$$

$$\begin{aligned}
&\leq b \liminf_{k \rightarrow \infty} \beta(e_{2m(k)}, e_{2n(k)+1}) \\
&\leq \liminf_{k \rightarrow \infty} b\beta(fe_{2m(k)-1}, ge_{2n(k)}) \\
&\leq \liminf_{k \rightarrow \infty} \alpha(e_{2m(k)-1}, e_{2n(k)}, fe_{2m(k)-1}, ge_{2n(k)})R(e_{2m(k)-1}, e_{2n(k)}) \\
&\leq \liminf_{k \rightarrow \infty} \frac{1}{b}\beta(e_{2m(k)-1}, e_{2n(k)}) \\
&\leq \liminf_{k \rightarrow \infty} \frac{1}{b}(b\epsilon) \\
&= \epsilon.
\end{aligned}$$

By executing an identical evaluation for the limit superior as $k \rightarrow \infty$, we consistently arrive at:

$$\begin{aligned}
\epsilon &= b \left(\frac{\epsilon}{b} \right) \\
&\leq b \limsup_{k \rightarrow \infty} \beta(e_{2m(k)}, e_{2n(k)+1}) \\
&\leq \limsup_{k \rightarrow \infty} b\beta(fe_{2m(k)-1}, ge_{2n(k)}) \\
&\leq \limsup_{k \rightarrow \infty} \alpha(e_{2m(k)-1}, e_{2n(k)}, fe_{2m(k)-1}, ge_{2n(k)})R(e_{2m(k)-1}, e_{2n(k)}) \\
&\leq \limsup_{k \rightarrow \infty} \frac{1}{b}\beta(e_{2m(k)-1}, e_{2n(k)}) \\
&\leq \limsup_{k \rightarrow \infty} \frac{1}{b}(b\epsilon) \\
&= \epsilon.
\end{aligned}$$

The preceding inequality chains demonstrate that both the limit inferior and limit superior of the sequences

$$b\beta(fe_{2m(k)-1}, ge_{2n(k)}) \quad \text{and} \quad \frac{1}{b}\beta(e_{2m(k)-1}, e_{2n(k)})$$

are squeezed strictly between ϵ and ϵ . By the squeeze theorem for limits, the exact limits exist, and we obtain:

$$\lim_{k \rightarrow \infty} b\beta(fe_{2m(k)-1}, ge_{2n(k)}) = \epsilon \implies \lim_{k \rightarrow \infty} \beta(e_{2m(k)}, e_{2n(k)+1}) = \frac{\epsilon}{b} > 0, \quad (3.4)$$

and

$$\lim_{k \rightarrow \infty} \frac{1}{b}\beta(e_{2m(k)-1}, e_{2n(k)}) = \epsilon \implies \lim_{k \rightarrow \infty} \beta(e_{2m(k)-1}, e_{2n(k)}) = b\epsilon > 0. \quad (3.5)$$

Furthermore, this strict squeezing forces all intermediate terms in our estimation chain to converge to the same boundary values. Consequently, we deduce:

$$\lim_{k \rightarrow \infty} \alpha(e_{2m(k)-1}, e_{2n(k)}, fe_{2m(k)-1}, ge_{2n(k)})R(e_{2m(k)-1}, e_{2n(k)}) = \epsilon. \quad (3.6)$$

Recall that the asymptotic behavior of $R(e_{2m(k)-1}, e_{2n(k)})$ is dominated by $\beta(e_{2m(k)-1}, e_{2n(k)})$, which the chain shows converges exactly to $b\epsilon$. Since $b\epsilon > 0$, we can algebraically separate the limits from (3.6) to isolate the behavior of the auxiliary function α :

$$\lim_{k \rightarrow \infty} \alpha(e_{2m(k)-1}, e_{2n(k)}, fe_{2m(k)-1}, ge_{2n(k)}) = \frac{\epsilon}{b\epsilon} = \frac{1}{b}.$$

At this critical juncture, we invoke the fundamental property of the auxiliary class $\mathbb{A}_b$. The condition $\lim_{k \rightarrow \infty} \alpha = \frac{1}{b}$ rigorously demands that either the distance between the domain points or their respective images must collapse to zero. That is, it requires either

$$\lim_{k \rightarrow \infty} \beta(e_{2m(k)-1}, e_{2n(k)}) = 0 \quad \text{or} \quad \lim_{k \rightarrow \infty} \beta(e_{2m(k)}, e_{2n(k)+1}) = 0.$$

However, we have already established in (3.4) and (3.5) that these limits rigidly evaluate to

$$\lim_{k \rightarrow \infty} \beta(e_{2m(k)-1}, e_{2n(k)}) = b\epsilon > 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \beta(e_{2m(k)}, e_{2n(k)+1}) = \frac{\epsilon}{b} > 0.$$

This represents a direct and inescapable mathematical contradiction, since neither limit can be zero. Therefore, our initial assumption must be false, confirming that $\{e_{2n}\}$ is indeed a Cauchy sequence.

Ultimately, coupling this with the hypothesis $\lim_{n \rightarrow \infty} \beta(e_n, e_{n+1}) = 0$ rigorously establishes that the entire sequence $\{e_n\}$ is Cauchy in (X, β) . To formally verify this, for any arbitrary large integers n, m , we can associate them with even integers $2p$ and $2q$ such that $n \in \{2p, 2p+1\}$ and $m \in \{2q, 2q+1\}$. By repeatedly applying the generalized b -metric triangle inequality, we obtain:

$$\begin{aligned} \beta(e_n, e_m) &\leq b\beta(e_n, e_{2p}) + b\beta(e_{2p}, e_m) \\ &\leq b\beta(e_n, e_{2p}) + b^2\beta(e_{2p}, e_{2q}) + b^2\beta(e_{2q}, e_m). \end{aligned}$$

As $n, m \rightarrow \infty$, it inherently follows that $p, q \rightarrow \infty$. Since we have proven that the even-indexed sequence $\{e_{2n}\}$ is Cauchy, the term $\beta(e_{2p}, e_{2q}) \rightarrow 0$. Additionally, the terms $\beta(e_n, e_{2p})$ and $\beta(e_{2q}, e_m)$ evaluate to either exactly 0 (e.g., when $n = 2p$) or the distance between consecutive terms (e.g., $\beta(e_{2p+1}, e_{2p})$). By the hypothesis, these consecutive distances also converge to 0. Consequently, taking the limit as $n, m \rightarrow \infty$ directly yields

$$\lim_{n, m \rightarrow \infty} \beta(e_n, e_m) = 0,$$

which rigorously confirms that the entire alternating sequence $\{e_n\}$ is indeed a Cauchy sequence. $\square$

To establish our main result, we introduce Property (E), which characterizes the behavior of the graph topology with respect to the limits of convergent sequences generated by f and g . Within the framework of dual G -orbital α - R -contractions, this condition is essential for establishing the existence and uniqueness of common fixed points.

Definition 3.11. *Within the environment (X, β, f, g, G) , we say that the pair (f, g) satisfies **Property (E)** provided that for any sequence $\{e_n\}$ in X such that $(e_n, e_{n+1}) \in E_G$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} e_n = e$, the following assertions remain valid:*

- from $(e_n, e_{n+1}) \in E(f, g)$, we can deduce the existence of a subsequence $\{e_{n(k)}\}$ satisfying $(e_{n(k)}, e) \in E(f, g)$;
- from $(e_{n+1}, e_n) \in E(f, g)$, we can deduce the existence of a subsequence $\{e_{m(k)}\}$ satisfying $(e, e_{m(k)}) \in E(f, g)$.

Theorem 3.12. *Let (X, β, f, g, G) be the underlying framework, and assume that (X, β) is a complete b -metric space. Suppose that the following conditions hold:*

- (i) the operator pair (f, g) satisfies the dual G -orbital transitive property;
- (ii) (f, g) forms a dual G -orbital α - R -contraction;
- (iii) (f, g) fulfills Property (E);
- (iv) there exists $e_0 \in X$ such that $(e_0, ge_0) \in E_G$.

Then, f and g admit at least one common fixed point, i.e., $Cm(f, g) \neq \emptyset$.

Furthermore, if $(x, y) \in E_G$ for all $x \in \text{Fix}(f)$ and $y \in \text{Fix}(g)$, then the common fixed point of f and g is uniquely determined (meaning $Cm(f, g)$ is a singleton).

Proof. Assume that there exists $e_0 \in X$ such that $(e_0, ge_0) \in E_G$. By invoking Lemma 3.9, we are assured of the existence of an even-odd G -sequence $\{e_n\}$ generated in X , strictly following the recursive pattern:

$$e_{2n+1} = ge_{2n} \quad \text{and} \quad e_{2n+2} = fe_{2n+1}$$

for all integers $n \geq 0$.

The evolutionary behavior of $\{e_n\}$ can be categorized into two distinct scenarios:

Scenario 1: The sequence becomes stationary at a certain index $n_0 \geq 0$, satisfying $e_{n_0} = e_{n_0+1}$.

We analyze this based on the parity of n_0 . If n_0 is odd, we can express it as $n_0 = 2k - 1$ for some $k \in \mathbb{N}$. The stationarity condition yields

$$e_{2k-1} = e_{2k} = fe_{2k-1},$$

indicating that e_{2k-1} is a fixed point of f . We must now prove that $fe_{2k-1} = ge_{2k}$. For the sake of contradiction, assume they are distinct, meaning $\beta(fe_{2k-1}, ge_{2k}) > 0$. We evaluate the structural function R at the coordinates $x = e_{2k-1}$ and $y = e_{2k}$:

$$R(e_{2k-1}, e_{2k}) = \max \left\{ \frac{\beta(e_{2k-1}, fe_{2k-1})[1 + \beta(e_{2k}, ge_{2k})]}{1 + \beta(e_{2k-1}, e_{2k})}, \right. \\ \frac{\beta(e_{2k}, ge_{2k})[1 + \beta(e_{2k-1}, fe_{2k-1})]}{1 + \beta(e_{2k-1}, e_{2k})}, \\ \frac{\beta(e_{2k-1}, ge_{2k}) + \beta(e_{2k}, fe_{2k-1})}{2b}, \\ \left. \beta(e_{2k-1}, e_{2k}) + |\beta(e_{2k-1}, fe_{2k-1}) - \beta(e_{2k}, ge_{2k})| \right\}.$$

Given that $\beta(e_{2k-1}, e_{2k}) = 0$ and $\beta(e_{2k-1}, fe_{2k-1}) = 0$, the first term naturally vanishes. For the third term, since $e_{2k-1} = e_{2k}$ and $fe_{2k-1} = e_{2k}$, it simplifies directly to:

$$\frac{\beta(e_{2k}, ge_{2k}) + 0}{2b} \leq \frac{\beta(e_{2k}, e_{2k+1})}{2}.$$

Thus, the maximum function elegantly collapses exactly to:

$$R(e_{2k-1}, e_{2k}) = \max \left\{ 0, \beta(e_{2k}, e_{2k+1}), \frac{\beta(e_{2k}, e_{2k+1})}{2b}, \beta(e_{2k}, e_{2k+1}) \right\} = \beta(e_{2k}, e_{2k+1}).$$

Applying the dual G -orbital α - R -contraction property with $b \geq 1$ leads to:

$$\beta(e_{2k}, e_{2k+1}) \leq b\beta(fe_{2k-1}, ge_{2k})$$

$$\begin{aligned}
&\leq \alpha(e_{2k-1}, e_{2k}, fe_{2k-1}, ge_{2k})R(e_{2k-1}, e_{2k}) \\
&< \frac{1}{b}\beta(e_{2k}, e_{2k+1}) \\
&\leq \beta(e_{2k}, e_{2k+1}).
\end{aligned}$$

This strict inequality $\beta(e_{2k}, e_{2k+1}) < \beta(e_{2k}, e_{2k+1})$ is a glaring absurdity. Thus, our initial assumption is invalid, dictating that

$$e_{2k-1} = e_{2k} = fe_{2k-1} = ge_{2k},$$

which firmly places $e_{2k-1} \in Cm(f, g)$. A perfectly symmetric argument confirms the same result if n_0 is even.

Scenario 2: Assume consecutive terms are always distinct, such that $e_n \neq e_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Let the sequence of metric displacements be defined as

$$\Sigma_n = \beta(e_n, e_{n+1}).$$

We intend to prove that $\{\Sigma_n\}$ is monotonically non-increasing. We accomplish this by dividing the analysis into two sub-cases based on parity:

Case I: Assume, for contradiction, that $\Sigma_{2a} < \Sigma_{2a+1}$ for some $a \in \mathbb{N} \cup \{0\}$. Evaluating the expanded four-term function $R(e_{2a+1}, e_{2a})$ gives:

$$\begin{aligned}
R(e_{2a+1}, e_{2a}) = \max \left\{ \frac{\Sigma_{2a+1}(1 + \Sigma_{2a})}{1 + \Sigma_{2a}}, \frac{\Sigma_{2a}(1 + \Sigma_{2a+1})}{1 + \Sigma_{2a}}, \right. \\
\left. \frac{\beta(e_{2a+1}, e_{2a+1}) + \beta(e_{2a}, e_{2a+2})}{2b}, \Sigma_{2a} + |\Sigma_{2a+1} - \Sigma_{2a}| \right\}.
\end{aligned}$$

Because $\Sigma_{2a} < \Sigma_{2a+1}$, the second term is inherently smaller than Σ_{2a+1} . Furthermore, the third term evaluates to

$$\frac{0 + \beta(e_{2a}, e_{2a+2})}{2b} \leq \frac{b(\Sigma_{2a} + \Sigma_{2a+1})}{2b} = \frac{\Sigma_{2a} + \Sigma_{2a+1}}{2} < \Sigma_{2a+1}.$$

Consequently, the expression strictly evaluates to $R(e_{2a+1}, e_{2a}) = \Sigma_{2a+1}$. Using the contraction property, we formulate the bound:

$$0 < \Sigma_{2a+1} = \beta(fe_{2a+1}, ge_{2a}) \leq \alpha(e_{2a+1}, e_{2a}, fe_{2a+1}, ge_{2a})R(e_{2a+1}, e_{2a}) < \Sigma_{2a+1}.$$

This presents a mathematical impossibility, securing the fact that $\Sigma_{2n+1} \leq \Sigma_{2n}$.

Case II: In a similar vein, assume $\Sigma_{2b-1} < \Sigma_{2b}$ for some integer $b \in \mathbb{N}$. Replicating the evaluation of $R(e_{2b-1}, e_{2b})$, the maximum resolves securely to Σ_{2b} . The contractive condition then dictates:

$$0 < \Sigma_{2b} = \beta(fe_{2b-1}, ge_{2b}) \leq \alpha(e_{2b-1}, e_{2b}, fe_{2b-1}, ge_{2b})R(e_{2b-1}, e_{2b}) < \Sigma_{2b},$$

which is identically absurd, enforcing $\Sigma_{2n} \leq \Sigma_{2n-1}$.

Combining both cases, we produce the unbroken inequality chain

$$\Sigma_{2n+1} \leq \Sigma_{2n} \leq \Sigma_{2n-1} \quad \text{for all } n \in \mathbb{N},$$

confirming that $\{\Sigma_n\}$ is a monotonically non-increasing sequence bounded below by zero. Thus, it converges to some limit $L \geq 0$. Assuming $L > 0$, taking the limit of the contractive inequality yields

$L \leq \lim_{n \rightarrow \infty} \alpha \cdot L$, which algebraically forces $1 \leq \lim_{n \rightarrow \infty} \alpha \leq \frac{1}{b}$. This directly dictates that $\lim_{n \rightarrow \infty} \alpha = \frac{1}{b}$, triggering the fundamental contradiction of $\mathbb{A}_b$ unless the metric limits vanish. Therefore, $L = 0$, meaning

$$\lim_{n \rightarrow \infty} \beta(e_n, e_{n+1}) = 0. \quad (3.7)$$

By virtue of Lemma 3.10, this limit ensures that $\{e_n\}$ is a Cauchy sequence. The completeness of the space (X, β) guarantees its convergence to a point $\tilde{e} \in X$:

$$\lim_{n \rightarrow \infty} e_{2n} = \lim_{n \rightarrow \infty} e_{2n+1} = \tilde{e}. \quad (3.8)$$

Since $(e_{2n+1}, e_{2n}) \in E(f, g)$ (as evidenced by $(ge_{2n}, fe_{2n+1}) \in E_G$), invoking Property (E) allows us to deduce the existence of a subsequence $\{e_{2n(k)}\}$ satisfying $(\tilde{e}, e_{2n(k)}) \in E(f, g)$.

It remains to verify that $\tilde{e} \in Cm(f, g)$ by showing $\tilde{e} = f\tilde{e}$ and $\tilde{e} = g\tilde{e}$.

Suppose, for the sake of contradiction, that $\beta(\tilde{e}, f\tilde{e}) > 0$. Evaluating the function R at $(\tilde{e}, e_{2n(k)})$, we have:

$$R(\tilde{e}, e_{2n(k)}) = \max \left\{ \frac{\beta(\tilde{e}, f\tilde{e})[1 + \beta(e_{2n(k)}, ge_{2n(k)})]}{1 + \beta(\tilde{e}, e_{2n(k)})}, \right. \\ \frac{\beta(e_{2n(k)}, ge_{2n(k)})[1 + \beta(\tilde{e}, f\tilde{e})]}{1 + \beta(\tilde{e}, e_{2n(k)})}, \\ \frac{\beta(\tilde{e}, ge_{2n(k)}) + \beta(e_{2n(k)}, f\tilde{e})}{2b}, \\ \left. \beta(\tilde{e}, e_{2n(k)}) + |\beta(\tilde{e}, f\tilde{e}) - \beta(e_{2n(k)}, ge_{2n(k)})| \right\}.$$

Taking the limit as $k \rightarrow \infty$ and noting that $e_{2n(k)} \rightarrow \tilde{e}$ and $ge_{2n(k)} \rightarrow \tilde{e}$, the second term evaluates to 0. Since $b \geq 1$, the third term is bounded above by $\frac{\beta(\tilde{e}, f\tilde{e})}{2b} \leq \frac{\beta(\tilde{e}, f\tilde{e})}{2}$. Thus, the limiting value of the maximum expression simplifies to:

$$\lim_{k \rightarrow \infty} R(\tilde{e}, e_{2n(k)}) = \max \left\{ \beta(\tilde{e}, f\tilde{e}), 0, \frac{\beta(\tilde{e}, f\tilde{e})}{2b}, \beta(\tilde{e}, f\tilde{e}) \right\} = \beta(\tilde{e}, f\tilde{e}).$$

Since (f, g) forms a dual G -orbital α - R -contraction with $\alpha \in \mathbb{A}_b$, we deduce the following sequence of inequalities:

$$\begin{aligned} \beta(f\tilde{e}, \tilde{e}) &= \lim_{k \rightarrow \infty} \beta(f\tilde{e}, ge_{2n(k)}) \leq \lim_{k \rightarrow \infty} \alpha(\tilde{e}, e_{2n(k)}, f\tilde{e}, ge_{2n(k)}) \lim_{k \rightarrow \infty} R(\tilde{e}, e_{2n(k)}) \\ &\leq \frac{1}{b} \lim_{k \rightarrow \infty} R(\tilde{e}, e_{2n(k)}) \\ &\leq \beta(\tilde{e}, f\tilde{e}). \end{aligned}$$

Dividing through by the strictly positive term $\beta(\tilde{e}, f\tilde{e})$ bounds the auxiliary function as $1 \leq \lim_{k \rightarrow \infty} \alpha \leq \frac{1}{b}$.

This mathematically forces $\lim_{k \rightarrow \infty} \alpha = \frac{1}{b}$ (which also intrinsically requires $b = 1$). This limit immediately triggers the asymptotic requirement of the class $\mathbb{A}_b$, ultimately dictating that $\beta(\tilde{e}, f\tilde{e}) = 0$. This outright contradicts our initial assumption, thereby confirming $\tilde{e} = f\tilde{e}$.

To establish that $\tilde{e} = g\tilde{e}$, we first observe that

$$(fe_{2n-1}, ge_{2n}) = (e_{2n}, e_{2n+1}) \in E_G,$$

which means $(e_{2n-1}, e_{2n}) \in E(f, g)$. By virtue of Property (E), we can directly deduce the existence of a subsequence $\{e_{2n(k)-1}\}$ satisfying

$$(e_{2n(k)-1}, \tilde{e}) \in E(f, g), \quad (3.9)$$

which inherently implies $(fe_{2n(k)-1}, g\tilde{e}) \in E_G$.

Following a similar reasoning, assume $\tilde{e} \neq g\tilde{e}$, meaning $\beta(\tilde{e}, g\tilde{e}) > 0$. Evaluating $R(e_{2n(k)-1}, \tilde{e})$ and recalling that $fe_{2n(k)-1} = e_{2n(k)}$, we construct:

$$R(e_{2n(k)-1}, \tilde{e}) = \max \left\{ \frac{\beta(e_{2n(k)-1}, e_{2n(k)}) [1 + \beta(\tilde{e}, g\tilde{e})]}{1 + \beta(e_{2n(k)-1}, \tilde{e})}, \right. \\ \frac{\beta(\tilde{e}, g\tilde{e}) [1 + \beta(e_{2n(k)-1}, e_{2n(k)})]}{1 + \beta(e_{2n(k)-1}, \tilde{e})}, \\ \frac{\beta(e_{2n(k)-1}, g\tilde{e}) + \beta(\tilde{e}, e_{2n(k)})}{2b}, \\ \left. \beta(e_{2n(k)-1}, \tilde{e}) + |\beta(e_{2n(k)-1}, e_{2n(k)}) - \beta(\tilde{e}, g\tilde{e})| \right\}.$$

Taking the limit as $k \rightarrow \infty$ and applying the convergence relations (3.7) and (3.8), we obtain:

$$\lim_{k \rightarrow \infty} R(e_{2n(k)-1}, \tilde{e}) = \max \left\{ 0, \beta(\tilde{e}, g\tilde{e}), \frac{\beta(\tilde{e}, g\tilde{e})}{2b}, \beta(\tilde{e}, g\tilde{e}) \right\} = \beta(\tilde{e}, g\tilde{e}).$$

Invoking condition (3.9) and replicating the contractive argument leads to a direct contradiction. Hence, $\tilde{e} = g\tilde{e}$, completing the proof that $\tilde{e} \in Cm(f, g)$.

To conclude with uniqueness, suppose a distinct common fixed point $\tilde{c} \in Cm(f, g)$ exists. According to the theorem's premises, the relation $(\tilde{e}, \tilde{c}) \in E_G$ is guaranteed. Applying Lemma 3.5 directly yields $R(\tilde{e}, \tilde{c}) = 0$, which strictly forces $\tilde{c} = \tilde{e}$. Consequently, $Cm(f, g)$ comprises exactly one unique element. $\square$

Remark 3.13. Regarding the uniqueness of the common fixed point, the global graph hypothesis requiring $(x, y) \in E_G$ for all $x \in \text{Fix}(f)$ and $y \in \text{Fix}(g)$ is a relatively strong but strictly necessary condition. In graph-endowed metric spaces, the contraction condition is evaluated exclusively along the edges of the graph. If this connectivity assumption is relaxed or omitted, two distinct common fixed points could exist in entirely disconnected components of the graph. In such a scenario, the contractive condition between these two points becomes vacuous, imposing no mathematical restriction on their metric distance. This absence of metric shrinkage inherently allows multiple distinct common fixed points to coexist, thereby demonstrating the essential role of graph edges in guaranteeing uniqueness.

To illustrate the functionality and validity of our proposed fixed-point framework, we present the following concrete example. By systematically verifying all the conditions of Theorem 3.12, we demonstrate that the chosen pair of operators successfully yields a unique common fixed point.

Example 3.14. Consider the space $X = [0, 7]$ endowed with the distance function

$$\beta(x, y) = |x - y|^2, \quad \text{for all } x, y \in X.$$

It is well known that (X, β) forms a complete b -metric space with the coefficient $b = 2$.

We define two self-mappings $f, g : X \rightarrow X$ as follows:

$$fx = \begin{cases} \frac{2x}{7}, & x \in [0, 1], \\ \frac{x^2 + 3}{9}, & x \in (1, 7], \end{cases} \quad gx = \begin{cases} \frac{2x}{7}, & x \in [0, 1], \\ \frac{3x}{7} + 4, & x \in (1, 7]. \end{cases}$$

Additionally, we construct the auxiliary function $\alpha : X^4 \rightarrow [0, \frac{1}{2})$ by the rule:

$$\alpha(x, y, u, v) = \begin{cases} \frac{1}{3}, & \text{if } \beta(x, y) = \beta(u, v) = 0, \\ \frac{1}{2(1 + \beta(x, y) + \beta(u, v))}, & \text{otherwise.} \end{cases}$$

By definition, it is evident that $\alpha \in \mathbb{A}_2$. Furthermore, we specify the edge set of the graph as

$$E_G = \left\{ (x, y) : x, y \in \left[0, \frac{2}{7}\right] \right\}.$$

We now verify that this configuration satisfies all the prerequisites of Theorem 3.12.

Step 1: Validating the dual G -orbital α - R -contraction property.

Assume there exists an edge $(x, y) \in E(f, g)$. This structural condition necessitates that $x, y \in [0, \frac{2}{7}] \subset [0, 1]$. Consequently, the mappings yield

$$fx = \frac{2x}{7} \quad \text{and} \quad gy = \frac{2y}{7}.$$

Evaluating the b -metric distance between these images gives

$$\beta(fx, gy) = \left| \frac{2x}{7} - \frac{2y}{7} \right|^2 = \frac{4}{49} |x - y|^2 = \frac{4}{49} \beta(x, y).$$

Since $x, y \in [0, \frac{2}{7}]$, the maximum possible value for $\beta(x, y)$ is precisely $\frac{4}{49}$. Therefore, for any valid edge where $x \neq y$ (i.e., $\beta(x, y) > 0$), we deduce a firm lower bound for α :

$$\alpha(x, y, fx, gy) = \frac{1}{2(1 + \beta(x, y) + \beta(fx, gy))} \geq \frac{1}{2\left(1 + \frac{4}{49} + \frac{16}{2401}\right)} = \frac{2401}{5226} > \frac{1}{3}.$$

Taking the space coefficient $b = 2$ into consideration, the contractive expression on the left-hand side evaluates to:

$$b\beta(fx, gy) = 2\left(\frac{4}{49}\beta(x, y)\right) = \frac{8}{49}\beta(x, y).$$

Since $\frac{8}{49} < \frac{1}{3}$, and recognizing the intrinsic property $\beta(x, y) \leq R(x, y)$ (derived inherently from the fourth term of the evaluation $R(x, y)$), we obtain

$$b\beta(fx, gy) \leq \frac{1}{3}\beta(x, y) \leq \alpha(x, y, fx, gy)R(x, y).$$

(In the trivial case where $\beta(x, y) = 0$, the inequality holds identically). Thus, the paired mappings (f, g) successfully form a dual G -orbital α - R -contraction.

Step 2: Confirming Property (E).

Let $\{e_n\} \subset X$ be a sequence such that $(e_n, e_{n+1}) \in E_G$ for all $n \in \mathbb{N}$, and suppose $\lim_{n \rightarrow \infty} e_n = e$. The definition of the edge set restricts the sequence such that $e_n \in \left[0, \frac{2}{7}\right]$ for all n . Because this interval is closed, the limiting value e inherently falls within $\left[0, \frac{2}{7}\right]$. Knowing that

$$fe_n = ge_n = \frac{2e_n}{7} \quad \text{for all } n,$$

it immediately follows that

$$(fe_n, ge_{n+1}) \in E_G \quad \text{and} \quad (ge_n, fe_{n+1}) \in E_G,$$

which implies that

$$(e_n, e_{n+1}) \in E(f, g) \quad \text{and} \quad (e_{n+1}, e_n) \in E(f, g).$$

By selecting the sequence itself as the required subsequence (i.e., setting $m_k = n_k = n$), we obtain

$$(fe_{n_k}, ge) \in E_G \quad \text{and} \quad (ge_{m_k}, fe) \in E_G.$$

This directly translates to

$$(e_{n_k}, e) \in E(f, g) \quad \text{and} \quad (e, e_{m_k}) \in E(f, g),$$

which conclusively verifies Property (E) for the pair (f, g) .

Step 3: Confirming the dual G -orbital transitive property.

Given that the operators coincide on the domain under consideration, we have

$$f(x) = g(x) = \frac{2x}{7}, \quad \text{for all } x \in [0, 1].$$

By the construction of G , any edge $(x, y) \in E_G$ necessitates that $x, y \in \left[0, \frac{2}{7}\right]$. Because f and g are identical, conditions (O_1) and (O_2) are equivalent, as are (T_1) and (T_2) . Thus, it suffices to verify only (O_1) and (T_1) to satisfy Definition 3.1:

(O_1) and (O_2) : Suppose $(a, fa) \in E_G$, which dictates $a \in \left[0, \frac{2}{7}\right]$. It follows that $fa = \frac{2a}{7} \in \left[0, \frac{4}{49}\right]$ and $g(fa) = \frac{4a}{49} \in \left[0, \frac{8}{343}\right]$. Since $\left[0, \frac{8}{343}\right] \subset \left[0, \frac{2}{7}\right]$, both fa and gfa reside in $\left[0, \frac{2}{7}\right]$. Hence, $(fa, gfa) \in E_G$.

(T_1) and (T_2) : Assume $(a, b) \in E_G$ and $(b, fb) \in E_G$. This implies $a, b \in \left[0, \frac{2}{7}\right]$. Evaluating f at b yields $fb = \frac{2b}{7} \in \left[0, \frac{4}{49}\right]$. Since both a and fb belong to $\left[0, \frac{2}{7}\right]$, we conclude that $(a, fb) \in E_G$.

With all required criteria met, the pair (f, g) naturally satisfies the dual G -orbital transitive property.

Finally, observe that there exists $0 \in X$ such that $(0, g0) = (0, 0) \in E_G$. Having verified all the hypotheses of Theorem 3.12, we conclude that the mappings f and g share a unique common fixed point. Indeed, simple calculation confirms that

$$Cm(f, g) = \{0\}.$$

Remark on numerical iteration behavior: To practically observe the alternating sequence generated by f and g , let us choose an arbitrary initial point $e_0 \in [0, \frac{2}{7}]$. Following the even-odd G -sequence construction, we systematically obtain $e_1 = ge_0 = \frac{2}{7}e_0$, $e_2 = fe_1 = \frac{2}{7}e_1 = (\frac{2}{7})^2 e_0$, and generally $e_n = (\frac{2}{7})^n e_0$. This concise formulation reveals that the numerical iteration exhibits a rapid geometric decay. It strictly preserves the directed graph connectivity at each alternating step and smoothly converges to the unique common fixed point 0.

To rigorously substantiate the claim that the dual G -orbital framework significantly relaxes existing literature, we present the following strict counterexample. This example demonstrates a scenario where standard connected-image contractions and classical graph-based auxiliary techniques completely fail due to localized distance expansion, whereas our proposed framework succeeds.

Example 3.15. Consider the finite set $X = \{0, 1, 2, 3\}$ equipped with the standard absolute difference metric $\beta(x, y) = |x - y|$. It is trivial that (X, β) forms a complete b -metric space with coefficient $b = 1$. Let the self-mappings $f, g : X \rightarrow X$ be identical, defined by:

$$f(x) = g(x) = \begin{cases} 0, & \text{if } x \in \{0, 1, 2\}, \\ 2, & \text{if } x = 3. \end{cases}$$

We equip the space with a directed graph G characterized by the edge set:

$$E_G = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0), (2, 3)\}.$$

Let us test this configuration against standard connected-image contractions or classical Jachymski graph contractions. Consider the valid graph edge $(2, 3) \in E_G$. Evaluating the mappings yields $f(2) = 0$ and $f(3) = 2$. The metric distance between the images is $\beta(f(2), f(3)) = |0 - 2| = 2$. However, the initial distance between the points is $\beta(2, 3) = |2 - 3| = 1$. Because the distance explicitly expanded from 1 to 2, any standard graph contraction condition requiring $\beta(f(x), f(y)) \leq k\beta(x, y)$ for some $k \in [0, 1)$ fundamentally fails ($2 \leq k \cdot 1$ is impossible). Thus, classical methods cannot resolve this system.

We now evaluate our proposed framework by systematically verifying the conditions of Theorem 3.12:

- (i) **Dual G -orbital transitive property:** For the operator pair (f, g) with $f = g$, condition (O_1) requires that if $(a, fa) \in E_G$, then $(fa, ffa) \in E_G$. Testing all elements in X , this holds identically (for instance, $(1, f(1)) = (1, 0) \in E_G \implies (0, f(0)) = (0, 0) \in E_G$). Similar verifications on X confirm that conditions (T_1) and (T_2) are also fully respected, thus fulfilling the transitive property.

(ii) **Dual G -orbital α - R -contraction:** The contraction is strictly assessed over the set $E(f, g) = \{(u, v) : (fu, gv) \in E_G \text{ or } (gv, fu) \in E_G\}$. Since the images of f and g are strictly mapped to $\{0, 2\}$, we analyze the corresponding pairs:

- Since $(0, 0) \in E_G$, any (x, y) producing $f(x) = 0$ and $g(y) = 0$ belongs to $E(f, g)$. This covers $\{0, 1, 2\} \times \{0, 1, 2\}$.
- Since $(2, 2) \in E_G$, any (x, y) producing $f(x) = 2$ and $g(y) = 2$ belongs to $E(f, g)$. This specifically covers the pair $(3, 3)$.
- Note that $(0, 2) \notin E_G$ and $(2, 0) \notin E_G$. Thus, pairs mapping to these combinations (e.g., $(2, 3)$ or $(3, 1)$) are rightfully excluded from $E(f, g)$.

Consequently, $E(f, g) = (\{0, 1, 2\} \times \{0, 1, 2\}) \cup \{(3, 3)\}$. For every single pair $(x, y) \in E(f, g)$, we inherently have $f(x) = g(y)$, which dictates that $\beta(f(x), g(y)) = 0$. Therefore, the contraction inequality $b\beta(fx, gy) \leq \alpha(x, y, fx, gy)R(x, y)$ reduces to $0 \leq \alpha \cdot R(x, y)$, which is trivially satisfied for any valid $\alpha \in \mathbb{A}_1$.

(iii) **Property (E):** Because X is a finite space, any convergent sequence with respect to the absolute difference metric must be eventually constant. Consequently, Property (E) is satisfied trivially.

(iv) **Initial edge:** Choosing $e_0 = 0$, we observe that $(0, g(0)) = (0, 0) \in E_G$, ensuring the existence of a starting element.

Having successfully verified conditions (i)–(iv) while bypassing the expanding edge $(2, 3)$ by evaluating strictly along orbital paths, Theorem 3.12 guarantees the existence of a common fixed point $(Cm(f, g) \neq \emptyset)$.

Finally, we check the uniqueness criterion. We observe that $\text{Fix}(f) = \text{Fix}(g) = \{0\}$. For all $x \in \text{Fix}(f)$ and $y \in \text{Fix}(g)$, the pair $(x, y) = (0, 0) \in E_G$. Therefore, Theorem 3.12 correctly dictates that the common fixed point is uniquely determined as a singleton, which is precisely $Cm(f, g) = \{0\}$.

To explicitly illustrate the broad applicability of Theorem 3.12, we derive the following corollary by specializing the parameters to the case of a single self-mapping on a complete metric space.

Corollary 3.16. Let (X, β) be a complete metric space (that is, $b = 1$), and let $f : X \rightarrow X$ be a self-mapping. Suppose that the following conditions hold:

- (i) f is edge-preserving and the pair (f, f) satisfies Property (E);
- (ii) there exists $x_0 \in X$ such that $(x_0, fx_0) \in E_G$;
- (iii) for all $x, y \in X$ with $(x, y) \in E_G$,

$$\beta(fx, fy) \leq k \max \left\{ \beta(x, fx), \beta(y, fy), \frac{\beta(x, fy) + \beta(y, fx)}{2}, \beta(x, y) \right\},$$

where $k \in [0, 1)$.

Then, f has a fixed point.

Proof. Setting $g = f$ and choosing the constant auxiliary function $\alpha(x, y, u, v) = k \in [0, 1)$ with $b = 1$, it is straightforward to verify that all hypotheses of Theorem 3.12 are directly fulfilled by conditions (i)–(iii). Specifically, the edge-preserving property and Property (E) satisfy the transitive and structural requirements, condition (ii) provides the starting edge, and the contractive inequality reduces precisely to condition (iii). Consequently, Theorem 3.12 guarantees that $\text{Fix}(f) = Cm(f, f) \neq \emptyset$, implying that f admits at least one fixed point. $\square$

When G is an arbitrary directed graph, this corollary provides a graph-theoretic fixed-point result for a single self-mapping on a complete metric space. Under the corresponding assumptions on the directed graph, this framework naturally extends and generalizes Jachymski's foundational graph-preserving contraction theorems.

4. Applications to nonlinear fractional differential equations

In this section, we explore practical implementations encompassing fractional differential equations to validate that our newly developed contraction methodology not only broadens the scope of current literature but also preserves its efficiency within classical analytical environments. We focus primarily on differential equations of the Caputo fractional type. These equations serve as highly robust instruments for formulating sophisticated mathematical paradigms, primarily due to their intrinsic capacity to incorporate historical memory characteristics and substantially boost the adaptability of dynamic models.

Driven by these advantages, our attention shifts to investigating a pair of independent Caputo fractional boundary value problems characterized by an order $\delta \in (n - 1, n]$, with $n \geq 2$.

For the remainder of this examination, let $\eta \in (0, 1)$ be a predefined constant, and suppose $f, g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ represent continuous mappings. Recalling the standard Gamma function denoted by $\Gamma(\delta) = \int_0^\infty s^{\delta-1} e^{-s} ds$, we establish a generalized integral operator $\hat{I}$, which takes the form:

$$\hat{I}(u, v, \varphi, h) = \int_0^u (u - v)^{n-\delta-1} \varphi(v, h(v)) dv, \quad (4.1)$$

where u represents the upper limit of integration and v serves as the dummy variable of integration. To simplify our theoretical investigation, we transform the differential configurations into their equivalent fixed-point frameworks. This is accomplished by establishing the integral operators $S, T : C[0, 1] \rightarrow C[0, 1]$. We partition the overarching structure into two isolated systems as described below:

System I: Primary Boundary Value Configuration

The dynamics of the first framework are dictated by the fractional differential relation:

$${}^c D^\delta x(t) = f(t, x(t)), \quad t \in [0, 1], \quad (4.2)$$

which is constrained by the boundary conditions:

$$x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0, \quad x(1) = \int_0^\eta x(s) ds. \quad (4.3)$$

The analytical solution corresponding to the setup (4.2)–(4.3) can be formulated as:

$$\begin{aligned} x(t) = & \frac{nt^n}{(n - \eta^n)\Gamma(n - \delta)} \int_0^\eta \hat{I}(s, r, f, x) ds \\ & - \frac{nt^n}{(n - \eta^n)\Gamma(n - \delta)} \hat{I}(1, s, f, x) \\ & + \frac{1}{\Gamma(n - \delta)} \hat{I}(t, s, f, x). \end{aligned}$$

Consequently, we construct the associated integral mapping S using the equation:

$$\begin{aligned}(Sx)(t) = & \frac{nt^n}{(n-\eta^n)\Gamma(n-\delta)} \int_0^\eta \hat{I}(s, r, f, x) ds \\ & - \frac{nt^n}{(n-\eta^n)\Gamma(n-\delta)} \hat{I}(1, s, f, x) \\ & + \frac{1}{\Gamma(n-\delta)} \hat{I}(t, s, f, x).\end{aligned}\quad (4.4)$$

System II: Secondary Boundary Value Configuration

In a parallel manner, the second framework is described by the equation:

$${}^c D^\delta y(t) = g(t, y(t)), \quad t \in [0, 1], \quad (4.5)$$

subject to the analogous boundary constraints:

$$y(0) = y'(0) = \dots = y^{(n-2)}(0) = 0, \quad y(1) = \int_0^\eta y(s) ds. \quad (4.6)$$

The exact solution associated with (4.5)–(4.6) is given by:

$$\begin{aligned}y(t) = & \frac{nt^n}{(n-\eta^n)\Gamma(n-\delta)} \int_0^\eta \hat{I}(s, r, g, y) ds \\ & - \frac{nt^n}{(n-\eta^n)\Gamma(n-\delta)} \hat{I}(1, s, g, y) \\ & + \frac{1}{\Gamma(n-\delta)} \hat{I}(t, s, g, y).\end{aligned}$$

Correspondingly, the integral operator T is generated as:

$$\begin{aligned}(Ty)(t) = & \frac{nt^n}{(n-\eta^n)\Gamma(n-\delta)} \int_0^\eta \hat{I}(s, r, g, y) ds \\ & - \frac{nt^n}{(n-\eta^n)\Gamma(n-\delta)} \hat{I}(1, s, g, y) \\ & + \frac{1}{\Gamma(n-\delta)} \hat{I}(t, s, g, y).\end{aligned}\quad (4.7)$$

Ultimately, discovering a shared analytical solution for the paired fractional differential systems translates directly into identifying a common fixed point for the newly constructed operators S and T .

To achieve this, we demonstrate the real-world applicability of our main common fixed-point result through its application to nonlinear fractional differential equations (FDEs). Because they excellently capture memory effects and hereditary characteristics across diverse scientific and engineering fields, FDEs have attracted considerable research attention. By reformulating these boundary value problems into equivalent fixed-point equations within an appropriate function space, our theoretical findings on the dual G -orbital α - R -contraction can be directly utilized to assert both the existence and uniqueness of their solutions.

Definition 4.1. Suppose X is a nonempty set accompanied by two self-operators S and T . Let $P(S, T)$ represent the family of all mappings

$$p : X \times X \rightarrow \mathbb{R}$$

satisfying the following conditions for any arbitrary $u, v \in X$:

- (P_1) if $p(u, Su) \geq 0$, then $p(Su, TSu) \geq 0$;
- (P_2) if $p(u, Tu) \geq 0$, then $p(Tu, STu) \geq 0$;
- (P_3) if $p(u, v) \geq 0$ and $p(v, Sv) \geq 0$, then $p(u, Sv) \geq 0$;
- (P_4) if $p(u, v) \geq 0$ and $p(v, Tv) \geq 0$, then $p(u, Tv) \geq 0$.

Given a specific condition-mapping $p \in P(S, T)$, a corresponding directed graph is formally established as

$$G_p = (X, E_{G_p}),$$

where the edge set is defined according to:

$$E_{G_p} = \{(u, v) \in X \times X : p(u, v) \geq 0\}.$$

To assist readers in visualizing how the graph edges are practically realized within the function space $C[0, 1]$, we provide a concrete, nontrivial example of a condition-mapping $p \in P(S, T)$.

Example 4.2. Consider the mapping $p : C[0, 1] \times C[0, 1] \rightarrow \mathbb{R}$ defined by the standard pointwise partial order:

$$p(x, y) = \inf_{t \in [0, 1]} (y(t) - x(t)).$$

In this scenario, a directed edge $(x, y) \in E_{G_p}$ exists if and only if $p(x, y) \geq 0$, which directly translates to $x(t) \leq y(t)$ for all $t \in [0, 1]$.

To ensure the integral operators fit this framework naturally, we assume that the nonlinear source functions $f(t, u)$ and $g(t, u)$ satisfy a cross-monotonicity condition: $f(t, u) \leq g(t, v)$ and $g(t, u) \leq f(t, v)$ whenever $u \leq v$. This inherently guarantees that for any $x, y \in C[0, 1]$ with $x(t) \leq y(t)$, the resulting integral operators satisfy $(Sx)(t) \leq (Ty)(t)$ and $(Tx)(t) \leq (Sy)(t)$.

Under this ordered behavior, we can systematically verify the four structural conditions of the family $P(S, T)$ for any arbitrary $u, v \in C[0, 1]$:

- (P_1) Suppose $p(u, Su) \geq 0$, meaning $u(t) \leq (Su)(t)$. By applying the cross-monotonicity property, it inherently follows that $(Su)(t) \leq (T(Su))(t)$, which directly implies $p(Su, TSu) \geq 0$.
- (P_2) Similarly, if $p(u, Tu) \geq 0$, meaning $u(t) \leq (Tu)(t)$, the cross-monotonicity yields $(Tu)(t) \leq (S(Tu))(t)$, ensuring that $p(Tu, STu) \geq 0$.
- (P_3) Assume $p(u, v) \geq 0$ and $p(v, Sv) \geq 0$. This implies $u(t) \leq v(t)$ and $v(t) \leq (Sv)(t)$. By the fundamental transitivity of the pointwise partial order, we obtain $u(t) \leq (Sv)(t)$, which means $p(u, Sv) \geq 0$.
- (P_4) Assume $p(u, v) \geq 0$ and $p(v, Tv) \geq 0$. This translates to $u(t) \leq v(t)$ and $v(t) \leq (Tv)(t)$. Again, transitivity guarantees $u(t) \leq (Tv)(t)$, safely verifying that $p(u, Tv) \geq 0$.

Because all conditions (P_1)–(P_4) are organically and robustly satisfied, we conclude that the condition-mapping p successfully belongs to the family $P(S, T)$.

Lemma 4.3. Consider the framework $(C[0, 1], \beta, S, T, G_p)$ equipped with an element $p \in P(S, T)$. Under these conditions, the operator pair (S, T) satisfies the dual G -orbital transitive property. Furthermore, the pair (S, T) inherently fulfills Property (E).

Proof. This outcome arises naturally as a direct mathematical consequence of Definition 4.1. By applying these criteria, it follows that the operator pair (S, T) satisfies the dual G -orbital transitive property and inherently fulfills Property (E). $\square$

Let $X = C([0, 1], \mathbb{R})$ be the space of all continuous real-valued functions defined on the interval $[0, 1]$. We define a distance function $\beta : X \times X \rightarrow [0, \infty)$ by

$$\beta(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|^2 = \|x - y\|_\infty^2 \quad (4.8)$$

for all $x, y \in X$. It is well known that (X, β) constitutes a complete b -metric space with the coefficient $b = 2$.

Following this, we define a vital inequality that will anchor the proof of our upcoming theorem.

Definition 4.4. In the context of $(C[0, 1], \beta, S, T, G_p)$ with $p \in P(S, T)$, let the integral operators S and T be specified via (4.4) and (4.7). We state that the nonlinearities f and g adhere to **Condition (I)** provided that the bounding relation

$$|f(t, x(t)) - g(t, y(t))|^2 \leq \frac{K}{12} \arctan(\beta(x, y))$$

holds for all $t \in [0, 1]$, where the structural constant K is precisely defined by:

$$K = \left[\sup_{t \in [0, 1]} \left(\left(\frac{n}{(n - \eta^n)\Gamma(n - \delta)} \right)^2 \frac{\eta^2}{2} \int_0^\eta \int_0^s (s - r)^{2(n-\delta-1)} dr ds \right. \right. \\ \left. \left. + \left(\frac{n}{(n - \eta^n)\Gamma(n - \delta)} \right)^2 \int_0^1 (1 - s)^{2(n-\delta-1)} ds \right. \right. \\ \left. \left. + \frac{1}{\Gamma(n - \delta)^2} \int_0^t (t - s)^{2(n-\delta-1)} ds \right) \right]^{-1}.$$

Lemma 4.5. Suppose we are working within $(C[0, 1], \beta, S, T, G_p)$ for some $p \in P(S, T)$, with integral operators S and T characterized by (4.4) and (4.7). If the nonlinear terms f and g conform to Condition (I), it is guaranteed that the pair (S, T) acts as a dual G -orbital α - R -contraction.

Proof. Let $x, y \in C[0, 1]$ be such that $(x, y) \in E(S, T)$. Using the definitions of S and T in (4.4) and (4.7), we first consider the squared difference multiplied by the b -metric constant $b = 2$:

$$2|(Sx)(t) - (Ty)(t)|^2 = 2 \left| \frac{nt^n}{(n - \eta^n)\Gamma(n - \delta)} \int_0^\eta \int_0^s (s - r)^{n-\delta-1} (f(r, x(r)) - g(r, y(r))) dr ds \right. \\ \left. - \frac{nt^n}{(n - \eta^n)\Gamma(n - \delta)} \int_0^1 (1 - s)^{n-\delta-1} (f(s, x(s)) - g(s, y(s))) ds \right. \\ \left. + \frac{1}{\Gamma(n - \delta)} \int_0^t (t - s)^{n-\delta-1} (f(s, x(s)) - g(s, y(s))) ds \right|^2.$$

To establish the bounds, we explicitly apply the elementary algebraic inequality $(A - B + C)^2 \leq 3(A^2 + B^2 + C^2)$. Multiplying this by our initial b -metric factor of 2 strictly yields the pre-factor 6:

$$2|(Sx)(t) - (Ty)(t)|^2 \leq 6 \left[\left(\frac{nt^n}{(n - \eta^n)\Gamma(n - \delta)} \int_0^\eta \int_0^s (s - r)^{n-\delta-1} |f(r, x(r)) - g(r, y(r))| dr ds \right)^2 \right. \\ \left. + \left(\frac{nt^n}{(n - \eta^n)\Gamma(n - \delta)} \int_0^1 (1 - s)^{n-\delta-1} |f(s, x(s)) - g(s, y(s))| ds \right)^2 \right. \\ \left. + \left(\frac{1}{\Gamma(n - \delta)} \int_0^t (t - s)^{n-\delta-1} |f(s, x(s)) - g(s, y(s))| ds \right)^2 \right].$$

Next, we formally apply the Cauchy-Schwarz inequality for integrals, namely

$$\left(\int u(s)v(s) ds \right)^2 \leq \left(\int u(s)^2 ds \right) \left(\int v(s)^2 ds \right),$$

independently to each of the three terms. This operation analytically squares the fractional constants and the integral kernel $(t - s)^{n-\delta-1}$, yielding the exact exponent $2(n - \delta - 1)$. Noting that $t \leq 1$ and $\int_0^\eta \int_0^s 1 dr ds = \frac{\eta^2}{2}$, we rigorously obtain:

$$2|(Sx)(t) - (Ty)(t)|^2 \leq 6 \left[\left(\frac{n}{(n - \eta^n)\Gamma(n - \delta)} \right)^2 \frac{\eta^2}{2} \left(\int_0^\eta \int_0^s (s - r)^{2(n-\delta-1)} dr ds \right) \right. \\ \left. + \left(\frac{n}{(n - \eta^n)\Gamma(n - \delta)} \right)^2 \left(\int_0^1 (1 - s)^{2(n-\delta-1)} ds \right) \right. \\ \left. + \frac{1}{\Gamma(n - \delta)^2} \left(\int_0^t (t - s)^{2(n-\delta-1)} ds \right) \right] \left(\sup_{s \in [0,1]} |f(s, x(s)) - g(s, y(s))|^2 \right).$$

By adhering to Condition (I), we substitute the upper bound

$$|f(s, x(s)) - g(s, y(s))|^2 \leq \frac{K}{12} \arctan(\beta(x, y))$$

into our inequality. Utilizing the precise formulation of the constant K defined in Definition 4.4, the entire bracketed expression simplifies significantly. We arrive at:

$$2|(Sx)(t) - (Ty)(t)|^2 \leq 6 \left(\frac{1}{K} \right) \left(\frac{K}{12} \arctan(\beta(x, y)) \right) \\ = \frac{1}{2} \arctan(\beta(x, y)) \\ = \frac{\arctan(\beta(x, y))}{2\beta(x, y)} \beta(x, y).$$

To formally establish the required contraction, we define the auxiliary function $\alpha : X^4 \rightarrow [0, \frac{1}{2})$ by:

$$\alpha(x, y, u, v) = \begin{cases} \frac{\arctan(\beta(x, y))}{2\beta(x, y)}, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Since $\beta(x, y) \leq R(x, y)$ inherently holds from the fourth component of the function R , taking the supremum over $t \in [0, 1]$ across the established inequality yields:

$$2\beta(Sx, Ty) \leq \alpha(x, y, Sx, Ty)R(x, y).$$

This rigorously confirms that the integral operator pair (S, T) acts as a dual G -orbital α - R -contraction, completing the proof. $\square$

Having established the necessary preliminary structures, we proceed to formulate the primary existence theorem concerning the fractional boundary conditions (4.2)–(4.3) and (4.5)–(4.6).

Theorem 4.6. *Let $(C[0, 1], \beta, S, T, G_p)$ be the underlying space equipped with some $p \in P(S, T)$. Suppose that the integral operators T and S , defined by (4.4) and (4.7), satisfy Condition (I). If there exists an initial function $u_0 \in C[0, 1]$ such that*

$$p(u_0, Tu_0) \geq 0,$$

then the operators T and S admit a common fixed point $u^ \in C[0, 1]$. Consequently, this u^* serves as a common solution to the nonlinear fractional boundary value problems (4.2)–(4.3) and (4.5)–(4.6).*

Proof. By Lemmas 4.3 and 4.5, it is established that the pair (S, T) forms a dual G -orbital α - R -contraction satisfying Property (E), and that (S, T) possesses the dual G -orbital transitive property. Furthermore, the assumption that there exists an initial function $u_0 \in C[0, 1]$ satisfying $p(u_0, Tu_0) \geq 0$ directly implies that $(u_0, Tu_0) \in E_{G_p}$. Since all the hypotheses of Theorem 3.12 are fulfilled, we conclude that the operators T and S admit a common fixed point $u^* \in C[0, 1]$. $\square$

5. Conclusions

In this paper, we introduced and established the novel mathematical framework of dual G -orbital α - R -contractions for a pair of non-commuting self-mappings within b -metric spaces endowed with directed graphs. By integrating the dual G -orbital transitive property with Property (E), we derived robust criteria guaranteeing the existence and uniqueness of common fixed points. This geometric and topological approach effectively relaxes the standard continuity and commutativity constraints typically required in the classical literature.

Beyond theoretical advancements, we validated our fundamental findings through a concrete numerical example, which demonstrated the practical computation and rapid convergence behavior of such fixed points. Most notably, we illustrated the broad applicability of our methodology by investigating a system of nonlinear fractional differential equations of the Caputo type, subject to nonlocal integral boundary conditions. By reformulating these fractional boundary value problems into equivalent integral operators within the complete function space $C[0, 1]$, we rigorously proved the existence of a common solution using our newly generalized contraction principles.

Overall, this study not only unifies and extends several existing fixed-point theorems but also provides a versatile analytical tool for modern fractional calculus. Looking forward, natural avenues for prospective extensions involve generalizing this framework to multi-valued mappings, fuzzy metric spaces, and complex coupled integro-differential systems. However, these prospective generalizations present distinct technical obstacles. For instance, transitioning to multi-valued

mappings necessitates the highly complex task of redefining graph edge connectivity strictly between subsets using Hausdorff metric generalizations. Similarly, adapting our novel auxiliary class $\mathbb{A}_b$ to fuzzy metric spaces requires completely reformulating the asymptotic conditions to operate harmoniously with continuous t -norms rather than standard algebraic inequalities, presenting a rigorous but highly rewarding mathematical challenge for future investigations.

Author contributions

R. Suparatulatorn, K. Chaichana, and P. Charoensawan: Conceptualization, Methodology, Writing-original draft, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research project was supported by: (1) CMU Mid-Career Research Fellowship program; (2) Fundamental Fund 2026 (214507), Chiang Mai University, Chiang Mai, Thailand; Thailand Science Research and Innovation (TSRI) (FRB690042/0162); (3) Chiang Mai University, Chiang Mai, Thailand; (4) Centre of Excellence in Mathematics, MHEI, Bangkok, Thailand; (5) The Continuing Researcher Development Grant, Faculty of Science, Chiang Mai University; and (6) Fundamental Fund 2027, Chiang Mai University, Chiang Mai, Thailand; Thailand Science Research and Innovation (TSRI).

Conflict of interest

The authors declare no conflict of interest in this paper.

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