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Research article

The star edge coloring of cubic Halin graphs with star chromatic index 5

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Abstract: The star chromatic index of a graph G, denoted by $\chi'_{st}(G)$, is the minimum number of colors needed to properly color the edges of G such that no path or cycle of length four is bi-colored. Casselgren et al. and Hou et al. independently proved that the star chromatic index of a cubic Halin graph, except in the case of a special graph, is at most 6. It remains an open problem to determine which of such graphs have star chromatic index 5. In this paper, we show that if $G \neq N_{e_2}$ is a cubic Halin graph whose tree is a caterpillar or a complete tree, then $\chi'_{st}(G) = 5$.

Keywords: star chromatic index; caterpillar; complete tree; Halin graph
Mathematics Subject Classification: 05C15

1. Introduction

All graphs in this paper are finite and simple. Given a graph G, let c: $E(G) \to [k]$ be a proper edge coloring of G, where $k \geq 1$ and $[k] = \{1, 2, \cdots, k\}$. We say that c is a star k-edge coloring of G if no path or cycle of length four in G is bi-colored under the coloring c; and G is star k-edge colorable if G admits a star k-edge coloring. The star chromatic index of G, denoted by $\chi'_{st}(G)$, is the smallest integer k such that G admits a star k-edge coloring. A strong edge coloring of a graph G is a proper edge coloring so that no edge can be adjacent to two edges with the same color. A strong edge coloring is a special case of star edge coloring, and the strong chromatic index of a graph G, denoted by $\chi'_s(G)$, is an upper bound of $\chi'_{st}(G)$.

Star edge coloring of a graph G was introduced by Liu and Deng [1], motivated by the star vertex coloring problem. Dvořák et al. [2] studied the star chromatic index of subcubic multigraphs and proved that every such graph G satisfied $\chi'_{st}(G) \leq 7$. As observed in [2], both $K_{3,3}$ and the Heawood graph admit star 6-edge colorings. Up to date, no subcubic multigraph with star chromatic index 7 is known. Dvořák et al. [2] therefore proposed the following conjecture.

Conjecture 1.1. [2] *If G is a subcubic graph, then $\chi'_{st}(G) \leq 6$.*

Although Conjecture 1.1 remains open, it was confirmed true for some special graphs, such as subcubic graphs with maximum average degree at most 5/2 [3]. Regarding a star 5-edge coloring of subcubic graphs, it was shown that every subcubic outer-planar graph is star 5-edge-colorable in [4]. Later, Lei et al. [5] proved that if G is a subcubic graph with maximum average degree at most $\frac{12}{5}$, then $\chi'_{\mathrm{st}}(G) \leq 5$. In addition, other graphs of star edge coloring have also been studied extensively, which was explicitly introduced in a survey [6].

A *Halin graph* $G = T \cup C$ is a planar graph that consists of a plane embedding of a tree T that has no vertices of degree two and a cycle C connecting all the leaves of T such that C is the boundary of the exterior face. The tree T is called the characteristic tree. If every non-leaf vertex in the tree T of a Halin graph G has degree exactly 3, then G is called a cubic Halin graph.

Recently, Conjecture 1.1 for cubic Halin graphs was independently proved true by Casselgren et al. [7] and Hou et al. [8].

For $l \geq 1$, a complete tree T_l is a tree of height $l + 1$ with a root vertex v_0 such that all its leaves are at the same distance l from v_0, and we say that T_l has l levels. A complete cubic Halin graph is a cubic Halin graph whose characteristic tree is a complete tree T_l, denoted by G_l. Clearly, $G_1 \cong K_4$.

A caterpillar is a tree such that the removal of all its leaves results in a path. Let $\mathcal{G}_h$ be the set of all cubic Halin graphs whose characteristic trees are caterpillars with $h+2$ leaves. In particular, $\mathcal{G}_1 = \{K_4\}$. For $h \geq 2$ and a Halin graph $G = T \cup C_{h+2} \in \mathcal{G}_h$, denote the spine of T by $P_h = v_1 v_2 \dots v_h$, and let u_0, u_1, and u_h, u_{h+1} be the neighbors of v_1 and v_h on C_{h+2}, respectively. For $2 \leq i \leq h-1$ ($h \geq 3$), let u_i be the neighbor of v_i on C_{h+2}, then $u_i v_i \in E(G)$; we call such an edge $u_i v_i$ a leaf-edge for $2 \leq i \leq h-1$. We draw the graph G on the plane such that the spine P_h in the middle and the leaf-edges $u_i v_i$ incident with v_i either on the left side or right side of P_{h+2} for all $i \in \{2, \dots, h-1\}$. For example, Figure 4 shows all graphs in $\mathcal{G}_4$ and $\mathcal{G}_5$. Furthermore, if $G \in \mathcal{G}_h$, and all the leaf-edges are on the same side of P_h (i.e., $C_{h+2} = u_0 u_1 \cdots u_h u_{h+1} u_0$), then G is called a necklace graph, denoted by N_{e_h}. For example, Figures 4(b),(e) show N_{e_4} and N_{e_5}, respectively.

There are many results about strong edge coloring of cubic Halin graphs. For a cubic Halin graph G different from N_{e_2} and N_{e_4}, Lih et al. [9] proved $\chi'_s(G) \leq 7$. For a Halin graph G other than a wheel W_n, N_{e_2}, and N_{e_4}, Yang et al. [10] determined $\chi'_s(G) \leq \chi'_s(T)+2$. For a complete cubic Halin graph G, Shiu et al. [11] determined $\chi'_s(G) = 6$ when $G \neq N_{e_2}$. For $G \in \mathcal{G}_h$, Shiu et al. [12] determined $\chi'_s(N_{e_h})$, showed that $6 \leq \chi'_s(G) \leq 8$ if $h \geq 4$, and conjectured that $\chi'_s(G) = 6$ if h is odd. Furthermore, Chang et al. [13] disproved the conjecture of Shiu et al. [12] and determined the values of strong chromatic indices for some special families of graphs $G \in \mathcal{G}_h$. Up to date, there are very few results about the star chromatic index for the above two kinds of cubic Halin graphs. Hou et al. [8] proved the following result.

Theorem 1.2. [8] *If h is odd, then* $4 \leq \chi'_{st}(N_{e_h}) \leq 5$.

Based on the above known results, we are interested to study star edge coloring of the above two kinds of cubic Halin graphs. In this paper, we determine the exact values of star chromatic indices for these graphs, which improves Theorem 1.2, and extends the known results. In Section 2, we prove that $\chi'_{st}(G) \geq 5$ for any cubic Halin graph G (see Theorem 2.1). In Section 3, we show that if $G \neq N_{e_2}$ is a complete cubic Halin graph, then $\chi'_{st}(G) = 5$ (see Theorem 3.2). In Section 4, we show that if $G \in \mathcal{G}_h$ and $G \neq N_{e_2}$, then $\chi'_{st}(G) = 5$ (see Theorem 4.1), and hence $\chi'_{st}(N_{e_h}) = 5$ for $h \neq 2$ (see Corollary 4.2).

2. Preliminaries

Theorem 2.1. *If G is a cubic Halin graph, then* $\chi'_{st}(G) \geq 5$.

Proof. Note that $\chi'_{st}(K_4) = 5$, so we may assume $G \ncong K_4$. We prove the result by contradiction. Assume φ is a star 4-edge coloring of G using the color set $\{1, 2, 3, 4\}$. It is easy to see that G contains a 5-cycle C_5. Let $C_5 = u_1u_2u_3u_4u_5u_1$. Note that C_5 is star 4-edge colorable. Therefore, there exist two non-adjacent edges in C_5 with the same color. Without loss of generality, we may assume $\varphi(u_1u_2) = \varphi(u_3u_4) = 1$, $\varphi(u_2u_3) = 2$, $\varphi(u_4u_5) = 3$, and $\varphi(u_5u_1) = 4$. Note that $d_G(u_1) = d_G(u_4) = 3$. Then $u_1u_4 \in E(G)$, or u_1, u_4 have other neighbors v_1, v_4 not on C_5, respectively (see Figure 1(a),(b)).

Figure 1. Two cases for the subgraphs of G: (a) $u_1u_4 \in E(T)$, (b) $u_1u_4 \notin E(T)$.

If $u_1u_4 \in E(G)$, then $\varphi(u_1u_4) \in [4] \setminus \{1, 3, 4\} = \{2\}$, since u_1u_4 is adjacent to the four edges of C_5 except u_2u_3. But $\varphi(u_1u_4) = \varphi(u_2u_3)$, which results in a bi-chromatic 4-cycle $u_1u_2u_3u_4u_1$ with $1-2-1-2$, a contradiction.

Suppose that $u_1v_1, u_4v_4 \in E(G)$. Then $\varphi(u_1v_1) \in \{2, 3\}$, $\varphi(u_4v_4) \in \{2, 4\}$. We claim that $2 \notin \{\varphi(u_1v_1), \varphi(u_4v_4)\}$. Otherwise, there is a bi-chromatic 4-path $v_1u_1u_2u_3u_4$ or $u_1u_2u_3u_4v_4$ with $2-1-2-1$ or $1-2-1-2$, respectively. Then, $\varphi(u_1v_1) = 3$, and $\varphi(u_4v_4) = 4$, which implies in a bi-chromatic 4-path $v_1u_1u_5u_4v_4$ with $3-4-3-4$, a contradiction.

From the above, the result follows. □

3. Characteristic trees are complete trees

Let T_0^i be a full binary tree of 3 levels, where the root vertex is v_0^i, the first-level vertices are x^i, y^i, the second-level vertices are sequentially $x_1^i, x_2^i, y_1^i, y_2^i$, and the third-level vertices are sequentially $x_{11}^i, x_{12}^i, x_{21}^i, x_{22}^i, y_{11}^i, y_{12}^i, y_{21}^i, y_{22}^i$, which are leaves. Define F^i as the graph obtained from T_0^i by sequentially connecting the leaves of T_0^i, such that there is a path $P_8 = x_{11}^i x_{12}^i x_{21}^i x_{22}^i y_{11}^i y_{12}^i y_{21}^i y_{22}^i$ in F^i (that will be applied in Lemma 3.1). For $i \geq 0$, let $G^i = T^i \cup C^i$ be a cubic Halin graph. For a vertex $v^i \in C^i$, let u^i, s^i, and t^i be the neighbors of v^i, where $u^i \in T^i$, $s^i, t^i \in C^i$. First we delete the two edges v^is^i and v^it^i in G^i; then, we attach F^i at v^i to u^i such that $v_0^i = v^i$; lastly, we add two new edges $x_{11}^i s^i$ and $y_{22}^i t^i$ in G^i. Let G^{i+1} be a graph obtained from G^i by the above three steps. It can be shown that if $G^0 = G_l$, then $G_{l+3} = G^k$, where $k = 3 \times 2^{l-1}$.

Lemma 3.1. *For* $i \geq 0$, *let* G^i *be a cubic Halin graph. If* $\chi'_{st}(G^i) \leq 5$, *then* $\chi'_{st}(G^{i+1}) \leq 5$.

Proof. Let φ be a star 5-edge coloring of G^i. Without loss of generality, assume that $\varphi(v_0^i u_1^i) = 1$, $\varphi(v_0^i s_1^i) = 2$, $\varphi(v_0^i t_1^i) = 3$. Now we extend the coloring ϕ of G^i to G^{i+1} using the colors shown in Figure 2. This gives a star 5-edge coloring of G^{i+1}. Thus the result follows. □

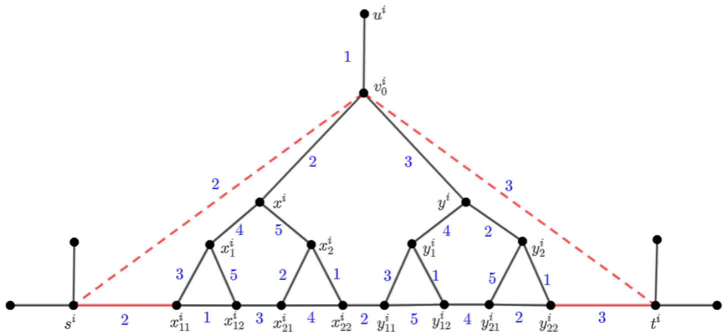
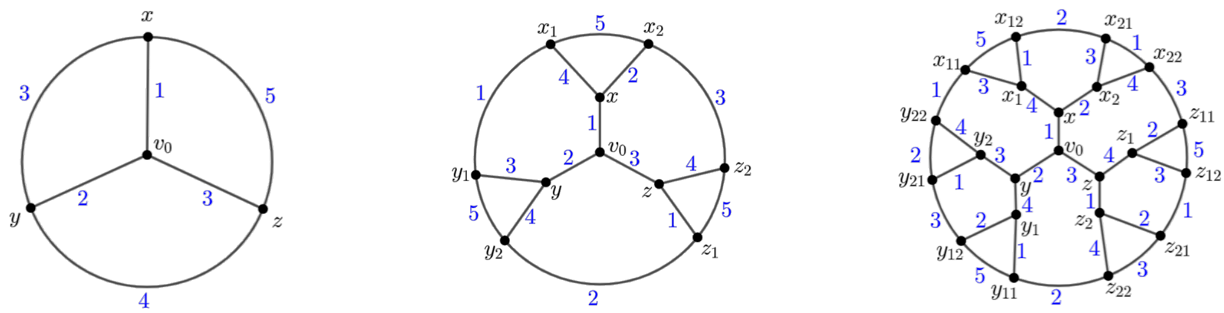


Figure 2. Structure of G^{i+1}.

Theorem 3.2. *If G is a complete cubic Halin graph, then* $\chi'_{st}(G) = 5$.

Proof. Let $G = G_l$. If $l = 1$, then $G = K_4$. From Figure 3(a), it is clear that $\chi'_{st}(G) = 5$. Suppose that $l \geq 2$. By Theorem 2.1, we only need to prove that $\chi'_{st}(G) \leq 5$. We will prove it by induction on the level l of G. If $l = 2, 3$, then $\chi'_{st}(G) \leq 5$, see Figure 3(b),(c). Suppose that for any graph G with level $l \leq k$, the result holds. Repeatedly using Lemma 3.1, we can directly prove that if $l = k + 3$, then $\chi'_{st}(G) \leq 5$. Thus the result follows. □



(a) **(b)** **(c)**

Figure 3. (a) $G_1 = K_4$, (b) G_2, (c) G_3.

4. Characteristic trees are caterpillars

Theorem 4.1. *Let* $G = T \cup C \in \mathcal{G}_h$ *and* $G \neq N_{e_2}$, *then* $\chi'_{st}(G) = 5$.

Proof. Note that $\mathcal{G}_h = \{N_{e_h}\}$ for $h = 1, 2, 3$. It is easy to check that $\chi'_{st}(N_{e_1}) = \chi'_{st}(N_{e_3}) = 5$ and $\chi'_{st}(N_{e_2}) = 6$. Suppose that $h \geq 4$. By Theorem 2.1, we only need to prove $\chi'_{st}(G) \leq 5$. We prove it by induction on h. For $h = 4, 5$, $\chi'_{st}(G) \leq 5$, all such colorings are supplied in Figure 4.

In the following, we may assume $h \geq 6$. Let $P = v_0 v_1 \cdots v_{h+1}$ be the longest path in T. Rename the vertices so that $u_1 = v_0$, $u = v_1$, $v = v_2$, $w = v_3$, $x = v_4$ and $y = v_5$. Apart from u_1 and v, let the other neighbor of u be a leaf u_2 on C. Apart from the edge $x_{i-1}x_i$ $(y_{i-1}y_i)$, let the other two edges incident to x_i (y_i) in G' be $x_i x'_i$ $(y_i y'_i)$ and $x_i x_{i+1}$ $(y_i y_{i+1})$ for $i \in \{1, 2, 3\}$ (let $x_0 = u_1$, $y_0 = u_2$). For convenience, let $x_i, y_i \in V(C)$, and $x'_i, y'_i \in V(P)$.

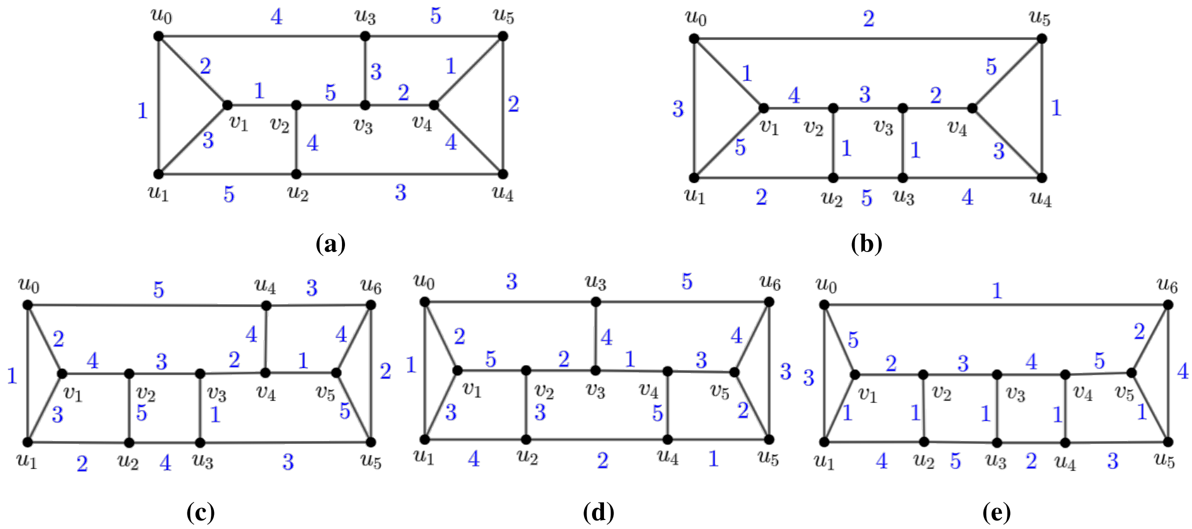


Figure 4. All graphs in $\mathcal{G}_h$: (a) and (b) with $h = 4$, (c)–(e) with $h = 5$.

Note that $x'_1 = v$ or $y'_1 = v$. Without loss of generality, we may assume $y'_1 = v$, i.e., $vy_1 \in E(T)$. In our later inductive steps, let $G' = G - u - u_1 - u_2 - y_1 + vx_1 + vy_2$. Note that $G' \in \mathcal{G}_{h-2}$. Then by the induction hypothesis, there exists a star 5-edge coloring f' for $E(G')$. For each edge e in $E(G')\backslash\{vx_1, vy_2\}$, let $f(e) = f'(e)$. Without loss of generality, assume that $f(vw) = 1$, $f(vx_1) = 2$, and $f(vy_2) = 3$. Denote by $C(v) = \{f(e) : e \in E(v)\}$. For convenience, let $\{t'_i\} = \{4, 5\} \setminus \{t_i\}$ if $t_i \in \{4, 5\}$ for $i \in \{0, 1, 2\}$; $\{\alpha'_0\} = \{1, 2\} \setminus \{\alpha_0\}$ if $\alpha_0 \in \{1, 2\}$.

In the following, we will consider two cases to extend f' of G' to the remaining edges of G to get a star 5-edge coloring f of G.

Case 1. $w = y'_2$, i.e., $wy_2 \in E(T)$ (see Figure 5(a)).

For convenience, let $f(xw) = t_0$, $f(y_2w) = t_1$, $f(y_2y_3) = t_2$, $f(x_1x'_1) = \lambda_1$, $f(x_1x_2) = \lambda'_1$, $f(y_3y'_3) = \mu_1$ and $f(y_3y_4) = \mu'_1$. We first let $f(u_1x_1) = 2$.

Subcase 1.1. $t_0 = 2$. Then $t_1 \in \{4, 5\}$, $t_2 \in \{1, 2, t'_1\}$. Let $f(y_1y_2) = f(uu_1) = 3$, $f(u_1u_2) = 1$, $f(vy_1) = 2$, $f(uu_2) \in \{t_1, t'_1\} \setminus \{f(uv)\}$, and

$$f(uv) = f(u_2y_1) = \begin{cases} t_1 & \text{if } t_2 = t'_1, \\ t'_1 & \text{if } t_2 \in \{1, 2\}. \end{cases}$$

Subcase 1.2. $t_0 = 3$. Then $t_1 \in \{4, 5\}$, $t_2 \in \{2, 4, 5\}$.

Suppose that $t_1 \in \{4, 5\}$. If $t_2 = 2$, or $t_2 = t'_1$ and $\{\mu_1, \mu'_1\} \neq \{1, 3\}$, then let $f(uv) = f(u_2y_1) = t'_1$, $f(uu_2) = t_1$, $f(vy_1) = 2$, $f(u_1u_2) \in \{1, 3\} \setminus \{f(uu_1)\}$, and

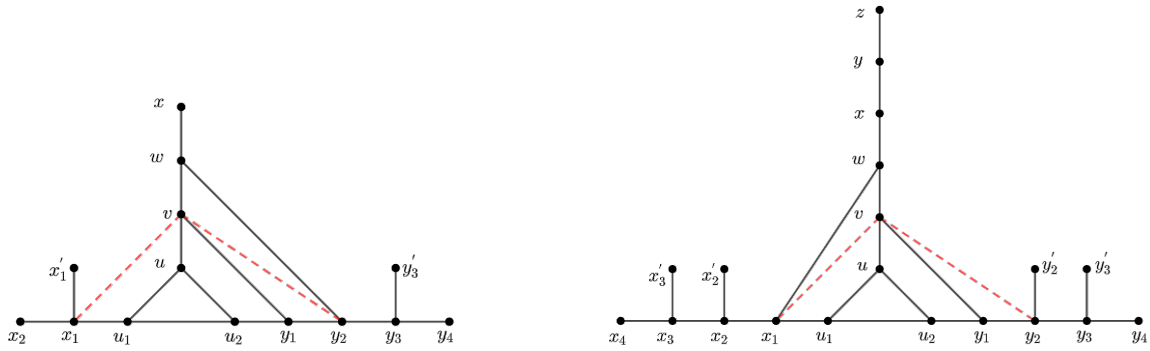
$$f(uu_1) = f(y_1y_2) = \begin{cases} 1 & \text{if } t_2 = t'_1, \text{ and } 1 \notin \{\mu_1, \mu'_1\}, \\ 3 & \text{otherwise.} \end{cases}$$

Otherwise, $t_2 = t'_1$ and $\{\mu_1, \mu'_1\} = \{1, 3\}$. Note that $C(x) \setminus \{t_0\} = \{2, t'_1\}$. Then $xx_1 \in E(T)$, $f(xy) = 2$, $f(xx_1) = \lambda_1 = t'_1$, and $\lambda'_1 \in \{1, 3, t_1\}$. First let $f(uu_1) = 3$. If $\lambda'_1 \in \{1, 3\}$, then let $f(u_2y_1) = 1$, $f(vy_1) = 2$, $f(y_1y_2) = 3$, $f(uv) = f(u_1u_2) = t_1$, and $f(uu_2) = t'_1$. If $\lambda'_1 = t_1$, then recolor the edge wv by setting $f(wv) = t'_1$, and let $f(u_1u_2) = f(vy_1) = 1$, $f(uv) = f(y_1y_2) = 2$, $f(u_2y_1) = t_1$, and $f(uu_2) = t'_1$.

Subcase 1.3. $t_0 \in \{4,5\}$. Then $t_1 \in \{2,4,5\}$ and $t_2 \in \{1,2,4,5\}$.
Subcase 1.3.1. $t_2 = 1$. Then $t_1 = t'_0$.

If $xx_1 \in E(T)$. Then $\{\lambda_1, \lambda'_1\} \neq \{4,5\}$. Let $f(uv) = f(y_1y_2) = 3$, $f(u_2y_1) = f(uu_1) = 1$, $f(vy_1) = 2$, $f(u_1u_2) \in \{4,5\} \setminus \{\lambda_1, \lambda'_1\}$, and $f(uu_2) \in \{4,5\} \setminus \{f(u_1u_2)\}$.

Now assume that $xy_3 \in E(T)$. First, let $f(uv) = f(u_2y_1) = t_0$, $f(y_1y_2) = 3$, $f(uu_2) = t_1$, and $f(vy_1) = 2$. If $s_0 \neq 1$, then let $f(uu_1) = 3$, and $f(u_1u_2) = 1$. Suppose that $s_0 = 1$, then $\{\mu_1, \mu'_1\} = \{2, t_0\}$. In this case, we recolor the edge wv in G by letting $f(wv) = 3$, and let $f(uu_1) = 1$, and $f(u_1u_2) = 3$.
Subcase 1.3.2. $t_2 \neq 1$. Then $t_2 = t_0$, and $t_1 \in \{2, t'_0\}$; or $\{t_1, t_2\} = \{2, t'_0\}$. First, let $f(uv) = t'_0$, $f(y_1y_2) = f(uu_1) = 3$, $f(u_1u_2) = 1$ and $f(vy_1) = 2$. If $t_2 = t_0$, and $t_1 \in \{2, t'_0\}$, then let $f(u_2y_1) = t'_0$, and $f(uu_2) = t_0$. If $\{t_1, t_2\} = \{2, t'_0\}$, then let $f(u_2y_1) = t_0$, $f(uu_2) = 2$.



(a) **(b)**

Figure 5. Two configurations of G: (a) $wy_2 \in E(T)$, (b) $wx_1 \in E(T)$.

Case 2. $w = x'_1$, i.e., $wx_1 \in E(T)$ (see Figure 5(b)).

For convenience, let $f(xy) = s_0$, $f(yz) = s_1$, $f(xw) = t_0$, $f(x_1w) = t_1$, $f(x_1x_2) = t_2$, $f(x_ix'_i) = \lambda_{i-1}$, $f(x_ix_{i+1}) = \lambda'_{i-1}$, $f(y_iy'_i) = \mu_{i-1}$ and $f(y_iy_{i+1}) = \mu'_{i-1}$, where $i \in \{2,3\}$.
Subcase 2.1. $t_0 = 3$. Then $t_1 \in \{4,5\}$, and $t_2 \in \{1,3,4,5\}$. Let $f(u_2y_1) = 1$, $f(u_1x_1) = f(vy_1) = 2$, $f(y_1y_2) = f(uu_1) = 3$, $f(uu_2) = \{4,5\} \setminus \{f(uv)\}$, and

$$f(uv) = f(u_1u_2) = \begin{cases} t'_1 & \text{if } t_2 = 1, \\ t_1 & \text{if } t_2 \neq 1. \end{cases}$$

Subcase 2.2. $t_0 \in \{4,5\}$.

Then $t_1 = t'_0$, $t_2 \in \{t_0, 1, 3\}$; or $t_1 = 3$, $t_2 \in \{t_0, t'_0\}$.
Subcase 2.2.1. $t_1 = t'_0$ and $t_2 = 1$.

Let $\alpha_0 \in \{1,2\}$. If $\alpha_0 \notin C(x)$, then let $f(u_1x_1) = 2$, and $f(u_1u_2) = f(vu) = t_0$, $f(uu_2) = t'_0$, $f(u_2y_1) = f(vw) = \alpha_0$, $f(vy_1) = \alpha'_0$, and $f(uu_1) = f(y_1y_2) = 3$.

Otherwise, $C(x) \setminus \{t_0\} = \{1,2\}$, which implies $xy_2 \in E(T)$. Then $\mu_1 \in \{1,2\}$, and $\mu'_1 \in \{1,2,4,5\}$. If $\mu'_1 \in \{4,5\}$, then let $f(y_1y_2) = f(vw) = f(uu_1) = 3$, $f(uu_2) \in \{4,5\} \setminus \{\mu'_1\}$, $f(u_1u_2) = \mu'_1$, $f(vu) = f(u_2y_1) = 1$, and $f(u_1x_1) = f(vy_1) = 2$. Suppose that $\{\mu_1, \mu'_1\} = \{1,2\}$, then $\{\lambda_1, \lambda'_1\} = \{3, t_0\}$. We can recolor the edge wx_1 by setting $f(wx_1) = 2$, and let $f(uu_1) = f(u_2y_1) = 2$, $f(u_1u_2) = 1$, $f(uv) = f(y_1y_2) = 3$, $f(uu_2) = t_0$, and $f(u_1x_1) = f(vy_1) = t'_0$.
Subcase 2.2.2. $t_1 = t'_0$, $t_2 \in \{t_0, 3\}$; or $t_1 = 3$, $t_2 \in \{t_0, t'_0\}$.

Note that $t_2 \neq 1$, then first let $f(u_1u_2) = f(vu) = t_1$, $f(u_2y_1) = 1$, $f(u_1x_1) = f(vy_1) = 2$, and $f(y_1y_2) = 3$. If $t_1 = t'_0$, $t_2 \in \{t_0, 3\}$, then let $f(uu_1) = 3$, and $f(uu_2) = t_0$. Otherwise, then let $f(uu_1) = t'_2$, and $f(uu_2) = t_2$.

Subcase 2.3. $t_0 = 2$. Then $t_1 \in \{4, 5\}$, $t_2 \in \{3, 4, 5\}$, and $C(x) \setminus \{t_0\} = \{3, t'_1\}$.

Subcase 2.3.1. $t_1 \in \{4, 5\}$, and $t_2 = 3$.

Note that $C(x) \setminus \{2\} = \{3, t'_1\}$ and $f(x_1x_2) = f(vy_2) = 3$. Then $f(xx_2) = \lambda_1 = t'_1$ and $\lambda'_1 \neq 2$ if $xx_2 \in E(T)$; $f(xy_2) = \mu_1 = t'_1$ and $\mu'_1 \neq 2$ if $xy_2 \in E(T)$. Thus we can recolor the edge vw such that $f(wv) = t'_1$, and let $f(u_1u_2) = t'_1$, $f(uv) = t_1$, $f(uu_2) = f(vy_1) = 1$, $f(x_1u_1) = f(u_2y_1) = 2$, and $f(uu_1) = f(y_1y_2) = 3$.

Subcase 2.3.2. $t_1 \in \{4, 5\}$, and $t_2 = t'_1$. Then $\{\mu_1, \mu'_1\} \subseteq \{1, 2, 4, 5\}$, and $t'_1 \notin \{\lambda_1, \lambda'_1\}$.

Subcase 2.3.2.1. $\{\mu_1, \mu'_1\} \neq \{4, 5\}$.

Let $f(u_1u_2) = 1$, $f(x_1u_1) = f(uu_2) = f(vy_1) = 2$, $f(uu_1) = f(y_1y_2) = 3$, $f(uv) \in \{4, 5\} \setminus \{\mu_1, \mu'_1\}$, $f(u_2y_1) \in \{4, 5\} \setminus \{f(uv)\}$.

Subcase 2.3.2.2. $\{\mu_1, \mu'_1\} = \{4, 5\}$.

Then $\{\mu_1, \mu'_1\} = \{t_1, t'_1\}$. For convenience, there are two remarks, one is that if there are some blue edges and blue numbers in G' (see Figures 6–9), then it implies that we have recolored those edges in G (see Figures 7(b),(c) and 9); the other is that blue bold faced numbers in Figures 6–9 means the colors of the remaining eight edges for $E(G) \setminus E(G')$.

First we consider the values of μ_2 and μ'_2.

If $3 \notin \{\mu_2, \mu'_2\}$, then the coloring is illustrated in Figure 6(a). Suppose that $1 \notin \{\mu_2, \mu'_2\}$. If $\mu'_1 = t_1$, then let the coloring be shown in Figure 7(a). Otherwise, $\mu'_1 = t'_1$, then $xx_2 \in E(T)$, which implies that $s_0 = t'_1$ and $\lambda_1 = 3$. If $\lambda'_1 = 1$, then see the coloring in Figure 7(b). If $\lambda'_1 \neq 1$, then see the coloring in Figure 7(c).

Thus we only need to consider the case when $\{\mu_2, \mu'_2\} = \{1, 3\}$. If $3 \notin \{\lambda_1, \lambda'_1\}$, the configuration is shown in Figure 6(b). Suppose that $1 \notin \{\lambda_1, \lambda'_1\}$. If $\mu'_1 = t_1$, then the remaining coloring is shown in Figure 8(a). Otherwise, $\mu'_1 = t'_1$, then the coloring is given in Figure 8(b).

In the following, we only need to consider the case when $\{\lambda_1, \lambda'_1\} = \{\mu_2, \mu'_2\} = \{1, 3\}$. Then $2, t_1 \notin \{\lambda_1, \lambda'_1, \mu_2, \mu'_2\}$. Since $s_0 \in \{3, t'_1\}$, we split into two cases.

First, assume $s_0 = 3$. Then $xy_2 \in E(T)$, so $f(xy_2) = \mu_1 = t'_1$ and $\mu'_1 = t_1$. Since $t_1 \notin \{\lambda_1, \lambda'_1\}$ and $2 \notin \{\mu'_1, \mu_2, \mu'_2\}$, we may swap the colors of xy_2 and wx in G', i.e., $f(xy_2) = 2$ and $f(wx) = t_0 = t'_1$. Then G' also has a star 5-edge coloring, and it has already been done in Subcase 2.2.2.

Now suppose $s_0 = t'_1$. Then $xx_2 \in E(T)$, thus $\lambda_1 = 3$ and $\lambda'_1 = 1$. We now consider the values of $\{\lambda_2, \lambda'_2\}$. First, consider that $t'_1 \notin \{\lambda_2, \lambda'_2\}$. If $\mu'_1 = t_1$, also see the coloring in Figure 8(a). Otherwise, $\mu'_1 = t'_1$, then the configuration is also given in Figure 8(b). If $\{\lambda_2, \lambda'_2\} = \{2, t'_1\}$, then we may recolor the two edges wx_1, x_1x_2 such that $f(wx_1) = 3$ and $f(x_1x_2) = t_1$, and extend the coloring in G, which was shown in Figure 9(a). If $2 \notin \{\lambda_2, \lambda'_2\}$, then we have $s_1 \in \{1, 2\}$, whether $yx_3 \in E(T)$ or $yy_2 \in E(T)$; see the coloring in Figures 9(b),(c).

From the above, it is easy to see that f is a star 5-edge coloring for G in the above two cases, then the result follows. □

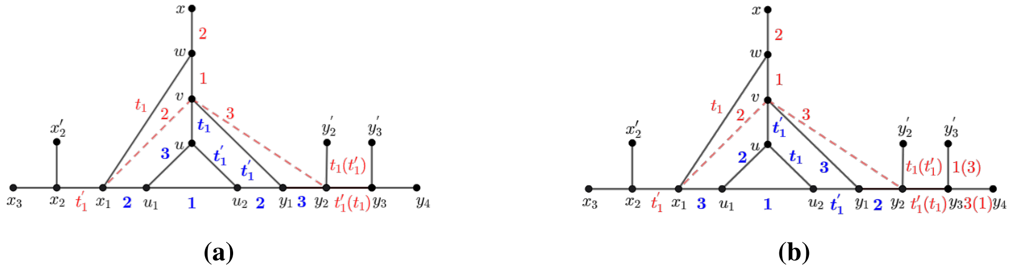


Figure 6. (a) $3 \notin \{\mu_2, \mu_2'\}$, (b) $3 \notin \{\lambda_1, \lambda_1'\}$.

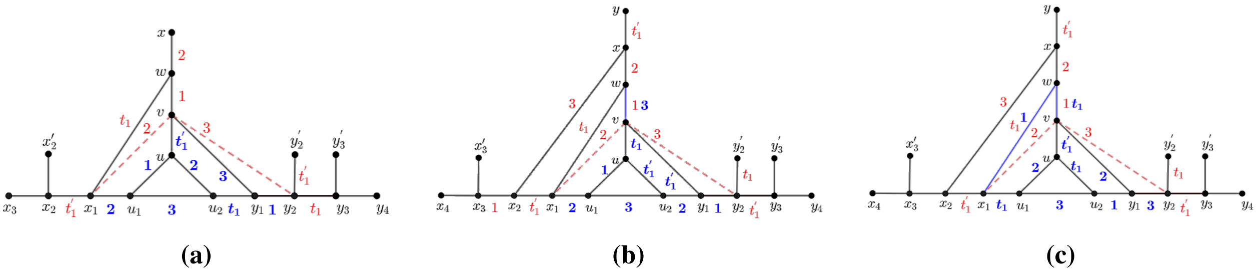


Figure 7. $1 \notin \{\lambda_1, \lambda_1'\}$: (a) $\mu_1' = t_1$, (b) $\mu_1' = t_1', \lambda_1' = 1$, (c) $\mu_1' = t_1', \lambda_1' \neq 1$.

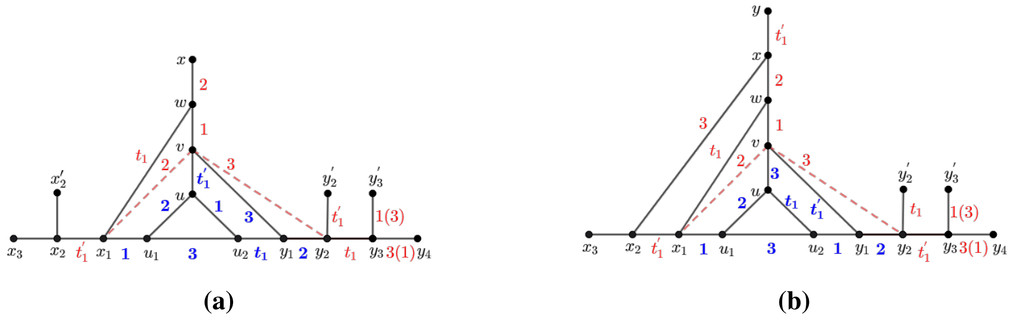


Figure 8. $1 \notin \{\lambda_1, \lambda_1'\}$; or $\lambda_1' = 1, t_1' \notin \{\lambda_2, \lambda_2'\}$: (a) $\mu_1' = t_1$, (b) $\mu_1' = t_1'$.

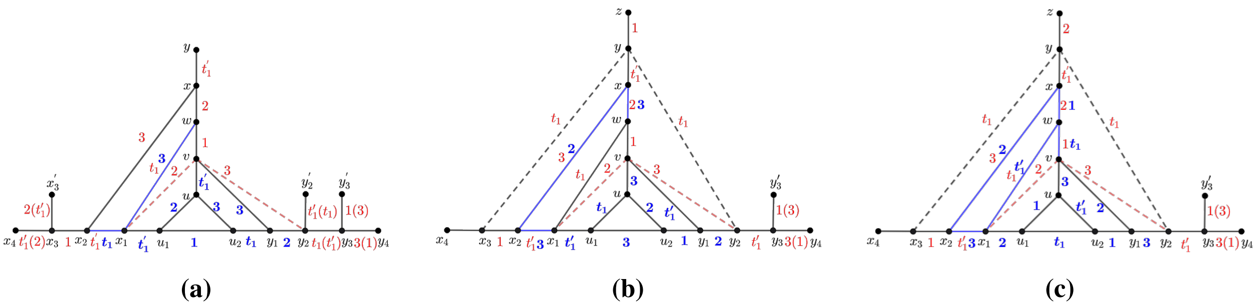
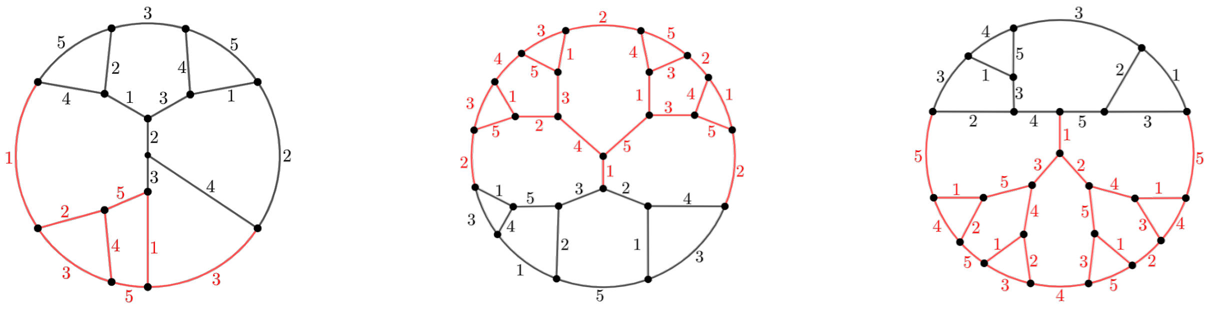


Figure 9. $s_0 = t_1'$: (a) $\{\lambda_2, \lambda_2'\} = \{2, t_1'\}$, (b) $2 \notin \{\lambda_2, \lambda_2'\}, s_1 = 1$, (c) $2 \notin \{\lambda_2, \lambda_2'\}, s_1 = 2$.

Corollary 4.2. *Let N_{e_h} be a necklace with $h \neq 2$, then $\chi'_{st}(N_{e_h}) = 5$.*

5. Conclusions

In this paper, we have determined star chromatic indices for the two kinds of cubic Halin graphs. For the other kinds of cubic Halin graphs whose characteristic trees are neither complete trees nor caterpillars, we believe that most of them obtain star chromatic index 5, some of which can be deduced from Theorems 2.1, 3.1, and 4.1, such as the three graphs shown in Figure 10. So it is interesting to characterize which of such graphs have star chromatic index 5.



(a) (b) (c)

Figure 10. Examples: Three cubic Halin graphs with star chromatic index 5.

Author contributions

Xingxing Hu: Investigation, visualization, writing-original draft; Yunfang Tang: Methodology, supervision, writing-review and editing, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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