



*Research article***Fixed-time zeroing neural networks for solving the singular Yang-Baxter-like matrix equation****Yiqiong Sun and Duanmei Zhou***

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Abstract: Solving singular Yang-Baxter-like matrix equations poses significant challenges due to ill-conditioned matrices and high computational costs. To address these issues, this work investigated the development of zeroing neural networks for singular problems. Initially, we constructed an enhanced improved zeroing neural network as a transitional model by directly integrating Tikhonov regularization into existing frameworks. However, theoretical analysis revealed that this direct regularization approach has limitations, as it only guarantees convergence to a residual limit set rather than the exact solution due to irreducible null spaces. To address this limitation, we proposed the effective zeroing neural network (EZNN). Under the assumption of an invertible auxiliary matrix, the EZNN utilizes full-rank factorization to equivalently transform the high-dimensional singular problem into a low-dimensional full-rank matrix equation. This structural reduction bypasses the singularity issue, avoiding the need for Tikhonov regularization while mitigating memory overflow, thus enabling the precise processing of high-dimensional matrices.

Keywords: Yang-Baxter-like matrix equation; fixed-time convergence; noise robustness; zeroing neural network model; Tikhonov regularization

Mathematics Subject Classification: 15A24, 65F45

1. Introduction

Nonlinear matrix equations hold a fundamental position in various scientific computing fields such as control theory [1,2] and statistical physics [3], and their solution methods have always been one of the core topics in numerical algebra research [3,4]. As a special class of nonlinear matrix equations, the Yang-Baxter equation has been an important research topic in mathematical physics since it was proposed by Professor Yang [5] in 1967 when studying one-dimensional many-body problems. Its classical form can be expressed as

$$A(u)B(u+v)A(v) = B(u)A(u+v)B(v),$$

where $A(u)$ and $B(v)$ are rational functions of the parameters. Subsequently, Baxter [6] independently derived the same equation in his research on two-dimensional classical statistical mechanics and successfully applied it to solve the eight-vertex model.

This equation has shown wide application value in various fields such as group theory [7], quantum field theory [8], statistical mechanics [9], and knot theory [10]. In a simplified case, this equation can be expressed as

$$XAX = AXA, \quad (1.1)$$

where $A \in \mathbb{R}^{n \times n}$ is a known matrix and X is the unknown matrix to be determined. We refer to Eq (1.1) as the Yang-Baxter-like matrix equation (YBLME) because its form is similar to the classical parameter-free Yang-Baxter equation. In the theoretical analysis of matrix equations, exploiting structural properties through factorization techniques is a common methodology for deriving explicit algebraic solutions. For instance, singular value decompositions have been used to determine consistent solutions within specific kernel spaces for general constrained matrix systems [11], while Jordan canonical forms have been applied to derive commuting solutions for rank-updated matrices in the specific context of static the YBLME [12]. Additionally, further algebraic methodologies have been extensively explored in the literature, such as determining commuting solutions for the YBLME [13] and identifying all exact solutions for diagonalizable coefficient matrices with distinct eigenvalues [14]. These algebraic methods provide exact structural characterizations for static problems. To facilitate real-time computation and tracking, static YBLME (1.1) is investigated in the continuous-time domain, yielding the following formulation:

$$X(t)AX(t) = AX(t)A, \quad 0 \leq t < \infty, \quad (1.2)$$

where $A \in \mathbb{R}_r^{n \times n}$ is an $n \times n$ time-invariant matrix with rank r , and $X(t) \in \mathbb{R}^{n \times n}$ is the continuous-time state matrix to be determined.

In the field of matrix equation solving, recurrent neural networks (RNNs), with their unique parallel processing mechanism [15], have demonstrated significant advantages in breaking through the efficiency bottleneck of traditional serial computing [16]. This characteristic makes RNNs particularly suitable for large-scale matrix operations and greatly facilitates hardware implementation [17, 18]. RNNs have been successfully applied in various scenarios such as Sylvester equation solving [19, 20] and generalized inverse computation [21]. As an important development of RNNs, zeroing neural networks (ZNNs) are specifically designed for real-time solving of time-varying problems and have achieved remarkable success [22]. This model was initially proposed by Zhang and Wang [20] for general time-varying matrix equations, and its application scope has rapidly expanded to include the time-invariant YBLME [23], linear equation systems [24, 25], generalized inverse problems [26], and quadratic optimization problems [27].

The convergence characteristics of ZNNs are key to their appeal. Studies have shown that ZNNs can achieve fixed-time convergence with a minimal upper bound [2, 28]. Recently, the continuous optimization of nonlinear activation functions has further improved ZNN performance; for example, activation functions based on integral upper bounds have been proposed to accelerate convergence in time-varying equations like the Sylvester equation [29]. Specifically in the context of solving the

YBLME, significant progress has also been made. Recent developments, particularly the introduction of various nonlinear activation functions, have greatly enhanced the convergence speed and noise robustness of ZNN models (e.g., IZNN_I [30] and INTZNN_I [31]). However, the theoretical derivation of these ZNN variants is primarily established upon the premise that the coefficient matrix A is non-singular. Consequently, their direct application to singular matrices is mathematically restricted, and the dynamic inversion process may encounter numerical instability associated with ill-posedness. Although traditional ZNN1 models can handle singular matrix problems [32], their convergence rate is limited to exponential convergence. This leads to overly conservative convergence times that struggle to meet the real-time computational requirements of modern hardware, especially for high-dimensional matrices. Furthermore, the widespread presence of noise in ZNN hardware implementations [33] poses severe challenges to model robustness, an aspect where traditional models for singular matrices remain insufficient. These collective limitations pose challenges to the practical application of existing ZNN methods in solving large-scale, high-dimensional singular time-invariant YBLME problems.

Given the above challenges, this paper aims to investigate the numerical solutions of time-invariant YBLME (1.2), especially for the case of high-dimensional singular matrix A . Motivated by the matrix factorization principles used in exact algebraic characterizations, we introduce full-rank decomposition into the ZNN design for effective dimensionality reduction of high-dimensional singular matrices. Concurrently, to mitigate the ill-posed instability and hardware-induced noise during the dynamic inversion process, Tikhonov regularization is integrated into the model formulation. Compared to matrix truncation methods such as truncated singular value decomposition (SVD), Tikhonov regularization preserves the continuity of the state matrix dynamics, which is crucial for continuous-time neural network modeling. Fully considering the inevitable noise interference in ZNN hardware implementations, we propose two improved models with fixed-time convergence and noise robustness to solve Eq (1.2). These new models are respectively named the enhanced improved zeroing neural network (EIZNN) and effective zeroing neural network (EZNN) models. The main contributions of this paper are explicitly summarized as follows:

- (1) **Novel model construction for the singular YBLME:** We systematically investigate the numerical treatment of the singular YBLME by proposing two structurally distinct models. We first establish the EIZNN model by directly incorporating Tikhonov regularization into traditional improved ZNNs. While this provides a preliminary workaround for singularity, our theoretical analysis reveals its inherent structural limitations, as it fails to achieve exact zero-error convergence due to irreducible null spaces. To resolve this fundamental limitation, we propose the novel EZNN model. By integrating full-rank factorization with a well-posedness assumption, the EZNN successfully bypasses singularity without relying on any regularization, ensuring convergence to the exact solution set. Consequently, the EIZNN serves primarily as a theoretical baseline for evaluating standard regularized dynamics, whereas the EZNN is formulated as a precise solver for practical deployment.
- (2) **Dimensionality reduction and computational superiority:** A key structural innovation is the formulation of the EZNN, which utilizes an offline full-rank factorization to equivalently transform the original $n \times n$ high-dimensional singular matrix equation into a significantly smaller $r \times r$ full-rank matrix problem. By decoupling this one-time offline preprocessing from the online real-time dynamic solving, this model significantly lowers the dimensionality of the network state variables, thereby mitigating the memory requirements during the dynamic

evolution phase. This enables the successful processing of high-dimensional matrices (e.g., up to 4096×4096). While the direct regularization in the EIZNN inevitably introduces scale-induced memory bottlenecks and steady-state errors, the structural dimensionality reduction in the EZNN provides a computationally efficient and feasible alternative for massive-scale computations. Therefore, the EZNN is particularly tailored for high-dimensional application scenarios.

- (3) **Rigorous guarantee of convergence and noise robustness:** Unlike traditional ZNN models that only offer asymptotic or exponential convergence, we mathematically prove that the proposed EZNN model achieves exact fixed-time convergence to the solution set, with an upper bound time strictly independent of the initial states, while the EIZNN model asymptotically converges to a specific residual limit set. Furthermore, we provide rigorous theoretical proofs and experimental validation demonstrating their superior robustness against both dynamically bounded vanishing and non-vanishing noises, ensuring high reliability for physical hardware implementations.

The rest of this paper is organized as follows: Necessary preliminaries are introduced in Section 2. The construction processes of the proposed EIZNN and EZNN models are detailed in Sections 3 and 4. We then provide a rigorous convergence analysis and prove the robustness of these models in Section 5. To validate the effectiveness of these models, numerical examples and comparisons with existing methods are presented in Section 6. Finally, concluding remarks are provided in the last section.

2. Preliminaries

Consider the following nonlinear dynamical system described by differential equations:

$$\frac{dx(t)}{dt} = g(x(t), t), \quad x(0) = x_0, \quad t \in [0, +\infty), \quad (2.1)$$

where $x(t) \in \mathbb{R}^n$ is the system state vector and $g(\cdot, \cdot) : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}^n$ is a nonlinear mapping. Assume that $x = 0$ is an equilibrium point of the system.

Definition 2.1. [34, 35] *The origin of system (2.1) is globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of (2.1) reaches the equilibria at some finite time moment, i.e., $x(t, x_0) = 0$ for all $t \geq T(x_0)$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$ is a settling-time function.*

Definition 2.2. [34, 35] *The origin of system (2.1) is globally fixed-time stable if it is globally finite-time stable and the settling-time function T is bounded, i.e., there exists a constant $T_{\max} > 0$ satisfying $T_{\max} \geq T(x_0)$ for all $x_0 \in \mathbb{R}^n$.*

3. Construction of the EIZNN model

3.1. IZNN_I model

To establish the EIZNN model, this section will systematically elaborate on how to construct an improved zeroing neural network model, namely the IZNN_I model, for solving Eq (1.2). The design of this model aims to optimize the convergence performance of the standard ZNN model by introducing nonlinear activation functions. The entire construction process follows the following four steps.

The first step is to establish the error function $\Theta(t)$ for YBLME (1.2):

$$\Theta(t) := X(t)AX(t) - AX(t)A. \quad (3.1)$$

The second step is to present the zeroing design formula

$$\dot{\Theta}(t) = -\gamma\Theta(t), \quad \gamma > 0, \quad (3.2)$$

where $\gamma > 0$ is a scaling parameter, which directly scales the convergence rate of the error matrix $\Theta(t)$ to zero. The third step is to combine (3.1) and (3.2) to obtain the ZNN model for solving (1.2):

$$\dot{X}(t)AX(t) + X(t)A\dot{X}(t) - A\dot{X}(t)A = -\gamma(X(t)AX(t) - AX(t)A). \quad (3.3)$$

Although the standard ZNN model theoretically guarantees asymptotic convergence, its convergence speed and anti-disturbance capability are often insufficient in practical applications. Research has shown that by introducing specific nonlinear activation functions into the ZNN model, its performance can be significantly enhanced, for example, by achieving fixed-time convergence, accelerating the convergence process, or improving robustness against noise [36, 37].

Leveraging this theoretical advantage, researchers propose the IZNN_I model [30] by incorporating a specific nonlinear activation function [38]. This activation function, which is applied element-wise to matrix variables, is defined as

$$\Psi_1(x) = c_1 \exp(|x|^q)|x|^{1-q}\text{sgn}(x)/q + c_2x + c_3\text{sgn}(x), \quad (3.4)$$

where $0 < q < 1$, $c_1 > 0$, $c_2, c_3 \geq 0$, and $\text{sgn}(\cdot)$ denotes the sign function. It is worth noting that while the parameters c_2 and c_3 can theoretically be set to zero without affecting the fixed-time convergence upper bound, they are intentionally retained. Specifically, the linear term c_2x accelerates the convergence rate when the initial error is large, and the sign-dependent term $c_3\text{sgn}(x)$ provides necessary robust compensation against non-vanishing noises. The IZNN_I model for solving YBLME (1.2) is then formulated as

$$\dot{X}(t)AX(t) + X(t)A\dot{X}(t) - A\dot{X}(t)A = -\gamma\Psi_1(X(t)AX(t) - AX(t)A). \quad (3.5)$$

Next, by applying the column-wise vectorization and Kronecker product formula $\text{vec}(JFV) = (V^T \otimes F)\text{vec}(J)$ to Eq (3.5), we obtain the following matrix identity:

$$[(AX(t))^T \otimes I_n + I_n \otimes X(t)A - A^T \otimes A]\text{vec}(\dot{X}(t)) = -\gamma\text{vec}[\Psi_1(X(t)AX(t) - AX(t)A)]. \quad (3.6)$$

3.2. EIZNN model

To simplify Eq (3.6), the following expressions are defined:

$$\begin{aligned} L(t) &\equiv (AX(t))^T \otimes I_n + I_n \otimes X(t)A - A^T \otimes A, \\ z(t) &\equiv -\gamma\text{vec}[\Psi_1(X(t)AX(t) - AX(t)A)], \\ \mathbf{x}(t) &\equiv \text{vec}(X(t)), \quad \dot{\mathbf{x}}(t) \equiv \text{vec}(\dot{X}(t)). \end{aligned}$$

Thus, the IZNN_I model can be concisely defined as

$$L(t)\dot{\mathbf{x}}(t) = z(t). \quad (3.7)$$

Next, we will discuss the existence of solutions for the linear system (3.7).

- (1) **No Solution:** If $L(t)$ is singular and the vector $z(t)$ is not in the column space of $L(t)$, then this linear system will have no solution. This means that no $\dot{x}(t)$ can be found to satisfy the equation. In numerical computation, this leads to failure in solving or meaningless results.
- (2) **Infinitely Many Solutions:** If $L(t)$ is singular and the vector $z(t)$ happens to be in the column space of $L(t)$, then this linear system will have infinitely many solutions. In this case, a unique solution $\dot{x}(t)$ cannot be determined. For a dynamic system (such as IZNN_I), a unique $X(t)$ is required at each step for numerical integration, and infinitely many solutions will prevent the model from determining the direction of evolution for the next step.

Evidently, when A is a singular matrix, $L(t)$ is also singular. Therefore, to extend the IZNN_I model [30] to solve singularity problems, we establish the EIZNN model by introducing Tikhonov regularization to address the singularity in (3.7). For a small regularization parameter $\lambda > 0$, the EIZNN model is

$$(L^T(t)L(t) + \lambda I_{n^2})\dot{x}(t) = L^T(t)z(t). \quad (3.8)$$

Meanwhile, to address singularity problems in solving time-varying matrix equations, researchers have constructed the ZNN1 neurodynamic model [32], which can also be reduced to solve time-invariant matrix problems. The difference between the ZNN1 and EIZNN models lies in the use of different activation functions. The ZNN1 model uses a linear activation function, as follows:

$$\Psi_2(x) = x. \quad (3.9)$$

That is, the ZNN1 model for solving the time-invariant YBLME is as follows:

$$(L(t) + \lambda I_{n^2})\dot{x}(t) = b(t), \quad (3.10)$$

where the activation function $\Psi_2(\cdot)$ is applied element-wise to the matrix, and $b(t) = -\gamma \text{vec}[\Psi_2(X(t)AX(t) - AX(t)A)]$.

Theorem 3.1. Assume that ZNN1 model (3.10) is used to solve the time-invariant YBLME (1.2), where the time-invariant matrix A is known. Then, starting from an arbitrary initial state $X(0)$, the state matrix $X(t)$ will exponentially converge to the exact solution set, rather than a unique theoretical solution, due to the non-uniqueness of the solutions for the singular YBLME.

The proof of Theorem 3.1 is similar to that of Theorem 1 [32], so we choose to omit it here.

4. Construction of the EZNN model

4.1. Review of full rank factorization theory

Since A is an $n \times n$ singular matrix, it satisfies $\text{rank}(A) = r < n$. It is well-known that any square matrix A of rank r has a decomposition in the sense of full rank factorization:

$$A = UG^T = u_1 g_1^T + \cdots + u_r g_r^T,$$

where $U = [u_1, \dots, u_r]$ and $G = [g_1, \dots, g_r]$ are both column full-rank matrices, and G^T is the transpose of G . Let $B = G^T U$.

Lemma 4.1. [39] Let $A = UG^T$ be an $n \times n$ singular matrix of rank r , where U and G are two $n \times r$ column full-rank matrices. Then $X = UEG^T$ is a YBLME solution of A if and only if E is a solution to the matrix equation

$$EB^2E = BEB. \quad (4.1)$$

Assumption 4.1. The auxiliary matrix $B = G^T U \in \mathbb{R}^{r \times r}$ is non-singular.

Remark 4.1. To address the dynamic equivalence between the original matrix equation (1.2) and the reduced matrix equation (4.1) during the continuous-time evolution, we rigorously evaluate the relationship between their respective residual errors. Substituting the parameterized state matrix $X(t) = UE(t)G^T$ and the full-rank factorization $A = UG^T$ into the original residual $\Theta(t)$ defined in (3.1), we obtain

$$\begin{aligned} \Theta(t) &= X(t)AX(t) - AX(t)A \\ &= (UE(t)G^T)(UG^T)(UE(t)G^T) - (UG^T)(UE(t)G^T)(UG^T) \\ &= UE(t)(G^T U)(G^T U)E(t)G^T - U(G^T U)E(t)(G^T U)G^T \\ &= U(E(t)B^2E(t) - BE(t)B)G^T \\ &= UH(t)G^T. \end{aligned}$$

Since U is a full column-rank matrix and G^T is a full row-rank matrix, their corresponding null spaces are trivial. Consequently, $\Theta(t) = 0$ holds if and only if the reduced residual $H(t) = 0$.

Lemma 4.1, Assumption 4.1, and the dynamic equivalence proven in Remark 4.1 jointly provide a rigorous theoretical mechanism for dimensionality reduction. The intractable $n \times n$ singular YBLME (1.2) is equivalently condensed into the $r \times r$ matrix equation (4.1).

Building on this equivalence, this section develops the proposed EZNN model to solve (4.1). Crucially, the guaranteed invertibility of B ensures that the subsequent zeroing dynamics are inherently well-posed. This structural advantage allows the EZNN formulation to fundamentally bypass singularity bottlenecks, avoiding the need for Tikhonov regularization.

4.2. EZNN model

The formulation of the EZNN model for solving matrix equation (4.1) comprises the following three steps.

The first step is to establish the error function $H(t)$ for the quadratic matrix equation (4.1):

$$H(t) := E(t)B^2E(t) - BE(t)B. \quad (4.2)$$

The second step is to present the zeroing design formula:

$$\dot{H}(t) = -\gamma\Psi_1(H(t)), \quad \gamma > 0. \quad (4.3)$$

The third step, by combining formulas (3.4), (4.2), and (4.3), yields the proposed EZNN model:

$$\dot{E}(t)B^2E(t) + E(t)B^2\dot{E}(t) - B\dot{E}(t)B = -\gamma\Psi_1(E(t)B^2E(t) - BE(t)B). \quad (4.4)$$

For computational convenience, by applying vectorization and the Kronecker product to (4.4), the EZNN model is thus equivalent to the following expression:

$$[(B^2 E(t))^T \otimes I_r + I_r \otimes E(t)B^2 - B^T \otimes B] \text{vec}(\dot{E}(t)) = -\gamma \text{vec}[\Psi_1(E(t)B^2 E(t) - BE(t)B)]. \quad (4.5)$$

To simplify Eq (4.5), we define the following expressions:

$$\begin{aligned} P(t) &\equiv [(B^2 E(t))^T \otimes I_r + I_r \otimes E(t)B^2 - B^T \otimes B], \\ k(t) &\equiv -\gamma \text{vec}[\Psi_1(E(t)B^2 E(t) - BE(t)B)], \\ \mathbf{E}(t) &\equiv \text{vec}(E(t)), \quad \dot{\mathbf{E}}(t) \equiv \text{vec}(\dot{E}(t)). \end{aligned}$$

Therefore, Eq (4.5) is equivalent to

$$P(t)\dot{\mathbf{E}}(t) = k(t). \quad (4.6)$$

5. Convergence and noise robustness analysis

This subsection will provide theoretical analysis of the EZNN and EIZNN models' convergence under noiseless conditions and their robustness to noise. These models are constructed based on nonlinear activation functions and Tikhonov regularization methods for solving the time-invariant YBLME (1.2).

5.1. Convergence analysis

This subsection first discusses the convergence of the EIZNN model for solving Eq (1.2), and then discusses the convergence of the EZNN model for solving Eq (4.1).

Theorem 5.1. Assume that EIZNN model (3.8) is used to solve the time-invariant YBLME (1.2), where the time-invariant matrix A is known. Then, starting from an arbitrary initial state $X(0)$, the state trajectory of the system remains globally bounded, and the error vector $e(t) = \text{vec}(\Theta(t))$ asymptotically converges to the limit set $\mathcal{M} = \{x \in \mathbb{R}^{n^2} \mid L^T(t)z(t) = 0\}$.

Proof. For any constant $\lambda > 0$, the regularized matrix is always strictly positive definite, i.e., $L^T(t)L(t) + \lambda I_{n^2} > 0$. Therefore, the EIZNN model possesses a unique evolution solution for any mapping matrix $L(t)$.

To analyze the convergence of the system, let the error vector be $e(t) = \text{vec}(\Theta(t))$ and define the following integral-type Lyapunov candidate function:

$$V(t) = \sum_{k=1}^{n^2} \int_0^{e_k(t)} \Psi_1(s) ds, \quad (5.1)$$

where $e_k(t)$ is the k -th element of the error vector $e(t)$. Since the nonlinear activation function $\Psi_1(s)$ consistently shares the same sign as the variable s , this integral function is strictly positive definite and satisfies $V(t) \geq 0$, with $V(t) = 0$ holding if and only if $e(t) = 0$.

Taking the time derivative of this Lyapunov function along the system trajectory, and applying the fundamental theorem of calculus alongside the chain rule $\dot{e}(t) = L(t)\dot{x}(t)$, we obtain

$$\dot{V}(t) = \Psi_1(e(t))^T \dot{e}(t) = \Psi_1(e(t))^T L(t)\dot{x}(t). \quad (5.2)$$

Substituting the evolution equation of the EIZNN model $\dot{x}(t) = (L^T(t)L(t) + \lambda I_{n^2})^{-1}L^T(t)z(t)$ and the definition $z(t) = -\gamma\Psi_1(e(t))$ into the above equation yields:

$$\dot{V}(t) = -\gamma\Psi_1(e(t))^T L(t)(L^T(t)L(t) + \lambda I_{n^2})^{-1}L^T(t)\Psi_1(e(t)). \quad (5.3)$$

Since the matrix $(L^T(t)L(t) + \lambda I_{n^2})^{-1}$ is strictly positive definite and the scaling parameter $\gamma > 0$, it inherently follows that $\dot{V}(t) \leq 0$. This proves that the energy function is monotonically non-increasing along the closed-loop system trajectory.

Given $\dot{V}(t) \leq 0$, the system's energy is strictly enveloped by its initial state, meaning $V(t) \leq V(0)$ for any $t \geq 0$. Further examining the spatial properties of $V(t)$, the definition of $\Psi_1(s)$ includes an exponential term $\exp(|s|^q)$. Consequently, as $\|e(t)\| \rightarrow \infty$, the integral result $V(t) \rightarrow +\infty$. This indicates that the Lyapunov function is radially unbounded. Thus, its sub-level set is necessarily compact, and the system's error trajectory is permanently confined within this compact set, rigorously proving that the system state is globally bounded.

Under the premise of bounded trajectories and continuous autonomous system dynamics, LaSalle's invariance principle dictates that as $t \rightarrow \infty$, the system's state trajectory will asymptotically converge to the largest invariant set where $\dot{V}(t) = 0$. Setting $\dot{V}(t) = 0$, and knowing the regularized inverse matrix is strictly positive definite, this equality holds if and only if $L^T(t)\Psi_1(e(t)) = 0$. Combined with the definition of $z(t)$, this is mathematically equivalent to $L^T(t)z(t) = 0$. This demonstrates that the system ultimately achieves asymptotic convergence to the limit set $\mathcal{M} = \{x \in \mathbb{R}^{n^2} \mid L^T(t)z(t) = 0\}$. This completes the proof. $\square$

Theorem 5.2. Assume that EZNN model (4.6) is used to solve the matrix equation (4.1), where matrix B is known. Then, starting from an arbitrary initial state $E(0)$, the state matrix $E(t)$ will converge to the solution set in finite time:

$$t(E(0)) \leq \frac{1}{\gamma c_1} = T_{\max}. \quad (5.4)$$

Proof. Model (4.4) can be reformulated as

$$\dot{H}(t) = -\gamma\Psi_1(H(t)). \quad (5.5)$$

Since $H(t)$ is an $r \times r$ matrix, we can consider its ij -th element $h_{ij}(t)$. Then the ij -th subsystem of Eq (5.5) is

$$\dot{h}_{ij}(t) = -\gamma\Psi_1(h_{ij}(t)), \quad i, j = 1, 2, \dots, r. \quad (5.6)$$

To analyze the convergence of subsystem (5.6), we define a Lyapunov candidate function as $v(t) = |h_{ij}(t)| \geq 0$. To rigorously handle the non-differentiable point at $h_{ij}(t) = 0$, we utilize the upper-right Dini derivative $D^+v(t)$. Its time derivative is

$$D^+v(t) = \dot{h}_{ij}(t)\text{sgn}(h_{ij}(t)). \quad (5.7)$$

Substituting $\dot{h}_{ij}(t) = -\gamma\Psi_1(h_{ij}(t))$ and the definition of $\Psi_1(x)$ into function (5.7), we get

$$\begin{aligned} D^+v(t) &= -\gamma \left(c_1 \exp(|h_{ij}(t)|^q) |h_{ij}(t)|^{1-q} \text{sgn}(h_{ij}(t)) / q + c_2 h_{ij}(t) + c_3 \text{sgn}(h_{ij}(t)) \right) \text{sgn}(h_{ij}(t)) \\ &= -\gamma \left(c_1 \exp(|h_{ij}(t)|^q) |h_{ij}(t)|^{1-q} / q + c_2 |h_{ij}(t)| + c_3 \right). \end{aligned}$$

Since $c_1 > 0$, $c_2 \geq 0$, and $c_3 \geq 0$, it is clear that

$$D^+v(t) \leq -\gamma c_1 \exp(|h_{ij}(t)|^q) |h_{ij}(t)|^{1-q} / q = -\gamma c_1 \exp(v^q(t)) v^{1-q}(t) / q \leq 0.$$

According to Lyapunov stability theory, $h_{ij}(t)$ will globally converge to zero over time.

We extract the differential form from the above inequality:

$$\gamma c_1 dt \leq -q v^{q-1}(t) \exp(-v^q(t)) dv(t).$$

If the convergence time of subsystem (5.6) is t_{ij} , we integrate the above expression from 0 to t_{ij} and from the initial value $v(0) = |h_{ij}(0)|$ to 0:

$$\int_0^{t_{ij}} \gamma c_1 d\tau \leq - \int_{v(0)}^0 q v^{q-1}(t) \exp(-v^q(t)) dv(t).$$

Net, we calculate the integral:

$$\begin{aligned} \gamma c_1 \tau \Big|_0^{t_{ij}} &\leq \exp(-v^q(t)) \Big|_{v(0)}^0, \\ \gamma c_1 t_{ij} &\leq \exp(0) - \exp(-v^q(0)), \\ t_{ij} &\leq \frac{1 - \exp(-v^q(0))}{\gamma c_1}. \end{aligned}$$

Note that $v(0) = |h_{ij}(0)|$, and $\exp(-|h_{ij}(0)|^q) \in (0, 1]$, thus $1 - \exp(-|h_{ij}(0)|^q) \in [0, 1)$. Taking the maximum value, i.e., when $|h_{ij}(0)| \rightarrow \infty$, $\exp(-|h_{ij}(0)|^q) \rightarrow 0$, at which point $1 - \exp(-|h_{ij}(0)|^q) \rightarrow 1$. Therefore, the upper bound for the convergence time t_{ij} of each element is $\frac{1}{\gamma c_1}$. Thus, the upper bound for the convergence time of model (4.6) is

$$T_{\max 2} = \frac{1}{\gamma c_1}.$$

□

5.2. Noise robustness analysis

In the real-world deployment on hardware platforms, the introduction of noise is inevitable. It primarily originates from measurement errors in experimental systems and random interference from the external environment, which directly affects the model performance and reliability (see [33]). To this end, this subsection will deeply analyze the anti-interference capability and robustness of three neural network models (i.e., EIZNN, EZNN, and ZNN1) in the presence of noise.

By adding the noise term $D(t)$ to the EIZNN, EZNN, and ZNN1 models, the models become

$$(L^T(t)L(t) + \lambda I_{n^2})\dot{\mathbf{x}}(t) = L^T(t)z(t) + \text{vec}(D(t)), \quad (5.8)$$

$$P(t)\dot{\mathbf{E}}(t) = k(t) + \text{vec}(D(t)), \quad (5.9)$$

$$(L(t) + \lambda I_{n^2})\dot{\mathbf{x}}(t) = b(t) + \text{vec}(D(t)), \quad (5.10)$$

where $D(t)$ is the time-varying noise.

In this paper, we will consider two types of noise. One is dynamically bounded vanishing noise, characterized as follows:

- EIZNN:

$$|d_{ij}(t)| \leq \varepsilon |\theta_{ij}(t)|, \quad \varepsilon \in (0, +\infty),$$

- EZNN:

$$|d_{ij}(t)| \leq \varepsilon |h_{ij}(t)|, \quad \varepsilon \in (0, +\infty),$$

- ZNN1:

$$|d_{ij}(t)| \leq \varepsilon |\theta_{ij}(t)|, \quad \varepsilon \in (0, +\infty),$$

where $d_{ij}(t)$ is the ij -th element of $D(t)$, and each model corresponds to its respective error function. The other is dynamically bounded non-vanishing noise, characterized as follows:

$$|d_{ij}(t)| \leq \varepsilon, \quad \varepsilon \in (0, +\infty).$$

Next, we will analyze the convergence of the above three models to noise interference under the influence of these two types of noise.

Theorem 5.3. *Let the noise-perturbed EIZNN model be used to solve the time-invariant YBLME (1.2), where the time-invariant matrix A is known. Then the following conclusions hold.*

- If $D(t)$ is a dynamically bounded vanishing noise (i.e., satisfying $\|\text{vec}(D(t))\|_2 \leq \varepsilon \|e(t)\|_2$), provided that the model scaling parameter γ is sufficiently large, the closed-loop error trajectory of the system will globally asymptotically converge to the steady-state limit set $\mathcal{M} = \{x \in \mathbb{R}^{n^2} \mid L^T(t)\Psi_1(e(t)) = 0\}$.
- If $D(t)$ is a dynamically bounded non-vanishing noise (i.e., satisfying $\|\text{vec}(D(t))\|_2 \leq \varepsilon$), the closed-loop error trajectory of the system is ultimately bounded. It converges to a steady-state error neighborhood surrounding the limit set $\mathcal{M}$, with its residual radius jointly determined by the scaling parameter γ , the regularization parameter λ , and the noise upper bound ε .

Proof. Upon introducing the Tikhonov regularization term λI_{n^2} , the actual state evolution equation of the EIZNN model perturbed by the noise $D(t)$ is defined as

$$(L^T(t)L(t) + \lambda I_{n^2})\dot{x}(t) = -\gamma L^T(t)\Psi_1(e(t)) + \text{vec}(D(t)). \quad (5.11)$$

Since the regularized matrix $(L^T(t)L(t) + \lambda I_{n^2})$ is strictly positive definite and invertible for any $\lambda > 0$, multiplying both sides by its inverse yields

$$\dot{x}(t) = (L^T(t)L(t) + \lambda I_{n^2})^{-1} (-\gamma L^T(t)\Psi_1(e(t)) + \text{vec}(D(t))). \quad (5.12)$$

By incorporating the derivative definition of the system residual vector, $\dot{e}(t) = L(t)\dot{x}(t)$, we obtain

$$\dot{e}(t) = -\gamma L(t)(L^T(t)L(t) + \lambda I_{n^2})^{-1} L^T(t)\Psi_1(e(t)) + L(t)(L^T(t)L(t) + \lambda I_{n^2})^{-1} \text{vec}(D(t)). \quad (5.13)$$

To simplify the analysis, we define the projection matrix induced by regularization as $P_\lambda(t) = L(t)(L^T(t)L(t) + \lambda I_{n^2})^{-1} L^T(t)$, and the state-dependent noise transfer matrix as $R_\lambda(t) = L(t)(L^T(t)L(t) + \lambda I_{n^2})^{-1}$. Thus, the residual equation can be equivalently written as

$$\dot{e}(t) = -\gamma P_\lambda(t)\Psi_1(e(t)) + R_\lambda(t)\text{vec}(D(t)). \quad (5.14)$$

According to the singular value decomposition (SVD), let the singular values of $L(t)$ be denoted by σ_i . It is easy to verify that $P_\lambda(t)$ is a positive semi-definite matrix with eigenvalues $\frac{\sigma_i^2}{\sigma_i^2 + \lambda} \in [0, 1)$. Similarly, the singular values of $R_\lambda(t)$ are given by the function $f(\sigma_i) = \frac{\sigma_i}{\sigma_i^2 + \lambda}$. Taking the analytical derivative reveals that $f(\sigma_i)$ achieves its global maximum of $\frac{1}{2\sqrt{\lambda}}$ at $\sigma_i = \sqrt{\lambda}$. Therefore, there exists a strict upper bound for the spectral norm (2-norm) of $R_\lambda(t)$:

$$\|R_\lambda(t)\|_2 \leq \frac{1}{2\sqrt{\lambda}}. \quad (5.15)$$

Constructing a Lyapunov candidate function $V(t) = \frac{1}{2}\|e(t)\|_2^2$ and taking its time derivative along the actual error trajectory yields

$$\dot{V}(t) = e^T(t)\dot{e}(t) = -\gamma e^T(t)P_\lambda(t)\Psi_1(e(t)) + e^T(t)R_\lambda(t)\text{vec}(D(t)). \quad (5.16)$$

(i) For the case of dynamically bounded vanishing noise:

In this case, the noise satisfies the condition $\|\text{vec}(D(t))\|_2 \leq \varepsilon\|e(t)\|_2$. By utilizing the Cauchy-Schwarz inequality and the properties of the matrix 2-norm, the noise perturbation term can be bounded as follows:

$$e^T(t)R_\lambda(t)\text{vec}(D(t)) \leq \|e(t)\|_2\|R_\lambda(t)\|_2\|\text{vec}(D(t))\|_2 \leq \frac{\varepsilon}{2\sqrt{\lambda}}\|e(t)\|_2^2. \quad (5.17)$$

Substituting this back into the Lyapunov derivative equation yields

$$\dot{V}(t) \leq -\gamma e^T(t)P_\lambda(t)\Psi_1(e(t)) + \frac{\varepsilon}{2\sqrt{\lambda}}\|e(t)\|_2^2. \quad (5.18)$$

Since the nonlinear activation function $\Psi_1(e(t))$ contains a linear growth term $c_2 e(t)$ (with $c_2 \geq 0$), the term $-\gamma e^T(t)P_\lambda(t)\Psi_1(e(t))$ provides a strong negative definite constraint in regions where the system state does not belong to the null space of the projection matrix $P_\lambda(t)$ (i.e., when deviating from the limit set $\mathcal{M}$). By selecting a sufficiently large scaling parameter γ , the dominant dissipation term can counteract the uncertainty introduced by $\frac{\varepsilon}{2\sqrt{\lambda}}$, thereby guaranteeing that $\dot{V}(t) < 0$ outside the domain of $\mathcal{M}$.

More importantly, as $e(t) \rightarrow 0$, the absolute intensity of the vanishing noise subsequently decays to zero. However, owing to the singularity of A , $P_\lambda(t)$ possesses an irreducible null space, causing the system to lose the strong driving force required for finite-time convergence. According to LaSalle's invariance principle, the error trajectory will globally asymptotically converge to the invariant set $\mathcal{M}$.

(ii) For the case of dynamically bounded non-vanishing noise:

In this scenario, the noise satisfies the condition $\|\text{vec}(D(t))\|_2 \leq \varepsilon$. Similarly, the derivative equation can be bounded as

$$\dot{V}(t) \leq -\gamma e^T(t)P_\lambda(t)\Psi_1(e(t)) + \frac{\varepsilon}{2\sqrt{\lambda}}\|e(t)\|_2. \quad (5.19)$$

When the system residual $\|e(t)\|_2$ is sufficiently large and far from the limit set $\mathcal{M}$, the negative definite growth rate of the dissipation term—driven by the activation function (including both exponential and linear terms)—is strictly greater than that of the linear perturbation term $\frac{\varepsilon}{2\sqrt{\lambda}}\|e(t)\|_2$. This ensures that $\dot{V}(t) < 0$ holds, forcing the system state to converge toward the solution set.

Nevertheless, since the non-vanishing noise maintains a constant-level intensity near the origin, while the term $e^T(t)P_\lambda(t)\Psi_1(e(t))$ approaches 0 as the state nears the limit set $\mathcal{M}$, the intrinsic convergence driving force of the system ultimately cannot resist the worst-case perturbations. Consequently, the derivative $\dot{V}(t)$ cannot maintain global negative definiteness, and the system error trajectory cannot precisely stay at the absolute zero point. Instead, it is confined within a bounded compact set surrounding $\mathcal{M}$. The size of this steady-state neighborhood is proportional to the noise upper bound ε and expands drastically as the regularization parameter λ decreases.

This completes the proof. $\square$

Theorem 5.4. *Let the noise-perturbed EZNN model (5.9) be used to solve the time-invariant matrix equation (4.1), where the time-invariant matrix B is known. Then the following conclusions hold:*

- (i) *If $D(t)$ is a dynamically bounded vanishing noise and $\gamma c_2 \geq \varepsilon$, then the state matrix $E(t)$, starting from an arbitrary initial state $E(0)$, converges to the solution set within a fixed time $T_{\max}$ (5.4).*
- (ii) *If $D(t)$ is a dynamically bounded non-vanishing noise and $\gamma c_3 \geq \varepsilon$, then the state matrix $E(t)$, starting from an arbitrary initial state $E(0)$, converges to the solution set within a fixed time $T_{\max}$ (5.4).*

Proof. Similar to the proof of Theorem 5.2, we reformulate the noise-perturbed EZNN model as

$$\dot{H}(t) = -\gamma\Psi_1(H(t)) + D(t). \quad (5.20)$$

Then, we can obtain the ij -th subsystem of model (5.20), expressed as

$$\dot{h}_{ij}(t) = -\gamma\Psi_1(h_{ij}(t)) + d_{ij}(t). \quad (5.21)$$

In the following analysis, we will focus only on subsystem (5.21). We define the Lyapunov candidate function as $u(t) = |h_{ij}(t)|^2 \geq 0$, and we have

$$\dot{u}(t) = 2h_{ij}(t)\dot{h}_{ij}(t).$$

- (i) Under dynamically bounded vanishing noise, we get

$$\begin{aligned} \dot{u}(t) &= 2h_{ij}(t)\dot{h}_{ij}(t) \\ &= -2\gamma \left(c_1 \exp(|h_{ij}(t)|^q) |h_{ij}(t)|^{2-q}/q + c_2 |h_{ij}(t)|^2 + c_3 |h_{ij}(t)| \right) + 2h_{ij}(t)d_{ij}(t) \\ &\leq -2\gamma \left(c_1 \exp(|h_{ij}(t)|^q) |h_{ij}(t)|^{2-q}/q + c_3 |h_{ij}(t)| \right) + 2(\varepsilon - \gamma c_2) |h_{ij}(t)|^2. \end{aligned}$$

Since $c_3 \geq 0$ and $\gamma c_2 \geq \varepsilon$, it is clear that

$$\dot{u}(t) \leq -2\gamma c_1 \exp(|h_{ij}(t)|^q) |h_{ij}(t)|^{2-q}/q = -\gamma c_1 \exp(u^{\frac{q}{2}}(t)) u^{\frac{2-q}{2}}(t)/(q/2) \leq 0. \quad (5.22)$$

- (ii) Under dynamically bounded non-vanishing noise, we get

$$\begin{aligned} \dot{u}(t) &= 2h_{ij}(t)\dot{h}_{ij}(t) \\ &= -2\gamma \left(c_1 \exp(|h_{ij}(t)|^q) |h_{ij}(t)|^{2-q}/q + c_2 |h_{ij}(t)|^2 + c_3 |h_{ij}(t)| \right) + 2h_{ij}(t)d_{ij}(t) \\ &\leq -2\gamma \left(c_1 \exp(|h_{ij}(t)|^q) |h_{ij}(t)|^{2-q}/q + c_2 |h_{ij}(t)|^2 \right) + 2(\varepsilon - \gamma c_3) |h_{ij}(t)|. \end{aligned}$$

Since $c_2 \geq 0$ and $\gamma c_3 \geq \varepsilon$, inequality (5.22) can be obtained. Therefore, Lyapunov stability theory indicates that $h_{ij}(t)$ still globally converges to zero over time under the influence of these two types of noise. Furthermore, according to inequality (5.22), we have

$$\gamma c_1 dt \leq -\frac{q}{2} u^{\frac{q-2}{2}}(t) \exp(-u^{\frac{q}{2}}(t)) du(t).$$

When the convergence time of subsystem (5.21) is t_{ij} , integrating the above expression from 0 to t_{ij} yields

$$\int_0^{t_{ij}} \gamma c_1 d\tau \leq - \int_{u(0)}^0 \frac{q}{2} u^{\frac{q-2}{2}}(t) \exp(-u^{\frac{q}{2}}(t)) du(t),$$

i.e.,

$$\gamma c_1 \tau \Big|_0^{t_{ij}} \leq \exp(-u^{\frac{q}{2}}(t)) \Big|_{u(0)}^0.$$

Based on this, we can directly calculate

$$t_{ij} \leq \frac{1 - \exp(-u^{\frac{q}{2}}(0))}{\gamma c_1}.$$

Therefore, the upper bound for the convergence time of the noise-perturbed EZNN model (5.9) is

$$T_{\max} = \frac{1}{\gamma c_1}.$$

□

6. Numerical experiments

This section presents three simulation experiments to verify the effectiveness of the proposed EIZNN and EZNN models.

Numerical simulations were conducted on a Micro-Star MS-7E44 desktop PC equipped with an Intel 12th Gen Core i5-12400F six-core processor, 32GB DDR4 3200MHz RAM, an NVIDIA GeForce RTX 4060 graphics card, and running the Windows 11 Pro 23H2 64-bit operating system. In these experiments, we utilized the `ode45` command in MATLAB 2023b to compute the EIZNN and EZNN models.

6.1. Example 6.1

Consider the coefficient matrix as follows [39]:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$\text{where } U = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}, G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, B = G^T U = 2 \operatorname{diag}(1, 2).$$

The first numerical example illustrates an optimal scenario for the EIZNN model, alongside the effectiveness of the EZNN model, in solving the time-invariant YBLME (1.2). Furthermore, compared with the ZNN1 model, the computational advantages of the EIZNN and EZNN models in terms of convergence time and noise robustness for solving matrix equation (1.2) will be shown. For the EIZNN and EZNN models, the matrix equation (1.2) is solved using model parameters from the following two cases:

Case 1: $c_1 = 4, c_2 = 5, c_3 = 2, q = 0.3, \gamma = 6, \lambda = 1e - 2$.

Case 2: $c_1 = 5, c_2 = 2.5, c_3 = 1.7, q = 0.36, \gamma = 4, \lambda = 1e - 2$.

6.1.1. Convergence analysis in a noise-free environment

First, we use the EIZNN and EZNN models to solve Example 6.1 in a noise-free environment. Detailed experimental results are shown below.

- (1) For the EIZNN model, we set the initial state matrices as the scalar matrix $X(0)_1 = 0.5I_4$, $X(0)_2 = 0.2I_4$, and $X(0)_3 = 1.2I_4$. Substituting the parameters from Cases 1 and 2 into Theorem 5.1, we can calculate $T_{\max 1} \approx 0.0417$ s and $T_{\max 1} = 0.0500$ s, respectively. Based on numerical simulations, the EIZNN steady-state solutions for Case 1 under $X(0)_1$, $X(0)_2$, and $X(0)_3$ are respectively:

$$X(t)_1 \approx \begin{pmatrix} 0.2562 & 0 & 0 & -0.2562 \\ 0 & 0.2501 & -0.2501 & 0 \\ 0 & -0.2501 & 0.2501 & 0 \\ -0.2562 & 0 & 0 & 0.2562 \end{pmatrix},$$

$$X(t)_2 \approx \begin{pmatrix} 0.1005 & 0 & 0 & -0.1005 \\ 0 & 0.1000 & -0.1000 & 0 \\ 0 & -0.1000 & 0.1000 & 0 \\ -0.1005 & 0 & 0 & 0.1005 \end{pmatrix},$$

$$X(t)_3 \approx \begin{pmatrix} 1.6000 & 0 & 0 & 0.4000 \\ 0 & 0.6162 & -0.6162 & 0 \\ 0 & -0.6162 & 0.6161 & 0 \\ 0.4000 & 0 & 0 & 1.6000 \end{pmatrix}.$$

Similarly, the solutions for Case 2 are

$$X(t)_1 \approx \begin{pmatrix} 0.2450 & 0 & 0 & -0.2451 \\ 0 & 0.2500 & -0.2500 & 0 \\ 0 & -0.2500 & 0.2500 & 0 \\ -0.2450 & 0 & 0 & 0.2451 \end{pmatrix},$$

$$X(t)_2 \approx \begin{pmatrix} 0.0999 & 0 & 0 & -0.0999 \\ 0 & 0.1000 & -0.1000 & 0 \\ 0 & -0.1000 & 0.1000 & 0 \\ -0.0999 & 0 & 0 & 0.0999 \end{pmatrix},$$

$$X(t)_3 \approx \begin{pmatrix} 1.6000 & 0 & 0 & 0.4000 \\ 0 & 0.5956 & -0.5956 & 0 \\ 0 & -0.5956 & 0.5956 & 0 \\ 0.4000 & 0 & 0 & 1.6000 \end{pmatrix}.$$

These essentially consistent results across different cases demonstrate that the model parameters (e.g., c_1 , c_2 , γ) primarily dictate the convergence rate rather than the final steady-state solutions.

- (2) For the EZNN model, we set the initial state matrices as $E(0)_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $E(0)_2 = \begin{pmatrix} 0.6 & 0 \\ 0.6 & 0 \end{pmatrix}$, and $E(0)_3 = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Substituting the parameters from Cases 1 and 2 into Theorem 5.2, we can calculate $T_{\max} \approx 0.0417$ s and $T_{\max} = 0.0500$ s, respectively. Under both Cases 1 and 2, the EZNN model converges to the identical steady-state solutions corresponding to their initial states:

$$E(t)_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E(t)_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E(t)_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This further confirms the models' robustness, as varying parameters only alters the convergence time without changing the final equilibrium.

It is important to emphasize that it can be easily verified by direct substitution that these varying steady-state matrices are indeed true analytical solutions satisfying (1.2) and (4.1), respectively. This dynamically illustrates the solving procedure of the presented models when multiple solutions exist, demonstrating that the network trajectories successfully converge to specific equilibrium points within the non-unique solution set depending on their initial states.

To visualize the convergence behavior, we define

$$E(t) = \begin{pmatrix} e_{11}(t) & e_{12}(t) \\ e_{21}(t) & e_{22}(t) \end{pmatrix}.$$

Figure 1 illustrates the convergence of the EIZNN model. For the EIZNN model using Case 1 parameters, as shown in Figure 1(a), the three absolute errors $\|X(t)AX(t) - AX(t)A\|_F$ corresponding to different initial states $X_\mu(0)$ (where $\mu = 1, 2, 3$) converge to zero within a fixed time not exceeding 0.0417 s. Here, $\|\cdot\|_F$ denotes the Frobenius norm of the matrix. From Figure 1(b), it can be clearly seen that the EIZNN model using Case 2 parameters converges in much less than 0.0500 s. It is crucial to note that in this specific instance, the EIZNN model's trajectory successfully minimizes the error to zero only because the residual limit set $\mathcal{M}$ (as defined in Theorem 5.1) perfectly aligns with the exact solution.

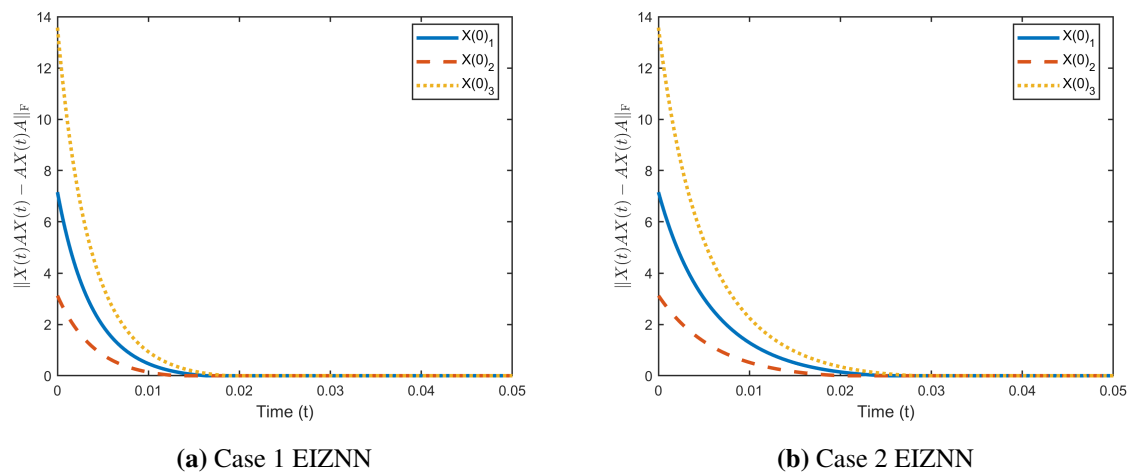


Figure 1. Convergence of the EIZNN model for solving Example 6.1.

Figure 2 illustrates how the convergence time of the EIZNN model is influenced by its key design parameters c_1 and γ , with their theoretical upper bounds provided in Tables 1 and 2, respectively. Figure 2(a) shows the model behavior when parameter c_1 is varied, while Figure 2(b) presents the model performance when parameter γ is varied. From the data in the figures and tables, it is evident that an increase in parameter c_1 is directly associated with a reduction in convergence time (Figure 2(a) and Table 1). Correspondingly, an increase in parameter γ also demonstrates a similar accelerating effect on convergence (Figure 2(b) and Table 2), which confirms the regulatory role of both parameters on the model's convergence performance.

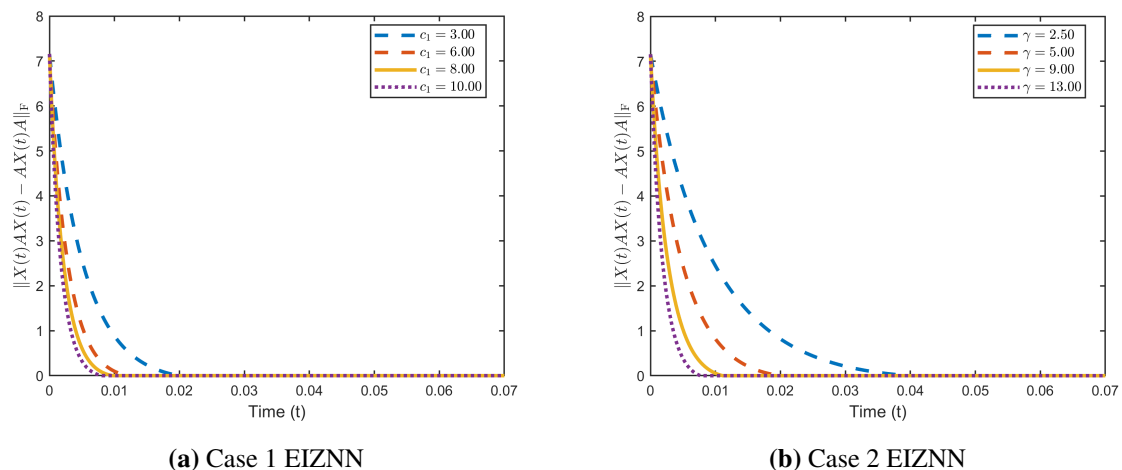


Figure 2. Influence of parameters on the convergence time of the EIZNN model for solving Example 6.1.

Table 1. Convergence time of the EIZNN model for different c_1 in Case 1.

parameters	$c_1 = 3$	$c_1 = 6$	$c_1 = 8$	$c_1 = 10$
$T_{\max 1}$	≈ 0.0556 s	≈ 0.0278 s	≈ 0.0208 s	≈ 0.0167

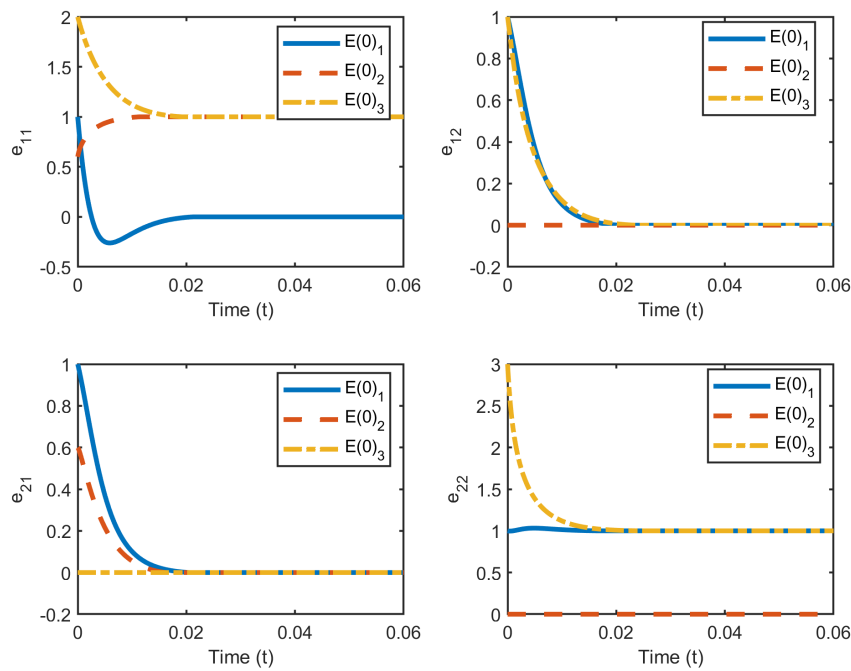
Table 2. Convergence time of the EIZNN model for different γ in Case 2.

parameters	$\gamma = 2.5$	$\gamma = 5$	$\gamma = 9$	$\gamma = 13$
$T_{\max 1}$	$= 0.0800 \text{ s}$	$= 0.0400 \text{ s}$	$\approx 0.0222 \text{ s}$	$\approx 0.0154 \text{ s}$

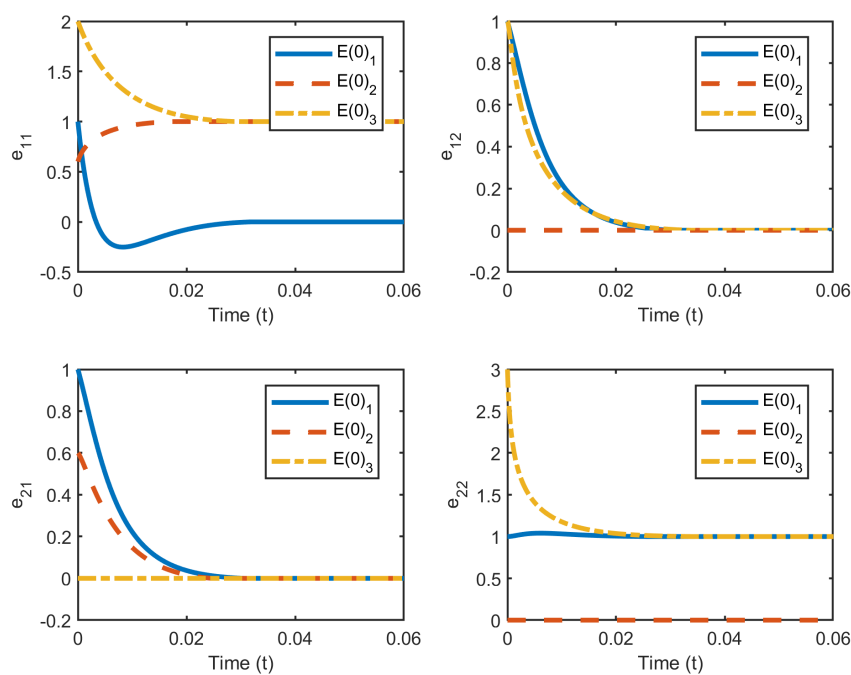
Figure 3 illustrates the trajectories of the state matrix $E(t)$ of the EZNN model for different initial values under Case 1(a) and Case 2(b) conditions. In each subplot, the curves originating from three distinct initial states converge to their respective steady-state solutions. As observed from Figure 3, all monitored elements within the EZNN model exhibit rapid dynamic responses, reaching a steady state in less than 0.035 second. This implies that $E(t)$ converges to a stable solution within a very short period under all tested conditions, demonstrating the effectiveness of the EZNN model in solving Example 6.1.

Figure 4 shows the convergence of the EZNN model. For the EZNN model using Case 1 parameters, as seen from Figure 4(a), the three absolute errors $\|H(t)\|_F$ corresponding to different initial states $E_\mu(0)$ (where $\mu = 1, 2, 3$) converge to zero within a fixed time not exceeding 0.0417 s. Here, $\|\cdot\|_F$ denotes the Frobenius norm of the matrix. From Figure 4(b), it can be clearly seen that the EZNN model using Case 2 parameters converges in much less than 0.0500 s, which verifies the effectiveness of Theorem 5.2.

Figure 5 visually demonstrates how the convergence time of the EZNN model is affected by its key parameters c_1 and γ . Figure 5(a) illustrates the EZNN model behavior where parameter c_1 is varied, with all other parameters fixed at Case 1. Conversely, Figure 5(b) presents the model's behavior where parameter γ is varied, with all other parameters fixed at Case 2. To provide a more comprehensive quantitative analysis, Tables 1 and 2 also present the theoretical upper bounds of the EZNN model's convergence time for different values of c_1 and γ , respectively. Specifically, according to the trend observed in the curves of Figure 5(a) and the theoretical data provided in Table 3, it can be observed that as the value of parameter c_1 increases, the convergence time of the EZNN model significantly decreases. Similarly, as shown in Figure 5(b) and Table 4, when the parameter γ increases, the convergence process of the EZNN model is also accelerated, and its convergence time decreases accordingly.



(a) Case 1 EZNN



(b) Case 2 EZNN

Figure 3. Trajectory of the state matrix $e_{ij}(t)$ for the EZNN model converging under the conditions for Cases 1 and 2.

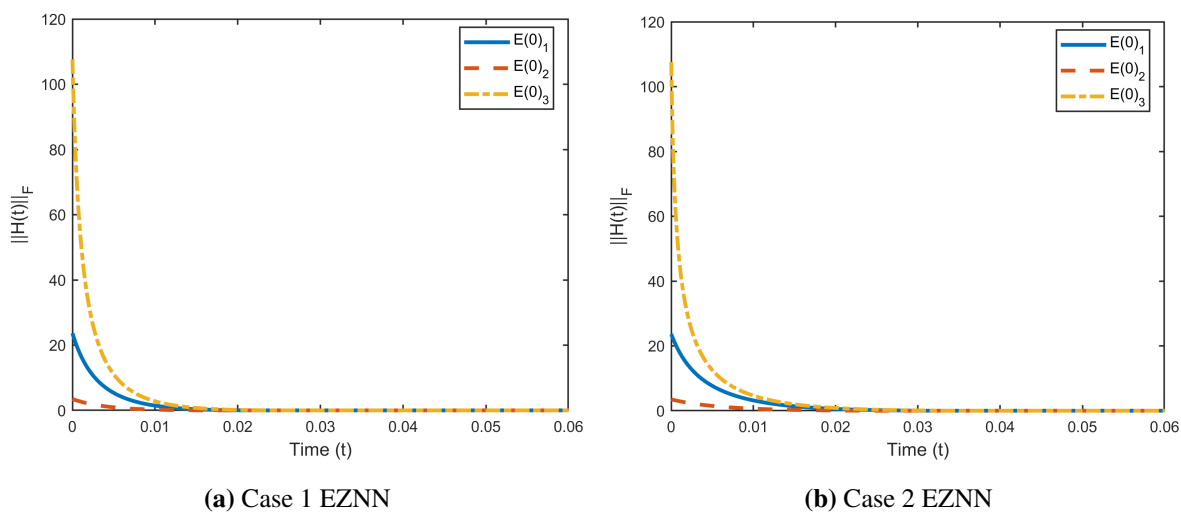


Figure 4. Convergence of the EZNN model under different initial values.

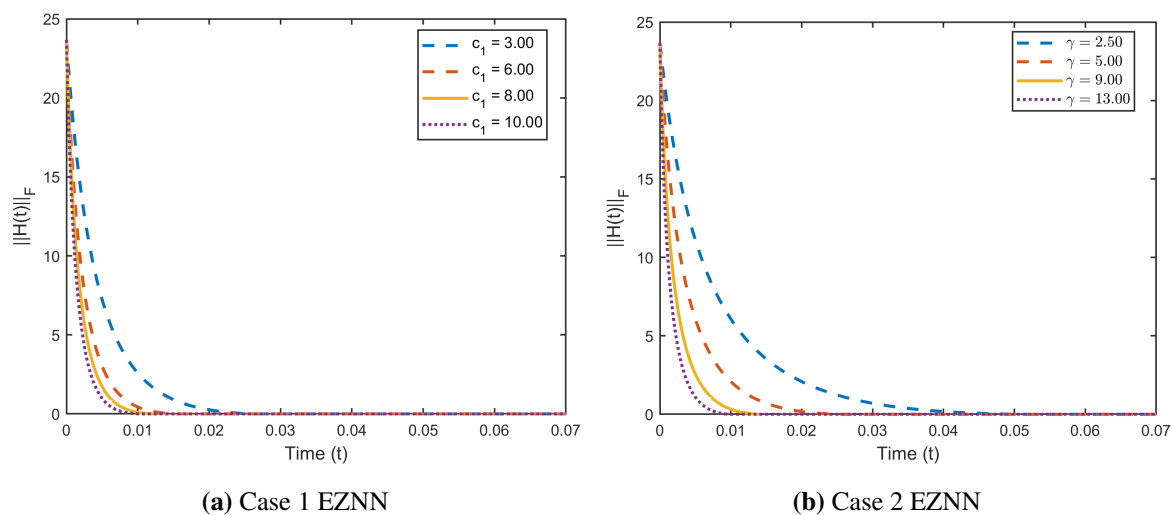


Figure 5. Influence of parameters on the convergence time of the EZNN model.

Table 3. Convergence time of EZNN model for different c_1 in Case 1.

parameters	$c_1 = 3$	$c_1 = 6$	$c_1 = 8$	$c_1 = 10$
$T_{\max 2}$	≈ 0.0556 s	≈ 0.0278 s	≈ 0.0208 s	≈ 0.0167

Table 4. Convergence time of EZNN model for different γ in Case 2.

parameters	$\gamma = 2.5$	$\gamma = 5$	$\gamma = 9$	$\gamma = 13$
$T_{\max 2}$	$= 0.0800$ s	$= 0.0400$ s	≈ 0.0222 s	≈ 0.0154 s

6.1.2. Robustness analysis under noise perturbations

In the following experiments, we aim to solve specific examples using the EZNN, EIZNN, and ZNN1 models, and systematically evaluate three different noise environment states: the first is the ideal case of no noise interference, i.e., $\Theta(t) = 0$; the second is dynamically bounded vanishing noise; and the third is dynamically bounded non-vanishing noise. Below, we will present in detail the specific experimental results and discussion.

For the EZNN, EIZNN, and ZNN1 models, YBLME (1.2) is solved in the presence of noise under two sets of parameters. The initial conditions for the EIZNN and ZNN1 models are set as $X(0) = 1.2I_4$. To ensure model comparability, the initial condition for the EZNN model is determined by solving the equation $X(0) = UE(0)G^T$ via the least squares method, yielding $E(0) = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.3 \end{pmatrix}$.

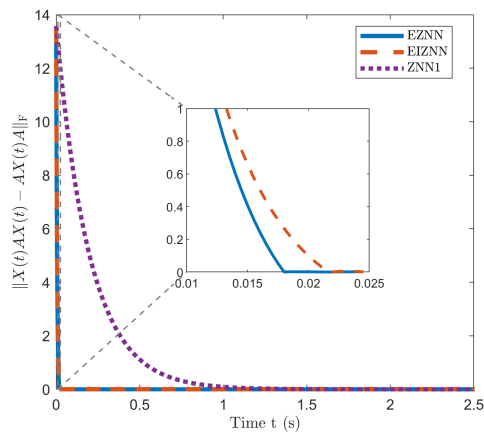
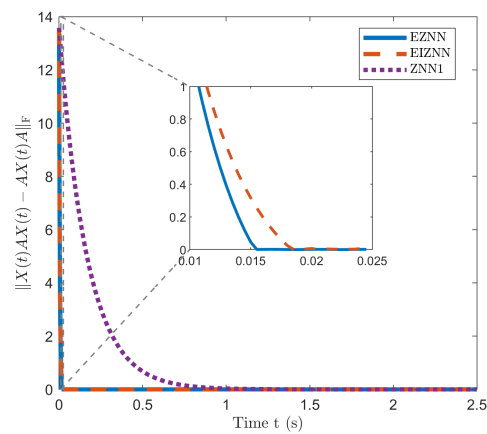
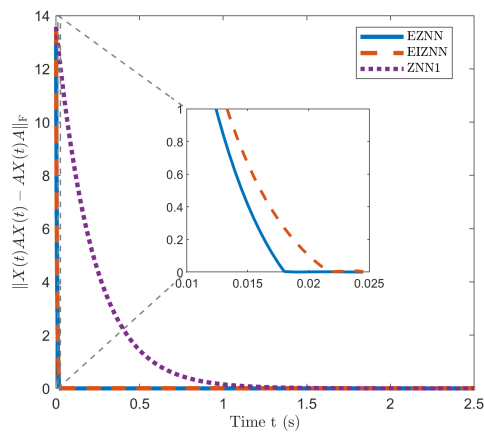
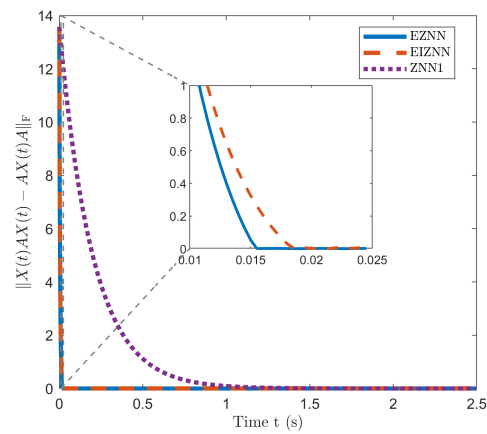
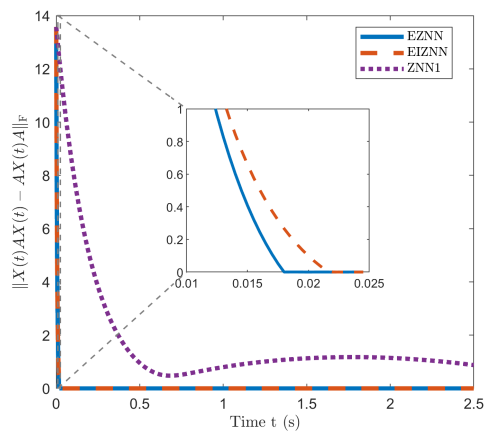
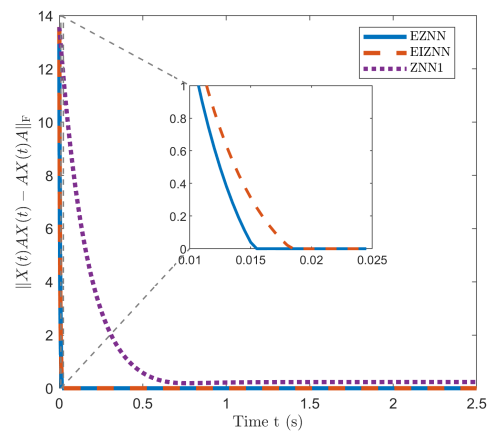
Case 3: $c_1 = 2, c_2 = 5, c_3 = 5, q = 0.18, \gamma = 5, \lambda = 1e - 2$.

Case 4: $c_1 = 3, c_2 = 6, c_3 = 5, q = 0.3, \gamma = 6, \lambda = 1e - 2$.

Figure 6 illustrates the noise tolerance of three models: EZNN, EIZNN, and ZNN1, under various noise conditions. In Figure 6, “Error” denotes the respective error functions. Initially, under ideal noise-free conditions ($D(t) = 0$, as depicted in Figures 6(a) and (b)), all models converge rapidly to zero, with the EZNN and EIZNN models exhibiting the fastest speeds.

However, when subjected to dynamically bounded vanishing and non-vanishing noise (as shown in Figures 6(c) to (f)), the EZNN and EIZNN models demonstrate superior robustness, precisely converging their absolute error to zero and effectively overcoming persistent noise interference. Notably, the EZNN model converges faster than the EIZNN model in all noisy scenarios. In stark contrast, the ZNN1 model fails to achieve precise convergence under these noisy conditions, with its absolute error settling into a non-zero steady-state error region (as shown in Figures 6(e), (f)).

In the simulations, the computational cost is measured by the CPU time using MATLAB’s `cputime` function. The ordinary differential equations are solved using the MATLAB solver `ode45` with relative and absolute error tolerances set to 10^{-6} and 10^{-8} , respectively, and the integration is terminated by an event function once the Frobenius norm of the error satisfies $\|\Theta(t)\|_F \leq 10^{-3}$.

(a) Case 3 and $D(t) = 0$ (b) Case 4 and $D(t) = 0$ (c) Case 3 and $D(t) = 0.5 \cdot \Theta(t)$ (d) Case 4 and $D(t) = 1 \cdot \Theta(t)$ (e) Case 3 and $D(t) = 1.5 \sin t$ (f) Case 4 and $D(t) = 0.35 \cdot \mathbf{1}_{n \times n}$ **Figure 6.** Noise robustness of the EIZNN and EZNN models when solving Example 6.1.

6.2. Example 6.2

To further observe the convergence behavior described in Theorem 5.1, particularly the limitations associated with irreducible null spaces, we consider the following singular coefficient matrix:

$$A = \text{diag}(1, 1, 1, 1, 0) \in \mathbb{R}^{5 \times 5},$$

where its full-rank factorization components are selected as

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the auxiliary matrix is calculated as $B = G^T U = I_4$.

To systematically evaluate the EIZNN and EZNN models under this highly singular condition, we adopt the same parameter configurations defined in Example 6.1. Specifically, we utilize the parameters of Case 1 for the noise-free environment and Case 3 for the noise-perturbed environment.

6.2.1. Convergence analysis in a noise-free environment

For the noise-free simulation using Case 1 parameters, we select three distinct initial states for the EIZNN model: $X(0)_1 = \text{diag}(0.5, 2.0, 2.0, 2.0, 0)$, $X(0)_2 = \text{diag}(0.5, 0.5, 2.0, 2.0, 0)$, and $X(0)_3 = \text{diag}(0.5, 0.5, 0.5, 2.0, 0)$. As predicted by Theorem 5.1, when the state trajectory falls into the irreducible null space, the model converges only to the residual limit set $\mathcal{M}$ rather than the exact zero-error solution.

For the EZNN model, the decomposed auxiliary matrix $B = I_4$ is invertible, which strictly satisfies Assumption 4.1. Utilizing the corresponding initial states $E(0)_1 = \text{diag}(2.0, 2.0, 2.0, 2.0)$, $E(0)_2 = \text{diag}(2.5, 2.5, 2.5, 2.5)$, and $E(0)_3 = \text{diag}(3.5, 3.5, 3.5, 3.5)$, the EZNN model effectively bypasses the singularity issue.

Figure 7 illustrates the convergence behaviors corresponding to the aforementioned theoretical analysis. As depicted in Figure 7(a), the error trajectories of the EIZNN model for the three initial states decrease initially but eventually stagnate at non-zero asymptotic values, which empirically verifies the residual limit set $\mathcal{M}$ induced by the irreducible null space. In contrast, Figure 7(b) demonstrates that the EZNN model successfully drives the error to zero within the theoretical fixed-time bound, maintaining convergence stability under all tested initial conditions.

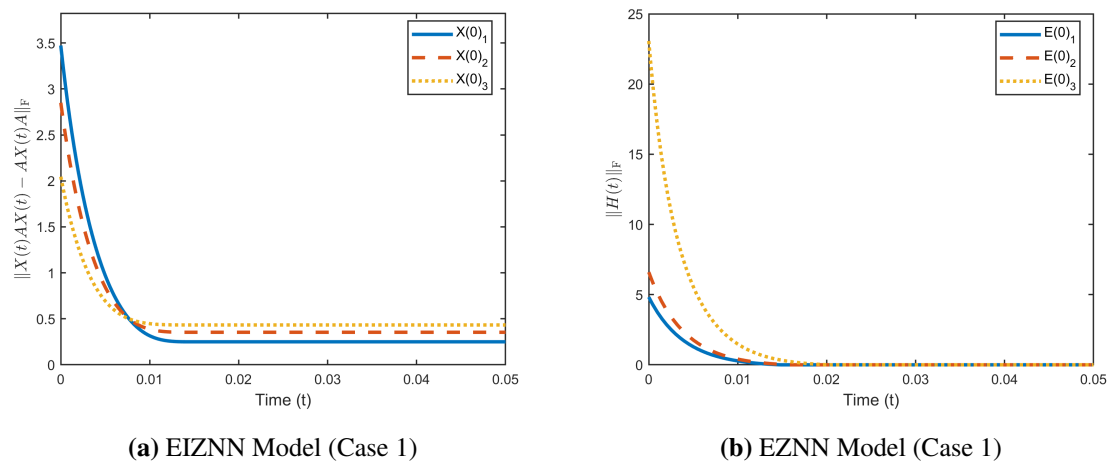


Figure 7. Comparison of convergence behaviors for singular matrix A in Example 6.2.

6.2.2. Robustness analysis under noise perturbations

To further verify the structural reliability of both models, we introduce dynamically bounded noise perturbations using the Case 3 parameter set. Under these noisy conditions, the residual error of the EIZNN model continues to stagnate within a non-zero steady-state region due to the inherent ill-posedness of matrix A . Conversely, benefiting from the full-rank equivalent transformation, the EZNN model maintains noise robustness, suppressing the bounded noise and converging to the exact solution.

This comparative example illustrates that the structural dimensionality reduction in the EZNN model provides an effective approach for solving singular YBLME cases, whereas direct Tikhonov regularization is theoretically and practically constrained by null-space limitations.

Figure 8 illustrates the convergence behaviors of the EIZNN and EZNN models under distinct noise perturbations. The performance of the EIZNN model exhibits dependency on the characteristics of the applied noise due to the singularity of matrix A . Under the vanishing noise ($D(t) = 1 \cdot \Theta(t)$), the state trajectory enters the irreducible null space, resulting in the residual error stagnating at a non-zero constant (Figure 8(a)). Under the non-vanishing constant noise ($D(t) = 0.35 \cdot \mathbf{1}_{n \times n}$), the continuous perturbation displaces the state trajectory from the null space, which subsequently allows the activation function to minimize the residual error to zero (Figure 8(b)).

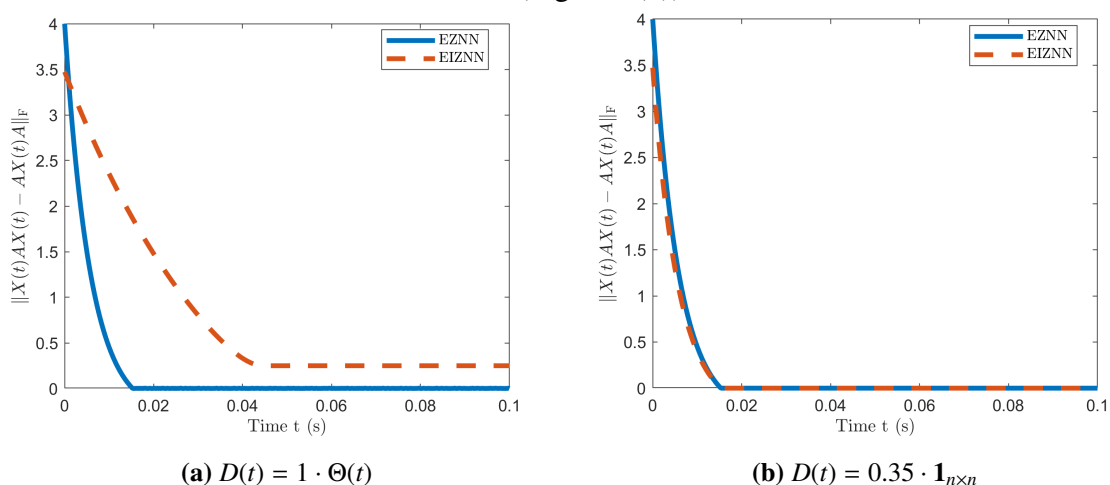


Figure 8. Comparison of convergence behaviors for singular matrix A in Example 6.2.

In contrast to the conditional convergence of the EIZNN model, the EZNN model maintains consistent convergence regardless of the noise type. By utilizing the full-rank equivalent transformation to bypass the singularity, the EZNN model avoids null-space limitations, suppresses both types of bounded noise, and drives the error trajectory to zero within the theoretical fixed-time bound.

6.2.3. Sensitivity analysis to the regularization parameter

In accordance with Theorem 5.3, the steady-state residual error of the EIZNN model under dynamically bounded non-vanishing noise exhibits dependency on the regularization parameter λ . To systematically evaluate this parameter sensitivity, we investigate the convergence behavior of the EIZNN model under the non-vanishing constant noise $D(t) = 0.35 \cdot \text{ones}$ across varying regularization values: $\lambda \in \{0.01, 0.1, 1, 10\}$.

Figure 9 illustrates the impact of λ on the steady-state accuracy. When a larger regularization parameter is applied (e.g., $\lambda = 10$), the regularization bias becomes prominent, causing the error trajectory to stabilize at a non-zero steady-state value. Conversely, as λ decreases ($\lambda = 0.1$ and 0.01), the structural deviation induced by Tikhonov regularization is mitigated. Although Theorem 5.3 establishes that a smaller λ theoretically enlarges the worst-case noise perturbation bounds, the continuous displacement provided by the constant noise in this specific scenario shifts the model out of the irreducible null space. Consequently, the reduced structural bias at smaller λ values facilitates a closer convergence toward the zero-error solution. This parameter dependency indicates that applying the EIZNN model requires a quantitative trade-off between bounding theoretical noise perturbations and minimizing steady-state errors. In contrast, the EZNN model, which relies on full-rank factorization rather than Tikhonov regularization, maintains exact zero-error convergence independently of such parameter constraints.

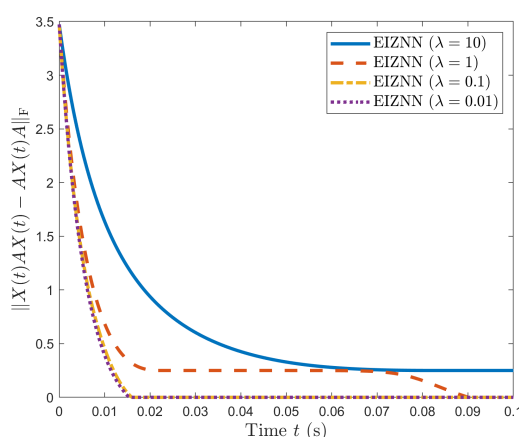


Figure 9. Sensitivity of the EIZNN model's steady-state error to the regularization parameter λ under constant noise $D(t) = 0.35 \cdot \text{ones}$.

6.3. Example 6.3

In this experiment, we randomly selected 5 types of matrices, and experiments were conducted in a noise-free environment. Their matrix dimensions were 32, 40, 48, 56, and 64, with each matrix having a rank of 2. We performed numerical experiments on these 5 types of matrices using the EZNN and

ZNN1 models with the following settings: $c_1 = 5$, $c_2 = 0$, $c_3 = 0$, $q = 0.1$, $\gamma = 8$, $\lambda = 1e - 2$.

Table 5 and Figure 10 jointly demonstrate the difference in CPU computational times between the EZNN and ZNN1 models across varying matrix dimensions. The ZNN1 model exhibits a relatively long computational time and lower efficiency, while its required time increases exponentially with growing matrix dimensions. This indicates practical limitations in the application of the ZNN1 model. In contrast, the EZNN model consistently achieves lower CPU computational times across all tested matrix dimensions.

Table 5. Average CPU running time comparison of the EZNN and ZNN1 models.

n	32	40	48	56	64
ZNN1	21.9844	51.0312	139.6250	252.3750	371.2132
EZNN	0.0068	0.0071	0.0078	0.0089	0.0112

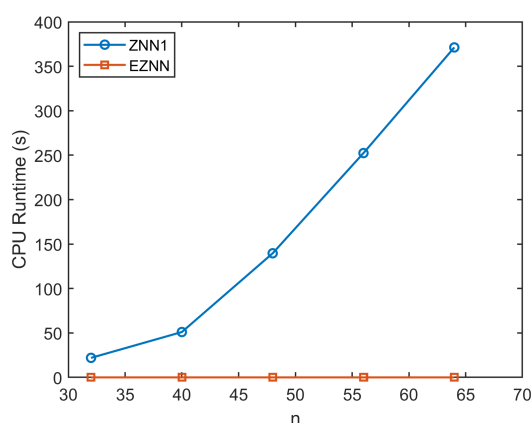


Figure 10. Average CPU running time comparison of the EZNN and ZNN1 models.

6.4. Example 6.4

In this experiment, we randomly selected five types of high-dimensional matrices with dimensions of 256, 512, 1024, 2048, and 4096, respectively. Each matrix had a rank of 2. Experiments were conducted in a noise-free environment, and numerical experiments were performed solely on these five types of matrices using the EZNN model with the following settings: $c_1 = 5$, $c_2 = 0$, $c_3 = 0$, $q = 0.1$, $\gamma = 8$.

This study evaluated the CPU runtime of the EZNN model on different high-dimensional matrices. Figure 11 graphically illustrates the consistent convergence of the runtime curves across five independent random matrix instances, providing strong evidence for the stability and reproducibility of the model's performance. Building upon this, Table 6 summarizes the average CPU running times obtained from extensive experiments, offering quantitative metrics for computational efficiency. The experimental results clearly indicate that the EZNN model exhibits high efficiency when processing high-dimensional low-rank matrices. However, when running these random matrices with the ZNN1 and EIZNN models, the required memory far exceeded the computer capacity. Consequently, we were unable to execute these five types of matrices using ZNN1 and EIZNN.

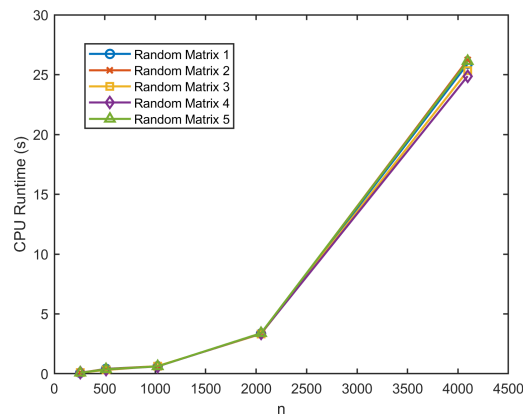


Figure 11. CPU runtime of the EZNN model.

Table 6. Average CPU running time of the EZNN model for different n values.

n	256	512	1024	2048	4096
CPU (s)	0.0840	0.3432	0.6250	3.3594	25.4062

7. Conclusions

In this paper, we investigated the numerical solutions of the time-invariant YBLME involving singular coefficient matrices. We proposed two structurally distinct zeroing neural network models: the EIZNN and the EZNN. The EIZNN model acts as a transitional approach by integrating Tikhonov regularization and a nonlinear activation function; however, theoretical analysis revealed that this method converges asymptotically to a residual limit set due to the irreducible null space of the singular matrix. To address this limitation, we developed the EZNN model, which utilizes full-rank factorization to equivalently transform the high-dimensional singular problem into a lower-dimensional full-rank matrix equation. This structural reduction bypasses the singularity issue without relying on Tikhonov regularization and mitigates memory overflow, enabling the computation of high-dimensional matrices (e.g., 4096×4096). Mathematical proofs and numerical simulations indicate that the EZNN model achieves fixed-time convergence and exhibits robustness against both dynamically bounded vanishing and non-vanishing noises. Conversely, the EIZNN model exhibits steady-state errors under non-vanishing noise. These findings confirm that the EZNN model provides a viable neurodynamic framework for solving the large-scale singular YBLME in practical hardware implementations.

Author contributions

Conceptualization and methodology: all authors. Software and visualization: all authors. Formal analysis: all authors. Writing: all authors.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. S. R. Garcia, A. L. Shoemaker, On the matrix equation $XA + AX^T = 0$, *Linear Algebra Appl.*, **438** (2013), 2740–2746. <https://doi.org/10.1016/j.laa.2012.10.041>
2. C. Han, B. Zheng, J. Xu, A modified noise-tolerant ZNN model for solving time-varying Sylvester equation with its application to robot manipulator, *J. Franklin Ins.*, **360** (2023), 8633–8650. <https://doi.org/10.1016/j.jfranklin.2023.06.045>
3. J. Hietarinta, Solving the two-dimensional constant quantum Yang-Baxter equation, *J. Math. Phys.*, **34** (1993), 1725–1756. <https://doi.org/10.1063/1.530185>
4. J. Ding, H. Tian, Solving the Yang-Baxter-like matrix equation for a class of elementary matrices, *Comput. Math. Appl.*, **72** (2016), 1541–1548. <https://doi.org/10.1016/j.camwa.2016.07.015>
5. C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, *Phys. Rev. Lett.*, **19** (1967), 1312. <https://doi.org/10.1103/PhysRevLett.19.1312>
6. R. J. Baxter, Partition function of the eight-vertex lattice model, *Ann. Phys.*, **70** (1972), 193–228. [https://doi.org/10.1016/0003-4916\(72\)90335-1](https://doi.org/10.1016/0003-4916(72)90335-1)
7. D. K. Matsumoto, Y. Shibukawa, Quantum Yang-Baxter equation, braided semigroups, and dynamical Yang-Baxter maps, *Tokyo J. Math.*, **38** (2015), 227–237. <https://doi.org/10.3836/tjm/1437506246>
8. Z. Tsuboi, Quantum groups, Yang-Baxter maps and quasi-determinants, *Nucl. Phys. B*, **926** (2018), 200–238. <https://doi.org/10.1016/j.nuclphysb.2017.11.005>
9. R. Vieira, A. Lima-Santos, Solutions of the Yang-Baxter equation for $(n+1)(2n+1)$ -vertex models using a differential approach, *J. Stat. Mech.*, **2021** (2021), 053103. <https://doi.org/10.1088/1742-5468/abf7be>
10. J. H. Przytycki, Knot theory: from fox 3-colorings of links to Yang-Baxter homology and Khovanov homology, In: *Knots, low-dimensional topology and applications*, Cham: Springer, 2016, 115–145. https://doi.org/10.1007/978-3-030-16031-9_5
11. Z. K. Lyu, W. R. Xu, A matrix nearness problem constrained on the kernel space of multilevel $(R, S \sigma)$ -commutative matrices, *Comput. Appl. Math.*, **44** (2025), 270. <https://doi.org/10.1007/s40314-025-03234-7>

12. H. Ren, X. Wang, T. Wang, Commuting solutions of the Yang-Baxter-like matrix equation for a class of rank-two updated matrices, *Comput. Math. Appl.*, **76** (2018), 1085–1098. <https://doi.org/10.1016/j.camwa.2018.05.042>
13. D. Shen, M. Wei, Z. Jia, On commuting solutions of the Yang-Baxter-like matrix equation, *J. Math. Anal. Appl.*, **462** (2018), 665–696. <https://doi.org/10.1016/j.jmaa.2018.02.030>
14. D. Shen, M. Wei, All solutions of the Yang-Baxter-like matrix equation for diagonalizable coefficient matrix with two different eigenvalues, *Appl. Math. Lett.*, **101** (2020), 106048. <https://doi.org/10.1016/j.aml.2019.106048>
15. L. Jin, S. Li, H. M. La, X. Zhang, B. Hu, Dynamic task allocation in multi-robot coordination for moving target tracking: a distributed approach, *Automatica*, **100** (2019), 75–81. <https://doi.org/10.1016/j.automatica.2018.11.001>
16. G. Wang, D. Li, X. Chen, H. Huang, Two modified Newton-Raphson iteration algorithms for Yang-Baxter-like matrix equation with step-size analyses, *Proceedings of 35th Youth Academic Annual Conference of Chinese Association of Automation (YAC)*, 2020, 259–263. <https://doi.org/10.1109/YAC51587.2020.9337656>
17. L. Jin, S. Li, L. Xiao, R. Lu, B. Liao, Cooperative motion generation in a distributed network of redundant robot manipulators with noises, *IEEE Trans. Syst. Man Cyb.*, **48** (2018), 1715–1724. <https://doi.org/10.1109/TSMC.2017.2693400>
18. L. Jin, Y. Zhang, Discrete-time Zhang neural network for online time-varying nonlinear optimization with application to manipulator motion generation, *IEEE Trans. Neur. Net. Lear.*, **26** (2015), 1525–1531. <https://doi.org/10.1109/TNNLS.2014.2342260>
19. Y. Zhang, S. S. Ge, Design and analysis of a general recurrent neural network model for time-varying matrix inversion, *IEEE Trans. Neural Networ.*, **16** (2005), 1477–1490. <https://doi.org/10.1109/TNN.2005.857946>
20. Y. Zhang, D. Jiang, J. Wang, A recurrent neural network for solving Sylvester equation with time-varying coefficients, *IEEE Trans. Neural Networ.*, **13** (2002), 1053–1063. <https://doi.org/10.1109/TNN.2002.1031938>
21. R. Manherz, B. Jordan, S. Hakimi, Analog methods for computation of the generalized inverse, *IEEE Trans. Automat. Contr.*, **13** (1968), 582–585. <https://doi.org/10.1109/TAC.1968.1098973>
22. M. Hüsken, P. Stagge, Recurrent neural networks for time series classification, *Neurocomputing*, **50** (2003), 223–235. [https://doi.org/10.1016/S0925-2312\(01\)00706-8](https://doi.org/10.1016/S0925-2312(01)00706-8)
23. T. Shi, Y. Tian, Z. Sun, K. Liu, L. Jin, J. Yu, Noise-tolerant neural algorithm for online solving Yang-Baxter-type matrix equation in the presence of noises: a control-based method, *Neurocomputing*, **424** (2021), 84–96. <https://doi.org/10.1016/j.neucom.2020.10.110>
24. V. N. Katsikis, S. D. Mourtas, P. S. Stanimirović, Y. Zhang, Solving complex-valued time-varying linear matrix equations via QR decomposition with applications to robotic motion tracking and on angle-of-arrival localization, *IEEE Trans. Neur. Net. Lear.*, **33** (2022), 3415–3424. <https://doi.org/10.1109/TNNLS.2021.3052896>
25. P. S. Stanimirović, V. N. Katsikis, S. Li, Hybrid GNN-ZNN models for solving linear matrix equations, *Neurocomputing*, **316** (2018), 124–134. <https://doi.org/10.1016/j.neucom.2018.07.058>
26. M. Kornilova, V. Kovalnogov, R. Fedorov, M. Zamaleev, V. N. Katsikis, S. D. Mourtas, et al., Zeroing neural network for pseudoinversion of an arbitrary time-varying matrix based on singular value decomposition, *Mathematics*, **10** (2022), 1208. <https://doi.org/10.3390/math10081208>

27. V. N. Katsikis, S. D. Mourtas, P. S. Stanimirović, S. Li, X. Cao, Time-varying mean–variance portfolio selection problem solving via LVI-PDNN, *Comput. Oper. Res.*, **138** (2022), 105582. <https://doi.org/10.1016/j.cor.2021.105582>
28. L. Xiao, Y. Zhang, Z. Hu, J. Dai, Performance benefits of robust nonlinear zeroing neural network for finding accurate solution of Lyapunov equation in presence of various noises, *IEEE Trans. Ind. Inform.*, **15** (2019), 5161–5171. <https://doi.org/10.1109/TII.2019.2900659>
29. J. Zhao, Y. Zhang, A novel zeroing neural network with integral upper-bound function for solving time-varying Sylvester equation, *Comput. Appl. Math.*, **44** (2025), 374. <https://doi.org/10.1007/s40314-025-03340-6>
30. T. Huang, S. X. Miao, An improved zeroing neural network model for solving the time-varying Yang-Baxter-like matrix equation, *J. Math. Anal. Appl.*, **545** (2025), 129095. <https://doi.org/10.1016/j.jmaa.2024.129095>
31. T. Huang, S. X. Miao, An improved noise-tolerant zeroing neural network model for solving the time-varying Yang-Baxter-like matrix equation, *Comput. Appl. Math.*, **45** (2026), 180. <https://doi.org/10.1007/s40314-025-03576-2>
32. W. Jiang, C. L. Lin, V. N. Katsikis, S. D. Mourtas, P. S. Stanimirović, T. E. Simos, Zeroing neural network approaches based on direct and indirect methods for solving the Yang-Baxter-like matrix equation, *Mathematics*, **10** (2022), 1950. <https://doi.org/10.3390/math10111950>
33. H. N. Nguyen, S. Olaru, P. O. Gutman, M. Hovd, Constrained control of uncertain, time-varying linear discrete-time systems subject to bounded disturbances, *IEEE Trans. Automat. Contr.*, **60** (2015), 831–836. <https://doi.org/10.1109/TAC.2014.2346872>
34. Y. Orlov, Finite time stability and robust control synthesis of uncertain switched systems, *SIAM J. Control Optim.*, **43** (2004), 1253–1271. <https://doi.org/10.1137/S0363012903425593>
35. A. Polyakov, Nonlinear feedback design for fixed-time stabilization of linear control systems, *IEEE Trans. Automat. Contr.*, **57** (2012), 2106–2110. <https://doi.org/10.1109/TAC.2011.2179869>
36. B. Qiu, Y. Zhang, Z. Yang, New discrete-time ZNN models for least-squares solution of dynamic linear equation system with time-varying rank-deficient coefficient, *IEEE Trans. Neur. Net. Lear.*, **29** (2018), 5767–5776. <https://doi.org/10.1109/TNNLS.2018.2805810>
37. H. Zhang, L. Wan, Zeroing neural network methods for solving the Yang-Baxter-like matrix equation, *Neurocomputing*, **383** (2020), 409–418. <https://doi.org/10.1016/j.neucom.2019.11.101>
38. L. Xiao, Y. Zhang, J. Dai, K. Chen, S. Yang, W. Li, et al., A new noise-tolerant and predefined-time ZNN model for time-dependent matrix inversion, *Neural Networks*, **117** (2019), 124–134. <https://doi.org/10.1016/j.neunet.2019.05.005>
39. L. Lu, Constructing solutions of the Yang-Baxter-like matrix equation for singular matrices, *J. Comput. Appl. Math.*, **436** (2024), 115403. <https://doi.org/10.1016/j.cam.2023.115403>



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