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Research article

Dynamics analysis and numerical simulations of a stochastic delayed epidemic model with double disease driven by Lévy noise

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Abstract: This paper investigates a stochastic delayed SI_1I_2 epidemic model with double epidemic and saturated incidence rate driven by Lévy noise. First, we used the stopping time theory to establish sufficient conditions for the existence of a unique positive solution. Furthermore, by the Lyapunov method and stochastic differential equation theory, we analyze the asymptotic properties of the stochastic delayed system around each equilibrium point. In addition, we show that both Lévy noise and time delay can affect the dynamics of the epidemic system. Finally, we use the Euler–Maruyama method to discretize the equations and perform numerical simulations to illustrate the theoretical results.

Keywords: Lévy noise; stochastic differential equation; delayed equation; Lyapunov function; existence and uniqueness; asymptotic behavior
Mathematics Subject Classification: 60G40, 60G51, 60H10

1. Introduction

The investigation of epidemic originated in the early 20th century. In 1927, Kermack and McKendrick [1] first proposed the classical Susceptible-Infected-Recovered (SIR) compartment model. In the classical SIR epidemic model, there is one epidemic disease. However, the simultaneous transmission of two diseases is a common occurrence in the real world. For instance, HIV/AIDS patients are frequently co-infected with Mycobacterium tuberculosis, forming an HIV-TB co-infection system with significant synergistic pathogenic effects; additionally, co-infection of influenza virus and Streptococcus pneumoniae is also widespread, which can exacerbate disease severity. Such phenomena not only affect individual clinical outcomes but also alter the disease transmission dynamics at the population level. To more accurately characterize such complex transmission patterns, scholars have studied epidemic models with the double epidemic hypothesis [2–4]. In [2], Boukanjime et al. investigated the epidemic model with double hypothesis, combined two different transmission mechanisms, and proved the local asymptotic stability of equilibria points. Zhang et al. (in the potential scenario of COVID-19 and influenza co-circulation) [4] indirectly confirmed the synergistic effect between influenza and COVID-19 influenza infection can enhance the receptor expression of respiratory cells in hosts for coronaviruses, increasing population susceptibility to COVID-19 and amplifying its transmission efficiency.

However, an individual may not be infectious until some time after becoming infected. Thus, realistic epidemic models often integrate delay effects to better capture the interpersonal transmission processes. In [5], Kumar et al. proposed and analyzed a deterministic time delayed SIR epidemic model incorporating a nonlinear saturated incidence rate and Holling type III treatment rate, and they studied local and global stability of the equilibria. In [6], Li further extended the framework by constructing a deterministic delayed double disease SI_1I_2 model with saturated incidence. The authors derived the basic reproduction number and classified equilibrium states, including disease-free equilibrium, single-disease endemic equilibrium, and double-disease coexistence equilibrium. By introducing saturation parameters α_1, α_2 and delay parameters τ_1, τ_2, the model captures saturation effects and delay characteristics in disease transmission, respectively. The SI_1I_2 model is as follows:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - \mu_0 S - \dfrac{\beta_1 S(t) I_1(t)}{1+\alpha_1 I_1(t)} - \dfrac{\beta_2 S(t) I_2(t)}{1+\alpha_2 I_2(t)}\right]\mathrm{d}t, \\ \mathrm{d}I_1(t) = \left[e^{-\mu_1\tau_1}\dfrac{\beta_1 S(t-\tau_1) I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)} - \mu_1 I_1(t)\right]\mathrm{d}t, \\ \mathrm{d}I_2(t) = \left[e^{-\mu_2\tau_2}\dfrac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - \mu_2 I_2(t)\right]\mathrm{d}t, \end{cases} \tag{1.1}$$

where $S(t), I_1(t)$ and $I_2(t)$ represent the number of susceptibles, infectors of infectious Disease I, and infectors of infectious Disease II at time t, respectively. The specific meanings of the parameters are provided in the Table 1.

Table 1. Model parameters.

Parameter	Description
Λ	The birth rate of the susceptible population.
μ_0	The natural death rate of the susceptible population.
μ_1	The death rate caused by Disease I.
μ_2	The death rate caused by Disease II.
β_1	The contact rate between infectives (Disease I) and susceptibles.
β_2	The contact rate between infectives (Disease II) and susceptibles.
α_1	The saturation parameter for Disease I.
α_2	The saturation parameter for Disease II.
τ_1	The time delay for a susceptible individual infected with Disease I to become an infective.
τ_2	The time delay for a susceptible individual infected with Disease II to become an infective.
$e^{-\mu_1\tau_1}$	The survival probability over the delay period τ_1 for individuals infected with Disease I.

Table 1 (Continued)

Parameter	**Description**
$e^{-\mu_2\tau_2}$	The survival probability over the delay period τ_2 for individuals infected with Disease II.

The threshold for the solution was derived, and the equilibrium states of the system were provided as follows:

$$R_1 = \frac{\beta_1 \Lambda e^{-\mu_1\tau_1}}{\mu_0\mu_1}, \quad R_2 = \frac{\beta_2 \Lambda e^{-\mu_2\tau_2}}{\mu_0\mu_2}.$$

(1) $R_1 < 1, R_2 < 1$, model (1.1) has a disease-free equilibrium point

$$E_0 = \left(\frac{\Lambda}{\mu_0}, 0, 0\right).$$

(2) $R_1 > 1, R_2 < 1$, model (1.1) has an equilibrium point

$$E_1 = (S_1^{'}, I_1^{'}, 0) = \left(\frac{\mu_1(\beta_1 + \mu_0\alpha_1) + \alpha_1\beta_1\Lambda e^{-\mu_1\tau_1} - \alpha_1\mu_0\mu_1}{\beta_1(\beta_1 + \mu_0\alpha_1)e^{-\mu_1\tau_1}}, \frac{\beta_1\Lambda e^{-\mu_1\tau_1} - \mu_0\mu_1}{\mu_1(\beta_1 + \mu_0\alpha_1)}, 0\right).$$

(3) $R_1 < 1, R_2 > 1$, model (1.1) has an equilibrium point

$$E_2 = (S_2^{'}, 0, I_2^{'}) = \left(\frac{\mu_2(\beta_2 + \mu_0\alpha_2) + \alpha_2\beta_2\Lambda e^{-\mu_2\tau_2} - \alpha_2\mu_0\mu_2}{\beta_2(\beta_2 + \mu_0\alpha_2)e^{-\mu_2\tau_2}}, \frac{\beta_2\Lambda e^{-\mu_2\tau_2} - \mu_0\mu_2}{\mu_2(\beta_2 + \mu_0\alpha_2)}, 0\right).$$

(4) $R_1 > 1, R_2 > 1$, model (1.1) has an equilibrium point

$$\begin{aligned} E_3 = (S^*, I_1^*, I_2^*) = \Bigg(& \frac{\Lambda\alpha_1\alpha_2 + \alpha_2\mu_1 e^{\mu_1\tau_1} + \alpha_1\mu_2 e^{\mu_2\tau_2}}{\mu_0\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_2\alpha_1}, \\ & \frac{\alpha_2(\beta_1\Lambda - \mu_0\mu_1 e^{\mu_1\tau_1}) + \beta_1\mu_2 e^{\mu_2\tau_2} - \beta_2\mu_1 e^{\mu_1\tau_1}}{\mu_1 e^{\mu_1\tau_1}(\mu_0\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_2\alpha_1)}, \\ & \frac{\alpha_1(\beta_2\Lambda - \mu_0\mu_2 e^{\mu_2\tau_2}) + \beta_2\mu_1 e^{\mu_1\tau_1} - \beta_1\mu_2 e^{\mu_2\tau_2}}{\mu_2 e^{\mu_2\tau_2}(\mu_0\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_2\alpha_1)}\Bigg). \end{aligned}$$

Deterministic delayed models can explain and simulate some phenomena in real life. They fail to account for the random fluctuations prevalent in natural environments. Minor changes in environmental factors and random variations in population contact patterns can exert non-negligible influences on disease spread [7–9]. To address this limitation, researchers commonly introduce environmental white noise into deterministic delayed models, constructing stochastic delayed models to characterize continuous small-scale random perturbations. White noise effectively simulates the impact of smooth environmental fluctuations on disease transmission, and related studies have yielded fruitful results [10, 11]. Zhang [10] et al. constructed a stochastic delayed model with a specific functional response to describe its epidemic dynamics. In [11], Liu et al. investigated a stochastic delayed Susceptible-Infectious-Susceptible (SIS) epidemic model with vaccination and double diseases and established sufficient conditions for extinction and persistence in the mean of the two diseases. They showed that time delay and environmental white noise have important effects on the persistence and extinction of the two diseases.

On the basis of the work of Li et al. [6], the delayed model incorporated by white noise can be described by

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - \mu_0 S - \dfrac{\beta_1 S(t) I_1(t)}{1+\alpha_1 I_1(t)} - \dfrac{\beta_2 S(t) I_2(t)}{1+\alpha_2 I_2(t)}\right]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t), \\ \mathrm{d}I_1(t) = \left[e^{-\mu_1\tau_1}\dfrac{\beta_1 S(t-\tau_1) I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)} - \mu_1 I_1(t)\right]\mathrm{d}t + \sigma_2 S(t)\mathrm{d}B_2(t), \\ \mathrm{d}I_2(t) = \left[e^{-\mu_2\tau_2}\dfrac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - \mu_2 I_2(t)\right]\mathrm{d}t + \sigma_3 S(t)\mathrm{d}B_3(t), \end{cases} \tag{1.2}$$

where $B_i(t)$ $(i = 1, 2, 3)$ is the standard Brownian motion defined on the complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and they are mutually independent. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ - null sets), and $\sigma_i > 0$ $(i = 1, 2, 3)$ represents the intensity of the Brownian motion $B_i(t)$ $(i = 1, 2, 3)$.

However, in real-world infectious disease transmission, besides continuous small scale random fluctuations, discontinuous severe perturbations caused by sudden environmental events such as earthquakes or floods may also occur. Such jump disturbances cannot be effectively described by stochastic models containing only white noise. The limitations of related research have driven the development of more realistic stochastic perturbation characterization methods. Bao et al. [12] first proposed that the Lévy process should be suitable to describe large occasional fluctuations. Consequently, researchers have incorporated Lévy noise into epidemic models to more accurately reflect the impact of such discontinuous, sudden environmental noise [13–15]. Zhang et al. [16] proposed and analyzed a stochastic Susceptible-Infected-Quarantined-Recovered (SIQR) epidemic model incorporating Lévy jumps and three distinct time delays. They investigated the existence and uniqueness of the global positive solution and showed that the solutions oscillate around the equilibria. In [17], Amine et al. investigated the dynamics of a stochastic SIR model with double epidemics, vaccination, and multiple time delays and established threshold conditions for disease extinction and persistence by analyzing the combined effects of white noise and Lévy jumps.

Li et al. [6] systematically analyzed the stability under different equilibrium states for the deterministic model (1.1). Building on this model, the stochastic epidemic models with white noise (1.2) have clarified the existence and uniqueness of solutions and the asymptotic behavior near equilibrium points under continuous random disturbances. However, natural and crucial scientific questions arise: When introducing Lévy noise to capture such discontinuous shocks into the models, does a unique global positive solution still exist? Will the solution still oscillate around various equilibrium points? What is the relationship between its oscillatory characteristics and Lévy noise parameters? Investigating these questions is of significant urgency and practical importance. Discontinuous perturbations triggered by sudden environmental events often disrupt conventional disease transmission patterns. If models cannot accurately characterize such disturbances, prediction results may significantly deviate from actual transmission situations, thereby affecting the scientific validity and effectiveness of control strategies.

Based on the above analysis, building on reference [6], which studied the dynamic behavior of the stochastic SI_1I_2 model (1.2) driven by white noise, this paper further investigates the dynamic behavior of the stochastic delayed SI_1I_2 epidemic model with double diseases driven by Lévy noise. The model is as follows:

$$
\begin{cases}
\mathrm{d}S(t) = \left[\Lambda - \mu_0 S - \dfrac{\beta_1 S(t) I_1(t)}{1+\alpha_1 I_1(t)} - \dfrac{\beta_2 S(t) I_2(t)}{1+\alpha_2 I_2(t)}\right]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t) \\
\qquad + \displaystyle\int_A D_1(y) S(t)\tilde{N}(\mathrm{d}t, \mathrm{d}y), \\
\mathrm{d}I_1(t) = \left[e^{-\mu_1\tau_1}\dfrac{\beta_1 S(t-\tau_1) I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)} - \mu_1 I_1(t)\right]\mathrm{d}t + \sigma_2 I_1(t)\mathrm{d}B_2(t) \\
\qquad + \displaystyle\int_A D_2(y) I_1(t)\tilde{N}(\mathrm{d}t, \mathrm{d}y), \\
\mathrm{d}I_2(t) = \left[e^{-\mu_2\tau_2}\dfrac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - \mu_2 I_2(t)\right]\mathrm{d}t + \sigma_3 I_2(t)\mathrm{d}B_3(t) \\
\qquad + \displaystyle\int_A D_3(y) I_2(t)\tilde{N}(\mathrm{d}t, \mathrm{d}y),
\end{cases}
\tag{1.3}
$$

where $D_i(y) > -1$, $y \in A$ $(i = 1, 2, 3)$ represents the intensity of the jump. $N(dt, dy)$ represents the Poisson counting measure with the Lévy measure $\nu(dy)$, $\tilde{N}(dt, dy)$ is the compensated Poisson random measure given by $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$, ν is defined on measurable subset A of $[0, \infty)$ satisfying $\nu(A) < \infty$, and A represents the support of the Lévy measure.

Define the initial value of model (1.3)

$$
\begin{cases}
S(\theta) = \varphi_1(\theta), I_1(\theta) = \varphi_2(\theta), I_2(\theta) = \varphi_3(\theta), \\
\varphi_i(\theta) \geq 0, \theta \in [-\tau, 0], i = 1, 2, 3, \\
(\varphi_1, \varphi_2, \varphi_3) \in C,
\end{cases}
\tag{1.4}
$$

where $C = C([-\tau, 0], \mathbb{R}_+^3)$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_+^3$ and $\tau = \max\{\tau_1, \tau_2\}$.

Our proposed stochastic delayed model, which uniquely integrates double diseases, time delays and Lévy noise, offers a quantitative framework for analyzing complex co-circulation patterns of respiratory pathogens. A pertinent application is in simulating the overlapping epidemic waves of COVID-19 and respiratory syncytial virus (RSV). In such a scenario, the co-infection mechanism captures the population-level risk during periods of concurrent transmission, while the incorporated time delays correspond realistically to the distinct incubation periods of each virus. Furthermore, the Lévy noise component effectively models sudden, large-scale perturbations in transmission rates, which can be triggered by superspreading events, mass gatherings, or holiday travel surges. This framework can serve as a tool for predicting the risk of co-infection outbreaks and informing and optimizing joint public health mitigation strategies, such as the timing of booster vaccinations or non-pharmaceutical interventions targeted at multiple pathogens.

This paper is organized as follows: In Section 2, we prove the existence and uniqueness of the positive solution of the stochastic delayed model (1.3). In Section 3, we describe the asymptotic behavior of the stochastic delayed model (1.3) at the equilibrium point. In Section 4, we present numerical simulations to validate our theoretical results. In Section 5, we give the future research directions.

2. Global positive solution

In order to investigate the dynamical behavior of the epidemic system, the existence and uniqueness of a global positive solution is necessary. In this section, we will obtain sufficient conditions to ensure that there exists a unique global solution for system (1.3). We impose the following hypotheses:

For each $N \geq 0$, there exists an $L_N \geq 0$ such as

(H1) $\int_A |K_i(x,y) - K_i(z,y)|^2 \nu(\mathrm{d}y) \leq L_N |x-z|^2$, $i = 1,2,3$, where $K_1(x,y) = D_1(y)S(t), K_2(x,y) = D_2(y)I_1(t), K_3(x,y) = D_3(y)I_2(t)$, and $|x| \vee |z| \leqslant N$.

(H2) $\int_A (1 + \ln(1 + D_i(y)))\, \nu(\mathrm{d}y) < +\infty$ for $1 + D_i(y) > 0$ $(i = 1, 2, 3)$.

Theorem 2.1. Suppose that conditions (H1) and (H2) hold. Then for any given initial value (1.4), the stochastic delayed system (1.3) has a unique positive solution $(S(t), I_1(t), I_2(t)) \in \mathbb{R}^3_+$ for $t \geq 0$, and this solution lies in $\mathbb{R}^3_+$ with probability 1.

Proof. Since the stochastic delayed system (1.3) satisfies the local Lipschitz condition (H1), for the given initial value $(S(\theta), I_1(\theta), I_2(\theta)) \in \mathbb{R}^3_+$, there exists a unique local solution $(S(t), I_1(t), I_2(t))$ for all $t \in [-\tau, \tau_e)$, where τ_e is the explosion time. To prove that there exists a unique global solution, we prove that $\tau_e = \infty$ a.s.. Let k_0 be a sufficiently large constant such that the initial values $S(\theta), I_1(\theta)$ and $I_2(\theta)$ all lie in the interval $[\frac{1}{k_0}, k_0]$.

For any integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\left\{ t \in [-\tau, \tau_e) : \min\{S(t), I_1(t), I_2(t)\} \leq \frac{1}{k} \text{ or } \max\{S(t), I_1(t), I_2(t)\} \geq k \right\},$$

which denotes the first exit time that $S(t)$, $I_1(t)$, or $I_2(t)$ escapes from $[\frac{1}{k}, k]$ for $t \in [-\tau, \tau_e)$. Set $\inf \emptyset = \infty$ (where $\emptyset$ is the empty set). It is easy to know that τ_k is monotonically increasing with respect to k. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$. Obviously, $\tau_\infty \leq \tau_e$ a.s.. If we can prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$. Using the method of contradiction, suppose that $\tau_\infty < \infty$. That is, there exists a constant T and $0 < \epsilon < 1$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \epsilon$ holds. Therefore, there exists a certain constant $k_1 \geq k_0$ satisfying

$$\mathbb{P}\{\tau_k \leq T\} \geq \epsilon, \quad \forall k \geq k_1. \tag{2.1}$$

Define a function $V(S, I_1, I_2) : \mathbb{R}^3_+ \to \mathbb{R}_+$

$$\begin{aligned} V(S, I_1, I_2) =& (S - 1 - \ln S) + (I_1 - 1 - \ln I_1) + (I_2 - 1 - \ln I_2) \\ &+ \mathrm{e}^{-\mu_1\tau_1} \int_{t-\tau_1}^{t} \frac{\beta_1 S(\mu) I_1(\mu)}{1 + \alpha_1 I_1(\mu)} \mathrm{d}\mu + \mathrm{e}^{-\mu_2\tau_2} \int_{t-\tau_2}^{t} \frac{\beta_2 S(\mu) I_2(\mu)}{1 + \alpha_1 I_2(\mu)} \mathrm{d}\mu. \end{aligned} \tag{2.2}$$

Obviously, the V-function is non-negative. According to Itô's formula, we have

$$\begin{aligned} \mathrm{d}V =& LV\mathrm{d}t + \left(1 - \frac{1}{S}\right)\sigma_1 S\,\mathrm{d}B_1(t) + \left(1 - \frac{1}{I_1}\right)\sigma_2 I_1 \mathrm{d}B_2(t) + \left(1 - \frac{1}{I_2}\right)\sigma_3 I_2 \mathrm{d}B_3(t) \\ &+ \int_A \Big[(S + D_1(y)S) - 1 - \ln(S + D_1(y)S) + (I_1 + D_2(y)I_1) - 1 - \ln(I_1 + D_2(y)I_1) \\ &+ (I_2 + D_3(y)I_2) - 1 - \ln((I_2 + D_3(y)I_2)) - (S - 1 - \ln S + I_1 - 1 - \ln I_1 \\ &+ I_2 - 1 - \ln I_2)\Big]\tilde{N}(\mathrm{d}t, \mathrm{d}y) \end{aligned}$$

$$
\begin{aligned}
&= LV\mathrm{d}t + \sigma_1(S-1)\mathrm{d}B_1(t) + \sigma_2(I_1-1)\mathrm{d}B_2(t) + \sigma_3(I_2-1)\mathrm{d}B_3(t) + \int_A \Big[D_1(y)S \\
&\quad - \ln(1+D_1(y)) + D_2(y)I_1 - \ln(1+D_2(y)) + D_3(y)I_2 - \ln(1+D_3(y))\Big]\tilde{N}(\mathrm{d}t,\mathrm{d}y),
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{aligned}
LV =& \left(1-\frac{1}{S}\right)\left(\Lambda - \mu_0 S - \frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)} - \frac{\beta_2 S(t)I_2(t)}{1+\alpha_2 I_2(t)}\right) + \left(1-\frac{1}{I_1}\right)\left(e^{-\mu_1\tau_1}\frac{\beta_1 S(t-\tau_1)I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)}\right. \\
&\left. - \mu_1 I_1(t)\right) + \left(1-\frac{1}{I_2}\right)\left(e^{-\mu_2\tau_2}\frac{\beta_2 S(t-\tau_2)I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - \mu_2 I_2(t)\right) + e^{-\mu_1\tau_1}\frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)} \\
&- e^{-\mu_1\tau_1}\frac{\beta_1 S(t-\tau_1)I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)} + e^{-\mu_2\tau_2}\frac{\beta_2 S(t)I_2(t)}{1+\alpha_2 I_2(t)} - e^{-\mu_2\tau_2}\frac{\beta_2 S(t-\tau_2)I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} \\
&+ \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right) + \int_A \left[D_1(y)S - \ln(1+D_1(y)) - \left(1-\frac{1}{S}\right)D_1(y)S + D_2(y)I_1\right. \\
&\left. - \ln(1+D_2(y) + D_3(y)I_2 - \ln(1+D_3(y))\left(1-\frac{1}{I_2}\right)D_3(y)I_2\right]\nu(\mathrm{d}y) \\
=& (e^{-\mu_1\tau_1}-1)\frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)} + (e^{-\mu_2\tau_2}-1)\frac{\beta_2 S(t)I_2(t)}{1+\alpha_2 I_2(t)} + \Lambda + \mu_0 + \frac{\beta_1 I_1(t)}{1+\alpha_1 I_1(t)} + \frac{\beta_2 I_2(t)}{1+\alpha_2 I_2(t)} \\
&+ \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2) - \mu_0 S - \frac{\Lambda}{S} - \mu_1 I_1(t) - \mu_2 I_2(t) - e^{-\mu_1\tau_1}\frac{\beta_1 S(t-\tau_1)I_1(t-\tau_1)}{(1+\alpha_1 I_1(t-\tau_1))I_1} \\
&- e^{-\mu_2\tau_2}\frac{\beta_2 S(t-\tau_2)I_2(t-\tau_2)}{(1+\alpha_2 I_2(t-\tau_2))I_2} + \mu_1 + \mu_2 + \int_A [D_1(y) - \ln(1+D_1(y))]\nu(\mathrm{d}y) \\
&+ \int_A [D_2(y) - \ln(1+D_2(y))]\nu(\mathrm{d}y) + \int_A [D_3(y) - \ln(1+D_3(y))]\nu(\mathrm{d}y) \\
\leq& \Lambda + \mu_0 + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \mu_1 + \mu_2 + \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2) + \int_A [D_1(y) - \ln(1+D_1(y))]\nu(\mathrm{d}y) \\
&+ \int_A [D_2(y) - \ln(1+D_2(y))]\nu(\mathrm{d}y) + \int_A [D_3(y) - \ln(1+D_3(y))]\nu(\mathrm{d}y).
\end{aligned}
$$

Using the inequality $x - \ln(1+x) \geq 0$ for all $x > -1$ and assumption (H2), we have

$$
LV \leq \Lambda + \mu_0 + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \mu_1 + \mu_2 + \frac{1}{2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right) + 3K_1 = K,
\tag{2.4}
$$

where

$$
\begin{aligned}
K_1 = \max\Bigg\{&\int_A [D_1(y) - \ln(1+D_1(y))]\,\nu(\mathrm{d}y), \int_A [D_2(u) - \ln(1+D_2(y))]\,\nu(\mathrm{d}y), \\
&\int_A [D_3(y) - \ln(1+D_3(y))]\,\nu(\mathrm{d}y)\Bigg\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{d}V \leq& K\mathrm{d}t + \sigma_1(S-1)\mathrm{d}B_1(t) + \sigma_2(I_1-1)\mathrm{d}B_2(t) + \sigma_3(I_2-1)\mathrm{d}B_3(t) \\
&+ \int_A \Big[D_1(y)S - \ln(1+D_1(y)) + D_2(y)I_1 - \ln(1+D_2(y))
\end{aligned}
$$

$$+ D_3(y)I_2 - \ln(1 + D_3(y))\Big]\tilde{N}(\mathrm{d}t, \mathrm{d}y). \tag{2.5}$$

Integrate from 0 to $\tau_k \wedge T$, then we have

$$\begin{aligned}\int_0^{\tau_k\wedge T} \mathrm{d}V \leq &\int_0^{\tau_k\wedge T} K\mathrm{d}\mu + \int_0^{\tau_k\wedge T} \sigma_1(S-1)\mathrm{d}B_1(t) + \int_0^{\tau_k\wedge T} \sigma_2(I_1-1)\mathrm{d}B_2(t)\\ &+ \int_0^{\tau_k\wedge T} \sigma_3(I_2-1)\mathrm{d}B_3(t) + \int_0^{\tau_k\wedge T}\left(\int_A \Big[D_1(y)S - \ln(1+D_1(y))\right.\\ &\left.+D_2(y)I_1 - \ln(1+D_2(y)) + D_3(y)I_2 - \ln(1+D_3(y))\Big]\tilde{N}(\mathrm{d}t,\mathrm{d}y)\right). \end{aligned} \tag{2.6}$$

Taking the expectation, we have

$$\begin{aligned}EV(S(\tau_k\wedge T), I_1(\tau_k\wedge T), I_2(\tau_k\wedge T)) &\leq EV(S(0), I_1(0), I_2(0)) + E\left(\int_0^{\tau_k\wedge T} K\mathrm{d}\mu\right)\\ &\leq EV(S(0), I_1(0), I_2(0)) + KT.\end{aligned}$$

Let $\Omega_k = \{\tau_k \leq T\}$. For any $k \geq k_1$, we have $P(\Omega_k) \geq \epsilon$. Notice that for each $\omega \in \Omega_k$, at least one of $S(\tau_k \wedge T), I_1(\tau_k \wedge T)$, and $I_2(\tau_k \wedge T)$ is equal to k or $\frac{1}{k}$. If one of $S(\tau_k \wedge T), I_1(\tau_k \wedge T)$ or $I_2(\tau_k \wedge T)$ is equal to k or $\frac{1}{k}$, then

$$V(S(\tau_k\wedge T), I_1(\tau_k\wedge T), I_2(\tau_k\wedge T)) \geq (k+1-\ln k) \wedge \left(\frac{1}{k} + 1 - \ln\frac{1}{k}\right),$$

thus

$$\begin{aligned}EV(S(0), I_1(0), I_2(0)) + KT &\geq E\left[\mathbf{1}_{\Omega_k} V(S(\tau_k\wedge T), I_1(\tau_k\wedge T), I_2(\tau_k\wedge T))\right]\\ &\geq \epsilon\left[(k+1-\ln k)\wedge\left(\frac{1}{k}+1-\ln\frac{1}{k}\right)\right],\end{aligned}$$

where $\mathbf{1}_{\Omega_k}$ is the indicator function of Ω_k. Let $k \to \infty$, then we have

$$\infty > V(S(0), I_1(0), I_2(0)) + KT = \infty.$$

This is a contradiction. Hence, $\tau_\infty = \infty$ a.s. This completes the proof, which shows that the solution of the system (1.3) will not explode in finite time, so it is global and unique. □

3. Asymptotic behavior of solutions near the equilibrium point

3.1. Asymptotic behavior of solutions near the equilibrium point E_0

When the basic reproduction numbers $R_1 < 1$ and $R_2 < 1$, the deterministic SI_1I_2 model has an equilibrium point $E_0\left(\frac{\Lambda}{\mu_0}, 0, 0\right)$, which is globally asymptotically stable. However, for the stochastic delayed SI_1I_2 model driven by Lévy noise, E_0 is not an equilibrium point. Therefore, the asymptotic properties of the solution of the stochastic delayed system (1.3) near the point E_0 will be discussed below.

Theorem 3.1. Suppose that conditions (H1) and (H2) hold. When $R_1 < 1$ and $R_2 < 1$ and the following conditions are satisfied:

$$\mu_0 > \sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y),$$
$$\mu_1 - \mu_0 > \sigma_2^2 + 3\int_A D_2^2(y)\nu(\mathrm{d}y),$$
$$\mu_2 - \mu_0 > \sigma_3^2 + 3\int_A D_3^2(y)\nu(\mathrm{d}y),$$

then for any given initial value (1.4), the solution $(S(t), I_1(t), I_2(t))$ of system (1.3) has the following properties:

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[\left(S(\mu) - \frac{\Lambda}{\mu_0}\right)^2\right] \mathrm{d}\mu \leq \frac{\left(\frac{\Lambda}{\mu_0}\right)^2 \left(\sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}{\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)},$$
$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[I_1^2(\mu) + I_2^2(\mu)\right] \mathrm{d}\mu \leq \frac{2M_1}{H_0}\left(\frac{\Lambda}{\mu_0}\right)^2,$$

where

$$M_1 = \left\{\left[\frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} + 1\right]\sigma_1^2 + \left[\frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} + 3\right]\int_A D_1^2(y)\nu(\mathrm{d}y)\right\}.$$

Proof. First, define the function

$$V_1(S) = \frac{1}{2}\left(S - \frac{\Lambda}{\mu_0}\right)^2.$$

According to Itô's formula, we get

$$\mathrm{d}V_1 = LV_1\mathrm{d}t + \left(S - \frac{\Lambda}{\mu_0}\right)\sigma_1 S\,\mathrm{d}B_1(t) + \int_A \left[\frac{1}{2}\left(S + D_1(y)S - \frac{\Lambda}{\mu_0}\right)^2 - \frac{1}{2}\left(S - \frac{\Lambda}{\mu_0}\right)^2\right]\tilde{N}(\mathrm{d}t, \mathrm{d}y),$$

where

$$\begin{aligned} LV_1 =& \left(S - \frac{\Lambda}{\mu_0}\right)\left(\Lambda - \mu_0 S - \frac{\beta_1 S(t) I_1(t)}{1 + \alpha_1 I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 I_2(t)}\right) + \frac{1}{2}\sigma_1^2 S^2 \\ &+ \int_A \left[\frac{1}{2}\left(S + D_1(y)S - \frac{\Lambda}{\mu_0}\right)^2 - \frac{1}{2}\left(S - \frac{\Lambda}{\mu_0}\right)^2 - D_1(y)S\left(S - \frac{\Lambda}{\mu_0}\right)\right]\nu(\mathrm{d}y) \\ =& \left(S - \frac{\Lambda}{\mu_0}\right)\left(-\mu_0\left(S - \frac{\Lambda}{\mu_0}\right) - \frac{\beta_1\left(S - \frac{\Lambda}{\mu_0}\right) I_1(t)}{1 + \alpha_1 I_1(t)} - \frac{\beta_2\left(S - \frac{\Lambda}{\mu_0}\right) I_2(t)}{1 + \alpha_2 I_2(t)}\right. \\ &\left. - \frac{\beta_1 \frac{\Lambda}{\mu_0} I_1(t)}{1 + \alpha_1 I_1(t)} - \frac{\beta_2 \frac{\Lambda}{\mu_0} I_2(t)}{1 + \alpha_2 I_2(t)}\right) + \frac{1}{2}\sigma_1^2 S^2 + \int_A \frac{1}{2} D_1^2(y) S^2 \nu(\mathrm{d}y) \end{aligned}$$

$$
\begin{aligned}
&= -\mu_0\left(S-\frac{\Lambda}{\mu_0}\right)^2-\frac{\beta_1 I_1(t)\left(S-\frac{\Lambda}{\mu_0}\right)^2}{1+\alpha_1 I_1(t)}-\frac{\beta_2 I_2(t)\left(S-\frac{\Lambda}{\mu_0}\right)^2}{1+\alpha_2 I_2(t)}-\frac{\beta_1\frac{\Lambda}{\mu_0}I_1(t)\left(S-\frac{\Lambda}{\mu_0}\right)}{1+\alpha_1 I_1(t)}\\
&\quad-\frac{\beta_2\frac{\Lambda}{\mu_0}I_2(t)\left(S-\frac{\Lambda}{\mu_0}\right)}{1+\alpha_2 I_2(t)}+\frac{1}{2}\sigma_1^2S^2+\int_A\frac{1}{2}D_1^2(y)S^2\nu(\mathrm{d}y).
\end{aligned}
$$

Using the inequality $(a+b)^2 \le 2a^2+2b^2$, we obtain

$$
\begin{aligned}
LV_1 \le& -\mu_0\left(S-\frac{\Lambda}{\mu_0}\right)^2-\frac{\beta_1\frac{\Lambda}{\mu_0}I_1(t)\left(S-\frac{\Lambda}{\mu_0}\right)}{1+\alpha_1 I_1(t)}-\frac{\beta_2\frac{\Lambda}{\mu_0}I_2(t)\left(S-\frac{\Lambda}{\mu_0}\right)}{1+\alpha_2 I_2(t)}+\sigma_1^2\left(S-\frac{\Lambda}{\mu_0}\right)^2\\
&+\sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2+\int_A D_1^2(y)\left(S-\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y)+\int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y)\\
=& -\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)\left(S-\frac{\Lambda}{\mu_0}\right)^2-\frac{\beta_1\frac{\Lambda}{\mu_0}I_1(t)\left(S-\frac{\Lambda}{\mu_0}\right)}{1+\alpha_1 I_1(t)}\\
&-\frac{\beta_2\frac{\Lambda}{\mu_0}I_2(t)\left(S-\frac{\Lambda}{\mu_0}\right)}{1+\alpha_2 I_2(t)}+\sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2+\int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y).
\end{aligned}
$$

Define

$$
V_2(I_1)=I_1+e^{\mu_1\tau_1}\beta_1\int_{t-\tau_1}^{t}\frac{S(\mu)I_1(\mu)}{1+\alpha_1 I_1(\mu)}\mathrm{d}\mu.
$$

Then

$$
\mathrm{d}V_2=LV_2\mathrm{d}t+\sigma_2 I_1\mathrm{d}B_2(t)+\int_A D_2(y)I_1\tilde{N}(\mathrm{d}t,\mathrm{d}y),
$$

where

$$
\begin{aligned}
LV_2 =& e^{-\mu_1\tau_1}\frac{\beta_1 S(t-\tau_1)I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)}-\mu_1 I_1(t)+e^{-\mu_1\tau_1}\frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)}\\
&-e^{-\mu_1\tau_1}\frac{\beta_1 S(t-\tau_1)I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)}+\int_A\left[I_1+D_2(y)I_1-I_1-D_2(y)I_1\right]\nu(\mathrm{d}y)\\
=& e^{-\mu_1\tau_1}\frac{\beta_1\left(S-\frac{\Lambda}{\mu_0}\right)I_1(t)}{1+\alpha_1 I_1(t)}+e^{-\mu_1\tau_1}\frac{\beta_1\left(\frac{\Lambda}{\mu_0}\right)I_1(t)}{1+\alpha_1 I_1(t)}-\mu_1 I_1(t)\\
=& e^{-\mu_1\tau_1}\frac{\beta_1\left(S-\frac{\Lambda}{\mu_0}\right)I_1(t)}{1+\alpha_1 I_1(t)}+\mu_1 I_1\left(\frac{e^{-\mu_1\tau_1}\beta_1\Lambda}{\mu_0\mu_1\left(1+\alpha_1 I_1\right)}-1\right)\\
\le& e^{-\mu_1\tau_1}\frac{\beta_1\left(S-\frac{\Lambda}{\mu_0}\right)I_1(t)}{1+\alpha_1 I_1(t)}+\mu_1 I_1\left(\frac{e^{-\mu_1\tau_1}\beta_1\frac{\Lambda}{\mu_0}}{\mu_1}-1\right).
\end{aligned}
$$

Due to the basic reproduction number $R_1 = \dfrac{\beta_1 \Lambda e^{-\mu_1\tau_1}}{\mu_0\mu_1} < 1$, we know that $\dfrac{\beta_1 e^{-\mu_1\tau_1}\Lambda}{\mu_0\mu_1} - 1 < 0$. Thus,

$$LV_2 \le e^{-\mu_1\tau_1} \frac{\beta_1 \left(S - \dfrac{\Lambda}{\mu_0}\right) I_1(t)}{1 + \alpha_1 I_1(t)}.$$

Let

$$V_3(I_2) = I_2 + e^{-\mu_2\tau_2}\beta_2 \int_{t-\tau_2}^{t} \frac{S(\mu)I_2(\mu)}{1+\alpha_2 I_2(\mu)} \mathrm{d}\mu.$$

Then

$$\mathrm{d}V_3 = LV_3 \mathrm{d}t + \sigma_3 I_2 \mathrm{d}B_3(t) + \int_A D_3(y) I_2 \tilde{N}(\mathrm{d}t, \mathrm{d}y),$$

where

$$\begin{aligned} LV_3 =& e^{-\mu_2\tau_2} \frac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - \mu_2 I_2(t) + e^{-\mu_2\tau_2} \frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 I_2(t)} \\ & - e^{-\mu_2\tau_2} \frac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} + \int_A \left[I_2 + D_3(y) I_2 - I_2 - D_3(y) I_2 \right] \nu(\mathrm{d}y) \\ \le & e^{-\mu_2\tau_2} \frac{\beta_2 \left(S - \dfrac{\Lambda}{\mu_0}\right) I_2(t)}{1+\alpha_2 I_2(t)} + \mu_2 I_2(t) \left(\frac{e^{-\mu_2\tau_2}\beta_2 \dfrac{\Lambda}{\mu_0}}{\mu_2} - 1 \right). \end{aligned}$$

Here, the basic reproduction number $R_2 = \dfrac{\beta_2 \Lambda e^{-\mu_2\tau_2}}{\mu_0\mu_2} < 1$, then $\dfrac{\beta_2 \Lambda e^{-\mu_2\tau_2}}{\mu_0\mu_2} - 1 < 0$. We have

$$LV_3 \le e^{-\mu_2\tau_2} \frac{\beta_2 \left(S - \dfrac{\Lambda}{\mu_0}\right) I_2(t)}{1 + \alpha_2 I_2(t)}.$$

Let

$$\begin{aligned} V_4(S, I_1, I_2) =& V_1 + \frac{\Lambda}{\mu_0} e^{\mu_1\tau_1} V_2 + \frac{\Lambda}{\mu_0} e^{\mu_2\tau_2} V_3 \\ =& \frac{1}{2}\left(S - \frac{\Lambda}{\mu_0}\right)^2 + \frac{\Lambda}{\mu_0} e^{\mu_1\tau_1} I_1 + \frac{\Lambda}{\mu_0} e^{\mu_2\tau_2} I_2 \\ & + \frac{\Lambda\beta_1}{\mu_0} \int_{t-\tau_1}^{t} \frac{S(\mu) I_1(\mu)}{1+\alpha_1 I_1(\mu)} \mathrm{d}\mu + \frac{\Lambda\beta_2}{\mu_0} \int_{t-\tau_2}^{t} \frac{S(\mu) I_2(\mu)}{1+\alpha_2 I_2(\mu)} \mathrm{d}\mu. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{d}V_4 =& LV_4 \mathrm{d}t + \left(S - \frac{\Lambda}{\mu_0}\right) \sigma_1 S \,\mathrm{d}B_1(t) + \frac{\Lambda}{\mu_0} e^{\mu_1\tau_1} \sigma_2 I_1 \mathrm{d}B_2(t) + \frac{\Lambda}{\mu_0} e^{\mu_2\tau_2} \sigma_3 I_2 \mathrm{d}B_3(t) \\ & + \int_A \left[\frac{1}{2}\left(S + D_1(y) S - \frac{\Lambda}{\mu_0}\right)^2 - \frac{1}{2}\left(S - \frac{\Lambda}{\mu_0}\right)^2 \right] \tilde{N}(\mathrm{d}t, \mathrm{d}y) + \int_A \frac{\Lambda}{\mu_0} e^{\mu_1\tau_1} (I_1 \\ & + D_2(y) I_1 - I_1) \tilde{N}(\mathrm{d}t, \mathrm{d}y) + \int_A \frac{\Lambda}{\mu_0} e^{\mu_2\tau_2} (I_2 + D_3(y) I_2 - I_2) \tilde{N}(\mathrm{d}t, \mathrm{d}y) \end{aligned}$$

$$
\begin{aligned}
&= LV_4\mathrm{d}t + \left(S - \frac{\Lambda}{\mu_0}\right)\sigma_1 S\,\mathrm{d}B_1(t) + \frac{\Lambda}{\mu_0}e^{\mu_1\tau_1}\sigma_2 I_1\mathrm{d}B_2(t) + \frac{\Lambda}{\mu_0}e^{\mu_2\tau_2}\sigma_3 I_2\mathrm{d}B_3(t)\\
&+ \int_A\left(\frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)S\frac{\Lambda}{\mu_0}\right)\tilde{N}(\mathrm{d}t,\mathrm{d}y) + \int_A\frac{\Lambda}{\mu_0}e^{\mu_1\tau_1}D_2(y)I_1\tilde{N}(\mathrm{d}t,\mathrm{d}y)\\
&+ \int_A\frac{\Lambda}{\mu_0}e^{\mu_2\tau_2}D_3(y)I_2\tilde{N}(\mathrm{d}t,\mathrm{d}y), \qquad (3.1)
\end{aligned}
$$

where

$$
\begin{aligned}
LV_4 \leq& -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)\left(S - \frac{\Lambda}{\mu_0}\right)^2 - \frac{\beta_1\frac{\Lambda}{\mu_0}I_1(t)\left(S - \frac{\Lambda}{\mu_0}\right)}{1 + \alpha_1 I_1(t)} - \frac{\beta_2\frac{\Lambda}{\mu_0}I_2(t)\left(S - \frac{\Lambda}{\mu_0}\right)}{1 + \alpha_2 I_2(t)}\\
&+ \sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2 + \int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y) + \frac{\Lambda}{\mu_0}\cdot\frac{\beta_1\left(S - \frac{\Lambda}{\mu_0}\right)I_1(t)}{1 + \alpha_1 I_1(t)} + \frac{\Lambda}{\mu_0}\cdot\frac{\beta_2\left(S - \frac{\Lambda}{\mu_0}\right)I_2(t)}{1 + \alpha_2 I_2(t)}\\
=& -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)\left(S - \frac{\Lambda}{\mu_0}\right)^2 + \sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2 + \int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y). \qquad (3.2)
\end{aligned}
$$

Integrate from 0 to t, then

$$
\begin{aligned}
\int_0^t \mathrm{d}V_4 \leq& \left[-\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)\left(S - \frac{\Lambda}{\mu_0}\right)^2 + \sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2 + \int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y)\right]\mathrm{d}\mu\\
&+ \int_0^t\left(S - \frac{\Lambda}{\mu_0}\right)\sigma_1 S\,\mathrm{d}B_1(t) + \int_0^t\frac{\Lambda}{\mu_0}e^{\mu_1\tau_1}\sigma_2 I_1\mathrm{d}B_2(t) + \int_0^t\frac{\Lambda}{\mu_0}e^{\mu_2\tau_2}\sigma_3 I_2\mathrm{d}B_3(t)\\
&+ \int_0^t\left[\int_A\left(\frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)S\frac{\Lambda}{\mu_0}\right)\tilde{N}(\mathrm{d}t,\mathrm{d}y)\right.\\
&\left.+ \int_A\frac{\Lambda}{\mu_0}e^{\mu_1\tau_1}D_2(y)I_1\tilde{N}(\mathrm{d}t,\mathrm{d}y) + \int_A\frac{\Lambda}{\mu_0}e^{\mu_2\tau_2}D_3(y)I_2\tilde{N}(\mathrm{d}t,\mathrm{d}y)\right]. \qquad (3.3)
\end{aligned}
$$

Take the expectation, and we get

$$
EV_4(S(t),I_1(t),I_2(t)) \leq EV_4(S(0),I_1(0),I_2(0)) + E\int_0^t LV_4(S(\mu),I_1(\mu),I_2(\mu))\mathrm{d}\mu.
$$

Then

$$
\begin{aligned}
0 \leq& E\int_0^t\left[-\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)\left(S - \frac{\Lambda}{\mu_0}\right)^2\right]\mathrm{d}\mu\\
&+ EV_4(S(0),I_1(0),I_2(0)) + \left[\sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2 + \int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y)\right]t.
\end{aligned}
$$

Hence, we have

$$
\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)\left(S - \frac{\Lambda}{\mu_0}\right)^2\right]\mathrm{d}\mu \leq \sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2 + \int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y).
$$

So,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left(S-\frac{\Lambda}{\mu_0}\right)^2\mathrm{d}\mu\le\frac{\sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2+\int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y)}{\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)}.$$

Define the function $V_5:\mathbb{R}_+^3\to\mathbb{R}_+$

$$V_5(S,I_1,I_2)=\frac{1}{2}\left[\left(S-\frac{\Lambda}{\mu_0}\right)+e^{\mu_1\tau_1}I_1(t+\tau_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)\right]^2.$$

We obtain

$$\begin{aligned}\mathrm{d}V_5=&LV_5\mathrm{d}t+\left(S-\frac{\Lambda}{\mu_0}+e^{\mu_1\tau_1}I_1(t+\tau_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)\right)(\sigma_1S\,\mathrm{d}B_1(t)+e^{\mu_1\tau_1}\sigma_2I_1\mathrm{d}B_2(t)\\&+e^{\mu_2\tau_2}\sigma_3I_2\mathrm{d}B_3(t))+\int_A\left[\frac{1}{2}\left(S+D_1(y)S-\frac{\Lambda}{\mu_0}+e^{\mu_1\tau_1}\left(I_1(t+\tau_1)+D_2(y)I_1(t+\tau_1)\right)\right.\right.\\&\left.\left.+e^{\mu_2\tau_2}\left(I_2(t+\tau_2)+D_3(y)I_2(t+\tau_2)\right)\right)^2-\frac{1}{2}\left(S-\frac{\Lambda}{\mu_0}+e^{\mu_1\tau_1}I_1(t+\tau_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)\right)^2\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),\end{aligned}$$

where

$$\begin{aligned}LV_5=&\left(S-\frac{\Lambda}{\mu_0}+e^{\mu_1\tau_1}I_1(t+\tau_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)\right)\left(\Lambda-\mu_0S-\frac{\beta_1S(t)I_1(t)}{1+\alpha_1I_1(t)}-\frac{\beta_2S(t)I_2(t)}{1+\alpha_2I_2(t)}\right.\\&+e^{\mu_1\tau_1}e^{-\mu_1\tau_1}\frac{\beta_1S(t+\tau_1-\tau_1)I_1(t+\tau_1-\tau_1)}{1+\alpha_1I_1(t+\tau_1-\tau_1)}+e^{\mu_2\tau_2}e^{-\mu_2\tau_2}\frac{\beta_2S(t+\tau_2-\tau_2)I_2(t+\tau_2-\tau_2)}{1+\alpha_2I_2(t+\tau_2-\tau_2)}\\&\left.-e^{\mu_1\tau_1}\mu_1I_1(t+\tau_1)-e^{\mu_2\tau_2}\mu_2I_2(t+\tau_2)\right)+\frac{1}{2}\sigma_1^2S^2+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2I_2^2(t+\tau_2)\\&+\int_A\left[\frac{1}{2}\left(S+D_1(y)S-\frac{\Lambda}{\mu_0}+e^{\mu_1\tau_1}(I_1(t+\tau_1)+D_2(y)I_1(t+\tau_1))+e^{\mu_2\tau_2}(I_2(t+\tau_2)\right.\right.\\&\left.+D_3(y)I_2(t+\tau_2))\right)^2-\frac{1}{2}\left(S-\frac{\Lambda}{\mu_0}+e^{\mu_1\tau_1}I_1(t+\tau_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)\right)^2-\left(S-\frac{\Lambda}{\mu_0}\right.\\&\left.\left.+e^{\mu_1\tau_1}I_1(t+\tau_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)\right)\left(D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)\right)\right]\nu(\mathrm{d}y)\\=&-\mu_0\left(S-\frac{\Lambda}{\mu_0}\right)^2-\mu_1e^{2\mu_1\tau_1}I_1^2(t+\tau_1)-\mu_2e^{2\mu_2\tau_2}I_2^2(t+\tau_2)-(\mu_0+\mu_1)e^{\mu_1\tau_1}\left(S-\frac{\Lambda}{\mu_0}\right)I_1(t+\tau_1)\\&-(\mu_0+\mu_2)e^{\mu_2\tau_2}\left(S-\frac{\Lambda}{\mu_0}\right)I_2(t+\tau_2)-(\mu_1+\mu_2)e^{\mu_1\tau_1+\mu_2\tau_2}I_1(t+\tau_1)I_2(t+\tau_2)+\frac{1}{2}\sigma_1^2S^2\\&+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2I_2^2(t+\tau_2)+\int_A\left\{\frac{1}{2}D_1^2(y)S^2+e^{\mu_1\tau_1}D_1(y)D_2(y)SI_1(t+\tau_1)\right.\\&+\frac{1}{2}e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)+e^{\mu_2\tau_2}D_1(y)D_3(y)SI_2(t+\tau_2)+\frac{1}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)\\&\left.+e^{\mu_1\tau_1+\mu_2\tau_2}D_2(y)D_3(y)I_1(t+\tau_1)I_2(t+\tau_2)\right\}\nu(\mathrm{d}y)\end{aligned}$$

$$
\begin{aligned}
\leq & -\mu_0\left(S-\frac{\Lambda}{\mu_0}\right)^2-\mu_1 e^{2\mu_1\tau_1}I_1^2(t+\tau_1)-\mu_2 e^{2\mu_2\tau_2}I_2^2(t+\tau_2)+\frac{\mu_0+\mu_1}{2}\left(S-\frac{\Lambda}{\mu_0}\right)^2\\
&+\frac{e^{2\mu_1\tau_1}(\mu_0+\mu_1)}{2}I_1^2(t+\tau_1)+\frac{\mu_0+\mu_2}{2}\left(S-\frac{\Lambda}{\mu_0}\right)^2+\frac{e^{2\mu_2\tau_2}(\mu_0+\mu_2)}{2}I_2^2(t+\tau_2)\\
&+\sigma_1^2\left(S-\frac{\Lambda}{\mu_0}\right)^2+\sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2I_2^2(t+\tau_2)\\
&+\int_A\left\{\frac{1}{2}D_1^2(y)S^2+\frac{1}{2}e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)+\frac{1}{2}D_1^2(y)S^2\right.\\
&+\frac{1}{2}e^{2\mu_1\tau_1}D_2^2I_1^2(t+\tau_1)+\frac{1}{2}D_1^2(y)S^2+\frac{1}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)+\frac{1}{2}e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)\\
&\left.+\frac{1}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)\right\}\nu(\mathrm{d}y)\\
= & -\mu_0\left(S-\frac{\Lambda}{\mu_0}\right)^2-\mu_1 e^{2\mu_1\tau_1}I_1^2(t+\tau_1)-\mu_2 e^{2\mu_2\tau_2}I_2^2(t+\tau_2)+\frac{2\mu_0+\mu_1+\mu_2}{2}\left(S-\frac{\Lambda}{\mu_0}\right)^2\\
&+\frac{e^{2\mu_1\tau_1}(\mu_0+\mu_1)}{2}I_1^2(t+\tau_1)+\frac{e^{2\mu_2\tau_2}(\mu_0+\mu_2)}{2}I_2^2(t+\tau_2)+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2I_1^2(t+\tau_1)\\
&+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2I_2^2(t+\tau_2)+\int_A\left\{\frac{3}{2}D_1^2(y)S^2+\frac{3}{2}e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)\right.\\
&\left.+\frac{3}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)\right\}\nu(\mathrm{d}y)\\
\leq & \frac{1}{2}\left(\mu_1+\mu_2+2\sigma_1^2\right)\left(S-\frac{\Lambda}{\mu_0}\right)^2-\frac{1}{2}e^{2\mu_1\tau_1}(\mu_1-\mu_0-\sigma_2^2)I_1^2(t+\tau_1)+\sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2\\
&-\frac{1}{2}e^{2\mu_2\tau_2}(\mu_2-\mu_0-\sigma_3^2)I_2^2(t+\tau_2)+\int_A\left\{3D_1^2(y)\left(S-\frac{\Lambda}{\mu_0}\right)^2+3D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\right.\\
&\left.+\frac{3}{2}e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)+\frac{3}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)\right\}\nu(\mathrm{d}y)\\
= & \frac{1}{2}\left(\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)\right)\left(S-\frac{\Lambda}{\mu_0}\right)^2+\sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2+3\int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y)\\
&-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-\sigma_2^2-3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^2(t+\tau_1)\\
&-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-\sigma_3^2-3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)I_2^2(t+\tau_2)\\
\leq & \frac{1}{2}\left(\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)\right)\left(S-\frac{\Lambda}{\mu_0}\right)^2+\sigma_1^2\left(\frac{\Lambda}{\mu_0}\right)^2+3\int_A D_1^2(y)\left(\frac{\Lambda}{\mu_0}\right)^2\nu(\mathrm{d}y)\\
&-\frac{1}{2}e^{\mu_2\tau_2}\left(\mu_2-\mu_0-\sigma_3^2-3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)I_2^2(t+\tau_2)\\
&-\frac{1}{2}e^{\mu_1\tau_1}\left(\mu_1-\mu_0-\sigma_2^2-3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^2(t+\tau_1).
\end{aligned}
$$

Construct the Lyapunov function $V_6 : \mathbb{R}_+^3 \to \mathbb{R}_+$

$$
\begin{aligned}
V_6(S, I_1, I_2) =& \frac{\left(\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} V_4 + V_5 \\
&+ \frac{1}{2} e^{\mu_1\tau_1} \left(\mu_1 - \mu_0 - \sigma_2^2 - 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right) \int_t^{t+\tau_1} I_1^2(\mu)\mathrm{d}\mu \\
&+ \frac{1}{2} e^{\mu_2\tau_2} \left(\mu_2 - \mu_0 - \sigma_3^2 - 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right) \int_t^{t+\tau_2} I_2^2(\mu)\mathrm{d}\mu.
\end{aligned}
$$

It is obvious that V_6 is positive-definite. By Itô's formula, we can obtain

$$
\begin{aligned}
\mathrm{d}V_6 =& \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} \mathrm{d}V_4 + \mathrm{d}V_5 \\
=& LV_6\mathrm{d}t + \Bigg\lfloor \left(\frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} + 1 \right) \left(S - \frac{\Lambda}{\mu_0}\right) + e^{\mu_1\tau_1} I_1(t+\tau_1) \\
&+ e^{\mu_2\tau_2} I_2(t+\tau_2) \Bigg\rfloor \sigma_1 S\,\mathrm{d}B_1(t) + \Bigg\lfloor \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} \cdot \frac{\Lambda}{\mu_0} + \left(S - \frac{\Lambda}{\mu_0}\right) \\
&+ e^{\mu_1\tau_1} I_1(t+\tau_1) + e^{\mu_2\tau_2} I_2(t+\tau_2) \Bigg\rfloor [\sigma_2 e^{\mu_1\tau_1} I_1(t+\tau_1) I_1 \mathrm{d}B_2(t) + \sigma_3 e^{\mu_2\tau_2} I_2(t+\tau_2) I_2 \mathrm{d}B_3(t)] \\
&+ \int_A \Bigg\lfloor \left(\frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} + 1 \right) \left[\frac{1}{2} D_1^2(y) S^2 + D_1(y) S \left(S - \frac{\Lambda}{\mu_0}\right) \right] \\
&+ e^{\mu_1\tau_1} D_2(y) \left(S - \frac{\Lambda}{\mu_0}\right) I_1(t+\tau_1) + e^{\mu_2\tau_2} D_3(y) \left(S - \frac{\Lambda}{\mu_0}\right) I_2(t+\tau_2) + e^{\mu_1\tau_1} D_1(y) S I_1(t+\tau_1) \\
&+ e^{2\mu_1\tau_1} D_2^2(y) I_1^2(t+\tau_1) + e^{\mu_1\tau_1 + \mu_2\tau_2} D_3(y) I_1(t+\tau_1) I_2(t+\tau_2) + \frac{1}{2} e^{2\mu_2\tau_2} D_3^2(y) I_2^2(t+\tau_2) \\
&+ e^{\mu_2\tau_2} D_1(y) S I_2(t+\tau_2) + e^{\mu_1\tau_1 + \mu_2\tau_2} D_2(y) I_1(t+\tau_1) I_2(t+\tau_2) + e^{2\mu_2\tau_2} D_3^2(y) I_2^2(t+\tau_2) \\
&+ e^{\mu_1\tau_1} D_1(y) D_2(y) S I_1(t+\tau_1) + e^{\mu_2\tau_2} D_1(y) D_3(y) S I_2(t+\tau_2) + \frac{1}{2} e^{2\mu_1\tau_1} D_2^2(y) I_1^2(t+\tau_1) \\
&+ e^{\mu_1\tau_1 + \mu_2\tau_2} D_2(y) D_3(y) I_1(t+\tau_1) I_2(t+\tau_2) \Bigg\rfloor \tilde{N}(\mathrm{d}t, \mathrm{d}y), \qquad (3.4)
\end{aligned}
$$

where

$$
\begin{aligned}
LV_6 \leq& \frac{\left(\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} \left[-\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right) \left(S - \frac{\Lambda}{\mu_0}\right)^2 \right. \\
&\left. + \sigma_1^2 \left(\frac{\Lambda}{\mu_0}\right)^2 + \int_A D_1^2(y)\nu(\mathrm{d}y) \right] + \frac{1}{2} \left(\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)\right) \left(S - \frac{\Lambda}{\mu_0}\right)^2 \\
&- \frac{1}{2} e^{2\mu_1\tau_1} \left(\mu_1 - \mu_0 - \sigma_2^2 - 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right) I_1^2(t+\tau_1) + 3\int_A D_1^2(y) \left(\frac{\Lambda}{\mu_0}\right)^2 \nu(\mathrm{d}y) \\
&- \frac{1}{2} e^{2\mu_2\tau_2} \left(\mu_2 - \mu_0 - \sigma_3^2 - 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right) I_2^2(t+\tau_2) + \sigma_1^2 \left(\frac{\Lambda}{\mu_0}\right)^2 + \frac{1}{2} e^{2\mu_1\tau_1} (\mu_1
\end{aligned}
$$

$$-\mu_0-\sigma_2^2-3\int_A D_2^2(y)\nu(\mathrm{d}y)\Big)I_1^2(t+\tau_1)-\frac{1}{2}e^{2\mu_1\tau_1}\Big(\mu_1-\mu_0-\sigma_2^2-3\int_A D_2^2(y)\nu(\mathrm{d}y)\Big)I_1^2(t)$$
$$+\frac{1}{2}e^{2\mu_2\tau_2}\Big(\mu_2-\mu_0-\sigma_3^2-3\int_A D_3^2(y)\nu(\mathrm{d}y)\Big)I_2^2(t+\tau_2)$$
$$-\frac{1}{2}e^{2\mu_2\tau_2}\Big(\mu_2-\mu_0-\sigma_3^2-3\int_A D_3^2(y)\nu(\mathrm{d}y)\Big)I_2^2(t)$$
$$=-\frac{1}{2}e^{2\mu_1\tau_1}\Big(\mu_1-\mu_0-\sigma_2^2-3\int_A D_2^2(y)\nu(\mathrm{d}y)\Big)I_1^2(t)+M_1\left(\frac{\Lambda}{\mu_0}\right)^2$$
$$-\frac{1}{2}e^{2\mu_2\tau_2}\Big(\mu_2-\mu_0-\sigma_3^2-3\int_A D_3^2(y)\nu(\mathrm{d}y)\Big)I_2^2(t), \tag{3.5}$$

where

$$M_1=\left\{\left[\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}+1\right]\sigma_1^2+\left[\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}+3\right]\int_A D_1^2(y)\nu(\mathrm{d}y)\right\}.$$

Integrate from 0 to t, then

$$\int_0^t \mathrm{d}V_6\le\int_0^t\Big[-\frac{1}{2}e^{\mu_1\tau_1}\Big(\mu_1-\mu_0-\sigma_2^2-3\int_A D_2^2(y)\nu(\mathrm{d}y)\Big)I_1^2-\frac{1}{2}e^{\mu_2\tau_2}\Big(-3\int_A D_3^2(y)\nu(\mathrm{d}y)$$
$$+\mu_2-\mu_0-\sigma_3^2\Big)I_2^2+M_1\left(\frac{\Lambda}{\mu_0}\right)^2\Big]\mathrm{d}\mu+\int_0^t\Big\{+e^{\mu_1\tau_1}I_1(t+\tau_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)$$
$$\left(\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}+1\right)\left(S-\frac{\Lambda}{\mu_0}\right)\Big\}\sigma_1 S\,\mathrm{d}B_1(t)+\Big\{\left(S-\frac{\Lambda}{\mu_0}\right)$$
$$+\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\cdot\frac{\Lambda}{\mu_0}+e^{\mu_1\tau_1}I_1(t+\tau_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)\Big\}\cdot$$
$$[\sigma_2 e^{\mu_1\tau_1}I_1(t+\tau_1)I_1\mathrm{d}B_2(t)+\sigma_3 e^{\mu_2\tau_2}I_2(t+\tau_2)I_2\mathrm{d}B_3(t)]$$
$$+\int_A\Big\{\left(\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}+1\right)\left[\frac{1}{2}D_1^2(y)S^2+D_1(y)S\left(S-\frac{\Lambda}{\mu_0}\right)\right]$$
$$+e^{\mu_1\tau_1}D_2(y)\left(S-\frac{\Lambda}{\mu_0}\right)I_1(t+\tau_1)+e^{\mu_2\tau_2}D_3(y)\left(S-\frac{\Lambda}{\mu_0}\right)I_2(t+\tau_2)$$
$$+e^{\mu_1\tau_1}D_1(y)SI_1(t+\tau_1)+e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)+e^{\mu_1\tau_1+\mu_2\tau_2}D_3(y)I_1(t+\tau_1)I_2(t+\tau_2)$$
$$+e^{\mu_2\tau_2}D_1(y)SI_2(t+\tau_2)+e^{\mu_1\tau_1+\mu_2\tau_2}D_2(y)I_1(t+\tau_1)I_2(t+\tau_2)+e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)$$
$$+e^{\mu_1\tau_1}D_1(y)D_2(y)SI_1(t+\tau_1)+e^{\mu_2\tau_2}D_1(y)D_3(y)SI_2(t+\tau_2)+\frac{1}{2}e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)$$
$$+e^{\mu_1\tau_1+\mu_2\tau_2}D_2(y)D_3(y)I_1(t+\tau_1)I_2(t+\tau_2)+\frac{1}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)\Big\}\tilde{N}(\mathrm{d}t,\mathrm{d}y). \tag{3.6}$$

Take the expectation, and we get

$$EV_6(S(t),I_1(t),I_2(t))\le EV_6(S(0),I_1(0),I_2(0))+E\int_0^t LV_6(S(\mu),I_1(\mu),I_2(\mu))\mathrm{d}\mu.$$

Then we have

$$\begin{aligned}0 \leq & E\int_0^t \left[-\frac{1}{2}e^{\mu_1\tau_1}\left(\mu_1-\mu_0-\sigma_2^2-3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^2(t)\right]\mathrm{d}\mu\\ &+E\int_0^t \left[-\frac{1}{2}e^{\mu_2\tau_2}\left(\mu_2-\mu_0-\sigma_3^2-3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)I_2^2(t)\right]\mathrm{d}\mu\\ &+EV_6(S(0),I_1(0),I_2(0))+M_1t.\end{aligned}$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t \left(I_1^2(\mu)+I_2^2(\mu)\right)\mathrm{d}\mu \leq \frac{2M_1}{H_0}\left(\frac{\Lambda}{\mu_0}\right)^2,$$

where

$$H_0=\min\left\{e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-\sigma_2^2-3\int_A D_2^2(y)\nu(\mathrm{d}y)\right),e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-\sigma_3^2-3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\right\}. \quad \square$$

Remark 1. Theorem 3.1 shows that when $R_1 < 1$ and $R_2 < 1$ and certain conditions are satisfied, the solution to the stochastic delayed system (1.3) will oscillate around the disease-free equilibrium point E_0. The intensity of these oscillations is related to the values of σ_i and $D_i(y)$, $(i = 1, 2, 3)$. The smaller the noise intensity, the weaker the oscillation intensity of the solution. In other words, as the random perturbations decrease, the solution of the system will approach the disease-free equilibrium point E_0, which implies that the disease will die out.

3.2. Asymptotic behavior of solutions near equilibrium point E_1

When $R_1 > 1$ and $R_2 < 1$, the deterministic SI_1I_2 model possesses an equilibrium point $E_1(S_1^{'}, I_1^{'}, 0)$, which is globally asymptotically stable. Therefore, the following discussion will focus on the asymptotic properties of the solutions to the stochastic delayed model (1.3) around the equilibrium point E_1.

Theorem 3.2. Assume that conditions (H1) and (H2) hold. When $R_1 > 1$ and $R_2 < 1$, if the following conditions are satisfied:

$$\begin{aligned}&\frac{\beta_2 S_1^{'}}{\mu_2}e^{-\mu_2\tau_2}\leq 1,\\ &\mu_0>\sigma_1^2+\int_A D_1^2(y)\nu(\mathrm{d}y),\\ &\mu_1>\mu_0+2\sigma_2^2+6\int_A D_2^2(y)\nu(\mathrm{d}y),\\ &\mu_2>\mu_0+\sigma_3^2+3\int_A D_3^2(y)\nu(\mathrm{d}y),\end{aligned}$$

then for any given initial values (1.4), the solution $(S(t), I_1(t), I_2(t))$ of system (1.3) has the following properties:

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[(S(\mu)-S_1')^2\right]\mathrm{d}\mu\le\frac{1}{\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)}M_2,$$

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[(I_1(\mu)-I_1')^2+I_2^2(\mu)\right]\mathrm{d}\mu\\&\le\frac{2}{H_1}\Bigg\{\left(\sigma_1^2+3\int_A D_1^2(y)\nu(\mathrm{d}y)\right)S_1'^2+e^{2\mu_1\tau_1}\left(\sigma_2^2+3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1'^2\\&+\frac{\left(\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}M_2\Bigg\},\end{aligned}$$

where

$$\begin{aligned}H_1&=\min\left\{e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2-6\int_A D_2^2(y)\nu(\mathrm{d}y)\right),e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-\sigma_3^2-3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\right\},\\M_2&=\left[\left(\frac{\beta_1 I_1'}{2\mu_0(1+\alpha_1 I_1'})+1\right)\sigma_1^2+\int_A D_1^2(y)\nu(\mathrm{d}y)\right]S_1'^2+\frac{e^{\mu_1\tau_1}}{2\mu_0}\left(\mu_0 S_1'+\frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'}\right)\sigma_2^2 I_1'\\&+\left(e^{\mu_1\tau_1}S_1'+\frac{e^{\mu_1\tau_1}}{\mu_0}\cdot\frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'}\right)\int_A\left[D_2(y)I_1'-I_1'\ln\left[1+D_2(y)\right]\right]\nu(\mathrm{d}y)\\&+\frac{\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1'})\int_A\left[D_1(y)S_1'-S_1'\ln\left[1+D_1(y)\right]\right]\nu(\mathrm{d}y).\end{aligned}$$

Proof. First, define the function $V_1:\mathbb{R}_+\to\mathbb{R}_+$

$$V_1(S)=S-S_1'-S_1'\ln\frac{S}{S_1'}.$$

According to Itô's formula, we have

$$\begin{aligned}\mathrm{d}V_1&=LV_1\mathrm{d}t+\left(1-S_1'\cdot\frac{1}{S_1'}\cdot\frac{S_1'}{S}\right)\sigma_1 S\,\mathrm{d}B_1(t)\\&+\int_A\left[\left(S+D_1(y)S-S_1'-S_1'\ln\frac{S+D_1(y)S}{S_1'}\right)-\left(S-S_1'-S_1'\ln\frac{S}{S_1'}\right)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y)\\&=LV_1\mathrm{d}t+\sigma_1(S-S_1')\,\mathrm{d}B_1(t)+\int_A\left[D_1(y)S-S_1'\ln(1+D_1(y))\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),\end{aligned}$$

where

$$\begin{aligned}LV_1&=\left(1-\frac{S_1'}{S}\right)\left(\Lambda-\mu_0 S-\frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)}-\frac{\beta_2 S(t)I_2(t)}{1+\alpha_2 I_2(t)}\right)+\frac{1}{2}\cdot\frac{S_1'}{S^2}\sigma_1^2 S^2+\int_A\Bigg[(S+D_1(y)S\\&-S_1'-S_1'\ln\frac{S+D_1(y)S}{S_1'}\Bigg)-\left(S-S_1'-S_1'\ln\frac{S}{S_1'}\right)-\left(1-\frac{S_1'}{S}\right)D_1(y)S\Bigg]\nu(\mathrm{d}y)\\&=\Lambda-\mu_0 S-\frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)}-\frac{\beta_2 S(t)I_2(t)}{1+\alpha_2 I_2(t)}-\frac{\Lambda S_1'}{S}+\mu_0 S_1'+\frac{\beta_1 S_1' I_1(t)}{1+\alpha_1 I_1(t)}+\frac{\beta_2 S_1' I_2(t)}{1+\alpha_2 I_2(t)}\\&+\frac{1}{2}\sigma_1^2 S_1'+\int_A\left[D_1(y)S'-S_1'\ln(1+D_1(y))\right]\nu(\mathrm{d}y).\end{aligned}$$

When $R_1 > 1$ and $R_2 < 1$, the equilibrium point $E_1(S'_1, I'_1, 0)$ of the deterministic SI_1I_2 model satisfies

$$\Lambda - \mu_0 S'_1 - \frac{\beta_1 S'_1 I'_1}{1 + \alpha_1 I'_1} = 0, \tag{3.7}$$

$$e^{-\mu_1 \tau_1} \frac{\beta_1 S'_1 I'_1}{1 + \alpha_1 I'_1} - \mu_1 I'_1 = 0. \tag{3.8}$$

Then

$$\mu_0 S'_1 = \Lambda - \frac{\beta_1 S'_1 I'_1}{1 + \alpha_1 I'_1},$$

$$\mu_0 = \frac{\Lambda}{S'_1} - \frac{\beta_1 I'_1}{1 + \alpha_1 I'_1}.$$

Thus,

$$\begin{aligned}
LV_1 =& \Lambda - \left(\frac{\Lambda}{S'_1} - \frac{\beta_1 I'_1}{1 + \alpha_1 I'_1}\right) S - \frac{\beta_1 S(t) I_1(t)}{1 + \alpha_1 I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 I_2(t)} - \frac{\Lambda S'_1}{S} + \Lambda - \frac{\beta_1 S'_1 I'_1}{1 + \alpha_1 I'_1} \\
&+ \frac{\beta_1 S'_1 I_1(t)}{1 + \alpha_1 I_1(t)} + \frac{\beta_2 S'_1 I_2(t)}{1 + \alpha_2 I_2(t)} + \frac{1}{2}\sigma_1^2 S'_1 + \int_A [D_1(y) S'_1 - S'_1 \ln(1 + D_1(y))] \nu(\mathrm{d}y) \\
=& \Lambda \left(2 - \frac{S}{S'_1} - \frac{S'_1}{S}\right) + \frac{\beta_1 S'_1 I_1(t)}{1 + \alpha_1 I'_1(t)} - \frac{\beta_1 S(t) I_1(t)}{1 + \alpha_1 I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 I_2(t)} - \frac{\beta_1 S'_1 I'_1}{1 + \alpha_1 I'_1} \\
&+ \frac{\beta_1 S'_1 I_1(t)}{1 + \alpha_1 I_1(t)} + \frac{\beta_2 S'_1 I_2(t)}{1 + \alpha_2 I_2(t)} + \frac{1}{2}\sigma_1^2 S'_1 + \int_A [D_1(y) S'_1 - S'_1 \ln(1 + D_1(y))] \nu(\mathrm{d}y) \\
=& \left(\mu_0 S'_1 - \frac{\beta_1 S'_1 I'_1}{1 + \alpha_1 I'_1}\right)\left(2 - \frac{S}{S'_1} - \frac{S'_1}{S}\right) + \frac{\beta_1 S I'_1(t)}{1 + \alpha_1 I'_1(t)} - \frac{\beta_1 S(t) I_1(t)}{1 + \alpha_1 I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 I_2(t)} \\
&- \frac{\beta_1 S'_1 I'_1(t)}{1 + \alpha_1 I'_1(t)} + \frac{\beta_1 S'_1 I_1(t)}{1 + \alpha_1 I_1(t)} + \frac{\beta_2 S'_1 I_2(t)}{1 + \alpha_2 I_2(t)} + \frac{1}{2}\sigma_1^2 S'_1 + \int_A [D_1(y) S' - S'_1 \ln(1 + D_1(y))] \nu(\mathrm{d}y) \\
\leq& -\left(\mu_0 + \frac{\beta_1 I'_1}{1 + \alpha_1 I'_1(t)}\right) \frac{\left(S - S'_1\right)^2}{S} - \beta_1 (S - S'_1)\left(\frac{I_1}{1 + \alpha_1 I_1(t)} - \frac{I'_1}{1 + \alpha_1 I'_1}\right) \\
&- \frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 I_2(t)} + \beta_2 S'_1 I_2(t) + \frac{1}{2}\sigma_1^2 S'_1 + \int_A [D_1(y) S' - S'_1 \ln(1 + D_1(y))] \nu(\mathrm{d}y).
\end{aligned}$$

Define the function $V_2 : \mathbb{R}_+^3 \to \mathbb{R}_+$

$$\begin{aligned}
V_2(S, I_1, I_2) &= V_1 + \frac{\beta_2 S'_1}{\mu_2} I_2 + \frac{\beta_2 S'_1}{\mu_2} \int_{t-\tau_2}^{t} e^{-\mu_2 \tau_2} \frac{\beta_2 S(\mu) I_2(\mu)}{1 + \alpha_2 I_2(\mu)} \mathrm{d}\mu \\
&= S - S'_1 - S'_1 \ln \frac{S}{S'_1} + \frac{\beta_2 S'_1}{\mu_2} I_2 + \frac{\beta_2 S'_1}{\mu_2} \int_{t-\tau_2}^{t} e^{-\mu_2 \tau_2} \frac{\beta_2 S(\mu) I_2(\mu)}{1 + \alpha_2 I_2(\mu)} \mathrm{d}\mu.
\end{aligned}$$

According to Itô's formula, we have

$$
\begin{aligned}
\mathrm{d}V_2 &= LV_2\mathrm{d}t + \left(1 - \frac{S_1'}{S}\right)\sigma_1 S\,\mathrm{d}B_1(t) + \frac{\beta_2 S_1'}{\mu_2}\sigma_3 I_2 \mathrm{d}B_3(t) + \int_A \frac{\beta_2 S_1'}{\mu_2} D_3(y) I_2 \tilde{N}(\mathrm{d}t, \mathrm{d}y) \\
&\quad + \int_A \left[\left(S + D_1(y)S - S_1' - S_1' \ln\frac{S + D_1(y)S}{S_1'}\right) - \left(S - S_1' - S_1' \ln\frac{S}{S_1'}\right)\right]\tilde{N}(\mathrm{d}t, \mathrm{d}y) \\
&= LV_2\mathrm{d}t + \sigma_1\left(S - S_1'\right)\mathrm{d}B_1(t) + \frac{\beta_2 S_1'}{\mu_2}\sigma_3 I_2 \mathrm{d}B_3(t) + \int_A \frac{\beta_2 S_1'}{\mu_2} D_3(y) I_2 \tilde{N}(\mathrm{d}t, \mathrm{d}y) \\
&\quad + \int_A \left[D_1(y)S - S_1' \ln\left(1 + D_1(y)\right)\right]\tilde{N}(\mathrm{d}t, \mathrm{d}y),
\end{aligned}
$$

where

$$
\begin{aligned}
LV_2 \leq & -\left(\mu_0 + \frac{\beta_1 I_1'}{1 + \alpha_1 I_1(t)}\right)\frac{\left(S - S_1'\right)^2}{S} - \beta_1\left(S - S_1'\right)\left(\frac{I_1}{1 + \alpha_1 I_1(t)} - \frac{I_1'}{1 + \alpha_1 I_1'(t)}\right) - \frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 I_2(t)} \\
& + \int_A \left[D_1(y)S' - S_1' \ln\left(1 + D_1(y)\right)\right]\nu(\mathrm{d}y) + \frac{\beta_2 S_1'}{\mu_2}\left[e^{-\mu_2\tau_2}\frac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1 + \alpha_2 I_2(t-\tau_2)} - \mu_2 I_2(t)\right] \\
& + \beta_2 S_1' I_2 + \frac{1}{2}\sigma_1^2 S_1' + \int_A \left[\frac{\beta_2 S_1'}{\mu_2}\left(I_2 + D_3(y)I_2\right) - \frac{\beta_2 S_1'}{\mu_2} I_2 - \frac{\beta_2 S_1'}{\mu_2} D_3(y) I_2\right]\nu(\mathrm{d}y) \\
& + \frac{\beta_2 S_1'}{\mu_2} e^{-\mu_2\tau_2}\frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 I_2(t)} - \frac{\beta_2 S_1'}{\mu_2} e^{-\mu_2\tau_2}\frac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1 + \alpha_2 I_2(t-\tau_2)} \\
= & -\left(\mu_0 + \frac{\beta_1 I_1'}{1 + \alpha_1 I_1'(t)}\right)\frac{\left(S - S_1'\right)^2}{S} - \beta_1\left(S - S_1'\right)\left(\frac{I_1}{1 + \alpha_1 I_1(t)} - \frac{I_1'}{1 + \alpha_1 I_1'}\right) \\
& + \left(\frac{\beta_2 S_1'}{\mu_2} e^{-\mu_2\tau_2} - 1\right)\frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 I_2(t)} + \frac{1}{2}\sigma_1^2 S_1' + \int_A \left[D_1(y)S_1' - S_1' \ln\left(1 + D_1(y)\right)\right]\nu(\mathrm{d}y) \\
\leq & -\left(\mu_0 + \frac{\beta_1 I_1'}{1 + \alpha_1 I_1(t)}\right)\frac{\left(S - S_1'\right)^2}{S} - \beta_1\left(S - S_1'\right)\left(\frac{I_1}{1 + \alpha_1 I_1(t)} - \frac{I_1'}{1 + \alpha_1 I_1'}\right) \\
& + \frac{1}{2}\sigma_1^2 S_1' + \int_A \left[D_1(y)S_1' - S_1' \ln\left(1 + D_1(y)\right)\right]\nu(\mathrm{d}y).
\end{aligned}
$$

Define

$$
V_3(I_1) = I_1(t + \tau_1) - I_1' - I_1' \ln\frac{I_1(t + \tau_1)}{I_1'}.
$$

According to Itô's formula, we have

$$
\begin{aligned}
\mathrm{d}V_3 =& LV_3\mathrm{d}t + \left(1 - I_1' \cdot \frac{1}{I_1'} \cdot \frac{I_1(t+\tau_1)}{I_1'}\right) + \sigma_2 I_1(t+\tau_1)\mathrm{d}B_2(t) + \int_A \left[\left(I_1(t+\tau_1) + D_2(y) I_1(t+\tau_1)\right.\right. \\
& \left. - I_1' - I_1' \ln\frac{I_1(t+\tau_1) + D_2(y) I_1(t+\tau_1)}{I_1'}\right) - \left.\left(I_1(t+\tau_1) - I_1' - I_1' \ln\frac{I_1(t+\tau_1)}{I_1'}\right)\right]\tilde{N}(\mathrm{d}t, \mathrm{d}y) \\
=& LV_3\mathrm{d}t + \sigma_2\left(I_1(t+\tau_1) - I_1'\right)\mathrm{d}B_2(t) + \int_A \left[D_2(y) I_1(t+\tau_1) - I_1' \ln\left(1 + D_2(y)\right)\right]\tilde{N}(\mathrm{d}t, \mathrm{d}y),
\end{aligned}
$$

where

$$
\begin{aligned}
LV_3 =& \left(1 - I_1' \frac{1}{I_1'} \cdot \frac{I_1'}{I_1(t+\tau_1)}\right)\left[e^{-\mu_1\tau_1} \frac{\beta_1 S(t+\tau_1-\tau_1) I_1(t+\tau_1-\tau_1)}{1+\alpha_1 I_1(t+\tau_1-\tau_1)} - \mu_1 I_1(t+\tau_1)\right] \\
&+ \int_A \left\{ \left[I_1(t+\tau_1) + D_2(y) I_1(t+\tau_1) - I_1' - I_1' \ln \frac{I_1(t+\tau_1) + D_2(y) I_1(t+\tau_1)}{I_1'} \right] \right. \\
&\left. - \left(I_1(t+\tau_1) - I_1' - I_1' \ln \frac{I_1(t+\tau_1)}{I_1'} \right) - \left[1 - \frac{I_1'}{I_1(t+\tau_1)} \right] D_2(y) I_1(t+\tau_1) \right\} \nu(\mathrm{d}y) \\
=& e^{-\mu_1\tau_1} \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 I_1(t)} - \mu_1 I_1(t+\tau_1) - e^{-\mu_1\tau_1} \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 I_1(t)} \cdot \frac{I_1'}{I_1(t+\tau_1)} + \mu_1 I_1'(t) \\
&+ \frac{1}{2}\sigma_2^2 {I_1'}^2 + \frac{1}{2} \frac{I_1'}{I_1^2(t+\tau_1)} \sigma_2^2 I_1^2(t+\tau_1) + \int_A \left[D_2(y) I_1' - I_1' \ln(1 + D_2(y)) \right] \nu(\mathrm{d}y).
\end{aligned}
$$

We know that

$$
\mu_1 I_1' = e^{-\mu_1\tau_1} \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'},
$$

$$
\mu_1 = e^{-\mu_1\tau_1} \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'} \cdot \frac{1}{I_1'}.
$$

Substitute into the above formula, and we have

$$
\begin{aligned}
LV_3 =& e^{-\mu_1\tau_1} \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 I_1(t)} - e^{-\mu_1\tau_1} \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'} \cdot \frac{I_1(t+\tau_1)}{I_1'} - e^{-\mu_1\tau_1} \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 I_1(t)} \cdot \frac{I_1'}{I_1(t+\tau_1)} \\
&+ e^{-\mu_1\tau_1} \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'} + \frac{1}{2}\sigma_2^2 {I_1'}^2 + \int_A \left[D_2(y) I_1' - I_1' \ln(1 + D_2(y)) \right] \nu(\mathrm{d}y) \\
=& e^{-\mu_1\tau_1} \beta_1 (S - S_1') \left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1'}{1+\alpha_1 I_1'} \right) + \frac{1}{2}\sigma_2^2 {I_1'}^2 + \int_A \left[D_2(y) I_1' - I_1' \ln(1 + D_2(y)) \right] \nu(\mathrm{d}y) \\
&+ e^{-\mu_1\tau_1} \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'} \left[\frac{S}{S_1'} - \frac{\dfrac{S I_1 I_1'}{1+\alpha_1 I_1}}{\dfrac{S_1' I_1' I_1(t+\tau_1)}{1+\alpha_1 I_1'}} - \frac{I_1(t+\tau_1)}{I_1'} + \frac{\dfrac{I_1}{1+\alpha_1 I_1}}{\dfrac{I_1'}{1+\alpha_1 I_1'}} \right].
\end{aligned}
$$

From $\omega \geq 1 + \ln\omega, \omega > 0$, we know that

$$
\frac{\dfrac{S I_1 I_1'}{1+\alpha_1 I_1}}{\dfrac{S_1' I_1' I_1(t+\tau_1)}{1+\alpha_1 I_1'}} \geq 1 + \ln \frac{\dfrac{S I_1 I_1'}{1+\alpha_1 I_1}}{\dfrac{S_1' I_1' I_1(t+\tau_1)}{1+\alpha_1 I_1'}} = 1 + \ln \frac{S}{S_1'} - \ln \frac{I_1(t+\tau_1)}{I_1'} + \ln \frac{\dfrac{I_1}{1+\alpha_1 I_1}}{\dfrac{I_1'}{1+\alpha_1 I_1'}}.
$$

Thus,

$$
LV_3 \leq e^{-\mu_1\tau_1} \beta_1 (S - S_1') \left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1'}{1+\alpha_1 I_1'} \right) + e^{-\mu_1\tau_1} \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'}
$$

$$
\begin{aligned}
&\left[\frac{S}{S'_1} - 1 - \ln\frac{S}{S'_1} + \ln\frac{I_1(t+\tau_1)}{I'_1} - \ln\frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I'_1}{1+\alpha_1 I_1}} - \frac{I_1(t+\tau_1)}{I'_1} + \frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I'_1}{1+\alpha_1 I'_1}}\right]\\
&+\frac{1}{2}\sigma_2^2 I'_1 + \int_A \left[D_2(y)I'_1 - I'_1\ln(1+D_2(y))\right]\nu(\mathrm{d}y)\\
=&e^{-\mu_1\tau_1}\beta_1(S-S'_1)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I'_1}{1+\alpha_1 I'_1}\right) + e^{-\mu_1\tau_1}\frac{\beta_1 S'_1 I'_1}{1+\alpha_1 I'_1}\\
&\left[\frac{S}{S'_1} - \ln\frac{S}{S'_1} - \left(\frac{I_1(t+\tau_1)}{I'_1} - \ln\frac{I_1(t+\tau_1)}{I'_1}\right) + \left(\frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I'_1}{1+\alpha_1 I'_1}} - \ln\frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I'_1}{1+\alpha_1 I'_1}}\right) - 1\right]\\
&+\frac{1}{2}\sigma_2^2 I'_1 + \int_A \left[D_2(y)I'_1 - I'_1\ln(1+D_2(y))\right]\nu(\mathrm{d}y).
\end{aligned}
$$

Define the function $V_4 : \mathbb{R}_+ \to \mathbb{R}_+$

$$
V_4(I_1) = V_3 + e^{-\mu_1\tau_1}\frac{\beta_1 S'_1 I'_1}{1+\alpha_1 I'_1}\int_t^{t+\tau_1}\left[\frac{I_1(\mu)}{I'_1} - \ln\frac{I_1(\mu)}{I'_1}\right]\mathrm{d}\mu.
$$

We obtain that

$$
\begin{aligned}
LV_4 \le& e^{-\mu_1\tau_1}\beta_1(S-S'_1)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I'_1}{1+\alpha_1 I'_1}\right) + e^{-\mu_1\tau_1}\frac{\beta_1 S'_1 I'_1}{1+\alpha_1 I'_1}\\
&\left[\frac{S}{S'_1} - \ln\frac{S}{S'_1} - \left(\frac{I_1(t+\tau_1)}{I'_1} - \ln\frac{I_1(t+\tau_1)}{I'_1}\right) + \left(\frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I'_1}{1+\alpha_1 I'_1}} - \ln\frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I'_1}{1+\alpha_1 I'_1}}\right) - 1\right]\\
&+ e^{-\mu_1\tau_1}\frac{\beta_1 S'_1 I'_1}{1+\alpha_1 I'_1}\left(\frac{I_1(t+\tau_1)}{I'_1} - \ln\frac{I_1(t+\tau_1)}{I'_1}\right) - e^{-\mu_1\tau_1}\frac{\beta_1 S'_1 I'_1}{1+\alpha_1 I'_1}\left(\frac{I_1(t)}{I'_1} - \ln\frac{I_1(t)}{I'_1}\right)\\
&+\frac{1}{2}\sigma_2^2 I'_1 + \int_A \left[D_2(y)I'_1 - I'_1\ln(1+D_2(y))\right]\nu(\mathrm{d}y).
\end{aligned}
$$

According to the logarithmic inequality, we can obtain

$$
\begin{aligned}
LV_4 \leq & e^{-\mu_1\tau_1}\beta_1\left(S-S_1'\right)\left(\frac{I_1}{1+\alpha_1 I_1}-\frac{I_1'}{1+\alpha_1 I_1'}\right)+\frac{1}{2}\sigma_2^2 I_1'+\int_A\left[D_2(y)I_1'-I_1'\ln\left(1+D_2(y)\right)\right]\nu(\mathrm{d}y)\\
&+e^{-\mu_1\tau_1}\frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'}\left[\frac{S}{S_1'}+\frac{S_1'}{S}-1-\frac{I_1(t)}{I_1'}+\frac{\frac{I_1 I_1'}{1+\alpha_1 I_1'}}{\frac{I_1' I_1}{1+\alpha_1 I_1}}-1+\frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I_1'}{1+\alpha_1 I_1'}}-1\right]\\
= & e^{-\mu_1\tau_1}\beta_1\left(S-S_1'\right)\left(\frac{I_1}{1+\alpha_1 I_1}-\frac{I_1'}{1+\alpha_1 I_1'}\right)+e^{-\mu_1\tau_1}\frac{\beta_1 I_1'}{1+\alpha_1 I_1'}\cdot\frac{\left(S-S_1'\right)^2}{S}\\
&+e^{-\mu_1\tau_1}\frac{\beta_1 S_1' I_1}{\frac{I_1}{1+\alpha_1 I_1}}\left(\frac{I_1}{1+\alpha_1 I_1}-\frac{I_1'}{1+\alpha_1 I_1'}\right)\left(\frac{\frac{I_1}{1+\alpha_1 I_1}}{I_1}-\frac{\frac{I_1'}{1+\alpha_1 I_1'}}{I_1'}\right)\\
&+\frac{1}{2}\sigma_2^2 I_1'+\int_A\left[D_2(y)I_1'-I_1'\ln\left(1+D_2(y)\right)\right]\nu(\mathrm{d}y).
\end{aligned}
$$

Let

$$
h(x)=\frac{x}{1+\alpha_1 x},
$$

then

$$
h'(x)=\frac{1+\alpha_1 x-\alpha_1 x}{(1+\alpha_1 x)^2}=\frac{1}{(1+\alpha_1 x)^2}>0,
$$

thus

$$
\left(\frac{I_1}{1+\alpha_1 I_1}-\frac{I_1'}{1+\alpha_1 I_1'}\right)\left(\frac{\frac{I_1}{1+\alpha_1 I_1}}{I_1}-\frac{\frac{I_1'}{1+\alpha_1 I_1'}}{I_1'}\right)<0.
$$

Therefore,

$$
\begin{aligned}
LV_4 \leq & e^{-\mu_1\tau_1}\beta_1\left(S-S_1'\right)\left(\frac{I_1}{1+\alpha_1 I_1}-\frac{I_1'}{1+\alpha_1 I_1'}\right)+e^{-\mu_1\tau_1}\frac{\beta_1 I_1'}{1+\alpha_1 I_1'}\cdot\frac{\left(S-S_1'\right)^2}{S}\\
&+\frac{1}{2}\sigma_2^2 I_1'+\int_A\left[D_2(y)I_1'-I_1'\ln\left(1+D_2(y)\right)\right]\nu(\mathrm{d}y).
\end{aligned}
$$

Define the function $V_5:\mathbb{R}_+\to\mathbb{R}_+$

$$
V_5(S)=\frac{1}{2}\left(S-S_1'\right)^2.
$$

From Itô's formula, we know that

$$\begin{aligned}
\mathrm{d}V_5 &= LV_5\mathrm{d}t + (S - S_1')\,\sigma_1 S\,\mathrm{d}B_1(t) + \int_A \left[\frac{1}{2}\,(S + D_1(y)S - S_1')^2 - \frac{1}{2}\,(S - S_1')^2\right]\tilde{N}(\mathrm{d}t, \mathrm{d}y)\\
&= LV_5\mathrm{d}t + (S - S_1')\,\sigma_1 S\,\mathrm{d}B_1(t) + \int_A \left[\frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_1'\right]\tilde{N}(\mathrm{d}t, \mathrm{d}y),
\end{aligned}$$

where

$$\begin{aligned}
LV_5 &= (S - S_1')\left(\Lambda - \mu_0 S - \frac{\beta_1 S I_1}{1+\alpha_1 I_1} - \frac{\beta_2 S I_2}{1+\alpha_2 I_2}\right) + \frac{1}{2}\sigma_1^2 S^2\\
&+ \int_A \left[\frac{1}{2}\,(S + D_1(y)S - S_1')^2 - \frac{1}{2}\,(S - S_1')^2 - D_1(y)S\,(S - S_1')\right]\nu(\mathrm{d}y)\\
&= (S - S_1')\left(\mu_0 S_1' + \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'} - \mu_0 S - \frac{\beta_1 S I_1}{1+\alpha_1 I_1} - \frac{\beta_2 S I_2}{1+\alpha_2 I_2}\right) + \frac{1}{2}\sigma_1^2 S^2 + \int_A \frac{1}{2}D_1^2(y)S^2\nu(\mathrm{d}y)\\
&= -\mu_0\,(S - S_1')^2 + \frac{1}{2}\sigma_1^2 S^2 + \int_A \frac{1}{2}D_1^2(y)S^2\nu(\mathrm{d}y) + \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'}\,(S - S_1')\\
&- \frac{\beta_1 S I_1}{1+\alpha_1 I_1}\,(S - S_1') - \frac{\beta_2 S I_2}{1+\alpha_2 I_2}\,(S - S_1')\\
&\leq -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S - S_1')^2 - \frac{\beta_1 I_1}{1+\alpha_1 I_1}\,(S - S_1')^2 - \frac{\beta_2 I_2}{1+\alpha_2 I_2}\,(S - S_1')^2\\
&\quad -\beta_1 S_1'\,(S - S_1')\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1'}{1+\alpha_1 I_1'}\right) - \frac{\beta_2 S I_2}{1+\alpha_2 I_2}S_1' + \beta_2 S_1'^2 I_2 + \left(\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\right)S_1'^2.
\end{aligned}$$

Define the function $V_6 : \mathbb{R}_+^2 \to \mathbb{R}_+$

$$V_6(S, I_2) = V_5 + \frac{\beta_2 S_1'^2}{\mu_2} I_2 + \frac{\beta_2 S_1'^2}{\mu_2}\int_{t-\tau_2}^{t} e^{-\mu_2(t-\mu)}\frac{\beta_2 S(\mu) I_2(\mu)}{1+\alpha_2 I_2(\mu)}\mathrm{d}\mu.$$

Then

$$\begin{aligned}
\mathrm{d}V_6 &= LV_5\mathrm{d}t + (S - S_1')\,\sigma_1 S\,\mathrm{d}B_1(t) + \int_A \left(\frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_1'\right)\tilde{N}(\mathrm{d}t, \mathrm{d}y)\\
&+ \frac{\beta_2 S_1'^2}{\mu_2}\left(e^{-\mu_2\tau_2}\frac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - \mu_2 I_2\right)\mathrm{d}t + \frac{\beta_2 S_1'^2}{\mu_2}\sigma_3 I_2 \mathrm{d}B_3(t) + \int_A \frac{\beta_2 S_1'^2}{\mu_2} D_3(y) I_2 \tilde{N}(\mathrm{d}t, \mathrm{d}y)\\
&= LV_6\mathrm{d}t + (S - S_1')\,\sigma_1 S\,\mathrm{d}B_1(t) + \frac{\beta_2 S_1'^2}{\mu_2}\sigma_3 I_2 \mathrm{d}B_3(t)\\
&+ \int_A \left(\frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_1' + \frac{\beta_2 S_1'^2}{\mu_2} D_3(y) I_2\right)\tilde{N}(\mathrm{d}t, \mathrm{d}y),
\end{aligned}$$

where

$$
\begin{aligned}
LV_6 \le & -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S - S'_1)^2 - \frac{\beta_1 I_1}{1+\alpha_1 I_1}(S - S'_1)^2 - \frac{\beta_2 I_2}{1+\alpha_2 I_2}(S - S'_1)^2 \\
& - \beta_1 S'_1 (S - S'_1)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1'}{1+\alpha_1 I_1'}\right) - \frac{\beta_2 S I_2}{1+\alpha_2 I_2} S'_1 + \beta_2 S_1'^2 I_2 + \left(\int_A D_1^2(y)\nu(\mathrm{d}y)\right. \\
& \left. + \sigma_1^2\right) S_1'^2 + \frac{\beta_2 S_1'^2}{\mu_2}\left(e^{-\mu_2\tau_2}\frac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - \mu_2 I_2\right) + \frac{\beta_2 S_1'^2}{\mu_2} e^{-\mu_2\tau_2}\frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 I_2(t)} \\
& - \frac{\beta_2 S_1'^2}{\mu_2} e^{-\mu_2\tau_2}\frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 I_2(t)} + \int_A \left(\frac{\beta_2 S_1'^2}{\mu_2}(I_2 + D_3(y) I_2) - \frac{\beta_2 S_1'^2}{\mu_2} I_2 - \frac{\beta_2 S_1'^2}{\mu_2} D_3(y) I_2\right)\nu(\mathrm{d}y) \\
\le & -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S - S'_1)^2 + \left(\frac{\beta_2 S'_1}{\mu_2} e^{-\mu_2\tau_2} - 1\right)\frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 I_2(t)} \\
& - \beta_1 S'_1 (S - S'_1)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1'}{1+\alpha_1 I_1'}\right) + \left(\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\right) S_1'^2.
\end{aligned}
$$

Since $R_2 < 1$, we know that $\frac{\beta_2 S'_1}{\mu_2} e^{-\mu_2\tau_2} < 1$. Thus,

$$
\begin{aligned}
LV_6 \le & -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S - S'_1)^2 - \beta_1 S'_1 (S - S'_1)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1'}{1+\alpha_1 I_1'}\right) \\
& + \left(\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\right) S_1'^2.
\end{aligned}
$$

Construct the Lyapunov function $V_7 : \mathbb{R}_+^3 \to \mathbb{R}_+$

$$
V_7(S, I_1, I_2) = \frac{\beta_1 S'_1 I_1'}{\mu_0\left(1+\alpha_1 I_1'\right)} V_2 + \frac{e^{\mu_1\tau_1}}{\mu_0}\left(\mu_0 S'_1 + \frac{\beta_1 S'_1 I_1'}{1+\alpha_1 I_1'}\right) V_4 + V_6.
$$

Then

$$
\begin{aligned}
\mathrm{d}V_7 = & \frac{\beta_1 S'_1 I_1'}{\mu_0(1+\alpha_1 I_1'})\left[LV_2 \mathrm{d}t + \sigma_1(S - S'_1)\mathrm{d}B_1(t) + \frac{\beta_2 S'_1}{\mu_2}\sigma_3 I_2 \mathrm{d}B_3(t) + \int_A \frac{\beta_2 S'_1}{\mu_2} D_3(y) I_2 \tilde{N}(\mathrm{d}t, \mathrm{d}y)\right. \\
& \left. + \int_A [D_1(y) S - S'_1 \ln(1 + D_1(y))]\, \tilde{N}(\mathrm{d}t, \mathrm{d}y)\right] + LV_6 \mathrm{d}t + \frac{e^{\mu_1\tau_1}}{\mu_0}\left(\mu_0 S'_1 + \frac{\beta_1 S'_1 I_1'}{1+\alpha_1 I_1'}\right)\{\mathrm{d}V_3 \\
& + \left[e^{-\mu_1\tau_1}\frac{\beta_1 S'_1 I_1'}{1+\alpha_1 I_1'}\left(\frac{I_1(t+\tau_1)}{I_1'} - \ln\frac{I_1(t+\tau_1)}{I_1'}\right) - e^{-\mu_1\tau_1}\frac{\beta_1 S'_1 I_1'}{1+\alpha_1 I_1'}\left(\frac{I_1(t)}{I_1'} - \ln\frac{I_1(t)}{I_1'}\right)\right]\Bigg\}\,\mathrm{d}t \\
& + (S - S'_1)\sigma_1 S\, \mathrm{d}B_1(t) + \frac{\beta_2 S_1'^2}{\mu_2}\sigma_3 I_2 \mathrm{d}B_3(t) + \int_A \left(\frac{1}{2} D_1^2(y) S^2 + D_1(y) S^2 - D_1(y) S S'_1\right. \\
& \left. + \frac{\beta_2 S_1'^2}{\mu_2} D_3(y) I_2\right)\tilde{N}(\mathrm{d}t, \mathrm{d}y) \\
= & LV_7 \mathrm{d}t + \left[\frac{\beta_1 S'_1 I_1'}{\mu_0(1+\alpha_1 I_1'})\sigma_1(S - S'_1) + \sigma_1 S(S - S'_1)\right]\mathrm{d}B_1(t) + \left(\frac{\beta_1 I_1'}{\mu_0(1+\alpha_1 I_1'}) + 1\right)\cdot
\end{aligned}
$$

$$\frac{\beta_2 S_1'^2}{\mu_2}\sigma_3 I_2 \mathrm{d}B_3(t) + \left(e^{\mu_1\tau_1}S_1' + \frac{e^{\mu_1\tau_1}\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\right)\sigma_2(I_1(t+\tau_1) - I_1')\mathrm{d}B_2(t)$$
$$+ \int_A \left\{\frac{\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\left[D_1(y)S - S_1'\ln(1+D_1(y)) + \frac{\beta_2 S_1'^2}{\mu_2}D_3(y)I_2\right] + \frac{1}{2}D_1^2(y)S^2\right.$$
$$+\left(e^{\mu_1\tau_1}S_1' + \frac{e^{\mu_1\tau_1}}{\mu_0}\frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'}\right)\left[D_2(y)I_1(t+\tau_1) - I_1'\ln(1+D_2(y))\right] + D_1(y)S^2$$
$$\left. -D_1(y)SS_1' + \frac{\beta_2 S_1'^2}{\mu_2}D_3(y)I_2\right\}\tilde{N}(\mathrm{d}t,\mathrm{d}y), \tag{3.9}$$

where

$$\begin{aligned}
LV_7 \le& \frac{\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\left[-\left(\mu_0 + \frac{\beta_1 I_1'}{1+\alpha_1 I_1'(t)}\right)\frac{(S-S_1')^2}{S} - \beta_1(S-S_1')\left(\frac{I_1}{1+\alpha_1 I_1(t)} - \frac{I_1'}{1+\alpha_1 I_1'}\right)\right.\\
&\left.+\frac{1}{2}\sigma_1^2 S_1' + \int_A [D_1(y)S' - S_1'\ln(1+D_1(y))]\,\nu(\mathrm{d}y)\right] + \left(e^{\mu_1\tau_1}S_1' + \frac{e^{\mu_1\tau_1}\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\right)\\
&\left[e^{-\mu_1\tau_1}\beta_1(S-S_1')\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1'}{1+\alpha_1 I_1'}\right) + e^{-\mu_1\tau_1}\frac{\beta_1 I_1'}{1+\alpha_1 I_1'}\cdot\frac{(S-S_1')^2}{S} + \frac{1}{2}\sigma_2^2 I_1'\right.\\
&\left.+\int_A\left[D_2(y)I_1' - I_1'\ln(1+D_2(y))\right]\nu(\mathrm{d}y)\right] - \left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S_1')^2\\
&-\beta_1 S_1'(S-S_1')\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1'}{1+\alpha_1 I_1'}\right) + \left(\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\right)S_1'^2\\
=& -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S_1')^2 + \left[\left(\frac{\beta_1 I_1'}{2\mu_0(1+\alpha_1 I_1')}) + 1\right)\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\right]S_1'^2\\
&+\frac{e^{\mu_1\tau_1}}{2\mu_0}\left(\mu_0 S_1' + \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'}\right)\sigma_2^2 I_1' + \frac{\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\int_A [D_1(y)S_1' - S_1'\ln(1+D_1(y))]\,\nu(\mathrm{d}y)\\
&+\left(e^{\mu_1\tau_1}S_1' + \frac{e^{\mu_1\tau_1}\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\right)\int_A\left[D_2(y)I_1' - I_1'\ln(1+D_2(y))\right]\nu(\mathrm{d}y)\\
=& -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S_1')^2 + M_2,
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
M_2 =& \left\{\left(\frac{\beta_1 I_1'}{2\mu_0(1+\alpha_1 I_1')}) + 1\right)\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\right\}S_1'^2 + \frac{e^{\mu_1\tau_1}}{2\mu_0}\left(\mu_0 S_1' + \frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'}\right)\sigma_2^2 I_1'\\
&+\left(e^{\mu_1\tau_1}S_1' + \frac{e^{\mu_1\tau_1}\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\right)\int_A\left[D_2(y)I_1' - I_1'\ln(1+D_2(y))\right]\nu(\mathrm{d}y)\\
&+\frac{\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\int_A [D_1(y)S_1' - S_1'\ln(1+D_1(y))]\,\nu(\mathrm{d}y).
\end{aligned}$$

Integrate from 0 to t, then

$$\int_0^t \mathrm{d}V_7 = \int_0^t LV_7\mathrm{d}\mu + \int_0^t\left[\frac{\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\sigma_1(S-S_1') + \sigma_1 S(S-S_1')\right]\mathrm{d}B_1(t)$$

$$
\begin{aligned}
&+\int_0^t\int_A\left\{\frac{\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\left[D_1(y)S-S_1'\ln(1+D_1(y))+\frac{\beta_2 S_1'^2}{\mu_2}D_3(y)I_2\right]\right.\\
&+\left(e^{\mu_1\tau_1}S_1'+\frac{e^{\mu_1\tau_1}}{\mu_0}\frac{\beta_1 S_1' I_1'}{1+\alpha_1 I_1'}\right)\left[D_2(y)I_1(t+\tau_1)-I_1'\ln(1+D_2(y))\right]\\
&\left.+\frac{1}{2}D_1^2(y)S^2+D_1(y)S^2-D_1(y)SS_1'+\frac{\beta_2 S_1'^2}{\mu_2}D_3(y)I_2\right\}\tilde{N}(\mathrm{d}t,\mathrm{d}y)\\
&+\int_0^t\left(e^{\mu_1\tau_1}S_1'+\frac{e^{\mu_1\tau_1}\beta_1 S_1' I_1'}{\mu_0(1+\alpha_1 I_1')})\right)\sigma_2(I_1(t+\tau_1)-I_1')\mathrm{d}B_2(t)\\
&+\int_0^t\left(\frac{\beta_1 I_1'}{\mu_0(1+\alpha_1 I_1')})+1\right)\frac{\beta_2 S_1'^2}{\mu_2}\sigma_3 I_2\mathrm{d}B_3(t).
\end{aligned}
\tag{3.11}
$$

Taking the expectation, we have

$$
EV_7(S(t),I_1(t),I_2(t))\le EV_7(S(0),I_1(0),I_2(0))+E\int_0^t LV_7(S(\mu),I_1(\mu),I_2(\mu))\mathrm{d}\mu.
$$

From (3.10), we have

$$
0\le E\int_0^t\left[-\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S_1')^2\right]\mathrm{d}\mu+EV_7(S(0),I_1(0),I_2(0))+M_2 t.
$$

Hence,

$$
\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[(S-S_1')^2\right]\mathrm{d}\mu\le\frac{1}{\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)}M_2.
$$

Define the function

$$
V_8(S,I_1,I_2)=\frac{1}{2}\left[(S-S_1')+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1')+e^{\mu_2\tau_2}I_2(t+\tau_2)\right]^2.
$$

We get

$$
\begin{aligned}
\mathrm{d}V_8=&LV_8\mathrm{d}t+\left[(S-S_1')+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1')+e^{\mu_2\tau_2}I_2(t+\tau_2)\right]\cdot[\sigma_1 S\,\mathrm{d}B_1(t)\\
&+e^{\mu_1\tau_1}\sigma_2 I_1\mathrm{d}B_2(t)+e^{\mu_2\tau_2}\sigma_3 I_2\mathrm{d}B_3(t)]+\int_A\left\{\frac{1}{2}\left[D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)\right.\right.\\
&+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)]^2+\left[(S-S_1')+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1')+e^{\mu_2\tau_2}I_2(t+\tau_2)\right]\cdot\\
&[D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)]\}\tilde{N}(\mathrm{d}t,\mathrm{d}y),
\end{aligned}
$$

where

$$
\begin{aligned}
LV_8=&\left[(S-S_1')+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1')+e^{\mu_2\tau_2}I_2(t+\tau_2)\right]\cdot\left\{\Lambda-\mu_0 S-\frac{\beta_1 SI_1}{1+\alpha_1 I_1}-\frac{\beta_2 SI_2}{1+\alpha_2 I_2}\right.\\
&+e^{\mu_1\tau_1}\left[e^{-\mu_1\tau_1}\frac{\beta_1 S(t+\tau_1-\tau_1)I_1(t+\tau_1-\tau_1)}{1+\alpha_1 I_1(t+\tau_1-\tau_1)}-\mu_1 I_1(t+\tau_1)\right]+e^{\mu_2\tau_2}\left[-\mu_2 I_2(t+\tau_2)+\right.\\
&\left.\left.e^{-\mu_2\tau_2}\frac{\beta_2 S(t+\tau_2-\tau_2)I_2(t+\tau_2-\tau_2)}{1+\alpha_2 I_2(t+\tau_2-\tau_2)}\right]\right\}+\frac{1}{2}\sigma_1^2 S^2+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2 I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2 I_2^2(t+\tau_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_A \left\{\frac{1}{2}\left[(S+D_1(y)S-S_1')+e^{\mu_1\tau_1}(I_1(t+\tau_1)+D_2(y)I_1(t+\tau_1)-I_1')+e^{\mu_2\tau_2}(I_2(t+\tau_2)\right.\right.\\
&\left.+D_3(y)I_2(t+\tau_2))\right]^2-\frac{1}{2}\left[(S-S_1')+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1')+e^{\mu_2\tau_2}I_2(t+\tau_2)\right]^2-\left[(S-S_1')\right.\\
&\left.\left.+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1')+e^{\mu_2\tau_2}I_2(t+\tau_2)\right]\left[D_1(y)S+D_2(y)I_1(t+\tau_1)+D_3(y)I_2(t+\tau_2)\right]\right\}\nu(\mathrm{d}y)\\
=&\left[(S-S_1')+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1')+e^{\mu_2\tau_2}I_2(t+\tau_2)\right]\cdot\left[-e^{\mu_1\tau_1}\mu_1I_1(t+\tau_1)-e^{\mu_2\tau_2}\mu_2I_2(t+\tau_2)\right.\\
&\left.+\Lambda-\mu_0S\right]+\frac{1}{2}\sigma_1^2S^2+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2I_2^2(t+\tau_2)\\
&+\int_A\frac{1}{2}\left[D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)\right]^2\nu(\mathrm{d}y).
\end{aligned}
$$

From (3.7) and (3.8), we know that

$$\Lambda=\mu_0S_1'+e^{\mu_1\tau_1}\mu_1I_1'.$$

Then

$$
\begin{aligned}
LV_8=&\left[(S-S_1')+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1')+e^{\mu_2\tau_2}I_2(t+\tau_2)\right]\cdot\left[\mu_0S_1'+e^{\mu_1\tau_1}I_1'-e^{\mu_1\tau_1}\mu_1I_1(t+\tau_1)\right.\\
&\left.-\mu_0S-e^{\mu_2\tau_2}\mu_2I_2(t+\tau_2)\right]+\frac{1}{2}\sigma_1^2S^2+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2I_2^2(t+\tau_2)\\
&+\int_A\frac{1}{2}\left[D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)\right]^2\nu(\mathrm{d}y)\\
=&-\mu_0(S-S_1')^2-e^{\mu_1\tau_1}\mu_1(I_1(t+\tau_1)-I_1')(S-S_1')-e^{\mu_2\tau_2}\mu_2(I_2(t+\tau_2)-I_2')(S-S_1')\\
&-e^{\mu_1\tau_1}\mu_0(I_1(t+\tau_1)-I_1')(S-S_1')-e^{2\mu_1\tau_1}\mu_1(I_1(t+\tau_1)-I_1')^2-e^{\mu_1\tau_1+\mu_2\tau_2}\mu_2(I_1(t+\tau_1)\\
&-I_1')\cdot I_2(t+\tau_2)-e^{\mu_2\tau_2}\mu_0I_2(t+\tau_2)(S-S_1')-e^{\mu_1\tau_1+\mu_2\tau_2}\mu_1(I_1(t+\tau_1)-I_1')I_2(t+\tau_2)\\
&-e^{2\mu_2\tau_2}\mu_2I_2^2(t+\tau_2)+\frac{1}{2}\sigma_1^2S^2+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2I_2^2(t+\tau_2)+\int_A\frac{1}{2}\left[D_1(y)S\right.\\
&\left.+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)\right]^2\nu(\mathrm{d}y)\\
\leq&-\mu_0(S-S_1')^2-e^{\mu_1\tau_1}(\mu_1+\mu_0)\mu_1(I_1(t+\tau_1)-I_1')(S-S_1')-e^{\mu_2\tau_2}(\mu_2+\mu_0)(I_2(t+\tau_2)-I_2')\cdot\\
&(S-S_1')-e^{2\mu_1\tau_1}\mu_1(I_1(t+\tau_1)-I_1')^2-e^{\mu_1\tau_1+\mu_2\tau_2}(\mu_1+\mu_2)(I_1(t+\tau_1)-I_1')I_2(t+\tau_2)-\\
&e^{2\mu_2\tau_2}\mu_2I_2^2(t+\tau_2)+\sigma_1^2(S-S_1')^2+\sigma_1^2S_1'^2+e^{2\mu_1\tau_1}\sigma_2^2(I_1(t+\tau_1)-I_1')^2+e^{2\mu_1\tau_1}\sigma_2^2I_1'^2\\
&+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2I_2^2(t+\tau_2)+\int_A\left\{\frac{1}{2}\left(D_1^2(y)+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)\right)^2+\frac{1}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)\right.\\
&\left.+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)\left(D_1^2(y)+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)\right)\right\}\nu(\mathrm{d}y)\\
\leq&\frac{1}{2}\left(\mu_1+\mu_2+2\sigma_1^2+6\int_AD_1^2(y)\nu(\mathrm{d}y)\right)(S-S_1')^2\\
&-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2-6\int_AD_2^2(y)\nu(\mathrm{d}y)\right)(I_1(t+\tau_1)-I_1')^2\\
&-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-\sigma_3^2-3\int_AD_3^2(y)\nu(\mathrm{d}y)\right)I_2^2(t+\tau_2)\\
&+\left(\sigma_1^2+3\int_AD_1^2(y)\nu(\mathrm{d}y)\right)S_1'^2+e^{2\mu_1\tau_1}\left(\sigma_2^2+3\int_AD_2^2(y)\nu(\mathrm{d}y)\right)I_1'^2.
\end{aligned}
$$

Define the function $V_9 : \mathbb{R}_+^3 \to \mathbb{R}_+$

$$V_9(S, I_1, I_2) = V_8 + \frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1 - \mu_0 - 2\sigma_2^2 - 6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)\int_t^{t+\tau_1}(I_1(\mu) - I_1')^2\mathrm{d}\mu$$
$$+ \frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2 - \mu_0 - \sigma_3^2 - 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\int_t^{t+\tau_2} I_2^2(\mu)\mathrm{d}\mu,$$

where

$$\begin{aligned}LV_9 \leq &\frac{1}{2}\left(\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S - S_1')^2\\
&- \frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1 - \mu_0 - 2\sigma_2^2 - 6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)(I_1(t) - I_1')^2\\
&- \frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2 - \mu_0 - \sigma_3^2 - 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)I_2^2(t)\\
&+ \left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right)S_1'^2 + e^{2\mu_1\tau_1}\left(\sigma_2^2 + 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1'^2.\end{aligned}$$

Let

$$V_{10}(S, I_1, I_2) = \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}V_7 + V_9.$$

We can get

$$\begin{aligned}\mathrm{d}V_{10} =& \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\mathrm{d}V_7 + \mathrm{d}V_9\\
=& LV_{10}\mathrm{d}t + \left\{\frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\cdot\left[\frac{\beta_1 S_1' I_1'}{\mu_0(1 + \alpha_1 I_1'}) \sigma_1(S - S_1') + \sigma_1 S(S - S_1')\right]\right.\\
&+ \left[(S - S_1') + e^{\mu_1\tau_1}(I_1(t + \tau_1) - I_1') + e^{\mu_2\tau_2}I_2(t + \tau_2)\right]\sigma_1 S\ \}\mathrm{d}B_1(t) + \left\{\left(\frac{e^{\mu_1\tau_1}\beta_1 S_1' I_1'}{\mu_0(1 + \alpha_1 I_1'})\right.\right.\\
&+ e^{\mu_1\tau_1}S_1')\sigma_2(I_1(t + \tau_1) - I_1') + \left[(S - S_1') + e^{\mu_1\tau_1}(I_1(t + \tau_1) - I_1') + e^{\mu_2\tau_2}I_2(t + \tau_2)\right]\cdot\\
&\sigma_2 e^{\mu_1\tau_1}I_1(t + \tau_1)\ \}\mathrm{d}B_2(t) + \left\{\left(\left(\frac{\beta_1 I_1'}{\mu_0(1 + \alpha_1 I_1'}) + 1\right)\frac{\beta_2 S_1'^2}{\mu_2}\sigma_3 I_2 + \left[e^{\mu_1\tau_1}(I_1(t + \tau_1) - I_1')\right.\right.\\
&+ (S - S_1') + e^{\mu_2\tau_2}I_2(t + \tau_2)]\sigma_3 e^{\mu_2\tau_2}I_2(t + \tau_2)\ \}\mathrm{d}B_3(t) + \int_A\left\{\frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\cdot\right.\\
&\left[\frac{\beta_1 S_1' I_1'}{\mu_0(1 + \alpha_1 I_1'})(D_1(y)S - S_1'\ln(1 + D_1(y))) + \frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_1'\right.\\
&\left.+\frac{\beta_2 S_1'^2}{\mu_2}D_3(y)I_2 + \left(e^{\mu_1\tau_1}S_1' + \frac{e^{\mu_1\tau_1}\beta_1 S_1' I_1'}{\mu_0(1 + \alpha_1 I_1'})\right)\left(D_2(y)I_1(t + \tau_1) - I_1'\ln(1 + D_2(y))\right)\right]\end{aligned}$$

$$+\frac{1}{2}\left[D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+D_3(y)I_2(t+\tau_2)\right]^2+\Big[e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1^{'})$$
$$+(S-S'_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)\Big]\cdot\left[D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+D_3(y)I_2(t+\tau_2)\right]\Big\}\tilde{N}(\mathrm{d}t,\mathrm{d}y),\tag{3.12}$$

where

$$LV_{10}\le\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\cdot\left[-\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S'_1)^2+M_2\right]$$
$$+\frac{1}{2}\left(\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S'_1)^2-\frac{1}{2}e^{2\mu_2\tau_2}\Big(\mu_2-\mu_0-\sigma_3^2$$
$$-3\int_A D_3^2(y)\nu(\mathrm{d}y)\Big)I_2^2(t)-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2-6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)(I_1(t)-I_1^{'})^2$$
$$+\left(\sigma_1^2+3\int_A D_1^2(y)\nu(\mathrm{d}y)\right)S_1'^2+e^{2\mu_1\tau_1}\left(\sigma_2^2+3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^{'2}$$
$$=-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2+6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)(I_1(t)-I_1^{'})^2+\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}M_2$$
$$+\left(\sigma_1^2+3\int_A D_1^2(y)\nu(\mathrm{d}y)\right)S_1'^2+e^{2\mu_1\tau_1}\left(\sigma_2^2+3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^{'2}$$
$$-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-\sigma_3^2-3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)I_2^2(t).\tag{3.13}$$

Integrate from 0 to t, then

$$\int_0^t\mathrm{d}V_{10}=\int_0^t LV_{10}\mathrm{d}\mu+\int_0^t\left\{\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\cdot\left[\frac{\beta_1S'_1I_1^{'}}{\mu_0(1+\alpha_1I_1^{'}}\right)\sigma_1(S-S'_1)\right.$$
$$+\sigma_1S(S-S'_1)\Big]+\Big[(S-S'_1)+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1^{'})+e^{\mu_2\tau_2}I_2(t+\tau_2)\Big]\sigma_1S\Big\}\mathrm{d}B_1(t)$$
$$+\int_0^t\left\{\left(\frac{e^{\mu_1\tau_1}\beta_1S'_1I_1^{'}}{\mu_0(1+\alpha_1I_1^{'}}\right)+e^{\mu_1\tau_1}S'_1\right)\sigma_2(I_1(t+\tau_1)-I_1^{'})+\Big[(S-S'_1)+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1^{'})$$
$$+e^{\mu_2\tau_2}I_2(t+\tau_2)\Big]\cdot\sigma_2e^{\mu_1\tau_1}I_1(t+\tau_1)\Big\}\mathrm{d}B_2(t)+\int_0^t\left\{\left(\frac{\beta_1I_1^{'}}{\mu_0(1+\alpha_1I_1^{'}}\right)+1\right)\frac{\beta_2S_1'^2}{\mu_2}\sigma_3I_2$$
$$+\Big[e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1^{'})+(S-S'_1)+e^{\mu_2\tau_2}I_2(t+\tau_2)\Big]\sigma_3e^{\mu_2\tau_2}I_2(t+\tau_2)\Big\}\mathrm{d}B_3(t)$$
$$+\int_0^t\int_A\left\{\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\left[\frac{\beta_1S'_1I_1^{'}}{\mu_0(1+\alpha_1I_1^{'}}\right)(D_1(y)S-S'_1\ln(1+D_1(y))\right.$$
$$\left.+\frac{\beta_2S_1'^2}{\mu_2}D_3(y)I_2\right)+\frac{1}{2}D_1^2(y)S^2+D_1(y)S^2-D_1(y)SS'_1+\left(e^{\mu_1\tau_1}S'_1+\frac{e^{\mu_1\tau_1}\beta_1S'_1I_1^{'}}{\mu_0(1+\alpha_1I_1^{'}}\right)\right)\cdot$$
$$\Big(D_2(y)I_1(t+\tau_1)-I_1^{'}\ln(1+D_2(y))\Big)\Big]+\frac{1}{2}\left[D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)\right.$$

$$+D_3(y)I_2(t+\tau_2)]^2 + \left[e^{\mu_1\tau_1}(I_1(t+\tau_1) - I_1') + (S - S_1') + e^{\mu_2\tau_2}I_2(t+\tau_2)\right]\cdot$$
$$[D_1(y)S + e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1) + D_3(y)I_2(t+\tau_2)]\}\,\tilde{N}(\mathrm{d}t,\mathrm{d}y). \tag{3.14}$$

Taking the expectation, we have

$$EV_{10}(S(t), I_1(t), I_2(t)) \leq EV_{10}(S(0), I_1(0), I_2(0)) + E\int_0^t LV_{10}(S(\mu), I_1(\mu), I_2(\mu))\mathrm{d}\mu.$$

Then

$$\begin{aligned}
0 \leq & E\int_0^t \Big[-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1 - \mu_0 - 2\sigma_2^2 - 6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)\left(I_1(\mu) - I_1'\right)^2 \\
& - \frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2 - \mu_0 - \sigma_3^2 - 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)I_2^2(\mu)\Big]\mathrm{d}\mu + EV_{10}(S(0), I_1(0), I_2(0)) \\
& + \left\{\left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right){S_1'}^2 + e^{2\mu_1\tau_1}\left(\sigma_2^2 + 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right){I_1'}^2 \right. \\
& \left. + \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}M_2\right\}t.
\end{aligned}$$

We get

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}E\int_0^t \left[(I_1(\mu) - I_1')^2 + I_2^2(\mu)\right]\mathrm{d}\mu \leq \frac{2}{H_1} & \left\{\left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right){S_1'}^2 \right. \\
& + e^{2\mu_1\tau_1}\left(\sigma_2^2 + 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right){I_1'}^2 \\
& \left. + \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}M_2\right\},
\end{aligned}$$

where
$H_1 = \min\left\{e^{2\mu_1\tau_1}\left(\mu_1 - \mu_0 - 2\sigma_2^2 - 6\int_A D_2^2(y)\nu(\mathrm{d}y)\right), e^{2\mu_2\tau_2}\left(\mu_2 - \mu_0 - \sigma_3^2 - 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\right\}$. □

Remark 2. Theorem 3.2 shows that when $R_1 > 1$ and $R_2 < 1$ and certain conditions are satisfied, the solution to the stochastic delayed system (1.3) will oscillate around the equilibrium point E_1. The intensity of oscillation is related to the values of σ_i and $D_i(y)(i = 1, 2, 3)$. The smaller the noise intensity, the closer the solution of the stochastic delayed system (1.3) is to the equilibrium point of E_1. This means that Disease I will prevail in the population, while Disease II will tend to die out.

3.3. Asymptotic behavior of solutions near equilibrium E_2

When $R_1 < 1$ and $R_2 > 1$, the deterministic SI_1I_2 model possesses an equilibrium point $E_2(S_2', 0, I_2')$ that is globally asymptotically stable. We will investigate the asymptotic behavior of solutions of the stochastic delayed model (1.3) near the equilibrium point E_2.

Theorem 3.3. Assume that conditions (H1) and (H2) hold. If $R_1 < 1$ and $R_2 > 1$ and the following conditions are satisfied:

$$\frac{\beta_1 S_2'}{\mu_1} e^{-\mu_1\tau_1} \leq 1,$$
$$\mu_0 > \sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y),$$
$$\mu_1 - \mu_0 > \sigma_2^2 + 3\int_A D_2^2(y)\nu(\mathrm{d}y),$$
$$\mu_2 - \mu_0 > 2\sigma_3^2 + 6\int_A D_3^2(y)\nu(\mathrm{d}y),$$

then for any given initial value (1.4), the solution $(S(t), I_1(t), I_2(t))$ of system (1.3) has the following properties:

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t (S(\mu) - S_2')^2 \mathrm{d}\mu \leq \frac{1}{\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)} M_3,$$

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[I_1^2(\mu) + (I_2(\mu) - I_2')^2\right] \mathrm{d}\mu \\ &\leq \frac{2}{H_2}\left\{\left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right) S_2' + e^{2\mu_2\tau_2}\left(\sigma_3^2 + 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right) I_2'^2 \right.\\ &\left. + \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} M_3\right\},\end{aligned}$$

where

$$\begin{aligned}M_3 = &\left\{\left(\frac{\beta_2 I_2'}{2\mu_0(1+\alpha_1 I_2')} + 1\right)\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\right\} S_2'^2 + \frac{e^{\mu_2\tau_2}}{2\mu_0}\left(\mu_0 S_2' + \frac{\beta_2 S_2' I_2'}{1+\alpha_1 I_2'}\right)\sigma_3^2 I_2' \\ &+ \frac{\beta_2 S_2' I_2'}{\mu_0(1+\alpha_2 I_2')}\int_A \left[D_3(y) I_2' - I_2' \ln[1 + D_3(y)]\right]\nu(\mathrm{d}y) \\ &+ \left(e^{\mu_2\tau_2} S_2' + \frac{e^{\mu_2\tau_2}\beta_2 S_2' I_2'}{\mu_0(1+\alpha_2 I_2')}\right)\int_A \left[D_3(y) I_2' - I_2' \ln(1 + D_3(y))\right]\nu(\mathrm{d}y).\end{aligned}$$

Proof. When $R_1 < 1$ and $R_2 > 1$, the equilibrium $E_2(S_2', 0, I_2')$ of the deterministic SI_1I_2 model satisfies the following equations:

$$\Lambda - \mu_0 S_2' - \frac{\beta_2 S_2' I_2'}{1+\alpha_2 I_2'} = 0, \tag{3.15}$$

$$e^{-\mu_2\tau_2}\frac{\beta_2 S_2' I_2'}{1+\alpha_2 I_2'} - \mu_2 I_2' = 0. \tag{3.16}$$

First, define the following functions:

$$V_1(S) = S - S_2' - S_2' \ln\frac{S}{S_2'},$$

$$
\begin{aligned}
V_2(S, I_1) &= V_1 + \frac{\beta_1 S_2^{'}}{\mu_1} I_1 + \frac{\beta_1 S_2^{'}}{\mu_1} \int_{t-\tau_1}^{t} e^{-\mu_1\tau_1} \frac{\beta_1 S_1(\mu) I_1(\mu)}{1+\alpha_1 I_1(\mu)} \mathrm{d}\mu \\
&= S - S_2^{'} - S_2^{'} \ln \frac{S}{S_2^{'}} + \frac{\beta_2 S_1^{'}}{\mu_2} + \frac{\beta_2 S_1^{'}}{\mu_2} \int_{t-\tau_1}^{t} e^{-\mu_1\tau_1} \frac{\beta_1 S_1(\mu) I_1(\mu)}{1+\alpha_1 I_1(\mu)} \mathrm{d}\mu, \\
V_3(I_2) &= I_2(t+\tau_2) - I_2^{'} - I_2^{'} \ln \frac{I_2(t+\tau_2)}{I_2^{'}}, \\
V_4(I_2) &= V_3 + e^{-\mu_2\tau_2} \frac{\beta_2 S_2^{'} I_2^{'}}{1+\alpha_2 I_2^{'}} \int_{t}^{t+\tau_2} \left[\frac{I_2(\mu)}{I_2^{'}} - \ln \frac{I_2(\mu)}{I_2^{'}} \right] \mathrm{d}\mu \\
&= I_2(t+\tau_2) - I_2^{'} - I_2^{'} \ln \frac{I_2(t+\tau_2)}{I_2^{'}} + e^{-\mu_2\tau_2} \frac{\beta_2 S_2^{'} I_2^{'}}{1+\alpha_2 I_2^{'}} \int_{t}^{t+\tau_2} \left[\frac{I_2(\mu)}{I_2^{'}} - \ln \frac{I_2(\mu)}{I_2^{'}} \right] \mathrm{d}\mu, \\
V_5(S) &= \frac{1}{2} \left(S - S_2^{'} \right)^2, \\
V_6(S, I_1) &= V_5 + \frac{\beta_1 S_2^{'2}}{\mu_1} I_1 + \frac{\beta_1 S_2^{'2}}{\mu_1} e^{-\mu_1\tau_1} \int_{t-\tau_1}^{t} \frac{\beta_1 S(\mu) I_1(\mu)}{1+\alpha_1 I_1(\mu)} \mathrm{d}\mu \\
&= \frac{1}{2} \left(S - S_2^{'} \right)^2 + \frac{\beta_1 S_2^{'2}}{\mu_1} I_1 + \frac{\beta_1 S_2^{'2}}{\mu_1} e^{-\mu_1\tau_1} \int_{t-\tau_1}^{t} \frac{\beta_1 S(\mu) I_1(\mu)}{1+\alpha_1 I_1(\mu)} \mathrm{d}\mu, \\
V_7(S, I_1, I_2) &= \frac{\beta_2 S_2^{'} I_2^{'}}{\mu_0 \left(1+\alpha_2 I_2^{'} \right)} V_2 + \frac{e^{\mu_2\tau_2}}{\mu_0} \left(\mu_0 S_2^{'} + \frac{\beta_2 S_2^{'} I_2^{'}}{1+\alpha_2 I_2^{'}} \right) V_4 + V_6, \\
V_8(S, I_1, I_2) &= \frac{1}{2} \left[\left(S - S_2^{'} \right) + e^{\mu_1\tau_1} I_1(t+\tau_1) + e^{\mu_2\tau_2} \left(I_2(t+\tau_2) - I_2^{'} \right) \right]^2, \\
V_9(S, I_1, I_2) &= V_8 + \frac{1}{2} e^{2\mu_2\tau_2} \left(\mu_2 - \mu_0 - 2\sigma_3^2 + 6 \int_A D_3^2(y)\nu(\mathrm{d}y) \right) \int_{t}^{t+\tau_2} \left(I_2(\mu) - I_2^{'} \right)^2 \mathrm{d}\mu \\
&\quad + \frac{1}{2} e^{2\mu_1\tau_1} \left(\mu_1 - \mu_0 - \sigma_2^2 - 3 \int_A D_2^2(y)\nu(\mathrm{d}y) \right) \int_{t}^{t+\tau_1} I_1^2(\mu) \mathrm{d}\mu, \\
V_{10}(S, I_1, I_2) &= \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6 \int_A D_1^2(y)\nu(\mathrm{d}y)}{2 \left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y) \right)} V_7 + V_9.
\end{aligned}
$$

From Itô's formula, we know that

$$
\begin{aligned}
\mathrm{d}V_1 &= LV_1 \mathrm{d}t + \sigma_1 (S - S_2') \mathrm{d}B_1(t) + \int_A \left[D_1(y) S - S_2' \ln(1 + D_1(y)) \right] \tilde{N}(\mathrm{d}t, \mathrm{d}y), \\
\mathrm{d}V_2 &= LV_2 \mathrm{d}t + \sigma_1 (S - S_2') \mathrm{d}B_1(t) + \frac{\beta_1 S_2'}{\mu_1} \sigma_2 I_1 \mathrm{d}B_2(t) + \int_A \frac{\beta_1 S_2'}{\mu_1} D_2(y) I_1 \tilde{N}(\mathrm{d}t, \mathrm{d}y) \\
&\quad + \int_A \left[D_1(y) S - S_2' \ln(1 + D_1(y)) \right] \tilde{N}(\mathrm{d}t, \mathrm{d}y), \\
\mathrm{d}V_3 &= LV_3 \mathrm{d}t + \sigma_3 (I_2(t+\tau_2) - I_2') \mathrm{d}B_3(t) + \int_A \left[D_3(y) I_2(t+\tau_2) - I_2' \ln(1 + D_3(y)) \right] \tilde{N}(\mathrm{d}t, \mathrm{d}y), \\
\mathrm{d}V_4 &= LV_4 \mathrm{d}t + \sigma_3 (I_2(t+\tau_2) - I_2') \mathrm{d}B_3(t) + \int_A \left[D_3(y) I_2(t+\tau_2) - I_2' \ln(1 + D_3(y)) \right] \tilde{N}(\mathrm{d}t, \mathrm{d}y),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}V_5 &= LV_5\mathrm{d}t + (S - S_2')\sigma_1 S\,\mathrm{d}B_1(t) + \int_A \left[\frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_2'\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),\\
\mathrm{d}V_6 &= LV_6\mathrm{d}t + (S - S_2')\sigma_1 S\,\mathrm{d}B_1(t) + \frac{\beta_1 S_2'^2}{\mu_1}\sigma_2 I_1\mathrm{d}B_2(t)\\
&\quad + \int_A \left[\frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_2' + \frac{\beta_2 S_2'^2}{\mu_1}D_2(y)I_1\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),\\
\mathrm{d}V_7 &= LV_7\mathrm{d}t + \left[\frac{\beta_2 S_2' I_2'}{\mu_0\left(1+\alpha_2 I_2'\right)}\sigma_1\left(S - S_2'\right) + \sigma_1 S\left(S - S_2'\right)\right]\mathrm{d}B_1(t) + \left(\frac{\beta_2 I_2'}{\mu_0\left(1+\alpha_2 I_2'\right)} + 1\right)\cdot\\
&\quad \frac{\beta_1 {S_2'}^2}{\mu_1}\sigma_2 I_1\mathrm{d}B_2(t) + \left(e^{\mu_2\tau_2}S_2' + \frac{e^{\mu_2\tau_2}}{\mu_0}\frac{\beta_2 S_2' I_2'}{1+\alpha_2 I_2'}\right)\sigma_3\left(I_2(t+\tau_2) - I_2'\right)\mathrm{d}B_3(t)\\
&\quad + \int_A\left\{\frac{\beta_2 S_2' I_2'}{\mu_0\left(1+\alpha_2 I_2'\right)}\left(D_1(y)S - S_2'\ln(1+D_1(y)) + \frac{\beta_1 S_2'}{\mu_1}D_2(y)I_1\right)\right.\\
&\quad + \left(e^{\mu_2\tau_2}S_2' + \frac{e^{\mu_2\tau_2}}{\mu_0}\frac{\beta_2 S_2' I_2'}{1+\alpha_2 I_2'}\right)\left[D_3(y)I_2(t+\tau_2) - I_2'\ln(1+D_3(y))\right]\\
&\quad \left. + \frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_2' + \frac{\beta_1 {S_2'}^2}{\mu_1}D_2(y)I_1\right\}\tilde{N}(\mathrm{d}t,\mathrm{d}y),
\end{aligned}
\tag{3.17}
$$

where

$$
LV_7 = -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S - S_2')^2 + M_3. \tag{3.18}
$$

Integrate formula (3.17) from 0 to t, then

$$
\begin{aligned}
\int_0^t \mathrm{d}V_7 &= \int_0^t LV_7\mathrm{d}\mu + \int_0^t\left[\frac{\beta_2 S_2' I_2'}{\mu_0\left(1+\alpha_2 I_2'\right)}\sigma_1\left(S - S_2'\right) + \sigma_1 S\left(S - S_2'\right)\right]\mathrm{d}B_1(t)\\
&\quad + \int_0^t\left(e^{\mu_2\tau_2}S_2' + \frac{e^{\mu_2\tau_2}}{\mu_0}\frac{\beta_2 S_2' I_2'}{1+\alpha_2 I_2'}\right)\sigma_3\left(I_2(t+\tau_2) - I_2'\right)\mathrm{d}B_3(t)\\
&\quad + \int_0^t\left(\frac{\beta_2 I_2'}{\mu_0\left(1+\alpha_2 I_2'\right)} + 1\right)\frac{\beta_1 {S_2'}^2}{\mu_1}\sigma_2 I_1\mathrm{d}B_2(t)\\
&\quad + \int_0^t\left\{\int_A\left\{\frac{\beta_2 S_2' I_2'}{\mu_0\left(1+\alpha_2 I_2'\right)}\left(D_1(y)S - S_2'\ln(1+D_1(y)) + \frac{\beta_1 S_2'}{\mu_1}D_2(y)I_1\right)\right.\right.\\
&\quad + \left(e^{\mu_2\tau_2}S_2' + \frac{e^{\mu_2\tau_2}}{\mu_0}\frac{\beta_2 S_2' I_2'}{1+\alpha_2 I_2'}\right)\left[D_3(y)I_2(t+\tau_2) - I_2'\ln(1+D_3(y))\right]\\
&\quad \left.\left. + \frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_2' + \frac{\beta_1 {S_2'}^2}{\mu_1}D_2(y)I_1\right\}\tilde{N}(\mathrm{d}t,\mathrm{d}y)\right\}.
\end{aligned}
\tag{3.19}
$$

Taking the expectation, we have

$$
EV_7(S(t), I_1(t), I_2(t)) \le EV_7(S(0), I_1(0), I_2(0)) + E\int_0^t LV_7(S(\mu), I_1(\mu), I_2(\mu))\mathrm{d}\mu.
$$

Then

$$0 \leq E \int_0^t \Big[-\Big(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\Big)\Big(S - S_2^{'}\Big)^2 \Big]\mathrm{d}\mu + EV_7(S(0), I_1(0), I_2(0)) + M_3 t.$$

Hence,

$$\limsup_{t\to\infty} \frac{1}{t} E \int_0^t \Big[\Big(S - S_2^{'}\Big)^2 \Big]\mathrm{d}\mu \leq \frac{1}{\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)} M_3,$$

where

$$\begin{aligned} M_3 = & \Big\{\Big(\frac{\beta_2 I_2'}{2\mu_0(1+\alpha_1 I_2')} + 1\Big)\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\Big\} S_2'^2 + \frac{e^{\mu_2\tau_2}}{2\mu_0}\Big(\mu_0 S_2' + \frac{\beta_2 S_2' I_2'}{1+\alpha_1 I_2'}\Big)\sigma_3^2 I_2' \\ & + \frac{\beta_2 S_2' I_2'}{\mu_0(1+\alpha_2 I_2')}\int_A [D_3(y) I_2' - I_2' \ln[1 + D_3(y)]]\, \nu(\mathrm{d}y) \\ & + \Big(e^{\mu_2\tau_2} S_2' + \frac{e^{\mu_2\tau_2}\beta_2 S_2' I_2'}{\mu_0(1+\alpha_2 I_2')}\Big)\int_A [D_3(y) I_2' - I_2' \ln(1 + D_3(y))]\, \nu(\mathrm{d}y). \end{aligned}$$

Using Itô's formula, we obtain

$$\begin{aligned} \mathrm{d}V_8 = & LV_8 \mathrm{d}t + \Big[\Big(S - S_2^{'}\Big) + e^{\mu_2\tau_2}\Big(I_2(t+\tau_2) - I_2^{'}\Big) + e^{\mu_1\tau_1} I_1(t+\tau_1)\Big][e^{\mu_1\tau_1}\sigma_2 I_1(t+\tau_1)\mathrm{d}B_2(t) \\ & + \sigma_1 S\,\mathrm{d}B_1(t) + e^{\mu_2\tau_2}\sigma_3 I_2(t+\tau_2)\mathrm{d}B_3(t)] + \int_A \Big\{\frac{1}{2}[D_1(y)S + e^{\mu_1\tau_1} D_2(y) I_1(t+\tau_1) \\ & + e^{\mu_2\tau_2} D_3(y) I_2(t+\tau_2)]^2 + \Big[\Big(S - S_2^{'}\Big) + e^{\mu_2\tau_2}\Big(I_2(t+\tau_2) - I_2^{'}\Big) + e^{\mu_1\tau_1} I_1(t+\tau_1)\Big] \\ & [D_1(y)S + e^{\mu_1\tau_1} D_2(y) I_1(t+\tau_1) + e^{\mu_2\tau_2} D_3(y) I_2(t+\tau_2)]\Big\}\tilde{N}(\mathrm{d}t, \mathrm{d}y), \end{aligned}$$

$$\begin{aligned} \mathrm{d}V_9 = & LV_9 \mathrm{d}t + \Big[\Big(S - S_2^{'}\Big) + e^{\mu_2\tau_2}\Big(I_2(t+\tau_2) - I_2^{'}\Big) + e^{\mu_1\tau_1} I_1(t+\tau_1)\Big][e^{\mu_1\tau_1}\sigma_2 I_1(t+\tau_1)\mathrm{d}B_2(t) \\ & + \sigma_1 S\,\mathrm{d}B_1(t) + e^{\mu_2\tau_2}\sigma_3 I_2(t+\tau_2)\mathrm{d}B_3(t)] + \int_A \Big\{\frac{1}{2}[D_1(y)S + e^{\mu_1\tau_1} D_2(y) I_1(t+\tau_1) \\ & + e^{\mu_2\tau_2} D_3(y) I_2(t+\tau_2)]^2 + \Big[\Big(S - S_2^{'}\Big) + e^{\mu_2\tau_2}\Big(I_2(t+\tau_2) - I_2^{'}\Big) + e^{\mu_1\tau_1} I_1(t+\tau_1)\Big] \\ & [D_1(y)S + e^{\mu_1\tau_1} D_2(y) I_1(t+\tau_1) + e^{\mu_2\tau_2} D_3(y) I_2(t+\tau_2)]\Big\}\tilde{N}(\mathrm{d}t, \mathrm{d}y), \end{aligned}$$

$$\begin{aligned} \mathrm{d}V_{10} = & LV_{10}\mathrm{d}t + \Big\{\frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\Big(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\Big)} \cdot \Big[\frac{\beta_2 S_2' I_2'}{\mu_0(1+\alpha_2 I_2')}\sigma_1(S - S_2') + \sigma_1 S(S - S_2')\Big] \\ & + [(S - S_2') + e^{\mu_2\tau_2}(I_2(t+\tau_2) - I_2') + e^{\mu_1\tau_1} I_1(t+\tau_1)]\,\sigma_1 S\Big\}\,\mathrm{d}B_1(t) + \Big\{\Big(\frac{e^{\mu_2\tau_2}\beta_2 S_2' I_2'}{\mu_0(1+\alpha_2 I_2')} \\ & + e^{\mu_2\tau_2} S_2'\Big)\sigma_3(I_2(t+\tau_2) - I_2') + [(S - S_2') + e^{\mu_2\tau_2}(I_2(t+\tau_2) - I_2') + e^{\mu_1\tau_1} I_1(t+\tau_1)] \cdot \\ & \sigma_3 e^{\mu_2\tau_2} I_2(t+\tau_2)\Big\}\,\mathrm{d}B_3(t) + \Big\{\Big(\frac{\beta_2 I_2'}{\mu_0(1+\alpha_2 I_2')} + 1\Big)\frac{\beta_1 S_2'^2}{\mu_1}\sigma_2 I_1 + [e^{\mu_2\tau_2}(I_2(t+\tau_2) - I_2') \\ & + (S - S_2') + e^{\mu_1\tau_1} I_1(t+\tau_1)]\,\sigma_2 e^{\mu_1\tau_1} I_1(t+\tau_1)\Big\}\,\mathrm{d}B_3(t) + \int_A \Big\{\frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\Big(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\Big)} \cdot \end{aligned}$$

$$\left[\frac{\beta_2 S_2' I_2'}{\mu_0(1+\alpha_2 I_2')}\left(D_1(y)S - S_2'\ln(1+D_1(y)) + \frac{\beta_1 S_2'}{\mu_1}D_2(y)I_1\right) + \frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_2'\right.$$
$$\left.+\frac{\beta_1 S_2'^2}{\mu_1}D_2(y)I_1 + \left(e^{\mu_2\tau_2}S_2' + \frac{e^{\mu_2\tau_2}\beta_2 S_2' I_2'}{\mu_0(1+\alpha_2 I_2')}\right)(D_3(y)I_2(t+\tau_2) - I_2'\ln(1+D_3(y)))\right]$$
$$+\frac{1}{2}\left[D_1(y)S + e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1) + D_3(y)I_2(t+\tau_2)\right]^2 + [e^{\mu_2\tau_2}(I_2(t+\tau_2) - I_2')$$
$$+(S - S_2') + e^{\mu_1\tau_1}I_1(t+\tau_1)] \cdot [D_1(y)S + e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1) + D_3(y)I_2(t+\tau_2)]\}\tilde{N}(\mathrm{d}t,\mathrm{d}y), \tag{3.20}$$

where

$$LV_{10} \le -\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1 - \mu_0 - 2\sigma_2^2 + 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^2(t) + \frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}M_3$$
$$-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2 - \mu_0 - 2\sigma_3^2 - 6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\left(I_2(t) - I_2'\right)^2$$
$$+\left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right)S_2'^2 + e^{2\mu_2\tau_2}\left(\sigma_3^2 + 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)I_2'^2. \tag{3.21}$$

Integrate from 0 to t on both sides of (3.20), then

$$\int_0^t \mathrm{d}V_{10} = \int_0^t LV_{10}\mathrm{d}\mu + \int_0^t\left\{\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\left[\frac{\beta_2 S_2' I_2'}{\mu_0\left(1+\alpha_2 I_2'\right)}\sigma_1\left(S - S_2'\right)\right.\right.$$
$$\left.\left.+\sigma_1 S\left(S - S_2'\right)\right] + \left[\left(S - S_2'\right) + e^{\mu_2\tau_2}\left(I_2(t+\tau_2) - I_2'\right) + e^{\mu_1\tau_1}I_1(t+\tau_1)\right]\sigma_1 S\right\}\mathrm{d}B_1(t)$$
$$+\int_0^t\left\{\left(e^{\mu_2\tau_2}S_2' + \frac{e^{\mu_2\tau_2}}{\mu_0}\frac{\beta_2 S_2' I_2'}{1+\alpha_2 I_2'}\right)\sigma_3\left(I_2(t+\tau_2) - I_2'\right) + \left[e^{\mu_2\tau_2}\left(I_2(t+\tau_2) - I_2'\right)\right.\right.$$
$$\left.\left.+\left(S - S_2'\right) + e^{\mu_1\tau_1}I_1(t+\tau_1)\right]\sigma_3 e^{\mu_2\tau_2}I_2(t+\tau_2)\right\}\mathrm{d}B_3(t) + \int_0^t\left\{\left[e^{\mu_2\tau_2}\left(I_2(t+\tau_2) - I_2'\right)\right.\right.$$
$$\left.\left.+\left(S - S_2'\right) + e^{\mu_1\tau_1}I_1(t+\tau_1)\right]\sigma_2 e^{\mu_1\tau_1}I_1(t+\tau_1) + \left(\frac{\beta_2 I_2'}{\mu_0\left(1+\alpha_2 I_2'\right)} + 1\right)\frac{\beta_1 S_2'^2}{\mu_1}\sigma_2 I_1\right\}\mathrm{d}B_2(t)$$
$$+\int_0^t\left\{\int_A\left\{\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\cdot\left[\frac{\beta_2 S_2' I_2'}{\mu_0\left(1+\alpha_2 I_2'\right)}\left(D_1(y)S - S_2'\ln(1+D_1(y))\right.\right.\right.\right.$$
$$\left.+\frac{\beta_1 S_2'}{\mu_1}D_2(y)I_1\right) + \left(e^{\mu_2\tau_2}S_2' + \frac{e^{\mu_2\tau_2}}{\mu_0}\frac{\beta_2 S_2' I_2'}{1+\alpha_2 I_2'}\right)\left(D_3(y)I_2(t+\tau_2) - I_2'\ln(1+D_3(y))\right)$$
$$\left.+\frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS_2' + \frac{\beta_1 S_2'^2}{\mu_1}D_2(y)I_1\right] + \frac{1}{2}\left[e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)\right.$$
$$\left.+D_1(y)S + D_3(y)I_2(t+\tau_2)\right]^2 + \left[\left(S - S_2'\right) + e^{\mu_2\tau_2}\left(I_2(t+\tau_2) - I_2'\right) + e^{\mu_1\tau_1}I_1(t+\tau_1)\right]\cdot$$
$$\left.\left.[D_1(y)S + e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1) + D_3(y)I_2(t+\tau_2)]\right\}\tilde{N}(\mathrm{d}t,\mathrm{d}y)\right\}. \tag{3.22}$$

Taking the expectation, we have

$$EV_{10}(S(t), I_1(t), I_2(t)) \leq EV_{10}(S(0), I_1(0), I_2(0)) + E\int_0^t LV_{10}(S(\mu), I_1(\mu), I_2(\mu))\mathrm{d}\mu.$$

Then

$$\begin{aligned}0 \leq & EV_{10}(S(0), I_1(0), I_2(0)) + E\int_0^t \Big[-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1 - \mu_0 - 2\sigma_2^2 - 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right) I_1^2(\mu) \\ & - \frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2 - \mu_0 - 2\sigma_3^2 - 6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\left(I_2(\mu) - I_2^{'}\right)^2 \Big]\mathrm{d}\mu \\ & + \left\{\left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right) {S_2^{'}}^2 + e^{2\mu_2\tau_2}\left(\sigma_3^2 + 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right) {I_2^{'}}^2 \right. \\ & \left. + \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} M_3 \right\} t.\end{aligned}$$

Hence,

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[\left(I_2(\mu) - I_2^{'}\right)^2 + I_1^2(\mu)\right]\mathrm{d}\mu \leq & \frac{2}{H_2}\left\{\left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right) {S_2^{'}}^2 \right. \\ & + e^{2\mu_2\tau_2}\left(\sigma_3^2 + 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right) {I_2^{'}}^2 \\ & \left. + \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} M_3 \right\},\end{aligned}$$

where

$H_2 = \min\left\{e^{2\mu_1\tau_1}\left(\mu_1 - \mu_0 - \sigma_2^2 - 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right), e^{2\mu_2\tau_2}\left(\mu_2 - \mu_0 - 2\sigma_3^2 - 6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\right\}$. □

Remark 3. Theorem 3.3 shows that when $R_1 < 1$ and $R_2 > 1$ and certain conditions are satisfied, the solution to the stochastic delayed system (1.3) will oscillate around the equilibrium point E_2. The intensity of these oscillations is related to the values of σ_i and $D_i(y)(i = 1, 2, 3)$. The smaller the noise intensity, the closer the solution of the stochastic delayed system (1.3) is to the equilibrium point E_2. This indicates that Disease II will prevail in the population, and Disease I will tend to be extinct.

3.4. Asymptotic behavior of solutions near equilibrium point E_3

When the basic reproduction numbers $R_1 > 1$ and $R_2 > 1$, the deterministic SI_1I_2 model possesses an equilibrium point $E_3(S, I_1^*, I_2^*)$. However, it should be noted that in the the stochastic delayed system (1.3), when incorporating stochastic disturbances and time delay, E_3 no longer satisfies the definition of an equilibrium point. Therefore, the subsequent analysis will focus on the stochastic delayed model (1.3), primarily discussing the asymptotic behavior of the solution near E_3.

Theorem 3.4. Suppose that conditions (H1) and (H2) hold. If $R_1 > 1$ and $R_2 > 1$ and the following conditions are satisfied:

$$\mu_0 > \sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y),$$
$$\mu_1 - \mu_0 > 2\sigma_2^2 + 6\int_A D_2^2(y)\nu(\mathrm{d}y),$$
$$\mu_2 - \mu_0 > 2\sigma_3^2 + 6\int_A D_3^2(y)\nu(\mathrm{d}y),$$

then for any given initial values (1.4), the solution $(S(t), I_1(t), I_2(t))$ of system (1.3) has the following properties:

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t (S(\mu) - S^*)^2 d\mu \le \frac{1}{\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)} M_4,$$
$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t [(I_1(\mu) - I_1^*)^2 + (I_2(\mu) - I_2^*)^2] d\mu$$
$$\le \frac{2}{H_3}\Bigg\{ \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y))} M_4$$
$$+ \left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right) S^{*2} + e^{2\mu_1\tau_1}\left(\sigma_2^2 + 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right) I_1^{*2}$$
$$+ e^{2\mu_2\tau_2}\left(\sigma_3^2 + 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right) I_2^{*2}\Bigg\}.$$

Proof. When $R_1 > 1$ and $R_2 > 1$, the equilibrium point $E_3(S^*, I_1^*, I_2^*)$ of the deterministic SI_1I_2 model satisfies the following system of equations:

$$\Lambda - \mu_0 S^* - \frac{\beta_1 S^* I_1^*}{1 + \alpha_1 I_1^*} - \frac{\beta_2 S^* I_2^*}{1 + \alpha_2 I_2^*} = 0, \tag{3.23}$$

$$e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1 + \alpha_1 I_1^*} - \mu_1 I_1^* = 0, \tag{3.24}$$

$$e^{-\mu_2\tau_2}\frac{\beta_2 S^* I_2^*}{1 + \alpha_2 I_2^*} - \mu_2 I_2^* = 0. \tag{3.25}$$

Define the function $V_1 : \mathbb{R}_+ \to \mathbb{R}_+$

$$V_1(S) = S - S^* - S^* \ln\frac{S}{S^*}.$$

According to Itô's formula, we can obtain

$$\mathrm{d}V_1 = LV_1 dt + \sigma_1(S - S^*)dB_1(t) + \int_A [D_1(y)S - S^* \ln(1 + D_1(y))]\, \tilde{N}(dt, dy),$$

where

$$LV_1 = \left(1 - \frac{S^*}{S}\right)\left(\Lambda - \mu_0 S - \frac{\beta_1 S(t) I_1(t)}{1 + \alpha_1 I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 I_2(t)}\right) + \frac{1}{2}\sigma_1^2 S^*$$

$$
\begin{aligned}
&+\int_A\Big[D_1(y)S^*-S^*\ln(1+D_1(y))\Big]\nu(\mathrm{d}y)\\
=&\Lambda-\mu_0S-\frac{\beta_1S(t)I_1(t)}{1+\alpha_1I_1(t)}-\frac{\beta_2S(t)I_2(t)}{1+\alpha_2I_2(t)}-\frac{\Lambda S^*}{S}+\mu_0S^*+\frac{\beta_1S^*I_1(t)}{1+\alpha_1I_1(t)}\\
&+\int_A\Big[D_1(y)S^*-S^*\ln(1+D_1(y))\Big]\nu(\mathrm{d}y)\\
=&\Lambda-\left(\frac{\Lambda}{S^*}-\frac{\beta_1I_1^*}{1+\alpha_1I_1^*}-\frac{\beta_2I_2^*}{1+\alpha_2I_2^*}\right)S-\frac{\beta_1S(t)I_1(t)}{1+\alpha_1I_1(t)}-\frac{\beta_2S(t)I_2(t)}{1+\alpha_2I_2(t)}-\frac{\Lambda S^*}{S}\\
&+\Lambda-\frac{\beta_1S^*I_1^*}{1+\alpha_1I_1^*}-\frac{\beta_2S^*I_2^*}{1+\alpha_2I_2^*}+\frac{\beta_1S^*I_1(t)}{1+\alpha_1I_1(t)}+\frac{\beta_2S^*I_2(t)}{1+\alpha_2I_2(t)}+\frac{1}{2}\sigma_1^2S^*\\
&+\int_A\Big[D_1(y)S^*-S^*\ln(1+D_1(y))\Big]\nu(\mathrm{d}y)\\
=&\Lambda\left(2-\frac{S}{S^*}-\frac{S^*}{S}\right)+\frac{\beta_1SI_1^*(t)}{1+\alpha_1I_1^*(t)}+\frac{\beta_2SI_2^*(t)}{1+\alpha_2I_2^*(t)}-\frac{\beta_1S(t)I_1(t)}{1+\alpha_1I_1(t)}-\frac{\beta_2S(t)I_2(t)}{1+\alpha_2I_2(t)}\\
&-\frac{\beta_1S^*I_1^*(t)}{1+\alpha_1I_1^*(t)}-\frac{\beta_2S^*I_2^*(t)}{1+\alpha_2I_2^*(t)}+\frac{\beta_1S^*I_1(t)}{1+\alpha_1I_1(t)}+\frac{\beta_2S^*I_2(t)}{1+\alpha_2I_2(t)}+\frac{1}{2}\sigma_1^2S^*\\
&+\int_A\Big[D_1(y)S^*-S^*\ln(1+D_1(y))\Big]\nu(\mathrm{d}y)\\
=&-\left(\mu_0S^*+\frac{\beta_1S^*I_1^*}{1+\alpha_1I_1^*}+\frac{\beta_2S^*I_2^*}{1+\alpha_2I_2^*}\right)\frac{(S-S^*)^2}{SS^*}+\frac{\beta_1SI_1^*(t)}{1+\alpha_1I_1^*(t)}+\frac{\beta_2SI_2^*(t)}{1+\alpha_2I_2^*(t)}\\
&-\frac{\beta_1S(t)I_1(t)}{1+\alpha_1I_1(t)}-\frac{\beta_2S(t)I_2(t)}{1+\alpha_2I_2(t)}-\frac{\beta_1S^*I_1^*(t)}{1+\alpha_1I_1^*(t)}-\frac{\beta_2S^*I_2^*(t)}{1+\alpha_2I_2^*(t)}+\frac{\beta_1S_1^*I_1(t)}{1+\alpha_1I_1(t)}\\
&+\frac{\beta_2S^*I_2(t)}{1+\alpha_2I_2(t)}+\frac{1}{2}\sigma_1^2S^*+\int_A\Big[D_1(y)S^*-S^*\ln(1+D_1(y))\Big]\nu(\mathrm{d}y)\\
\le&-\left(\mu_0+\frac{\beta_1I_1^*}{1+\alpha_1I_1^*}+\frac{\beta_2I_2^*}{1+\alpha_2I_2^*}\right)\frac{(S-S^*)^2}{S}-\beta_1(S-S^*)\left(\frac{I_1}{1+\alpha_1I_1(t)}-\frac{I_1^*}{1+\alpha_1I_1^*(t)}\right)\\
&-\beta_2(S-S^*)\left(\frac{I_2}{1+\alpha_2I_2(t)}-\frac{I_2^*}{1+\alpha_2I_2^*(t)}\right)+\frac{1}{2}\sigma_1^2S^*+\int_A\Big[D_1(y)S^*-S^*\ln(1+D_1(y))\Big]\nu(\mathrm{d}y).
\end{aligned}
$$

Define

$$V_2(I_1)=I_1(t+\tau_1)-I_1^*-I_1^*\ln\frac{I_1(t+\tau_1)}{I_1^*}+e^{-\mu_1\tau_1}\frac{\beta_1S^*I_1^*}{1+\alpha_1I_1^*}\int_t^{t+\tau_1}\left(\frac{I_1(\mu)}{I_1^*}-\ln\frac{I_1(\mu)}{I_1^*}\right)\mathrm{d}\mu.$$

Then

$$\mathrm{d}V_2=LV_2\mathrm{d}t+\sigma_2\left(I_1(t+\tau_1)-I_1^*\right)\mathrm{d}B_2(t)+\int_A\Big[D_2(y)I_1(t+\tau_1)-I_1^*\ln(1+D_2(y))\Big]\tilde{N}(\mathrm{d}t,\mathrm{d}y),$$

where

$$
\begin{aligned}
LV_2=&\left[1-\frac{I_1^*}{I_1(t+\tau_1)}\right]\left[e^{-\mu_1\tau_1}\frac{\beta_1S(t)I_1(t)}{1+\alpha_1I_1(t)}-\mu_1I_1(t+\tau_1)\right]+\int_A\left[D_2(y)I_1^*-I_1^*\ln(1+D_2(y))\right]\nu(\mathrm{d}y)\\
&+\frac{1}{2}\sigma_2^2I_1^*+e^{-\mu_1\tau_1}\frac{\beta_1S^*I_1^*}{1+\alpha_1I_1^*}\left(\frac{I_1(t+\tau_1)}{I_1^*}-\ln\frac{I_1(t+\tau_1)}{I_1^*}\right)-e^{-\mu_1\tau_1}\frac{\beta_1S^*I_1^*}{1+\alpha_1I_1^*}\left(\frac{I_1(t)}{I_1^*}-\ln\frac{I_1(t)}{I_1^*}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-\mu_1\tau_1}\frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)} - \mu_1 I_1(t+\tau_1) - e^{-\mu_1\tau_1}\frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)}\frac{I_1^*}{I_1(t+\tau_1)} + \mu_1 I_1^* + \frac{1}{2}\sigma_2^2 I_1^* \\
&\quad + e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left(\frac{I_1(t+\tau_1)}{I_1^*} - \ln\frac{I_1(t+\tau_1)}{I_1^*}\right) - e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left(\frac{I_1(t)}{I_1^*} - \ln\frac{I_1(t)}{I_1^*}\right) \\
&\quad + \int_A \left[D_2(y)I_1^* - I_1^*\ln(1+D_2(y))\right]\nu(\mathrm{d}y) \\
&= e^{-\mu_1\tau_1}\frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)} - e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\frac{I_1(t+\tau_1)}{I_1^*} - e^{-\mu_1\tau_1}\frac{\beta_1 S(t)I_1(t)}{1+\alpha_1 I_1(t)}\frac{I_1^*}{I_1(t+\tau_1)} \\
&\quad + e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left(\frac{I_1(t+\tau_1)}{I_1^*} - \ln\frac{I_1(t+\tau_1)}{I_1^*}\right) - e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left(\frac{I_1(t)}{I_1^*} - \ln\frac{I_1(t)}{I_1^*}\right) \\
&\quad + e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{1}{2}\sigma_2^2 I_1^* + \int_A \left[D_2(y)I_1^* - I_1^*\ln(1+D_2(y))\right]\nu(\mathrm{d}y) \\
&= e^{-\mu_1\tau_1}\beta_1 (S - S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_1 I_1^*}\right) + \frac{1}{2}\sigma_2^2 I_1^* + \int_A \left[D_2(y)I_1^* - I_1^*\ln(1+D_2(y))\right]\nu(\mathrm{d}y) \\
&\quad + e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left(\frac{I_1(t+\tau_1)}{I_1^*} - \ln\frac{I_1(t+\tau_1)}{I_1^*}\right) - e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left(\frac{I_1(t)}{I_1^*} - \ln\frac{I_1(t)}{I_1^*}\right) \\
&\quad + e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left[\frac{S}{S^*} - \frac{\dfrac{S I_1 I_1^*}{1+\alpha_1 I_1}}{\dfrac{S^* I_1^* I_1(t+\tau_1)}{1+\alpha_1 I_1^*}} - \frac{I_1(t+\tau_1)}{I_1^*} + \frac{\dfrac{I_1}{1+\alpha_1 I_1}}{\dfrac{I_1^*}{1+\alpha_1 I_1^*}}\right].
\end{aligned}
$$

From $\omega \geq 1 + \ln\omega, \omega > 0$, we know that

$$
\frac{\dfrac{S I_1 I_1^*}{1+\alpha_1 I_1}}{\dfrac{S^* I_1^* I_1(t+\tau_1)}{1+\alpha_1 I_1^*}} \geq 1 + \ln\frac{\dfrac{S I_1 I_1^*}{1+\alpha_1 I_1}}{\dfrac{S_1^* I_1^* I_1(t+\tau_1)}{1+\alpha_1 I_1^*}} = 1 + \ln\frac{S}{S_1^*} - \ln\frac{I_1(t+\tau_1)}{I_1^*} + \ln\frac{\dfrac{I_1}{1+\alpha_1 I_1}}{\dfrac{I_1^*}{1+\alpha_1 I_1^*}}.
$$

Thus,

$$
\begin{aligned}
LV_2 &\leq e^{-\mu_1\tau_1}\beta_1 (S - S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_1 I_1^*}\right) + \frac{1}{2}\sigma_2^2 I_1^* + \int_A \left[D_2(y)I_1^* - I_1^*\ln(1+D_2(y))\right]\nu(\mathrm{d}y) \\
&\quad + e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left[\frac{S}{S^*} - 1 - \ln\frac{S}{S^*} + \ln\frac{I_1(t+\tau_1)}{I_1^*} - \ln\frac{\dfrac{I_1}{1+\alpha_1 I_1}}{\dfrac{I_1^*}{1+\alpha_1 I_1^*}} - \frac{I_1(t+\tau_1)}{I_1^*} + \frac{\dfrac{I_1}{1+\alpha_1 I_1}}{\dfrac{I_1^*}{1+\alpha_1 I_1^*}}\right] \\
&\quad + e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left(\frac{I_1(t+\tau_1)}{I_1^*} - \ln\frac{I_1(t+\tau_1)}{I_1^*}\right) - e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left(\frac{I_1(t)}{I_1^*} - \ln\frac{I_1(t)}{I_1^*}\right) \\
&= e^{-\mu_1\tau_1}\beta_1 (S - S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_1 I_1^*}\right) + \frac{1}{2}\sigma_2^2 I_1^* + \int_A \left[D_2(y)I_1^* - I_1^*\ln(1+D_2(y))\right]\nu(\mathrm{d}y)
\end{aligned}
$$

$$
\begin{aligned}
&+ e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left[\frac{S}{S^*} + \ln\frac{S^*}{S} - \frac{I_1(t)}{I_1^*} + \frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I_1^*}{1+\alpha_1 I_1^*}} + \ln\frac{I_1(t)}{I_1^*} - \ln\frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I_1^*}{1+\alpha_1 I_1^*}} - 1\right]\\
\leq & e^{-\mu_1\tau_1}\beta_1 (S-S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_1 I_1^*}\right) + \frac{1}{2}\sigma_2^2 I_1^* + \int_A \left[D_2(y) I_1^* - I_1^*\ln(1+D_2(y))\right]\nu(\mathrm{d}y)\\
&+ e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\left[\frac{S}{S^*} + \frac{S^*}{S} - 2 - \frac{I_1(t)}{I_1^*} + \frac{\frac{I_1}{1+\alpha_1 I_1}}{\frac{I_1^*}{1+\alpha_1 I_1^*}} + \frac{\frac{I_1 I_1^*}{1+\alpha_1 I_1^*}}{\frac{I_1^* I_1}{1+\alpha_1 I_1}} - 1\right]\\
= & e^{-\mu_1\tau_1}\beta_1 (S-S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_1 I_1^*}\right) + e^{-\mu_1\tau_1}\frac{\beta_1 I_1^*}{1+\alpha_1 I_1^*}\frac{(S-S^*)^2}{S}\\
&+ e^{-\mu_1\tau_1}\frac{\beta_1 S^* I_1}{\frac{I_1}{1+\alpha_1 I_1}}\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_1 I_1^*}\right)\left(\frac{\frac{I_1}{1+\alpha_1 I_1}}{I_1} - \frac{\frac{I_1^*}{1+\alpha_1 I_1^*}}{I_1^*}\right)\\
&+ \frac{1}{2}\sigma_2^2 I_1^* + \int_A \left[D_2(y) I_1^* - I_1^*\ln(1+D_2(y))\right]\nu(\mathrm{d}y)\\
\leq & e^{-\mu_1\tau_1}\beta_1 (S-S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_1 I_1^*}\right) + e^{-\mu_1\tau_1}\frac{\beta_1 I_1^*}{1+\alpha_1 I_1^*}\frac{(S-S^*)^2}{S}\\
&+ \frac{1}{2}\sigma_2^2 I_1^* + \int_A \left[D_2(y) I_1^* - I_1^*\ln(1+D_2(y))\right]\nu(\mathrm{d}y).
\end{aligned}
$$

Define the function $V_3 : \mathbb{R}_+ \to \mathbb{R}_+$

$$
V_3(I_2) = I_2(t+\tau_2) - I_2^* - I_2^*\ln\frac{I_2(t+\tau_2)}{I_2^*} + e^{-\mu_2\tau_2}\frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\int_t^{t+\tau_2}\left(\frac{I_2(\mu)}{I_2^*} - \ln\frac{I_2(\mu)}{I_2^*}\right)\mathrm{d}\mu.
$$

From Itô's formula, we know that

$$
\mathrm{d}V_3 = LV_3\mathrm{d}t + \sigma_3\left(I_2(t+\tau_2) - I_2^*\right)\mathrm{d}B_3(t) + \int_A \left[D_3(y) I_2(t+\tau_2) - I_2^*\ln(1+D_3(y))\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y),
$$

where

$$
\begin{aligned}
LV_3 \leq & e^{-\mu_2\tau_2}\beta_2 (S-S^*)\left(\frac{I_2}{1+\alpha_2 I_2} - \frac{I_2^*}{1+\alpha_2 I_2^*}\right) + e^{-\mu_2\tau_2}\frac{\beta_2 I_2^*}{1+\alpha_2 I_2^*}\frac{(S-S^*)^2}{S}\\
&+ \frac{1}{2}\sigma_3^2 I_2^* + \int_A \left[D_3(y) I_2^* - I_2^*\ln(1+D_3(y))\right]\nu(\mathrm{d}y).
\end{aligned}
$$

Define the function

$$
V_4(S) = \frac{1}{2}(S-S^*)^2.
$$

Then

$$dV_4 = LV_4 dt + \sigma_1 S\,(S - S^*)\,dB_1(t) + \int_A \left[\frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS^*\right]\tilde{N}(dt, dy),$$

where

$$\begin{aligned}
LV_4 =& (S - S^*)\left(\Lambda - \mu_0 S - \frac{\beta_1 S I_1}{1+\alpha_1 I_1} - \frac{\beta_2 S I_2}{1+\alpha_2 I_2}\right) + \frac{1}{2}\sigma_1^2 S^2 \\
&+ \int_A \left[\frac{1}{2}(S + D_1(y)S - S^*)^2 - \frac{1}{2}(S - S^*)^2 - D_1(y)S\,(S - S^*)\right]\nu(dy) \\
=& (S - S^*)\left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*} - \mu_0 S - \frac{\beta_1 S I_1}{1+\alpha_1 I_1} - \frac{\beta_2 S I_2}{1+\alpha_2 I_2}\right) \\
&+ \frac{1}{2}\sigma_1^2 S^2 + \int_A \frac{1}{2}D_1^2(y)S^2\nu(dy) \\
=& -\mu_0 (S - S^*)^2 + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}(S - S^*) + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}(S - S^*) - \frac{\beta_1 S I_1}{1+\alpha_1 I_1}(S - S^*) \\
&- \frac{\beta_2 S I_2}{1+\alpha_2 I_2}(S - S^*) + \frac{1}{2}\sigma_1^2 S^2 + \int_A \frac{1}{2}D_1^2(y)S^2\nu(dy) \\
=& -\mu_0 (S - S^*)^2 + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}(S - S^*) + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}(S - S^*) - \frac{\beta_1 S^* I_1}{1+\alpha_1 I_1}(S - S^*) + \frac{1}{2}\sigma_1^2 S^2 \\
&- \frac{\beta_1 I_1}{1+\alpha_1 I_1}(S - S^*)^2 - \frac{\beta_2 S^* I_2}{1+\alpha_2 I_2}(S - S^*) - \frac{\beta_2 I_2}{1+\alpha_2 I_2}(S - S^*)^2 + \int_A \frac{1}{2}D_1^2(y)S^2\nu(dy) \\
\le& -\mu_0 (S - S^*)^2 - \frac{\beta_1 I_1}{1+\alpha_1 I_1}(S - S^*)^2 - \frac{\beta_2 I_2}{1+\alpha_2 I_2}(S - S^*)^2 + \sigma_1^2 (S - S^*)^2 + \sigma_1^2 S^{*2} \\
&- \beta_1 S^*(S - S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_2 I_2^*}\right) - \beta_2 S^*(S - S^*)\left(\frac{I_2}{1+\alpha_2 I_2} - \frac{I_2^*}{1+\alpha_2 I_2^*}\right) \\
&+ \int_A D_1^2(y)\nu(dy)\,(S - S^*)^2 + \int_A D_1^2(y)\nu(dy)S^{*2} \\
\le& -\mu_0 (S - S^*)^2 - \beta_1 S^*(S - S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_2 I_2^*}\right) + \sigma_1^2 (S - S^*)^2 + \sigma_1^2 S^{*2} \\
&- \beta_2 S^*(S - S^*)\left(\frac{I_2}{1+\alpha_2 I_2} - \frac{I_2^*}{1+\alpha_2 I_2^*}\right) + \int_A D_1^2(y)\nu(dy)\,(S - S^*)^2 + \int_A D_1^2(y)\nu(dy)S^{*2} \\
=& -\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(dy)\right)(S - S^*)^2 - \beta_1 S^*(S - S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_2 I_2^*}\right) \\
&- \beta_2 S^*(S - S^*)\left(\frac{I_2}{1+\alpha_2 I_2} - \frac{I_2^*}{1+\alpha_2 I_2^*}\right) + \left(\sigma_1^2 + \int_A D_1^2(y)\nu(dy)\right)S^{*2}.
\end{aligned}$$

Construct the Lyapunov function

$$V_5(S, I_1, I_2) = \frac{1}{\mu_0}\left(\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)V_1 + \left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\left(\frac{e^{\mu_1\tau_1}}{\mu_0}V_2 + \frac{e^{\mu_2\tau_2}}{\mu_0}V_3\right) + V_4.$$

Then

$$dV_5 = LV_5 dt + \left[\frac{1}{\mu_0}\left(\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\sigma_1 (S - S^*) + \sigma_1 S\,(S - S^*)\right]dB_1(t)$$

$$
\begin{aligned}
&+\left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\frac{e^{\mu_1\tau_1}}{\mu_0}\sigma_2\left(I_1(t+\tau_1) - I_1^*\right)\mathrm{d}B_2(t)\\
&+\left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\frac{e^{\mu_2\tau_2}}{\mu_0}\sigma_3\left(I_2(t+\tau_2) - I_2^*\right)\mathrm{d}B_3(t)\\
&+\int_A\left\{\left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\left[\frac{e^{\mu_1\tau_1}}{\mu_0}[D_2(y)I_1(t+\tau_1) - I_1^*\ln(1+D_2(y))]\right.\right.\\
&+\left.\left.\frac{e^{\mu_2\tau_2}}{\mu_0}[D_3(y)I_2(t+\tau_2) - I_2^*\ln(1+D_3(y))]\right] + \frac{1}{2}D_1^2(y)S^2 + D_1(y)S^2 - D_1(y)SS^*\right\}\tilde{N}(\mathrm{d}t,\mathrm{d}y),
\end{aligned}
\tag{3.26}
$$

where

$$
\begin{aligned}
LV_5 \le &\left(\frac{\beta_1 S^* I_1^*}{\mu_0\left(1+\alpha_1 I_1^*\right)} + \frac{\beta_2 S^* I_2^*}{\mu_0\left(1+\alpha_2 I_2^*\right)}\right)\left[-\left(\mu_0 + \frac{\beta_1 I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 I_2^*}{1+\alpha_2 I_2^*}\right)\frac{(S-S^*)^2}{S}\right.\\
&-\beta_1(S-S^*)\left(\frac{I_1}{1+\alpha_1 I_1(t)} - \frac{I_1^*}{1+\alpha_1 I_1^*(t)}\right) - \beta_2(S-S^*)\left(\frac{I_2}{1+\alpha_2 I_2(t)} - \frac{I_2^*}{1+\alpha_2 I_2^*(t)}\right)\\
&\left.+\frac{1}{2}\sigma_1^2 S^* + \int_A\Big[D_1(y)S^* - S^*\ln(1+D_1(y))\Big]\nu(\mathrm{d}y)\right] + \left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\cdot\\
&\left\{\frac{e^{\mu_1\tau_1}}{\mu_0}\left[e^{-\mu_1\tau_1}\beta_1(S-S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_1 I_1^*}\right) + e^{-\mu_1\tau_1}\frac{\beta_1 I_1^*}{1+\alpha_1 I_1^*}\frac{(S-S^*)^2}{S}\right.\right.\\
&\left.+\frac{1}{2}\sigma_2^2 I_1^* + \int_A[D_2(y)I_1^* - I_1^*\ln(1+D_2(y))]\,\nu(\mathrm{d}y)\right] + \frac{e^{\mu_2\tau_2}}{\mu_0}\left[e^{-\mu_2\tau_2}\beta_2(S-S^*)\cdot\right.\\
&\left(\frac{I_2}{1+\alpha_2 I_2} - \frac{I_2^*}{1+\alpha_2 I_2^*}\right) + e^{-\mu_2\tau_2}\frac{\beta_2 I_2^*}{1+\alpha_2 I_2^*}\frac{(S-S^*)^2}{S} + \frac{1}{2}\sigma_3^2 I_2^* + \int_A[D_3(y)I_2^*\\
&\left.\left.-I_2^*\ln(1+D_3(y))]\,\nu(\mathrm{d}y)\right]\right\} - \left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S^*)^2\\
&-\beta_1 S^*(S-S^*)\left(\frac{I_1}{1+\alpha_1 I_1} - \frac{I_1^*}{1+\alpha_2 I_2^*}\right) + \left(\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\right)S^{*2}\\
&-\beta_2 S^*(S-S^*)\left(\frac{I_2}{1+\alpha_2 I_2} - \frac{I_2^*}{1+\alpha_2 I_2^*}\right)\\
= &-\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S^*)^2 + \left\{\left[\frac{1}{2\mu_0}\left(\frac{\beta_1 I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 I_2^*}{1+\alpha_2 I_2^*}\right) + 1\right]\sigma_1^2\right.\\
&\left.+\int_A D_1^2(y)\nu(\mathrm{d}y)\right\}S^{*2} + \frac{1}{\mu_0}\left(\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\int_A\Big[D_1(y)S^* - S^*\ln(1+D_1(y))\Big]\nu(\mathrm{d}y)\\
&+\left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\cdot\left[\frac{e^{\mu_1\tau_1}}{2\mu_0}\sigma_2^2 I_1^* + \frac{e^{\mu_2\tau_2}}{2\mu_0}\sigma_3^2 I_2^* + 2\int_A[D_2(y)I_1^*\right.\\
&\left.-I_1^*\ln(1+D_2(y))]\,\nu(\mathrm{d}y) + 2\int_A[D_3(y)I_2^* - I_2^*\ln(1+D_3(y))]\,\nu(\mathrm{d}y)\right]\\
= &-\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S^*)^2 + M_4.
\end{aligned}
\tag{3.27}
$$

Integrate from 0 to t on both sides of (3.26), then

$$\begin{aligned}\int_0^t \mathrm{d}V_5 =& \int_0^t LV_5 \mathrm{d}\mu + \int_0^t \left[\frac{1}{\mu_0}\left(\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\sigma_1 (S - S^*) + \sigma_1 S (S - S^*)\right] \mathrm{d}B_1(t)\\ &+ \int_0^t \left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\frac{e^{\mu_1\tau_1}}{\mu_0}\sigma_2 (I_1(t+\tau_1) - I_1^*)\,\mathrm{d}B_2(t)\\ &+ \int_0^t \left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\frac{e^{\mu_2\tau_2}}{\mu_0}\sigma_3 (I_2(t+\tau_2) - I_2^*)\,\mathrm{d}B_3(t)\\ &+ \int_0^t \left\{\int_A \left\{\left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\left[\frac{e^{\mu_1\tau_1}}{\mu_0}[D_2(y) I_1(t+\tau_1)\right.\right.\right.\\ &\left. - I_1^* \ln(1+D_2(y))] + \frac{e^{\mu_2\tau_2}}{\mu_0}[D_3(y) I_2(t+\tau_2) - I_2^* \ln(1+D_3(y))]\right]\\ &\left.\left. + \frac{1}{2}D_1^2(y) S^2 + D_1(y) S^2 - D_1(y) S S^*\right\}\tilde{N}(\mathrm{d}t, \mathrm{d}y)\right\}. \end{aligned} \tag{3.28}$$

Taking the expectation, we have

$$EV_5(S(t), I_1(t), I_2(t)) \le EV_5(S(0), I_1(0), I_2(0)) + E\int_0^t LV_5(S(\mu), I_1(\mu), I_2(\mu))\mathrm{d}\mu.$$

From (3.27), we have

$$EV_5(S(t), I_1(t), I_2(t)) - EV_5(S(0), I_1(0), I_2(0)) \le E\int_0^t \left[-\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S - S^*)^2\right]\mathrm{d}\mu + M_4 t.$$

Hence,

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \left[(S - S^*)^2\right]\mathrm{d}\mu \le \frac{1}{\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)} M_4,$$

where

$$\begin{aligned} M_4 =& \left\{\left[\frac{1}{2\mu_0}\left(\frac{\beta_1 I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 I_2^*}{1+\alpha_2 I_2^*}\right) + 1\right]\sigma_1^2 + \int_A D_1^2(y)\nu(\mathrm{d}y)\right\} S^{*2} + \frac{1}{\mu_0}\left(\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\right.\\ &\left. + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\int_A \left[D_1(y) S^* - S^* \ln(1+D_1(y))\right]\nu(\mathrm{d}y) + \left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}\right.\\ &\left. + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\cdot\left[\frac{e^{\mu_1\tau_1}}{2\mu_0}\sigma_2^2 I_1^* + \frac{e^{\mu_2\tau_2}}{2\mu_0}\sigma_3^2 I_2^* + 2\int_A [D_2(y) I_1^* - I_1^* \ln(1+D_2(y))]\,\nu(\mathrm{d}y)\right.\\ &\left. + 2\int_A [D_3(y) I_2^* - I_2^* \ln(1+D_3(y))]\,\nu(\mathrm{d}y)\right]. \end{aligned}$$

Define the function $V_6 : \mathbb{R}_+^3 \to \mathbb{R}_+$

$$V_6(S, I_1, I_2) = \frac{1}{2}\left[(S - S^*) + e^{\mu_1\tau_1}(I_1(t+\tau_1) - I_1^*) + e^{\mu_2\tau_2}(I_2(t+\tau_2) - I_2^*)\right]^2.$$

Then

$$
\begin{aligned}
LV_6 =& \left[(S-S^*)+e^{\mu_1\tau_1}\left(I_1(t+\tau_1)-I_1^*\right)+e^{\mu_2\tau_2}\left(I_2(t+\tau_2)-I_2^*\right)\right]\Bigg[\Lambda-\mu_0 S-\frac{\beta_1 S I_1}{1+\alpha_1 I_1}\\
&-\frac{\beta_2 S I_2}{1+\alpha_2 I_2}+e^{\mu_1\tau_1}\left[e^{-\mu_1\tau_1}\frac{\beta_1 S(t+\tau_1-\tau_1)I_1(t+\tau_1-\tau_1)}{1+\alpha_1 I_1(t+\tau_1-\tau_1)}-\mu_1 I_1(t+\tau_1)\right]\\
&+e^{\mu_2\tau_2}\left[e^{-\mu_2\tau_2}\frac{\beta_2 S(t+\tau_2-\tau_2)I_2(t+\tau_2-\tau_2)}{1+\alpha_2 I_2(t+\tau_2-\tau_2)}-\mu_2 I_2(t+\tau_2)\right]\Bigg]+\frac{1}{2}\sigma_1^2 S^2\\
&+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2 I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2 I_2^2(t+\tau_2)+\int_A\Bigg\{\frac{1}{2}\left[(S+D_1(y)S-S^*)\right.\\
&+e^{\mu_1\tau_1}\left(I_1(t+\tau_1)+D_2(y)I_1(t+\tau_1)-I_1^*\right)+e^{\mu_2\tau_2}\left.\left(I_2(t+\tau_2)+D_3(y)I_2(t+\tau_2)-I_2^*\right)\right]^2\\
&-\frac{1}{2}\left[(S-S^*)+e^{\mu_1\tau_1}\left(I_1(t+\tau_1)-I_1^*\right)+e^{\mu_2\tau_2}\left(I_2(t+\tau_2)-I_2^*\right)\right]^2-\left[(S-S^*)\right.\\
&+e^{\mu_1\tau_1}\left(I_1(t+\tau_1)-I_1^*\right)+e^{\mu_2\tau_2}\left.\left(I_2(t+\tau_2)-I_2^*\right)\right]\left[D_1(y)S+D_2(y)I_1(t+\tau_1)\right.\\
&\left.+D_3(y)I_2(t+\tau_2)\right]\Bigg\}\nu(\mathrm{d}y)\\
=&-\mu_0\left(S-S^*\right)^2-e^{\mu_1\tau_1}\mu_1\left(S-S^*\right)\left(I_1(t+\tau_1)-I_1^*\right)+\frac{1}{2}e^{2\mu_1\tau_1}\sigma_2^2 I_1^2(t+\tau_1)\\
&-e^{\mu_2\tau_2}\mu_2\left(S-S^*\right)\left(I_2(t+\tau_2)-I_2^*\right)-e^{\mu_1\tau_1}\mu_0\left(S-S^*\right)\left(I_1(t+\tau_1)-I_1^*\right)\\
&-e^{2\mu_1\tau_1}\mu_1\left(I_1(t+\tau_1)-I_1^*\right)^2-e^{\mu_1\tau_1+\mu_2\tau_2}\mu_2\left(I_1(t+\tau_1)-I_1^*\right)\left(I_2(t+\tau_2)-I_2^*\right)\\
&-e^{\mu_2\tau_2}\mu_0\left(S-S^*\right)\left(I_2(t+\tau_2)-I_2^*\right)-e^{2\mu_2\tau_2}\mu_2\left(I_2(t+\tau_2)-I_2^*\right)^2+\frac{1}{2}\sigma_1^2 S^2\\
&-e^{\mu_1\tau_1+\mu_2\tau_2}\mu_1\left(I_1(t+\tau_1)-I_1^*\right)\left(I_2(t+\tau_2)-I_2^*\right)+\frac{1}{2}e^{2\mu_2\tau_2}\sigma_3^2 I_2^2(t+\tau_2)\\
&+\int_A\frac{1}{2}\left[D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)\right]^2\nu(\mathrm{d}y).
\end{aligned}
$$

From the basic inequalities $-2ab \le a^2+b^2$ and $(a+b)^2 \le 2a^2+2b^2$, we get

$$
\begin{aligned}
LV_6 \le& -\mu_0\left(S-S^*\right)^2-e^{\mu_1\tau_1}\left(\mu_0+\mu_1\right)\left(S-S^*\right)\left(I_1(t+\tau_1)-I_1^*\right)+e^{2\mu_2\tau_2}\sigma_3^2\left(I_2(t+\tau_2)-I_2^*\right)^2\\
&-e^{\mu_2\tau_2}\left(\mu_0+\mu_2\right)\left(S-S^*\right)\left(I_2(t+\tau_2)-I_2^*\right)-e^{2\mu_1\tau_1}\mu_1\left(I_1(t+\tau_1)-I_1^*\right)^2+e^{2\mu_2\tau_2}\sigma_3^2 I_2^{*2}\\
&-e^{\mu_1\tau_1+\mu_2\tau_2}\left(\mu_1+\mu_2\right)\left(I_1(t+\tau_1)-I_1^*\right)\left(I_2(t+\tau_2)-I_2^*\right)-e^{2\mu_2\tau_2}\mu_2\left(I_2(t+\tau_2)-I_2^*\right)^2\\
&+\sigma_1^2\left(S-S^*\right)^2+\sigma_1^2 S^{*2}+e^{2\mu_1\tau_1}\sigma_2^2\left(I_1(t+\tau_1)-I_1^*\right)^2+e^{2\mu_1\tau_1}\sigma_2^2 I_1^{*2}\\
&+\int_A\frac{1}{2}\left[D_1(y)S+e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)\right]^2\nu(\mathrm{d}y)\\
\le&\frac{1}{2}\left(\mu_1+\mu_2+2\sigma_1^2\right)\left(S-S^*\right)^2-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2\right)\left(I_1(t+\tau_1)-I_1^*\right)^2\\
&-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-2\sigma_3^2\right)\left(I_2(t+\tau_2)-I_2^*\right)^2+\sigma_1^2 S^{*2}+e^{2\mu_1\tau_1}\sigma_2^2 I_1^{*2}\\
&e^{2\mu_2\tau_2}\sigma_3^2 I_2^{*2}+\int_A\Bigg\{\frac{1}{2}D_1^2(y)S^2+\frac{1}{2}D_1^2(y)S^2+\frac{1}{2}e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)\\
&+\frac{1}{2}e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)+\frac{1}{2}D_1^2(y)S^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}e^{2\mu_1\tau_1}D_2^2(y)I_1^2(t+\tau_1)+\frac{1}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)+\frac{1}{2}e^{2\mu_2\tau_2}D_3^2(y)I_2^2(t+\tau_2)\Big\}\nu(\mathrm{d}y)\\
\leq&\frac{1}{2}\left(\mu_1+\mu_2+2\sigma_1^2\right)(S-S^*)^2-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2\right)(I_1(t+\tau_1)-I_1^*)^2\\
&-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-2\sigma_3^2\right)(I_2(t+\tau_2)-I_2^*)^2+\sigma_1^2S^{*2}+e^{2\mu_1\tau_1}\sigma_2^2I_1^{*2}\\
&+e^{2\mu_2\tau_2}\sigma_3^2I_2^{*2}+\int_A\Big\{3D_1^2(y)(S-S^*)^2+3D_1^2(y)S^{*2}+3e^{2\mu_1\tau_1}D_2^2(y)(I_1(t+\tau_1)-I_1^*)^2\\
&+3e^{2\mu_1\tau_1}D_2^2(y)I_1^{*2}+3e^{2\mu_2\tau_2}D_3^2(y)(I_2(t+\tau_2)-I_2^*)^2+3e^{2\mu_2\tau_2}D_3^2(y)I_2^{*2}\Big\}\nu(\mathrm{d}y)\\
=&\frac{1}{2}\left(\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S^*)^2\\
&-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2-6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)(I_1(t+\tau_1)-I_1^*)^2\\
&-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-2\sigma_3^2-6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)(I_2(t+\tau_2)-I_2^*)^2\\
&+\left(\sigma_1^2+3\int_A D_1^2(y)\nu(\mathrm{d}y)\right)S^{*2}+e^{2\mu_1\tau_1}\left(\sigma_2^2+3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^{*2}\\
&+e^{2\mu_2\tau_2}\left(\sigma_3^2+3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\sigma_3^2I_2^{*2}.
\end{aligned}
$$

Construct the Lyapunov function $V_7:\mathbb{R}_+^3\to\mathbb{R}_+$

$$
\begin{aligned}
V_7(S,I_1,I_2)=&\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}V5+V_6\\
&+\frac{1}{2}\left(\mu_1-\mu_0-2\sigma_2^2-6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)\int_t^{t+\tau_1}(I_1(\mu)-I_1^*)^2\,\mathrm{d}\mu\\
&+\frac{1}{2}\left(\mu_1-\mu_0-2\sigma_3^2-6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\int_t^{t+\tau_2}(I_2(\mu)-I_2^*)^2\,\mathrm{d}\mu.
\end{aligned}
$$

According to Itô's formula, we know that

$$
\begin{aligned}
\mathrm{d}V_7=&LV_7\mathrm{d}t+\Bigg\{\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\Bigg[\frac{1}{\mu_0}\left(\frac{\beta_1S^*I_1^*}{1+\alpha_1I_1^*}+\frac{\beta_2S^*I_2^*}{1+\alpha_2I_2^*}\right)\sigma_1(S-S^*)\\
&+\sigma_1S(S-S^*)\Bigg]+\left[(S-S^*)+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1^*)+e^{\mu_2\tau_2}(I_2(t+\tau_2)-I_2^*)\right]\sigma_1S\Bigg\}\mathrm{d}B_1(t)\\
&+\Bigg\{\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\left(\mu_0S^*+\frac{\beta_1S^*I_1^*}{1+\alpha_1I_1^*}+\frac{\beta_2S^*I_2^*}{1+\alpha_2I_2^*}\right)\frac{e^{\mu_1\tau_1}}{\mu_0}\sigma_2(I_1(t+\tau_1)-I_1^*)\\
&+\left[(S-S^*)+e^{\mu_1\tau_1}(I_1(t+\tau_1)-I_1^*)+e^{\mu_2\tau_2}(I_2(t+\tau_2)-I_2^*)\right]\sigma_2e^{\mu_1\tau_1}I_1(t+\tau_1)\Bigg\}\mathrm{d}B_2(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg\{\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\left(\mu_0 S^*+\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}+\frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\frac{e^{\mu_2\tau_2}}{\mu_0}\sigma_3\left(I_2(t+\tau_2)-I_2^*\right)\\
&+\left[(S-S^*)+e^{\mu_1\tau_1}\left(I_1(t+\tau_1)-I_1^*\right)+e^{\mu_2\tau_2}\left(I_2(t+\tau_2)-I_2^*\right)\right]\sigma_3 e^{\mu_2\tau_2}I_2(t+\tau_2)\Bigg\}\mathrm{d}B_3(t)\\
&+\int_A\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\Bigg\{\left(\mu_0 S^*+\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*}+\frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\cdot\\
&\left[\frac{e^{\mu_1\tau_1}}{\mu_0}\left[D_2(y)I_1(t+\tau_1)-\ln(1+D_2(y))\right]+\frac{e^{\mu_2\tau_2}}{\mu_0}\left[D_3(y)I_2(t+\tau_2)-\ln(1+D_3(y))\right]\right]\\
&+\frac{1}{2}D_1^2(y)S^2+D_1(y)S^2-D_1(y)SS^*\Bigg\}+\frac{1}{2}\left[e^{\mu_1\tau_1}D_2(y)I_1(t+\tau_1)+e^{\mu_2\tau_2}D_3(y)I_2(t+\tau_2)\right.\\
&\left.+D_1(y)S\right]^2+\left[(S-S^*)+e^{\mu_1\tau_1}\left(I_1(t+\tau_1)-I_1^*\right)+e^{\mu_2\tau_2}\left(I_2(t+\tau_2)-I_2^*\right)\right]\cdot\\
&\left[D_1(y)S+D_2(y)I_1(t+\tau_1)+D_3(y)I_2(t+\tau_2)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}y), \qquad (3.29)
\end{aligned}
$$

where

$$
\begin{aligned}
LV_7\le&\frac{\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)}\left(-\left(\mu_0-\sigma_1^2-\int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S^*)^2+M_4\right)\\
&+\frac{1}{2}\left(\mu_1+\mu_2+2\sigma_1^2+6\int_A D_1^2(y)\nu(\mathrm{d}y)\right)(S-S^*)^2+e^{2\mu_2\tau_2}\left(\sigma_3^2+3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\sigma_3^2 I_2^{*2}\\
&-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2-6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)\left(I_1(t+\tau_1)-I_1^*\right)^2\\
&-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-2\sigma_3^2-6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\left(I_2(t+\tau_2)-I_2^*\right)^2\\
&+\left(\sigma_1^2+3\int_A D_1^2(y)\nu(\mathrm{d}y)\right)S^{*2}+e^{2\mu_1\tau_1}\left(\sigma_2^2+3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^{*2}\\
&+\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2-6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)\left(I_1(t+\tau_1)-I_1^*\right)^2\\
&-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2-6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)\left(I_1(t)-I_1^*\right)^2\\
&+\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-2\sigma_3^2-6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\left(I_2(t+\tau_2)-I_2^*\right)^2\\
&-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-2\sigma_3^2-6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\left(I_2(t)-I_2^*\right)^2\\
=&-\frac{1}{2}e^{2\mu_1\tau_1}\left(\mu_1-\mu_0-2\sigma_2^2-6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)\left(I_1(t)-I_1^*\right)^2\\
&-\frac{1}{2}e^{2\mu_2\tau_2}\left(\mu_2-\mu_0-2\sigma_3^2-6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\left(I_2(t)-I_2^*\right)^2\\
&+\left(\sigma_1^2+3\int_A D_1^2(y)\nu(\mathrm{d}y)\right)S^{*2}+e^{2\mu_1\tau_1}\left(\sigma_2^2+3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^{*2}
\end{aligned}
$$

$$+ e^{2\mu_2\tau_2}\left(\sigma_3^2 + 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\sigma_3^2 I_2^{*2} + \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} M_4. \tag{3.30}$$

Then

$$\begin{aligned}
\int_0^t \mathrm{d}V_7 = &\int_0^t LV_7 \mathrm{d}\mu + \int_0^t \left\{ \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} \left[\frac{1}{\mu_0}\left(\frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right)\sigma_1 (S - S^*)\right.\right. \\
&\left.\left. + \sigma_1 S (S - S^*)\right] + \left[(S - S^*) + e^{\mu_1\tau_1}(I_1(t+\tau_1) - I_1^*) + e^{\mu_2\tau_2}(I_2(t+\tau_2) - I_2^*)\right]\sigma_1 S \right\} \mathrm{d}B_1(t) \\
&+ \int_0^t \left\{ \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} \left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right) \frac{e^{\mu_1\tau_1}}{\mu_0}\sigma_2 (I_1(t+\tau_1) - I_1^*) \right. \\
&\left. + \left[(S - S^*) + e^{\mu_1\tau_1}(I_1(t+\tau_1) - I_1^*) + e^{\mu_2\tau_2}(I_2(t+\tau_2) - I_2^*)\right]\sigma_2 e^{\mu_1\tau_1} I_1(t+\tau_1) \right\} \mathrm{d}B_2(t) \\
&+ \int_0^t \left\{ \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} \left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right) \frac{e^{\mu_2\tau_2}}{\mu_0}\sigma_3 (I_2(t+\tau_2) - I_2^*) \right. \\
&\left. + \left[(S - S^*) + e^{\mu_1\tau_1}(I_1(t+\tau_1) - I_1^*) + e^{\mu_2\tau_2}(I_2(t+\tau_2) - I_2^*)\right]\sigma_3 e^{\mu_2\tau_2} I_2(t+\tau_2) \right\} \mathrm{d}B_3(t) \\
&+ \int_0^t \left\{ \int_A \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)} \left\{ \left(\mu_0 S^* + \frac{\beta_1 S^* I_1^*}{1+\alpha_1 I_1^*} + \frac{\beta_2 S^* I_2^*}{1+\alpha_2 I_2^*}\right) \cdot \right.\right. \\
&\left[\frac{e^{\mu_1\tau_1}}{\mu_0}\left[D_2(y) I_1(t+\tau_1) - \ln(1 + D_2(y))\right] + \frac{e^{\mu_2\tau_2}}{\mu_0}\left[D_3(y) I_2(t+\tau_2) - \ln(1 + D_3(y))\right]\right] \\
&\left. + \frac{1}{2} D_1^2(y) S^2 + D_1(y) S^2 - D_1(y) S S^* \right\} + \frac{1}{2}\left[e^{\mu_1\tau_1} D_2(y) I_1(t+\tau_1) + e^{\mu_2\tau_2} D_3(y) I_2(t+\tau_2)\right. \\
&\left. + D_1(y) S\right]^2 + \left[(S - S^*) + e^{\mu_1\tau_1}(I_1(t+\tau_1) - I_1^*) + e^{\mu_2\tau_2}(I_2(t+\tau_2) - I_2^*)\right] \cdot \\
&\left.\left[D_1(y) S + D_2(y) I_1(t+\tau_1) + D_3(y) I_2(t+\tau_2)\right] \tilde{N}(\mathrm{d}t, \mathrm{d}y) \right\}.
\end{aligned} \tag{3.31}$$

Taking the expectation, we have

$$EV_7(S(t), I_1(t), I_2(t)) \le EV_7(S(0), I_1(0), I_2(0)) + E\int_0^t LV_7(S(\mu), I_1(\mu), I_2(\mu)) \mathrm{d}\mu.$$

Then

$$\begin{aligned}
0 \le & E\int_0^t \left[-\frac{1}{2} e^{2\mu_1\tau_1}\left(\mu_1 - \mu_0 - 2\sigma_2^2 - 6\int_A D_2^2(y)\nu(\mathrm{d}y)\right)(I_1(\mu) - I_1^*)^2 \right. \\
& \left. - \frac{1}{2} e^{2\mu_2\tau_2}\left(\mu_2 - \mu_0 - 2\sigma_3^2 - 6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)(I_2(\mu) - I_2^*)^2 \right] \mathrm{d}\mu + EV_7(S(0), I_1(0), I_2(0)) \\
& + \left\{ \left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right) S^{*2} + e^{2\mu_1\tau_1}\left(\sigma_2^2 + 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right) I_1^{*2} \right.
\end{aligned}$$

$$+ e^{2\mu_2\tau_2}\left(\sigma_3^2 + 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)I_2^{*2} + \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}M_4\Bigg\}t.$$

Hence,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[(I_1(\mu) - I_1^*)^2 + (I_2(\mu) - I_2^*)^2\right]\mathrm{d}\mu \\
&\le \frac{2}{H_3}\Bigg\{\left(\sigma_1^2 + 3\int_A D_1^2(y)\nu(\mathrm{d}y)\right)S^{*2} + e^{2\mu_1\tau_1}\left(\sigma_2^2 + 3\int_A D_2^2(y)\nu(\mathrm{d}y)\right)I_1^{*2} \\
&+ e^{2\mu_2\tau_2}\left(\sigma_3^2 + 3\int_A D_3^2(y)\nu(\mathrm{d}y)\right)I_2^{*2} + \frac{\mu_1 + \mu_2 + 2\sigma_1^2 + 6\int_A D_1^2(y)\nu(\mathrm{d}y)}{2\left(\mu_0 - \sigma_1^2 - \int_A D_1^2(y)\nu(\mathrm{d}y)\right)}M_4\Bigg\},
\end{aligned}$$

where
$H_3 = \min\left\{e^{2\mu_1\tau_1}\left(\mu_1 - \mu_0 - 2\sigma_2^2 - 6\int_A D_2^2(y)\nu(\mathrm{d}y)\right), e^{2\mu_2\tau_2}\left(\mu_2 - \mu_0 - 2\sigma_3^2 - 6\int_A D_3^2(y)\nu(\mathrm{d}y)\right)\right\}$.

Remark 4. Theorem 3.4 shows that when $R_1 > 1$ and $R_2 > 1$ and certain conditions are satisfied, the solution to the stochastic delayed system (1.3) will oscillate around the equilibrium point E_3. The intensity of these oscillations is related to the values of σ_i and $D_i(y)(i = 1, 2, 3)$. The smaller the noise intensity, the closer the solution of the stochastic delayed system (1.3) is to the equilibrium point E_3. This means that both Disease I and Disease II will prevail in the population simultaneously.

4. Numerical simulation

Both white noise and Lévy noise induce oscillations of the solutions for the epidemic models around the corresponding equilibrium points, with oscillation intensity increasing with noise magnitude. However, Lévy noise causes more significant and abrupt oscillations compared to white noise. Its disturbance on epidemic transmission trajectories is far more pronounced, highlighting the crucial value of incorporating Lévy noise into stochastic delayed model research.

In this chapter, we will use the parameters set in the table to perform numerical simulations, so as to verify the theoretical results we obtained for the stochastic delayed system (1.3). In the following four examples, we apply the Euler–Maruyama algorithm proposed in references [18, 19] to discretize the system and use the compound Poisson process proposed in reference [19] to approximate Lévy noise.

In the following simulation of the stochastic delayed system (1.3), we consider that the jump part is the compound Poisson process $X_t = \sum_{i=1}^{N_t} Y_i$, where N_t is a Poisson process with mean λt and the jump size Y_i is independent and an identically distributed random variable with distribution function F. The Lévy measure of X_t is given by $\nu(A) = \lambda\int_A dF(x)$. Here, we take the jump size to follow the standard normal distribution; the jump intensity λ is 1, $\nu(dx) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx$, and $D_i(y) = y$, $i = 1, 2, 3$.

The complete discretized equations are now presented as

$$S(t + \Delta t) = S(t) + \left[\Lambda - \mu_0 S(t) - \frac{\beta_1 S(t)I_1(t)}{1 + \alpha_1 I_1(t)} - \frac{\beta_2 S(t)I_2(t)}{1 + \alpha_2 I_2(t)} - \lambda_1\mathbb{E}[D_1]S(t)\right]\Delta t$$

$$+\sigma_1 S(t)\sqrt{\Delta t}\epsilon_1(t) + S(t)\sum_{j=1}^{N_1(\Delta t)} D_1(y_j),$$

$$I_1(t+\Delta t) = I_1(t) + \left[e^{-\mu_1\tau_1} \frac{\beta_1 S(t-\tau_1) I_1(t-\tau_1)}{1+\alpha_1 I_1(t-\tau_1)} - \mu_1 I_1(t) - \lambda_2 \mathbb{E}[D_2] I_1(t) \right] \Delta t$$

$$+\sigma_2 I_1(t)\sqrt{\Delta t}\epsilon_2(t) + I_1(t) \sum_{j=1}^{N_2(\Delta t)} D_2(y_j),$$

$$I_2(t+\Delta t) = I_2(t) + \left[e^{-\mu_2\tau_2} \frac{\beta_2 S(t-\tau_2) I_2(t-\tau_2)}{1+\alpha_2 I_2(t-\tau_2)} - \mu_2 I_2(t) - \lambda_3 \mathbb{E}[D_3] I_2(t) \right] \Delta t$$

$$+\sigma_3 I_2(t)\sqrt{\Delta t}\epsilon_3(t) + I_2(t) \sum_{j=1}^{N_3(\Delta t)} D_3(y_j),$$

where $\Delta t = 0.01$ is the time step, $\epsilon_i(t) \sim N(0,1)$ are independent standard normal variables, $N_i(\Delta t) \sim \text{Poisson}(\lambda_i \Delta t)$ are Poisson counting processes, and y_j are jump sizes with $-\lambda_i \mathbb{E}[D_i]X(t)\Delta t$ for proper drift correction.

First, set the initial value of the system (1.3) as $(S(0), I_1(0), I_2(0)) = (3, 2, 1)$. Next, set the relevant parameters of the system (1.3). See details in Tables 2 and 3.

Table 2. System parameter settings.

Parameter	Figure 1	Figure 2	Figure 3	Figure 4
Λ	0.80	1.00	1.00	1.20
μ_0	0.30	0.25	0.25	0.20
β_1	0.25	0.28	0.20	0.28
β_2	0.20	0.20	0.28	0.26
μ_1	0.50	0.42	0.50	0.45
μ_2	0.55	0.50	0.42	0.42
α_1	1.00	1.00	1.00	1.00
α_2	0.80	0.80	0.80	0.80
τ_1	2.00	1.20	2.20	1.80
τ_2	2.50	2.20	1.20	1.50

Table 3. Value of the $(\sigma_i, D_i), i = 1, 2, 3$.

	Figure 1			Figure 2			Figure 3			Figure 4		
	(b)	(c)	(d)	(b)	(c)	(d)	(b)	(c)	(d)	(b)	(c)	(d)
σ_1	0.12	0.12	0.20	0.12	0.12	0.20	0.12	0.12	0.20	0.12	0.12	0.22
σ_2	0.10	0.10	0.18	0.10	0.10	0.16	0.10	0.10	0.15	0.10	0.10	0.18
σ_3	0.10	0.10	0.18	0.10	0.10	0.15	0.10	0.10	0.16	0.10	0.10	0.18
D_1	0.02	0.08	0.08	0.02	0.08	0.08	0.02	0.08	0.08	0.02	0.08	0.08
D_2	0.02	0.06	0.06	0.02	0.05	0.05	0.02	0.06	0.06	0.02	0.06	0.06
D_3	0.02	0.06	0.06	0.02	0.06	0.06	0.02	0.05	0.05	0.02	0.06	0.06

Example 1: The parameter settings of the system are as shown in Column 1 of Table 2. Then there is an equilibrium point $E_0 = (2.667, 0, 0)$, with $R_1 = 0.49 < 1$ and $R_2 = 0.25 < 1$, indicating that the diseases will eventually die out.

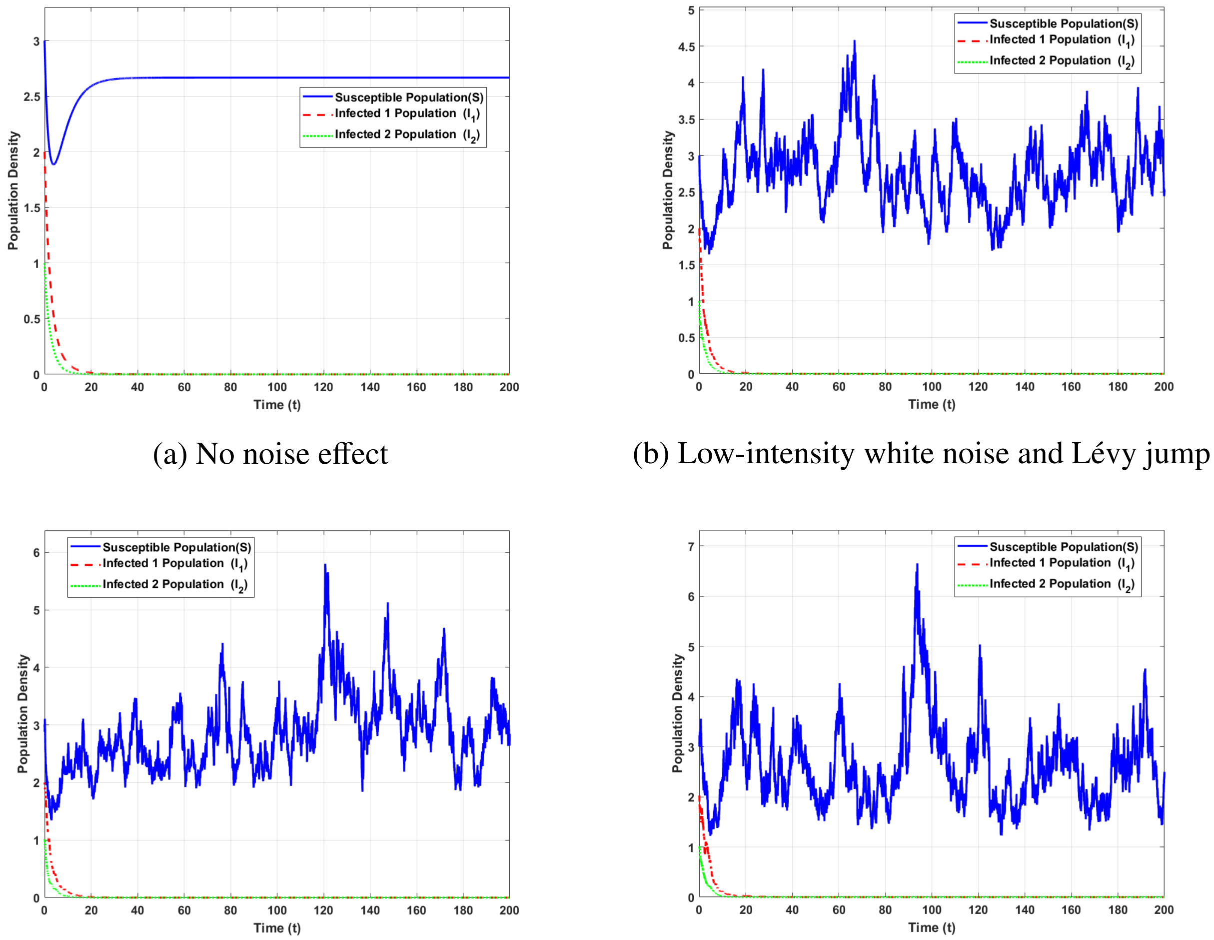
To illustrate the influence of stochastic perturbations on system dynamics, we present four simulation scenarios with different noise intensities, all satisfying the conditions of Theorem 3.1.

In Figure 1(a), we set the noise intensities as $(\sigma_i, D_i) = (0, 0)$ for $i = 1, 2, 3$. The solutions converge smoothly to the disease-free equilibrium $E_0 = (2.667, 0, 0)$. The susceptible population stabilizes at the theoretical value, while the two infected populations gradually approach zero without any fluctuations, fully consistent with the disease extinction prediction of the deterministic model.

In Figure 1(b), the noise intensities are set as listed in Column 1 of Table 3, where low-intensity white noise and the Lévy jump are introduced simultaneously. The solutions exhibit small-amplitude oscillations around E_0. The susceptible population fluctuates slightly near 2.667, and although the two infected populations show transient small increases, they generally tend to zero. The combined disturbance of the two noises is weak, and the long-term trend of eventual disease extinction remains unchanged.

In Figure 1(c), the noise intensities are set as listed in Column 2 of Table 3, where the white noise intensity is increased. The oscillation intensity is significantly larger than that in Figure 1(b). The fluctuation range of the susceptible population expands, and the short-term fluctuations of the infected populations become more frequent, but the fluctuations are relatively stable without sudden peaks. The core trend of disease extinction is not affected; only the volatility of transient dynamics increases.

In Figure 1(d), the noise intensities are set as listed in Column 3 of Table 3, where the Lévy jump intensity is further increased. The oscillation intensity is remarkably higher than that in Figure 1(c), with obvious sudden fluctuations. The susceptible population may deviate from the equilibrium point abruptly, and the infected populations also experience transient sudden increases followed by rapid declines. Despite the long-term tendency toward disease extinction, the jump-like disturbances caused by the Lévy jump greatly enhance the uncertainty of transmission trajectories, which is far beyond the stable fluctuation effect of white noise.



(a) No noise effect

(b) Low-intensity white noise and Lévy jump

(c) Increased white noise

(d) Increased Lévy jump

Figure 1. Dynamic behavior of model (1.3) near the equilibrium point E_0 for S, I_1 and I_2 under different noise intensities (parameters in Table 2, Col. 1).

Example 2: The parameter settings of the system are as shown in Column 2 of Table 2. Then there is a equilibrium point $E_1 = (3.199, 0.2881, 0)$, with $R_1 = 1.61 > 1$ and $R_2 = 0.53 < 1$, indicating that only the Disease I strain persists while Disease II dies out.

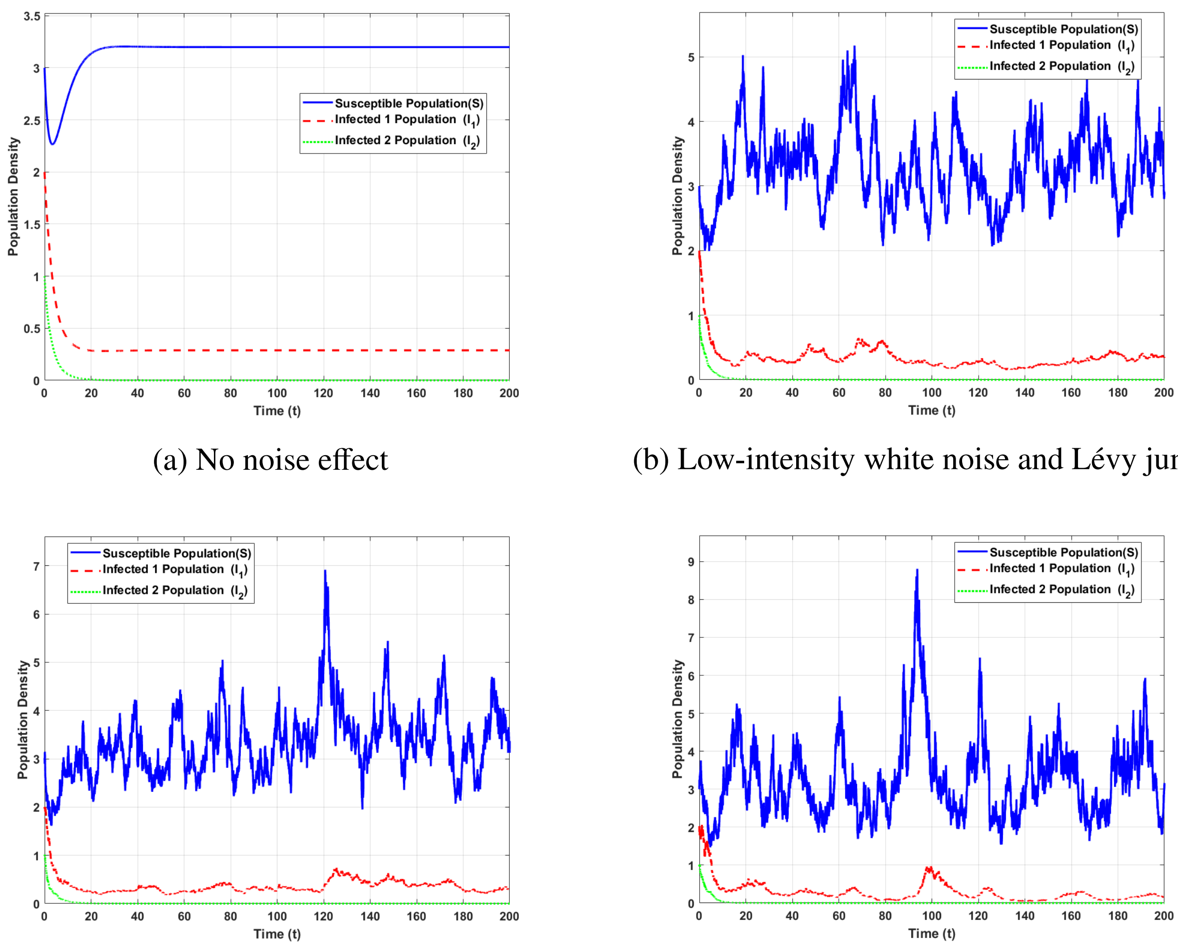
To illustrate the influence of stochastic perturbations on system dynamics, we present four simulation scenarios with different noise intensities, all satisfying the conditions of Theorem 3.2.

In Figure 2(a), the noise intensities are set as $(\sigma_i, D_i) = (0, 0)$ for $i = 1, 2, 3$. The solutions stably converge to the equilibrium $E_1 = (3.199, 0.2881, 0)$. The susceptible population stabilizes at the corresponding value, the infected population of Disease I maintains around 0.2881, and the infected population of Disease II approaches zero without fluctuations, reflecting the deterministic result of Disease I persisting and Disease II becoming extinct.

In Figure 2(b), the noise intensities are set as listed in Column 4 of Table 3, where low-intensity white noise and the Lévy jump are introduced simultaneously. The solutions show small-amplitude oscillations around E_1. The infected population of Disease I fluctuates slightly near 0.2881, and the infected population of Disease II tends to zero. The combined effect of the two noises does not change the long-term pattern of "Disease I persists and Disease II becomes extinct".

In Figure 2(c), the noise intensities are set as listed in Column 5 of Table 3. The white noise intensity is increased, the fluctuation frequency of the infected population of Disease I rises, and the fluctuation range of the susceptible population expands, but the fluctuation process is stable without sudden extreme values. In the long run, the trend of Disease I persisting and Disease II becoming extinct is still maintained.

In Figure 2(d), the noise intensities are set as listed in Column 6 of Table 3. Here, the Lévy jump intensity is increased, and the oscillations exhibit strong suddenness and large-amplitude fluctuation characteristics. The infected population of Disease I may deviate from the equilibrium point of 0.2881 abruptly, showing transient peaks, and the susceptible population also undergoes sudden fluctuations. Compared with increased white noise, the jump effect of Lévy noise significantly improves the unpredictability of transmission trajectories. Even though the long-term trend remains unchanged, short-term sudden fluctuations may pose greater challenges to disease prevention and control.



(a) No noise effect

(b) Low-intensity white noise and Lévy jump

(c) Increased white noise

(d) Increased Lévy jump

Figure 2. Dynamic behavior of model (1.3) near the equilibrium point E_1 for S, I_1 and I_2 under different noise intensities (parameters in Table 2, Col. 2).

Example 3: System parameters are set according to Column 3 of Table 2. For the deterministic case, the system possesses a boundary equilibrium $E_2 = (3.115, 0, 0.3180)$, with $R_1 = 0.53 < 1$ and $R_2 = 1.61 > 1$, indicating that only the Disease II strain persists while the Disease I dies out.

To investigate the effect of stochastic perturbations, we simulate the system under four noise configurations, each satisfying Theorem 3.3.

In Figure 3(a), the solutions converge smoothly to the equilibrium $E_2 = (3.115, 0, 0.3180)$. The susceptible population stabilizes at the corresponding value, the infected population of Disease II maintains around 0.3180, and the infected population of Disease I approaches zero, fully consistent with the prediction of "Disease II persists and Disease I becomes extinct" in the deterministic model.

In Figure 3(b), the noise intensities are set as listed in Column 7 of Table 3, where low-intensity white noise and the Lévy jump are introduced simultaneously. The solutions present continuous small-amplitude oscillations around E_2. The infected population of Disease II fluctuates near 0.3180, and the infected population of Disease I tends to zero. The disturbance of the two noises is weak, and the transmission pattern remains unchanged.

In Figure 3(c), white noise intensity is increased, and the noise intensities are set as listed in Column 8 of Table 3. The oscillation amplitude is significantly larger than that in Figure 3(b). The fluctuation range of the infected population of Disease II expands, and the fluctuation frequency increases, but the overall fluctuations are relatively stable without sudden drastic changes. In the long run, the trend of Disease II persisting and Disease I becoming extinct is still maintained.

In Figure 3(d), the Lévy jump is further increased, and the noise intensities are set as listed in Column 9 of Table 3. the oscillations exhibit strong suddenness and large-amplitude fluctuation characteristics. The infected population of Disease II may deviate from the equilibrium point of 0.3180 abruptly, showing transient peaks, and the susceptible population also undergoes sudden fluctuations. Compared with increased white noise, the jump effect of the Lévy jump significantly improves the unpredictability of transmission trajectories. Even though the long-term trend remains unchanged, short-term sudden fluctuations may pose greater challenges to disease prevention and control.

Example 4: System parameters are chosen according to Table 2. For the deterministic case, the system admits an interior equilibrium $E_3 = (3.9720, 0.0995, 0.3867)$, with $R_1 = 1.66 > 1$ and $R_2 = 1.98 > 1$. This indicates the coexistence of both disease strains in the long term.

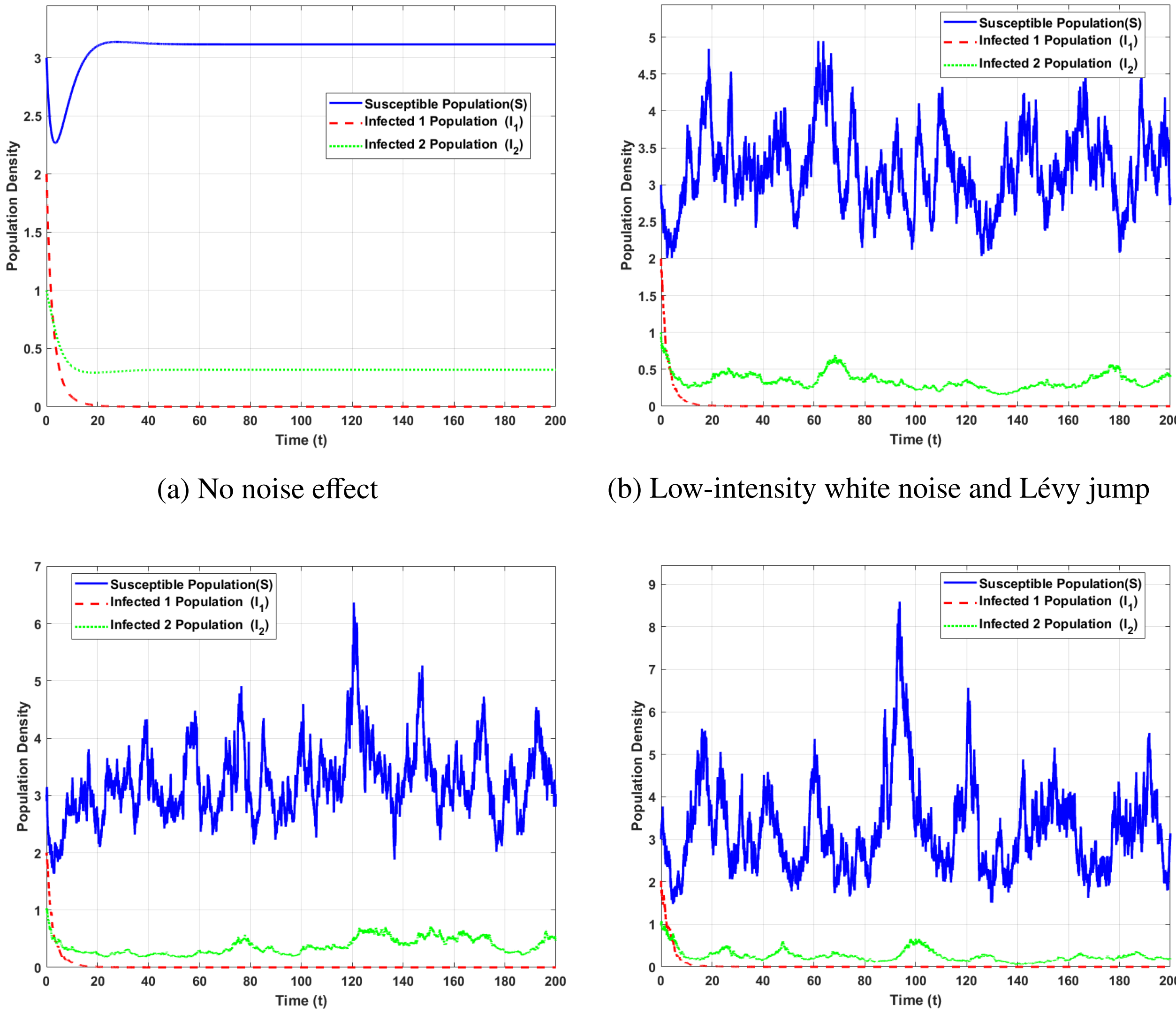
To examine how stochastic perturbations influence this endemic coexistence state, we simulate the system under four noise scenarios, all of which satisfy Theorem 3.4.

In Figure 4(a), we set the noise intensities as $(\sigma_i, D_i) = (0, 0)$ for $i = 1, 2, 3$. The solutions converge smoothly to the equilibrium $E_3 = (3.9720, 0.0995, 0.3867)$. The susceptible population stabilizes at the theoretical value, while the two infected populations remain steady near 0.0995 and 0.3867, respectively, without fluctuations, fully consistent with the prediction of the deterministic model.

In Figure 4(b), the noise intensities are set as listed in Column 10 of Table 3, where low-intensity white noise and the Lévy jump are introduced simultaneously. The solutions exhibit small-amplitude oscillations around E_3. The susceptible population fluctuates slightly near 3.9720, and although the two infected populations show transient small increases or decreases, they generally tend to their respective equilibrium values. The combined disturbance of the two noises is weak, and the long-term trend of persistence of the two diseases remains unchanged.

In Figure 4(c), the noise intensities are set as listed in Column 11 of Table 3, and the white noise intensity is increased. The oscillation intensity is significantly larger than that in Figure 4(b). The fluctuation range of the susceptible population expands, and the short-term fluctuations of the two infected populations become more frequent, but the fluctuations are relatively stable without sudden peaks. The trend of persistence of the two diseases remains unchanged.

In Figure 4(d), the Lévy jump intensity is increased, and the noise intensities are set as listed in Column 12 of Table 3. The oscillations show strong suddenness and large-amplitude fluctuation characteristics. The two infected populations may deviate from their respective equilibrium values abruptly, showing transient peaks, and the fluctuation amplitude is far greater than that in the increased white noise scenario. The jump effect of the Lévy jump makes the presistence trajectory of the two diseases highly uncertain. Its disturbance effect on the system is much greater than that of white noise.



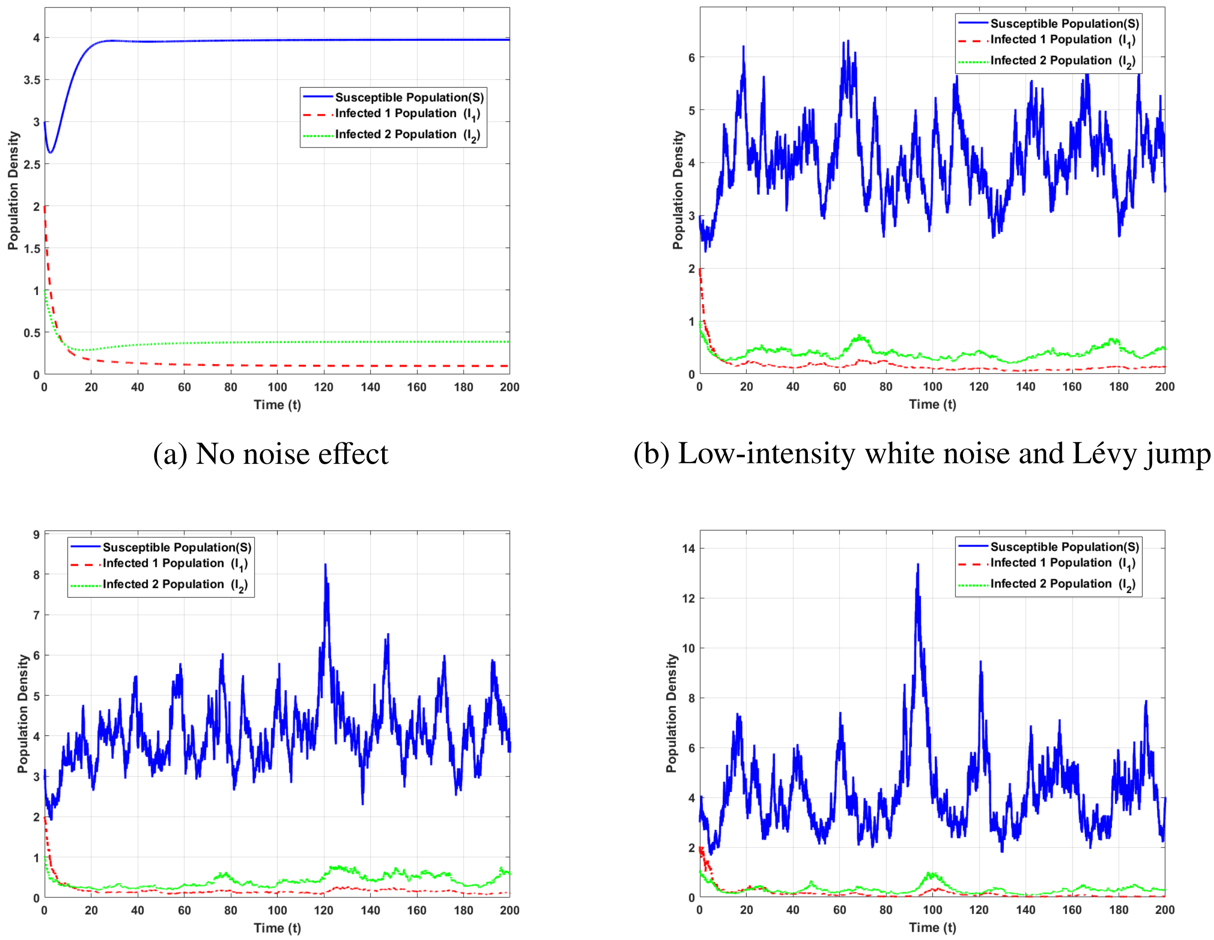
(a) No noise effect

(b) Low-intensity white noise and Lévy jump

(c) Increased white noise

(d) Increased Lévy jump

Figure 3. Dynamic behavior of model (1.3) near the equilibrium point E_2 for S, I_1 and I_2 under different noise intensities (parameters in Table 2, Col. 3).



(a) No noise effect

(b) Low-intensity white noise and Lévy jump

(c) Increased white noise

(d) Increased Lévy jump

Figure 4. Dynamic behavior of model (1.3) near the equilibrium point E_3 for S, I_1 and I_2 under different noise intensities (parameters in Table 2, Col. 4).

With MATLAB software, we plot the solution time series of a stochastic delayed model under different threshold conditions, verify the theoretical result that the solutions oscillate around the corresponding equilibrium points, and intuitively demonstrate the influence of noise intensity on the spread trajectory.

5. Future research directions

We intend to construct a stochastic delayed SI_1I_2 epidemic model that incorporates both Markov switching and Lévy noise simultaneously, and then conduct an in-depth analysis of its dynamical behaviors. Furthermore, we will further carry out in-depth research on the persistence and extinction of the stochastic delayed double epidemic model. The systematic analysis of the persistence and extinction conditions of the two infectious diseases will provide more targeted theoretical support for the prediction and control of double epidemic outbreaks in practical scenarios.

Author contributions

Xueqing Li: Conceptualization, Formal analysis, Writing–original draft; Yingjia Guo: Methodology, Validation, Writing–review and editing. Both authors have read and approved the final version of the manuscript for publication.

Use of generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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