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Research article

Weak Pareto–Nash equilibria in generalized interval-valued multiobjective games with fuzzy constraint mappings and applications to price competition

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Abstract: In this study, we examine a generalized interval-valued multiobjective game with fuzzy constraint mappings (GIMGFCM). By employing interval support functions from convex geometry, the set of all d-dimensional interval vectors is embedded into the space of real-valued continuous functions defined on the unit sphere in $\mathbb{R}^d$. This embedding yields a closed convex cone $\mathcal{I}(\mathbb{R}^d_+)$ within that function space. Using this cone, we define a partial order for interval vectors and establish the semi-continuity and generalized $\mathcal{I}(\mathbb{R}^d_+)$-quasi-concavity of interval-vector-valued functions. On this basis, we propose a weak Pareto–Nash equilibrium concept for a GIMGFCM and prove an existence theorem for such equilibria. Finally, we apply the relevant theoretical framework to analyze a price competition problem between two firms that sell heterogeneous goods.

Keywords: generalized interval-valued multiobjective game; weak Pareto–Nash equilibrium; interval-vector-valued function; fuzzy constraint mappings; price competition
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1. Introduction

Multiobjective game theory was initially introduced by Zeleny [1], who analyzed two-person multiobjective zero-sum games using parameter vectors and weighted coefficients, with solutions explored through parameter variation. Subsequently, Wang [2] established several sufficient conditions for the existence of Pareto equilibria in such games. Yu and Yuan [3] further investigated the existence of Pareto equilibria by applying the Ky Fan minimax inequality and the Fan–Glicksberg fixed point theorem. Additionally, Yang and Yu [4] examined the essential components of the set of weak Pareto–Nash equilibrium points in multiobjective games. Yuan and Tarafdar [5] applied the Ky

Fan minimax principle to investigate the existence of weighted Nash equilibria and Pareto equilibria in non-compact multiobjective games, deriving several existence theorems under non-compact conditions. More recently, building on the intersection theorem, Chen and Jia [6] established the existence of weak Pareto–Nash equilibria in multiobjective games with infinitely many players under a compactness assumption. Subsequently, by extending the intersection theorem to non-compact settings, they further demonstrated the existence of such equilibria even in the absence of compactness constraints.

Ding [7] pioneered the introduction of the feasible strategy set-valued mapping into multiobjective games, establishing the theoretical foundation for generalized multiobjective games. He demonstrated the existence of Pareto equilibria in such games and extended related findings from [3, 8]. Subsequent research has further explored the existence of equilibria and the stability of equilibrium sets in generalized multiobjective games. Lin [9] studied the existence and essential components of the set of weak Pareto–Nash equilibrium points for generalized multiobjective games in two different uniform topological spaces. Song and Wang [10] studied the stability of both weak Pareto–Nash and Pareto–Nash equilibria under payoff perturbations, showing that in most cases, at least one essential Pareto–Nash equilibrium exists. Recently, Hung and Keller [11] investigated the existence and convergence of solutions for generalized multiobjective game control systems and their optimal control problems, applying the theoretical results to traffic network control systems with notable progress.

In classical generalized multiobjective game theory, the feasible strategy set-valued mappings and the payoff functions of players are typically defined as deterministic sets and vectors. In real-world and economic contexts, however, complex environments and various fuzzy or uncertain factors often cause the values of feasible strategy set-valued mappings to be represented as fuzzy sets, while the values of payoff functions become interval vectors. As a result, the classical multiobjective game model is insufficient for fully capturing the complexity of real-world multiobjective game problems. There is a need to develop a more general multiobjective game framework that better reflects actual conditions.

Moore [12] first introduced the concepts of interval numbers and interval-valued functions, establishing the groundwork for interval analysis. Subsequently, numerous scholars have conducted extensive research on both the theoretical and applied aspects of interval analysis, as documented in [13–15] and the references cited therein. In [16], it was noted that interval systems based on Moore's operations do not form a group due to the lack of inverses for nonzero intervals. To address this limitation, Hukuhara [17] proposed the notion of interval difference, known as the H-difference. However, the H-difference is not clearly defined for both narrow and broad intervals. To resolve this ambiguity, Stefanini and Bede [18] introduced the generalized H-difference (gH-difference), which extends the H-difference concept. Building on the gH-difference, they further developed the theory of the generalized Hukuhara derivative (gH-derivative) for interval-valued functions. Currently, the gH-difference and gH-derivative have been widely utilized in various domains, including interval optimization [19–21], interval optimal control [22], and interval differential equations [18, 23, 24]. However, applications of the gH-difference in multiobjective interval games remain relatively unexplored.

Zadeh introduced the theory of fuzzy sets in his seminal work [25], and since its inception, the theory has found broad applications across diverse scientific disciplines, including operations analysis [26], social sciences [27], fuzzy control [28, 29], and artificial intelligence [30, 31].

Butnariu [32] pioneered the incorporation of fuzzy sets into non-cooperative games, establishing a foundational framework for such fuzzy game models. Campos [33] later developed a solution approach for two-person zero-sum matrix games with fuzzy elements using fuzzy linear programming. Clemente [34] introduced a standardized fuzzy order to compare fuzzy payoffs and investigated Pareto equilibria in fuzzy matrix games. Based on the fuzzy decision-making principle proposed by Bellman and Zadeh [35], Buckley [36] formulated a multiobjective non-cooperative game model within a fuzzy environment. For additional studies on fuzzy games, see [37–39]. In the study of generalized multiobjective games, the fuzziness in strategy selection has increasingly attracted scholarly attention. Hung et al. [40] developed a class of generalized multiobjective game models incorporating fuzzy strategies, in which each payoff function takes values in a Hausdorff topological vector space, and further examined the existence of fuzzy equilibria in such games.

Building on previous studies [4, 9, 18, 40] and seeking a more rigorous characterization of the objective world and economic behavior, this paper investigates a class of generalized interval-valued multiobjective games with fuzzy constraint mappings (GIMGFCMs). Such games are characterized as follows: Each player's strategy set is contained within a locally convex Hausdorff topological vector space, the feasible strategy mapping is modeled as a fuzzy mapping, and the payoff function is represented by an interval-valued vector function. Since the space of interval vectors constitutes a quasi-linear space, rather than a conventional linear space, the equilibrium results from [40] cannot be directly extended to a GIMGFCM. It is therefore necessary to introduce a comparison method for interval vectors. Using the interval support function, this paper elucidates the relationship between the support function norm of a d-dimensional interval vector and the support function norms of its components. We then define a norm on the quasi-linear space $\mathcal{I}(\mathbb{R}^d)$ of all d-dimensional interval vectors. Under the condition of norm invariance, we demonstrate that $\mathcal{I}(\mathbb{R}^d)$ can be embedded into $C(\mathbb{S}^{d-1})$. In particular, the subset $\mathcal{I}(\mathbb{R}^d_+)$ of all non-negative d-dimensional interval vectors forms a closed convex cone in $C(\mathbb{S}^{d-1})$. Utilizing this cone, we introduce a partial order on $\mathcal{I}(\mathbb{R})$ and employ it to analyze the weak Pareto–Nash equilibrium.

This paper is structured as follows. Section 2 provides an overview of fundamental concepts related to intervals, interval vectors, fuzzy mappings, and associated results. In Section 3, we introduce interval support functions and define a partial order for interval vectors. As will be demonstrated, this new tool proves instrumental in Sections 4 and 5. To analyze the weak Pareto–Nash equilibrium in the GIMGFCM, Section 4 examines the semi-continuity and generalized quasi-concavity of interval vector-valued functions in the context of cones. Section 5 establishes an existence theorem for weak Pareto–Nash equilibria for the GIMGFCM. This result yields two corollaries regarding the existence of such equilibria in generalized interval-valued games with fuzzy constraint mappings and in generalized multiobjective games with fuzzy constraint mappings. Section 6 illustrates a practical application of the weak Pareto–Nash equilibrium in heterogeneous market price competition. Finally, Section 7 summarizes the main findings and suggests avenues for future research.

2. Preliminaries and terminology

In this section, we review the basic terminology of intervals, interval vectors, fuzzy mappings, and some related conclusions.

Nomenclature

$\mathbb{R}$	the set of all real numbers
$\mathbb{R}_+$	the set of all nonnegative real numbers
$\mathbb{R}^d$	the d-dimensional Euclidean space
$\mathbb{S}^{d-1}$	the unit sphere in $\mathbb{R}^d$
$C(\mathbb{S}^{d-1})$	the space of real-valued continuous functions defined on $\mathbb{S}^{d-1}$
$\mathcal{I}(\mathbb{R})$	the class of all compact intervals in $\mathbb{R}$
$\mathcal{I}(\mathbb{R}_+)$	the class of all compact intervals in $[0, +\infty)$
$\mathcal{I}(\mathbb{R}^d)$	the class of all d-dimensional interval vectors
$\mathcal{I}(\mathbb{R}^d_+)$	the class of all d-dimensional interval vectors where each component is in $\mathcal{I}(\mathbb{R}_+)$
$\mathcal{F}(Y)$	the class of all fuzzy sets on Y
$\widetilde{a}^\alpha$	the α-cut of the fuzzy set $\widetilde{a}$
$\widetilde{G}$	the fuzzy mapping
$h_{\boldsymbol{I}}(\cdot)$	the support function of interval vector $\boldsymbol{I}$
$\|\cdot\|_{\mathcal{I}(\mathbb{R}^d)}$	the norm of interval vector

2.1. Interval arithmetic

Denote by $\mathcal{I}(\mathbb{R})$ the class of all compact intervals in $\mathbb{R}$, i.e.,

$$\mathcal{I}(\mathbb{R}) = \left\{[\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \overline{a} \geq \underline{a}\right\}.$$

And denote by $\mathcal{I}(\mathbb{R}_+)$ the class of all compact intervals contained in $[0, +\infty)$, i.e.,

$$\mathcal{I}(\mathbb{R}_+) = \left\{[\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \overline{a} \geq \underline{a} \geq 0\right\}.$$

For $I_1 = [\underline{a}_1, \overline{a}_1],\ I_2 = [\underline{a}_2, \overline{a}_2] \in \mathcal{I}(\mathbb{R})$ and $s \in \mathbb{R}$, the following operations are considered:

$$I_1 + I_2 = [\underline{a}_1 + \underline{a}_2, \overline{a}_1 + \overline{a}_2] \tag{2.1}$$

and

$$s \cdot I_1 = \begin{cases} [s\underline{a}_1, s\overline{a}_1], & \text{if } s \geq 0, \\ [s\overline{a}_1, s\underline{a}_1], & \text{if } s < 0. \end{cases} \tag{2.2}$$

Form Eqs (2.1) and (2.2), it follows that

$$-I_1 = [-\overline{a}_1, -\underline{a}_1] \quad \text{and} \quad I_2 - I_1 = [\underline{a}_2 - \overline{a}_1, \overline{a}_2 - \underline{a}_1].$$

Since a compact interval may not possess an inverse element, the space $\mathcal{I}(\mathbb{R})$ equipped with the operations $\{+, -, .\}$ does not qualify as a linear space. Hence, the subtraction $I_2 - I_1$ does not have adequate properties (see [15, 18]). As a supplement, Stefanini and Bede [18] put forward the following gH-difference of two compact intervals.

Definition 2.1. *(gH-difference [18]) Let I_1 and I_2 be two compact intervals. The gH-difference between I_1 and I_2, denoted $I_1 \ominus I_2$, is a compact interval I_3 such that*

$$I_1 = I_2 + I_3 \quad \text{or} \quad I_2 = I_1 + (-1) \cdot I_3.$$

It is noted that in some articles (e.g., [15, 18]), the notation $\ominus_{gH}$ rather than $\ominus$ is taken for the gH-difference. However, for the sake of simplicity, we adopt the latter in the article. For an interval $I = [\underline{a}, \overline{a}]$, its width is $\omega(I) = \overline{a} - \underline{a}$. Then for all

$$I_1 = [\underline{a}_1, \overline{a}_1], \quad I_2 = [\underline{a}_2, \overline{a}_2] \in \mathcal{I}(\mathbb{R}),$$

we have

$$I_1 \ominus I_2 = \begin{cases} [\underline{a}_1 - \underline{a}_2, \overline{a}_1 - \overline{a}_2], & \text{if } \omega(I_1) \geq \omega(I_2), \\ [\overline{a}_1 - \overline{a}_2, \underline{a}_1 - \underline{a}_2], & \text{if } \omega(I_2) > \omega(I_1). \end{cases} \tag{2.3}$$

The gH-difference of two compact intervals has some very interesting properties. For example, we always have $I_1 \ominus I_1 = [0, 0]$. Moreover, the gH-difference between I_1 and I_2 always exists and is equal to (see [18])

$$I_1 \ominus I_2 = [\min\{\underline{a}_1 - \underline{a}_2, \overline{a}_1 - \overline{a}_2\}, \max\{\underline{a}_1 - \underline{a}_2, \overline{a}_1 - \overline{a}_2\}]. \tag{2.4}$$

In particular, $(-1) \cdot (I_1 \ominus I_2) = I_2 \ominus I_1$.

The Hausdorff metric H, which is defined in $\mathcal{I}(\mathbb{R})$, is expressed as

$$H(I_1, I_2) = \max\{|\underline{a}_1 - \underline{a}_2|, |\overline{a}_1 - \overline{a}_2|\}.$$

It is well known that the metric space $(\mathcal{I}(\mathbb{R}), H)$ is complete; see [18].

Definition 2.2. *(Norm on $\mathcal{I}(\mathbb{R})$ [12]) Let $I = [\underline{a}, \overline{a}]$ be a compact interval. The function $\|\cdot\|_{\mathcal{I}(\mathbb{R})}$: $\mathcal{I}(\mathbb{R}) \to \mathbb{R}_+$, defined as*

$$\|I\|_{\mathcal{I}(\mathbb{R})} = \max\{|\underline{a}|, |\overline{a}|\},$$

is called a norm on $\mathcal{I}(\mathbb{R})$.

The space $\mathcal{I}(\mathbb{R})$ equipped with the norm $\|\cdot\|_{\mathcal{I}(\mathbb{R})}$ is a normed quasilinear space. Moreover, the space $(\mathcal{I}(\mathbb{R}), \|\cdot\|_{\mathcal{I}(\mathbb{R})})$ is complete; see [23]. Endowed with the topology induced by the norm $\|\cdot\|_{\mathcal{I}(\mathbb{R})}$ on $\mathcal{I}(\mathbb{R})$, we obtain

$$\operatorname{int} \mathcal{I}(\mathbb{R}_+) = \left\{[\underline{a}, \overline{a}] \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \overline{a} \geq \underline{a} > 0\right\}.$$

2.2. Interval vector arithmetic

An ordered d-tuple of intervals, $(I_1, I_2, \ldots, I_d)$, is referred to as a d-dimensional interval vector. Let $\mathcal{I}(\mathbb{R}^d)$ denote the class of all d-dimensional interval vectors. Moreover, let $\mathcal{I}(\mathbb{R}^d_+)$ denote the class of all d-dimensional interval vectors in which each component belongs to $\mathcal{I}(\mathbb{R}_+)$, that is,

$$\mathcal{I}(\mathbb{R}_+) = \{\boldsymbol{I} = (I_1, \ldots, I_d) \mid I_i \in \mathcal{I}(\mathbb{R}_+), i = 1, \ldots, d\}.$$

With appropriate modifications, many concepts of compact intervals can be extended to interval vectors. For $s \in \mathbb{R}$, and

$$\boldsymbol{I} = (I_1, \ldots, I_d), \boldsymbol{I}' = (I'_1, \ldots, I'_d) \in \mathcal{I}(\mathbb{R}^d),$$

the following operations on interval vectors are considered:

$$\boldsymbol{I} + \boldsymbol{I}' = (I_1 + I'_1, \ldots, I_d + I'_d) \tag{2.5}$$

and

$$s \cdot \boldsymbol{I} = (s \cdot I_1, \ldots, s \cdot I_d). \tag{2.6}$$

The gH-difference between $\boldsymbol{I}$ and $\boldsymbol{I}'$, denoted $\boldsymbol{I} \ominus \boldsymbol{I}'$, is defined by

$$\boldsymbol{I} \ominus \boldsymbol{I}' = (I_1 \ominus I_1', \ldots, I_d \ominus I_d'). \tag{2.7}$$

In particular,

$$(-1) \cdot (\boldsymbol{I} \ominus \boldsymbol{I}') = \boldsymbol{I}' \ominus \boldsymbol{I}.$$

2.3. Set valued analysis

Assume that X and Y are two Hausdorff topological spaces, $2^Y = \{A \mid A \subset Y\}$ is the power set of Y, and F is a set-valued mapping from X to 2^Y. The set-valued mapping F is upper semi-continuous (lower semi-continuous) in the following sense: For any $x \in X$ and any open set $G \subset Y$ such that $F(x) \subset G$ ($F(x) \cap G \neq \emptyset$), there exists a neighborhood $o(x)$ of x with the property that for all $x' \in o(x)$, it holds that $F(x') \subset G$ ($F(x') \cap G \neq \emptyset$). The set-valued mapping F is regarded as continuous when it is both lower semi-continuous and upper semi-continuous at the same time. In particular, F is called a USCO (or CO) set-valued mapping if, F is upper semi-continuous (or continuous) with nonempty and compact values. The set-valued mapping F is said to be closed if the graph of F, denoted as

$$\text{Graph}(F) = \{(x, y) \in X \times Y \mid y \in F(x)\},$$

is a closed set in $X \times Y$. For more detailed information, refer to [41–43].

Lemma 2.3. *([43]) Assume that F and G are two set-valued mappings from X to 2^Y, and that $F(x) \cap G(x) \neq \emptyset$ for every $x \in X$. If F is closed and G is USCO, then the set-valued mapping $F \cap G$ is upper semi-continuous.*

Lemma 2.4. *([43]) For each $i \in \{1, \ldots, n\}$, assume that X and Y_i are two Hausdorff topological spaces. If F_i: $X \to 2^{Y_i}$ is a USCO set-valued mapping, then the set-valued mapping F: $X \to 2^Y$, defined as*

$$F(x) = \prod_{i=1}^{n} F_i(x)$$

for all $x \in X$, is also a USCO set-valued mapping. Here,

$$Y = \prod_{i=1}^{n} Y_i.$$

Theorem 2.5. *(Fan–Glicksberg [44, 45]) Assume that X is a nonempty compact subset of a locally convex Hausdorff topological vector space $\mathcal{V}$. If a set-valued mapping Φ: $X \to 2^X$ is convex-valued and USCO, then there is $x^* \in X$ such that $x^* \in \Phi(x^*)$.*

Lemma 2.6. *([43]) Assume that C is a nonempty compact subset of a Hausdorff topological space $\mathcal{X}$ and that $P = \{p_1, \ldots, p_d\}$ is a vector-valued function from C to $\mathbb{R}^d$. If a certain p_i is upper semi-continuous on C, then there exists an $x^* \in C$ such that for all $x \in C$, $P(x) - P(x^*) \notin \text{int}\, \mathbb{R}^d_+$.*

2.4. Fuzzy mappings

Suppose that X and Y are two nonempty convex subsets of two Hausdorff topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. A mapping $\widetilde{G}$ from X to $\mathcal{F}(Y)$ is defined as a fuzzy mapping. For every $x \in X$, the $\widetilde{G}(x)$ (which can be denoted as $\widetilde{G}_x$) is a fuzzy set and $\widetilde{G}_x(y)$ is the membership function of y in $\widetilde{G}_x$. The fuzzy mapping $\widetilde{G}$ is convex if, for $x \in X$, $y, z \in Y$, and for $t \in [0, 1]$, we have

$$\widetilde{G}_x(ty + (1-t)z) \geq \min\{\widetilde{G}_x(y), \widetilde{G}_x(z)\}.$$

For the related content of fuzzy sets and fuzzy mappings, see [25, 46, 47].

Lemma 2.7. *([46]) Assume that g is a function from X to $[0, 1]$, and that $\widetilde{G}$ is a convex fuzzy mapping from X to $\mathcal{F}(Y)$. For each $x \in X$, define the set by*

$$G(x) = (\widetilde{G}_x)^{g(x)} = \{y \in Y \mid \widetilde{G}_x(y) \geq g(x)\}.$$

Then $G\colon X \to 2^Y$ is a set-valued mapping with convex values.

3. The partial order relation of interval vectors

In this section, we present a novel tool that originates from Banach geometry and convex geometry [48]. Such a tool will play an effective and crucial role in Sections 4 and 5. Our idea is as follows: By identifying each interval vector with its support function, we embed $\mathcal{I}(\mathbb{R}^d)$ into $C(\mathbb{S}^{d-1})$. As an important consequence, $\mathcal{I}(\mathbb{R}^d_+)$ forms a convex cone. Finally, we obtain the partial order in $\mathcal{I}(\mathbb{R}^d)$ by making use of the convex cone $\mathcal{I}(\mathbb{R}^d_+)$.

3.1. Interval support functions

A d-dimensional interval vector can be interpreted as a box in $\mathbb{R}^d$, that is, as a convex body in $\mathbb{R}^d$. The following subsection provides a rigorous analytical characterization of this concept using the support function from convex geometry.

For each $\boldsymbol{I} = (I_1, \ldots, I_d) \in \mathcal{I}(\mathbb{R}^d)$, its support function $h_{\boldsymbol{I}}\colon \mathbb{R}^d \to \mathbb{R}$ is defined by

$$h_{\boldsymbol{I}}(x) = \max\left\{\sum_{j=1}^{d} a_j x_j \mid (a_1, \ldots, a_d) \in \boldsymbol{I}\right\},$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. In particular, if $d = 1$, $I = [\underline{a}, \overline{a}] \in \mathcal{I}(\mathbb{R})$, then

$$h_I(x) = \begin{cases} \overline{a}x, & \text{if } x \geq 0, \\ \underline{a}x, & \text{if } x < 0. \end{cases}$$

Therefore,

$$h_{\boldsymbol{I}}(x) = \sum_{j=1}^{d} h_{I_j}(x_j).$$

According to the definition of $h_{\boldsymbol{I}}$, we have the following facts:

(1) $h_{\boldsymbol{I}}$ is positively homogeneous; i.e., for $x \in \mathbb{R}^d$ and $s > 0$, $h_{\boldsymbol{I}}(sx) = s \cdot h_{\boldsymbol{I}}(x)$.
(2) $h_{\boldsymbol{I}}$ is subadditive; i.e., for $x, y \in \mathbb{R}^d$, $h_{\boldsymbol{I}}(x + y) \leq h_{\boldsymbol{I}}(x) + h_{\boldsymbol{I}}(y)$.

In other words, $h_{\boldsymbol{I}}$ is sublinear. Conversely, a crucial fact reads as follows: For a sublinear function h: $\mathbb{R}^d \to \mathbb{R}$, assume that

$$h(x) = \sum_{j=1}^{d} h_j(x_j)$$

for every $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, where all the functions $h_1, \ldots, h_d$ are sublinear functions from $\mathbb{R}$ to $\mathbb{R}$, then there is a unique interval vector $\boldsymbol{I}$ such that $h = h_{\boldsymbol{I}}$.

In this article, the support function of an interval vector is called an *interval support function*. For the sake of discussion, we restrict each interval support function to the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. Since each interval vector $\boldsymbol{I}$ corresponds to a unique interval support function $h_{\boldsymbol{I}}$, this fact allows us to embed the class of all interval vectors $\mathcal{I}(\mathbb{R}^d)$ into $C(\mathbb{S}^{d-1})$.

Lemma 3.1. *Assume that $\boldsymbol{I} = (I_1, \ldots, I_d)$ is a d-dimensional interval vector. Then*

$$\|h_{\boldsymbol{I}}\|_{\max} = \left(\sum_{j=1}^{d} \|h_{I_j}\|_{\max}^2 \right)^{1/2}.$$

Proof. See Appendix A. □

3.2. A norm on $\mathcal{I}(\mathbb{R}^d)$

Given the correspondence between the interval vector and its support functions, Lemma 3.1 enables us to introduce Definition 3.2.

Definition 3.2. *Assume that $\boldsymbol{I} = (I_1, \ldots, I_d)$ is a d-dimensional interval vector. The function $\|\cdot\|_{\mathcal{I}(\mathbb{R}^d)}$: $\mathcal{I}(\mathbb{R}^d) \to \mathbb{R}_+$, which is defined as*

$$\|\boldsymbol{I}\|_{\mathcal{I}(\mathbb{R}^d)} = \left(\sum_{j=1}^{d} \|I_j\|_{\mathcal{I}(\mathbb{R})}^2 \right)^{1/2},$$

is said to be a norm on $\mathcal{I}(\mathbb{R}^d)$.

Based on Definition 3.2, it can be observed that $(\mathcal{I}(\mathbb{R}^d), \|\cdot\|_{\mathcal{I}(\mathbb{R}^d)})$ is a normed quasilinear space. Because of the completeness of $\mathcal{I}(\mathbb{R})$, it is easy to verify that $(\mathcal{I}(\mathbb{R}^d), \|\cdot\|_{\mathcal{I}(\mathbb{R}^d)})$ is complete. Based on the topology induced by norm $\|\cdot\|_{\mathcal{I}(\mathbb{R}^d)}$ on $\mathcal{I}(\mathbb{R}^d)$, we have

$$\operatorname{int} \mathcal{I}(\mathbb{R}_+^d) = \left\{ (I_1, \ldots, I_d) \mid I_j \in \operatorname{int} \mathcal{I}(\mathbb{R}_+), j = 1, \ldots, d \right\}.$$

Lemma 3.3. *Assume that $\boldsymbol{I} = (I_1, \ldots, I_d)$ is a d-dimensional interval vector. Then*

$$\|h_{\boldsymbol{I}}\|_{\max} = \|\boldsymbol{I}\|_{\mathcal{I}(\mathbb{R}^d)}.$$

Proof. If $d = 1$, then $I = [\underline{a}, \overline{a}]$ and

$$h_I(x) = \begin{cases} \overline{a}, & \text{if } x = 1, \\ -\underline{a}, & \text{if } x = -1. \end{cases} \tag{3.1}$$

According to Definition 2.2 and Eq (3.1), it follows that

$$\|h_I\|_{\max} = \max\{|\underline{a}|, |\overline{a}|\} = \|I\|_{\mathcal{I}(\mathbb{R})}. \tag{3.2}$$

If $d > 1$, then $\boldsymbol{I} = (I_1, \cdots, I_d)$. By Lemma 3.1, Definition 3.2, and Eq (3.2), we can see that

$$\|h_{\boldsymbol{I}}\|_{\max} = \left(\sum_{j=1}^{d} \|h_{I_i}\|_{\max}^2 \right)^{1/2} = \|\boldsymbol{I}\|_{\mathcal{I}(\mathbb{R}^d)}.$$

This completes the proof. □

Lemma 3.3 guarantees that the normed quasilinear space $(\mathcal{I}(\mathbb{R}^d), \|\cdot\|_{\mathcal{I}(\mathbb{R}^d)})$ can be embedded into the normed linear space $(C(\mathbb{S}^{d-1}), \|\cdot\|_{max})$ with the norm unchanged.

Lemma 3.4. *Assume that $\{\boldsymbol{I}_1^m\}$ and $\{\boldsymbol{I}_2^m\}$ are two convergent sequences in $\mathcal{I}(\mathbb{R}^d)$. Then*

$$\lim_{m\to\infty} (\boldsymbol{I}_1^m \ominus \boldsymbol{I}_2^m) = \left(\lim_{m\to\infty} \boldsymbol{I}_1^m\right) \ominus \left(\lim_{m\to\infty} \boldsymbol{I}_2^m\right).$$

Proof. See Appendix B. □

3.3. A partial order in $\mathcal{I}(\mathbb{R}^d)$

In order to explore a partial order in the normed quasilinear space $\mathcal{I}(\mathbb{R}^d)$, we begin with the following fact.

Lemma 3.5. *$\mathcal{I}(\mathbb{R}_+^d)$ is a convex, closed and pointed cone in $C(\mathbb{S}^{d-1})$.*

Proof. See Appendix C. □

Definition 3.6. *Assume that $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are two d-dimensional interval vectors.*

(1) If $\boldsymbol{I}_1 \ominus \boldsymbol{I}_2 \in \mathcal{I}(\mathbb{R}_+^d)$, then $\boldsymbol{I}_1$ is said to be dominated by $\boldsymbol{I}_2$ from below, and we write $\boldsymbol{I}_2 \preceq \boldsymbol{I}_1$. Otherwise, we write $\boldsymbol{I}_2 \npreceq \boldsymbol{I}_1$.
(2) If $\boldsymbol{I}_1 \ominus \boldsymbol{I}_2 \in \operatorname{int} \mathcal{I}(\mathbb{R}_+^d)$, then $\boldsymbol{I}_1$ is said to be strictly dominated by $\boldsymbol{I}_2$ from below, and we write $\boldsymbol{I}_2 \prec \boldsymbol{I}_1$. Otherwise, we write $\boldsymbol{I}_2 \nprec \boldsymbol{I}_1$.
(3) If $\boldsymbol{I}_2 \npreceq \boldsymbol{I}_1$ and $\boldsymbol{I}_1 \npreceq \boldsymbol{I}_2$, then it is said that $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are not mutually dominated, or that $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are incomparable.

According to Definition 3.6, it is straightforward to confirm that $\preceq$ is a partial order on $\mathcal{I}(\mathbb{R}^d)$, but binary relation $\prec$ is not partial order on $\mathcal{I}(\mathbb{R}^d)$. Let

$$\boldsymbol{I}_1 = ([\underline{a}_{11}, \overline{a}_{11}], \ldots, [\underline{a}_{1d}, \overline{a}_{1d}]), \quad \boldsymbol{I}_2 = ([\underline{a}_{21}, \overline{a}_{21}], \ldots, [\underline{a}_{2d}, \overline{a}_{2d}]),$$

and from Definition 3.6 and Eq (2.7), we can obtain that $\boldsymbol{I}_2 \preceq \boldsymbol{I}_1$ ($\boldsymbol{I}_2 \prec \boldsymbol{I}_1$) if and only if $\underline{a}_{2i} \preceq \underline{a}_{1i}$ and $\overline{a}_{2i} \preceq \overline{a}_{1i}$ ($\underline{a}_{2i} \prec \underline{a}_{1i}$ and $\overline{a}_{2i} \prec \overline{a}_{1i}$) for every i.

4. Interval-vector-valued functions

To investigate the existence of weak Pareto–Nash equilibria for GIMGFCMs, this section discusses the semi-continuity and generalized $\mathcal{I}(\mathbb{R}^d_+)$-quasi-concavity of interval-vector-valued functions. Suppose that $\mathcal{X}$ is a Hausdorff topological space and $\mathcal{Y}$ is a Hausdorff topological vector space.

4.1. Upper and lower semi-continuity

We start with the upper and lower semi-continuity of interval-vector-valued functions. For $\epsilon > 0$, let

$$V^d_\epsilon = \{(I_1, \ldots, I_d) \in \mathcal{I}(\mathbb{R}^d) \mid I_j \subset (-\epsilon, \epsilon), j = 1, \ldots, d\}.$$

Definition 4.1. *Assume that $P = \{p_1, \ldots, p_d\}$ from $\mathcal{X}$ to $\mathcal{I}(\mathbb{R}^d)$ is an interval-vector-valued function. For any $x \in \mathcal{X}$ and any $\epsilon > 0$, there is an open neighborhood $o(x)$ of x with the property that for every $x' \in o(x)$, it holds that*

$$P(x') \ominus P(x) \in V^d_\epsilon - \mathcal{I}(\mathbb{R}^d_+).$$

Then P is upper semi-continuous on $\mathcal{X}$.

Analogously, the *lower semi-continuity* of P is defined in the same way as described above, but with $V^d_\epsilon + \mathcal{I}(\mathbb{R}^d_+)$ instead of $V^d_\epsilon - \mathcal{I}(\mathbb{R}^d_+)$. Moreover, P is continuous when it is both upper semi-continuous and lower semi-continuous.

Note that for $x \in \mathcal{X}$, we have the representation

$$P(x) = ([\underline{p}_1(x), \overline{p}_1(x)], \cdots, [\underline{p}_d(x), \overline{p}_d(x)])$$

where $\underline{P} = \{\underline{p}_1, \ldots, \underline{p}_d\}$ and $\overline{P} = \{\overline{p}_1, \ldots, \overline{p}_d\}$ are vector-valued functions from $\mathcal{X}$ to $\mathbb{R}^d$.

Remark 4.2. *The interval-vector-valued function $P = \{p_1, \ldots, p_d\}$ is upper semi-continuous (lower semi-continuous) on $\mathcal{X}$ if and only if every interval-valued function p_i is upper semi-continuous (lower semi-continuous) on $\mathcal{X}$.*

Example 4.3. *For $x \in [30, 60]$, let*

$$p_1(x) = [100 - 2x, 120 - 2x],$$

and

$$p_2(x) = [(x - 25)(100 - 2x), (x - 25)(120 - 2x)],$$

then $P(x) = (p_1(x), p_2(x))$ is an interval-vector-valued function from $[30, 60]$ to $\mathcal{I}(\mathbb{R}^2)$. Since

$$\underline{p}_1(x) = 100 - 2x, \quad \overline{p}_1(x) = 120 - 2x$$

and

$$\underline{p}_2(x) = 150x - 2x^2 - 2500, \quad \overline{p}_2(x) = 170x - 2x^2 - 3000$$

are continuous on $\mathbb{R}$, we conclude that P is a continuous interval-valued function.

Lemma 4.4. *An interval-vector-valued function $P = \{p_1, \ldots, p_d\}$ from $\mathcal{X}$ to $\mathcal{I}(\mathbb{R}^d)$ is upper semi-continuous (lower semi-continuous) if and only if two vector-valued functions $\underline{P}$ and $\overline{P}$ are $\mathbb{R}^d_+$-upper semi-continuous ($\mathbb{R}^d_+$-lower semi-continuous).*

Proof. See Appendix D. □

Let P_1 and P_2 be two interval-vector-valued functions from $\mathcal{X}$ to $\mathcal{I}(\mathbb{R}^d)$. Then the function $P_1 \ominus P_2$: $\mathcal{X} \to \mathcal{I}(\mathbb{R}^d)$ is defined as follows:

$$(P_1 \ominus P_2)(x) = P_1(x) \ominus P_2(x), \text{for } x \in \mathcal{X}.$$

An application of Lemma 3.4 gives the next fact.

Lemma 4.5. *Suppose that P_1, P_2: $\mathcal{X} \to \mathcal{I}(\mathbb{R}^d)$ are both continuous on $\mathcal{X}$. Then $P_1 \ominus P_2$ is continuous on $\mathcal{X}$.*

Lemma 4.6. *Suppose that C is a nonempty compact subset of $\mathcal{X}$ and that $P = \{p_1, \ldots, p_d\}$ is an interval-vector-valued function from C to $\mathcal{I}(\mathbb{R}^d)$. If a certain p_i is upper semi-continuous on C, then there exists an element $x^* \in C$ such that for all $x \in C$, $P(x^*) \nless P(x)$.*

Proof. Let

$$\underline{P} = \{\underline{p}_1, \ldots, \underline{p}_d\} : C \to \mathbb{R}^d.$$

If a certain p_i is upper semi-continuous on C, according to the proof process of Lemma 4.4, it can be deduced that $\underline{p}_i$ is upper semi-continuous. And by Lemma 2.6, there is $x^* \in C$ such that for all $x \in C$, $\underline{P}(x) - \underline{P}(x^*) \notin \text{int}\, \mathbb{R}^d_+$. From (2.4) and (2.7), it can be deduced that $P(x) \ominus P(x^*) \notin \text{int}\, \mathcal{I}(\mathbb{R}_+)^d$, that is, $P(x^*) \nless P(x)$ for all $x \in C$. □

4.2. Generalized $\mathcal{I}(\mathbb{R}^d_+)$-quasi-concavity

Definition 4.7. *Suppose that $\mathcal{K}$ is a nonempty convex subset of $\mathcal{Y}$ and that P is an interval-vector-valued function from $\mathcal{K}$ to $\mathcal{I}(\mathbb{R}^d)$. For $x_1, x_2 \in \mathcal{K}$, $t \in [0, 1]$ and for $\boldsymbol{I} \in \mathcal{I}(\mathbb{R}^d)$, if $\boldsymbol{I} \ominus P(x_i) \notin \text{int}\, \mathcal{I}(\mathbb{R}^d_+)$ for $i = 1, 2$, we obtain*

$$\boldsymbol{I} \ominus P(tx_1 + (1-t)x_2) \notin \text{int}\, \mathcal{I}(\mathbb{R}^d_+).$$

Then P is said to be generalized $\mathcal{I}(\mathbb{R}^d_+)$-quasi-concave on $\mathcal{K}$.

Let P be a vector-valued function from $\mathcal{K} \subset \mathbb{R}$ to $\mathbb{R}^d$. Then the generalized $\mathbb{R}^d_+$-quasi-concavity of P degenerate into the following implication: For $x_1, x_2 \in \mathcal{K}$, $t \in [0, 1]$ and for $\boldsymbol{a} \in \mathbb{R}^d$, $P(x_i) \notin a - \text{int}\, \mathbb{R}^d_+$ with $i = 1, 2$ implies that

$$P(tx_1 + (1-t)x_2) \notin \boldsymbol{a} - \text{int}\, \mathbb{R}^d_+.$$

In Definition 4.7, let $P = \{p_1, \ldots, p_d\}$ be an interval-vector-valued function from $\mathcal{K}$ to $\mathcal{I}(\mathbb{R}^d)$. For a fixed $\boldsymbol{I} = ([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_d, \overline{a}_d])$, we define

$$S(\boldsymbol{I}) = \{x \in \mathcal{K} \mid \boldsymbol{I} \ominus P(x) \notin \text{int}\, \mathcal{I}(\mathbb{R}^d_+)\}.$$

From Eqs (2.4) and (2.7), it is easy to obtain that

$$S(\boldsymbol{I}) = \bigcup_{j=1}^{d} (\{x \in \mathcal{K} \mid \underline{p}_j(x) \geq \underline{a}_j\} \cup \{x \in \mathcal{K} \mid \overline{p}_j(x) \geq \overline{a}_j\}).$$

Remark 4.8. *A mapping $P = \{p_1, \ldots, p_d\}$ from $\mathcal{K}$ to $\mathcal{I}(\mathbb{R}^d)$ is generalized $\mathcal{I}(\mathbb{R}^d_+)$-quasi-concave on $\mathcal{K}$ if and only if for every $\boldsymbol{I} = ([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_d, \overline{a}_d])$, the set $S(\boldsymbol{I})$ is either empty, a singleton, or convex.*

In Example 4.3, for $x \in [42.5, 60]$,

$$\underline{p}_1(x) = 100 - 2x, \quad \overline{p}_1(x) = 120 - 2x,$$

and

$$\underline{p}_2(x) = 150x - 2x^2 - 2500, \quad \overline{p}_2(x) = 170x - 2x^2 - 3000,$$

and for a fixed $\boldsymbol{I} = ([\underline{a}_1, \overline{a}_1], [\underline{a}_2, \overline{a}_2])$, we obtain

$$S(\boldsymbol{I}) = \bigcup_{j=1}^{2} (\{x \in [42.5, 60] \mid \underline{p}_j(x) \geq \underline{a}_j\} \cup \{x \in [42.5, 60] \mid \overline{p}_j(x) \geq \overline{a}_j\}).$$

Since the functions $\underline{p}_1$, $\overline{p}_1$, $\underline{p}_2$, and $\overline{p}_2$ are all non-increasing on $[42.5, 60]$, the set $S(\boldsymbol{I})$ is either empty, a singleton, or a closed interval contained in $[42.5, 60]$. Therefore, P is generalized $\mathcal{I}(\mathbb{R}^2_+)$-quasi-concave on $[42.5, 60]$.

Lemma 4.9. *Suppose that $\mathcal{K}$ is a nonempty convex subset of $\mathcal{Y}$, and that $P = \{p_1, \ldots, p_d\}\colon \mathcal{K} \to \mathcal{I}(\mathbb{R}^d)$ is generalized $\mathcal{I}(\mathbb{R}^d_+)$-quasi-concave. Then $\overline{P} = \{\overline{p}_1, \ldots, \overline{p}_d\}\colon \mathcal{K} \to \mathbb{R}^d$ is generalized $\mathbb{R}^d_+$-quasi-concave.*

Proof. Let $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d$, let $x_1, x_2 \in \mathcal{K}$, and let $t \in [0, 1]$. Moreover, take

$$\boldsymbol{I} = ([a_1, a_1], \ldots, [a_d, a_d]).$$

Since

$$\boldsymbol{I} \ominus P(x_i) \notin \operatorname{int} \mathcal{I}(\mathbb{R}^d_+)$$

with $i = 1, 2$, then generalized $\mathcal{I}(\mathbb{R}^d_+)$-quasi-concavity of P implies that

$$\boldsymbol{I} \ominus P(tx_1 + (1 - t)x_2) \notin \operatorname{int} \mathcal{I}(\mathbb{R}^d_+).$$

And thus, $\boldsymbol{a} - \overline{P}(x_1) \notin \operatorname{int} \mathbb{R}^d_+$ and $\boldsymbol{a} - \overline{P}(x_2) \notin \operatorname{int} \mathbb{R}^d_+$ imply

$$\boldsymbol{a} - \overline{P}(tx_1 + (1 - t)x_2) \notin \operatorname{int} \mathbb{R}^d_+.$$

Therefore, $\overline{P}$ is generalized $\mathbb{R}^d_+$-quasi-concave. □

5. Existence of weak Pareto–Nash equilibria for GIMGFCMs

Let us turn to discuss the weak Pareto–Nash equilibria of GIMGFCMs. Based on the preparations in Sections 3 and 4, we initially construct the best response set-valued mapping for each player. Subsequently, on this foundation, we utilize Fan–Glicksberge's fixed point theorem to prove the existence theorem of weak Pareto–Nash equilibria for GIMGFCMs. Interestingly, this theorem directly yields two existence theorems of weak Pareto–Nash equilibria for generalized interval-valued games with fuzzy constraint mappings (GIGFCMs) and generalized multiobjective games with fuzzy constraint mappings (GMGFCMs).

5.1. The definition of weak Pareto–Nash equilibrium

A GIMGFCM in normal form is defined by the tuple

$$\Gamma = (X_i, \widetilde{G}_i, g_i, P_i)_{i\in N},$$

in which $N = \{1, 2, \ldots, n\}$ denotes the set of players. Here, X_i represents the strategy space of Player i, and

$$X = \prod_{i\in N} X_i$$

denotes the joint strategy space of Γ. Let

$$-i = N \setminus \{i\} \quad \text{and} \quad X_{-i} = \prod_{j\neq i} X_j.$$

The fuzzy constraint mapping of Player i is given by $\widetilde{G}_i$: $X_{-i} \to \mathcal{F}(X_i)$, while the feasibility function g_i: $X_{-i} \to [0, 1]$ specifies the degree of feasibility for Player i under a given strategy profile $x_{-i} \in X_{-i}$. The payoff function of Player i is defined as $P_i = \{p^i_1, \ldots, p^i_{d_i}\}$: $X \to \mathcal{I}(\mathbb{R}^{d_i})$, where d_i denotes the number of objectives for Player i. Throughout this section, for $i \in N$, write $\mathcal{V}_i$ for a locally convex Hausdorff topological vector space.

Definition 5.1. *Let* $\Gamma = (X_i, \widetilde{G}_i, g_i, P_i)_{i\in N}$ *be a GIMGFCM. Suppose there is* $x^* = (x^*_i, x^*_{-i}) \in X$ *such that for every* $i \in N$, $\widetilde{G}_{ix^*_{-i}}(x^*_i) \geq g_i(x^*_{-i})$. *Moreover, for any* u_i *that satisfies* $\widetilde{G}_{ix^*_{-i}}(u_i) \geq g_i(x^*_{-i})$, *we have*

$$P_i(u_i, x^*_{-i}) \ominus P_i(x^*_i, x^*_{-i}) \notin \operatorname{int} \mathcal{I}(\mathbb{R}^{d_i}_+).$$

Then x^* *is called a weak Pareto–Nash equilibrium of* Γ.

Each GIMGFCM $(X_i, \widetilde{G}_i, g_i, P_i)_{i\in N}$ can yield 2^N different generalized multiobjective games with fuzzy constraint mappings (GMGFCGs) $(X_i, \widetilde{G}_i, g_i, P^*_i)_{i\in N}$, where each P^*_i can be either $\underline{P}_i$ or $\overline{P}_i$. Among these GMGFCGs, the most special ones are the following two models: $(X_i, \widetilde{G}_i, g_i, \underline{P}_i)_{i\in N}$ and $(X_i, \widetilde{G}_i, g_i, \overline{P}_i)_{i\in N}$. Based on Eqs (2.4) and (2.7), the condition

$$P_i(u_i, x^*_{-i}) \ominus P_i(x^*_i, x^*_{-i}) \notin \operatorname{int} \mathcal{I}(\mathbb{R}^{d_i}_+)$$

holds if and only if either

$$\underline{P}_i(u_i, x^*_{-i}) - \underline{P}_i(x^*_i, x^*_{-i}) \notin \operatorname{int} \mathbb{R}^{d_i}_+$$

or

$$\overline{P}_i(u_i, x^*_{-i}) - \overline{P}_i(x^*_i, x^*_{-i}) \notin \operatorname{int} \mathbb{R}^{d_i}_+.$$

Remark 5.2. *If a strategy profile* $x^* \in X$ *is a weak Pareto–Nash equilibrium of GMGFCG* $(X_i, \widetilde{G}_i, g_i, P^*_i)_{i\in N}$, *then it is also a weak Pareto–Nash equilibrium of GIMGFCM* $(X_i, \widetilde{G}_i, g_i, P_i)_{i\in N}$.

With the aid of the fuzzy mapping $\widetilde{G}_i$ and the feasible function g_i, a set-valued mapping G_i: $X_{-i} \to 2^{X_i}$ can be defined as follows: for each $x_{-i} \in X_{-i}$,

$$G_i(x_{-i}) = (\widetilde{G}_{ix_{-i}})^{g_i(x_{-i})} = \{x_i \in X_i \mid \widetilde{G}_{ix_{-i}}(x_i) \geq g_i(x_{-i})\}.$$

Subsequently, we refer to G_i as a set-valued mapping induced by $\widetilde{G}_i$. Simultaneously, $(x_i^*, x_{-i}^*) \in X$ is a weak Pareto–Nash equilibrium of Γ if for any $i \in N$, $x_i^* \in G_i(x_{-i}^*)$, and for all $u_i \in G_i(x_{-i}^*)$, it holds that

$$P_i(u_i, x_{-i}^*) \ominus P_i(x_i^*, x_{-i}^*) \notin \operatorname{int} \mathcal{I}(\mathbb{R}_+^{d_i}).$$

For each $i \in N$, the best response set-valued mapping Φ_i: $X_{-i} \to 2^{X_i}$ is defined for $x_{-i} \in X_{-i}$ as follows:

$$\Phi_i(x_{-i}) = \{a_i \in G_i(x_{-i}) \mid P_i(a_i, x_{-i}) \nprec P_i(u_i, x_{-i}), \text{ for all } u_i \in G_i(x_{-i})\}. \tag{5.1}$$

For $x = (x_i, x_{-i}) \in X$, define a set

$$\Phi(x) = \prod_{i \in N} \Phi_i(x_{-i}),$$

then Φ is a set-valued mapping from X to 2^X. If $x^* = (x_i^*, x_{-i}^*) \in X$ is a fixed point of Φ, then we have $x^* \in \Phi(x^*)$. That is, for each $i \in N$, $x_i^* \in G_i(x_{-i}^*)$, and for all $u_i \in G_i(x_{-i}^*)$, it holds that

$$P_i(x_i^*, x_{-i}^*) \nprec P_i(u_i, x_{-i}^*).$$

Therefore, x^* is a weak Pareto–Nash equilibrium of Γ.

5.2. The existence theorem of weak Pareto–Nash equilibrium

Next, we introduce the existence theorem for the weak Pareto–Nash equilibrium of GIMGFCM.

Theorem 5.3. *For each $i \in N$, suppose that X_i is a nonempty, convex, and compact subset of $\mathcal{V}_i$. If the following three conditions hold:*

(1) $\widetilde{G}_i$: $X_{-i} \to \mathcal{F}(X_i)$ is a convex fuzzy mapping, and G_i is a CO set-valued mapping;
(2) P_i: $X = \prod_{i \in \mathrm{N}} X_i \to \mathcal{I}(\mathbb{R}^{d_i})$ is a continuous interval-vector-valued function;
(3) $x_i \mapsto P_i(x_i, x_{-i})$ is generalized $\mathcal{I}(\mathbb{R}_+^{d_i})$-quasi-concave for $x_{-i} \in X_{-i}$;

then there exists at least one weak Pareto–Nash equilibrium of the GIMGFCM $(X_i, \widetilde{G}_i, g_i, P_i)_{i \in N}$.

Proof. For each $i \in N$, since G_i is a CO set-valued mapping, for $x_{-i} \in X_{-i}$, the set $G_i(x_{-i})$ is both nonempty and compact. From the continuity of P_i and Lemma 4.6, it follows that $\Phi_i(x_{-i}) \neq \emptyset$ for $x_{-i} \in X_{-i}$. Subsequently, it is necessary for us to prove that the set-valued mapping Φ_i: $X_{-i} \to 2^{X_i}$ is USCO and convex-valued.

First, we need to verify the compactness of $\Phi_i(x_{-i})$ for $x_{-i} \in X_{-i}$. Noting that $\Phi_i(x_{-i}) \subset G_i(x_{-i})$ and $G_i(x_{-i})$ are compact, it suffices to show that $\Phi_i(x_{-i})$ is closed. To achieve this goal, we employ proof by contradiction. We assume that there exists a net $\{x_i^\tau\}_{\tau \in D} \subset \Phi_i(x_{-i})$ such that $x_i^\tau \to x_i$, yet $x_i \notin \Phi_i(x_{-i})$. Then there exists $u_0 \in G_i(x_{-i})$ such that

$$P_i(x_i, x_{-i}) \prec P_i(u_0, x_{-i}). \tag{5.2}$$

On the other hand, as $x_i^\tau \in \Phi_i(x_{-i})$, we infer that for all $u \in G_i(x_{-i})$,

$$P_i(u, x_{-i}) \ominus P_i(x_i^\tau, x_{-i}) \notin \operatorname{int} \mathcal{I}(\mathbb{R}_+^{d_i}). \tag{5.3}$$

Since $G_i(x_{-i})$ is compact, for a fixed point $u_0 \in G_i(x_{-i})$, there exists a net $\{u^\tau\}_{\tau \in D}$ such that $u^\tau \in G_i(x_{-i})$ and $u^\tau \to u_0$. Together with the continuity of P_i and (5.3), it immediately follows that

$$P_i(u_0, x_{-i}) \ominus P_i(x_i, x_{-i}) \notin \operatorname{int} \mathcal{I}(\mathbb{R}_+^{d_i}),$$

which contradicts (5.2).

Second, our objective is to prove the convexity of $\Phi_i(x_{-i})$ for $x_{-i} \in X_{-i}$. Suppose that $a_1, a_2 \in \Phi_i(x_{-i})$ and $t \in [0, 1]$. Evidently, $a_1, a_2 \in G_i(x_{-i})$, and for every $u_i \in G_i(x_{-i})$, we have

$$P_i(u_i, x_{-i}) \ominus P_i(a_1, x_{-i}) \notin \text{int } \mathcal{I}(\mathbb{R}_+^{d_i})$$

and

$$P_i(u_i, x_{-i}) \ominus P_i(a_2, x_{-i}) \notin \text{int } \mathcal{I}(\mathbb{R}_+^{d_i}).$$

Noting that $\widetilde{G}_i$ is convex for every $i \in N$, according to Lemma 2.7, it can be deduced that $G_i(x_{-i})$ is convex. Therefore,

$$ta_1 + (1 - t)a_2 \in G_i(x_{-i}).$$

Since $x_i \mapsto P_i(x_i, x_{-i})$ is generalized $\mathcal{I}(\mathbb{R}_+^{d_i})$-quasi-concave, we obtain

$$P_i(u_i, x_{-i}) \ominus P_i(ta_1 + (1 - t)a_2, x_{-i}) \notin \text{int } \mathcal{I}(\mathbb{R}_+^{d_i}).$$

That is,

$$ta_1 + (1 - t)a_2 \in \Phi_i(x_{-i}).$$

Hence, $\Phi_i(x_{-i})$ is convex for each $i \in N$.

Third, we turn to show the upper semi-continuity of Φ_i. For $x_{-i} \in X_{-i}$, recall that

$$\Phi_i(x_{-i}) = G_i(x_{-i}) \cap T_i(x_{-i}),$$

where

$$T_i(x_{-i}) = \{x_i \in X_i \mid P_i(x_i, x_{-i}) \nprec P_i(u, x_{-i}), \text{for all } u \in G_i(x_{-i})\}.$$

According to Lemma 2.3, we now verify the closedness of the set-valued mapping $T_i \colon X_{-i} \to 2^{X_i}$. This amounts to showing that

$$\text{Graph}(T_i) = \{(x_i, x_{-i}) \in X \mid x_i \in T_i(x_{-i})\}$$

is closed in X. To this end, consider nets $\{y_{-i}^\tau\}_{\tau\in D} \subset X_{-i}$ and $\{x_i^\tau\} \subset X_i$ such that $y_{-i}^\tau \to y_{-i}$, $x_i^\tau \in T_i(y_{-i}^\tau)$, and $x_i^\tau \to x_i$. It suffices to show that $x_i \in T_i(y_{-i})$. We employ proof by contradiction and assume that $x_i \notin T_i(y_{-i})$. Then there is $u_0 \in G_i(y_{-i})$ such that

$$P_i(x_i, y_{-i}) \prec P_i(u_0, y_{-i}). \tag{5.4}$$

On the other hand, from $x_i^\tau \in T_i(y_{-i}^\tau)$, it follows that $x_i^\tau \in X_i$ and for all $u \in G_i(y_{-i}^\tau)$,

$$P_i(u, y_{-i}^\tau) \ominus P_i(x_i^\tau, y_{-i}^\tau) \notin \text{int } \mathcal{I}(\mathbb{R}_+^{d_i}). \tag{5.5}$$

Due to the lower semi-continuity of G_i, there exists a net $\{u^\tau\}_{\tau\in D}$ such that $u_i^\tau \in G_i(y_{-i}^\tau)$ and $u_i^\tau \to u_0$. By the continuity of P_i and (5.5), we find that

$$P_i(u_0, y_{-i}) \ominus P_i(x_i, y_{-i}) \notin \text{int } \mathcal{I}(\mathbb{R}_+^{d_i}),$$

which contradicts (5.4).

Finally, Lemma 2.4 implies that the set-valued mapping Φ is USCO and convex-valued. By Theorem 2.5, there is $x^* \in X$ such that $x^* \in \Phi(x^*)$. Therefore, x^* is a weak Pareto–Nash equilibrium of $(X_i, \widetilde{G}_i, g_i, P_i)_{i\in N}$. □

5.3. Special cases of GIMGFCM and corresponding conclusions

In Theorem 5.3, for each $i \in N$ and $x_{-i} \in X_{-i}$, if $g_i(x_{-i}) = 1$, then

$$G_i(x_{-i}) = \{x_i \in X_i \mid \widetilde{G}_{ix_{-i}}(x_i) \geq 1\}.$$

At this moment, the GIMGFCM $(X_i, \widetilde{G}_i, g_i, P_i)_{i\in N}$ transforms into a generalized interval-valued multiobjective game (GIMG) $(X_i, G_i, P_i)_{i\in N}$. For $i \in N$ and $x_{-i} \in X_{-i}$, if

$$\widetilde{G}_{ix_{-i}}(x_i) = 1$$

for each $x_i \in X_i$, then

$$G_i(x_{-i}) = X_i.$$

Consequently, the GIMG $(X_i, G_i, P_i)_{i\in N}$ transforms into an interval-valued multiobjective game (IMG) $(X_i, P_i)_{i\in N}$. Meanwhile, $\widetilde{G}_1$ and $\widetilde{G}_2$ are two convex fuzzy mappings, and G_1 and G_2 are two CO set-valued mappings. According to Theorem 5.3, we can directly obtain the next corollary.

Corollary 5.4. *For each $i \in N$, assume that X_i is a nonempty, convex, and compact subset of $\mathcal{V}_i$. Assume that P_i is a continuous interval-vector-valued function from $\prod_{i\in \mathrm{N}} X_i$ to $\mathcal{I}(\mathbb{R}^{d_i})$ and that $x_i \mapsto P_i(x_i, x_{-i})$ is generalized $\mathcal{I}(\mathbb{R}^{d_i}_+)$-quasi-concave for $x_{-i} \in X_{-i}$. Then there exists at least one weak Pareto-Nash equilibrium of the IMG $(X_i, P_i)_{i\in N}$.*

Let us now consider the following GIGFCM:

$$\Gamma' = (X_i, \widetilde{G}_i, g_i, p_i)_{i\in N},$$

where the payoff function of Player i is defined as $p_i\colon X \to \mathcal{I}(\mathbb{R})$. A strategy profile $x^* \in X$ is a weak Pareto–Nash equilibrium of Γ' if, for every $i \in N$, $x^*_i \in G_i(x^*_{-i})$ and

$$p_i(u_i, x^*_{-i}) \ominus p_i(x^*_i, x^*_{-i}) \notin \operatorname{int} \mathcal{I}(\mathbb{R}_+) \text{ for all } u_i \in G_i(x^*_{-i}).$$

Let $d = 1$ in Theorem 5.3. Then we can derive the following corollary.

Corollary 5.5. *For each $i \in N$, assume that X_i is a nonempty, convex, and compact subset of $\mathcal{V}_i$. If the following three conditions are satisfied:*

(1) $\widetilde{G}_i\colon X_{-i} \to \mathcal{F}(X_i)$ is a convex fuzzy mapping, and G_i is a CO set-valued mapping;
(2) $p_i\colon X = \prod_{i\in \mathrm{N}} X_i \to \mathcal{I}(\mathbb{R})$ is a continuous interval-valued function;
(3) $x_i \mapsto p_i(x_i, x_{-i})$ is generalized $\mathcal{I}(\mathbb{R}_+)$-quasi-concave for $x_{-i} \in X_{-i}$;

then there exists at least one weak Pareto–Nash equilibrium of the GIGFCM $(X_i, \widetilde{G}_i, g_i, p_i)_{i\in N}$.

It is noteworthy that if $g_i(x_{-i}) = 1$ for every $i \in N$ and $x_{-i} \in X_{-i}$, then the GIGFCM $(X_i, \widetilde{G}_i, g_i, p_i)_{i\in N}$ transforms into a generalized interval-valued game (GIG) $(X_i, G_i, p_i)_{i\in N}$. If $G_i(x_{-i}) = X_i$ for every $i \in N$ and $x_{-i} \in X_{-i}$, then the GIG $(X_i, G_i, p_i)_{i\in N}$ transforms into an interval-valued game (IG) $(X_i, p_i)_{i\in N}$.

Next, consider the following GMGFCM:

$$\Gamma'' = (X_i, \widetilde{G}_i, g_i, P^*_i)_{i\in N},$$

where Player i's payoff function is given by

$$P_i^* = \{p_1^i, \dots, p_{d_i}^i\} : X = \prod_{i \in \mathrm{N}} X_i \to \mathbb{R}^{d_i}.$$

$x^* \in X$ is a weak Pareto–Nash equilibrium of Γ'' if, for each $i \in N$, $x_i^* \in G_i(x_{-i}^*)$ and

$$P_i^*(u_i, x_{-i}^*) - P_i^*(x_i^*, x_{-i}^*) \notin \operatorname{int} \mathbb{R}_+^{d_i}, \text{ for all } u_i \in G_i(x_{-i}^*).$$

According to Theorem 5.3, we can directly obtain the next corollary.

Corollary 5.6. *For every $i \in N$, assume that X_i is a nonempty, convex, and compact subset of $\mathcal{V}_i$. If the following three conditions hold:*

(1) $\widetilde{G}_i$: $X_{-i} \to \mathcal{F}(X_i)$ is a convex fuzzy mapping, and G_i is a CO set-valued mapping;
(2) P_i^*: $X = \prod_{i \in \mathrm{N}} X_i \to \mathbb{R}^{d_i}$ is a continuous vector-valued function;
(3) $x_i \mapsto P_i^*(x_i, x_{-i})$ is generalized $\mathbb{R}_+^{d_i}$-quasi-concave for $x_{-i} \in X_{-i}$;

then there exists at least one weak Pareto–Nash equilibrium of the GMGFCM $(X_i, \widetilde{G}_i, g_i, P_i^*)_{i \in N}$.

It is important to note that if $g_i(x_{-i}) = 1$ for each $i \in N$ and $x_{-i} \in X_{-i}$, then the GMGFCM $(X_i, \widetilde{G}_i, g_i, P_i^*)_{i \in N}$ transforms into a generalized multiobjective game $(X_i, G_i, P_i^*)_{i \in N}$ (GMG, see [9]). If $G_i(x_{-i}) = X_i$ for every $i \in N$ and $x_{-i} \in X_{-i}$, then the GMG $(X_i, G_i, P_i^*)_{i \in N}$ transforms into a multiobjective game $(X_i, P_i^*)_{i \in N}$ (MG, see [4]). Figure 1 illustrates the relationship between the GIMGFCM and its special case game models involved in Section 5.

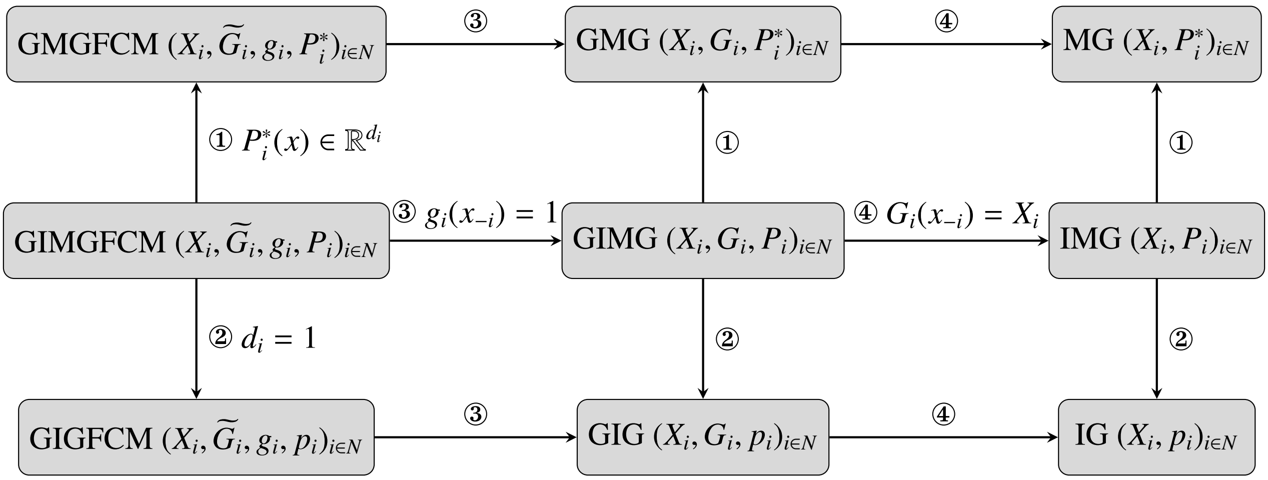

Figure 1. The relationship between the GIMGFCM and its special case game models.

6. Applications in heterogeneous market price competition

Product differentiation markets represent a central area of study in industrial organization theory. While the classic Bertrand model and its extensions typically assume complete information and single-objective profit maximization, they often overlook the multi-objective nature (such as pursuing profit, sales volume, and risk control) and the presence of parameter uncertainty that characterize real-world pricing decisions. This section develops a fuzzy game model of differentiated price competition that incorporates dual objectives (profit and sales volume) and treats cross-price effects as interval-valued parameters.

Example 6.1. *This study examines two firms (Players 1 and 2) offering differentiated products, each required to set prices within a feasible market range. For a given competitor's price, each firm evaluates the rationality and satisfaction of different pricing levels by constructing a fuzzy restriction mapping* $\widetilde{G}_i$ *(*$i = 1, 2$*). The payoff function for each firm incorporates two objectives: sales volume (*p_{i1}*) and profit (*p_{i2}*). The demand function is modeled in an asymmetric linear form* $\alpha - \beta x_1 + \gamma x_2$*, and due to uncertainties in the cross-price effect* γ*, its influence is expressed as an interval, i.e.,* $\gamma \in \mathcal{I}(\mathbb{R})$*. As a result, each firm's payoff function is represented as a two-dimensional interval vector-valued function. The problem is formulated as a two-player generalized interval-valued two-objective game with fuzzy constraint mappings, denoted by* $(X_i, \widetilde{G}_i, g_i, P_i)_{i\in\{1,2\}}$*. The specifics are detailed as follows:*

(1) $X_1 = [30, 60]$ *and* $X_2 = [25, 55]$ *are the strategy sets of Players* 1 *and* 2*.*

(2) Player 1*'s fuzzy constraint mapping* $\widetilde{G}_1$*:* $[30, 60] \to \mathcal{F}([30, 60])$ *is defined for* $x_2 \in [30, 60]$*,*

$$\widetilde{G}_{1x_2}(x_1) = \begin{cases} 0.8, & \text{if } x_1 \in [32.5 + 0.15x_2, 42.5 + 0.15x_2], \\ 0, & \text{otherwise}, \end{cases}$$

and Player 2*'s fuzzy constraint mapping* $\widetilde{G}_2$*:* $[25, 55] \to \mathcal{F}([25, 55])$ *is defined for* $x_1 \in [25, 55]$*,*

$$\widetilde{G}_{2x_1}(x_2) = \begin{cases} 0.7, & \text{if } x_2 \in [26 + 0.1x_1, 34 + 0.1x_1], \\ 0, & \text{otherwise}. \end{cases}$$

(3) Player 1*'s degree of feasibility* g_1*:* $[25, 55] \to [0, 1]$ *is given by*

$$g_1(x_2) = \frac{2}{3}, \ \textit{for } x_2 \in [25, 55],$$

and Player 2*'s degree of feasibility* g_2*:* $[30, 60] \to [0, 1]$ *is given by*

$$g_2(x_1) = \frac{1}{2}, \ \textit{for } x_1 \in [30, 60].$$

(4) For $i = 1, 2$*,* P_i*:* $[30, 60] \times [25, 55] \to \mathcal{I}(\mathbb{R}^2)$ *is Player* i*'s payoff function given for* $(x_1, x_2) \in [30, 60] \times [25, 55]$ *by*

$$P_1(x_1, x_2) = (p_{11}(x_1, x_2), p_{12}(x_1, x_2)),$$

$$P_2(x_1, x_2) = (p_{21}(x_1, x_2), p_{22}(x_1, x_2)),$$

where $p_{11}(x_1, x_2) = 100 - 2x_1 + [0.6, 1.0]x_2$*,* $p_{12}(x_1, x_2) = (x_1 - 25)(100 - 2x_1 + [0.6, 1.0]x_2)$*,* $p_{21}(x_1, x_2) = 100 - 2.5x_2 + [0.5, 0.9]x_1$*, and* $p_{22}(x_1, x_2) = (x_2 - 20)(100 - 2.5x_2 + [0.5, 0.9]x_1)$*.*

Next, we confirm that the game meets the three conditions of Theorem 5.3. First, for $x_2 \in [30, 60]$, $x_1', x_1'' \in [25, 55]$, and $t \in [0, 1]$, it holds that

$$\widetilde{G}_{1x_2}(tx_1' + (1 - t)x_1'') \geq \min\{\widetilde{G}_{1x_2}(x_1'), \widetilde{G}_{1x_2}(x_1'')\}.$$

From the definition of convexity, it follows that the fuzzy mapping $\widetilde{G}_1$ is convex. Similarly, we can verify that $\widetilde{G}_2$ is convex. Two set-valued mappings, G_1 and G_2, induced by $\widetilde{G}_1$ and $\widetilde{G}_2$, are given explicitly as follows:

$$G_1(x_2) = (\widetilde{G}_{1x_2})^{g_1(x_2)} = [32.5 + 0.15x_2, 42.5 + 0.15x_2], \text{ for } x_2 \in [25, 55],$$
$$G_2(x_1) = (\widetilde{G}_{2x_1})^{g_2(x_1)} = [26 + 0.1x_1, 34 + 0.1x_1], \text{ for } x_1 \in [30, 60].$$

Obviously, G_1 and G_2 are CO and convex-valued. Hence, condition (1) of Theorem 5.3 is satisfied.

Second, for each $(x_1, x_2) \in [30, 60] \times [25, 55]$, we observe that

$$\underline{p}_{11}(x_1, x_2) = 100 - 2x_1 + 0.6x_2, \quad \overline{p}_{11}(x_1, x_2) = 100 - 2x_1 + x_2,$$

and

$$\underline{p}_{12}(x_1, x_2) = 100 - 2.5x_2 + 0.5x_1, \quad \overline{p}_{12}(x_1, x_2) = 100 - 2.5x_2 + 0.9x_1$$

are both continuous. So

$$\underline{P}_1 = (\underline{p}_{11}, \underline{p}_{12}) \text{ and } \overline{P}_1 = (\overline{p}_{11}, \overline{p}_{12})$$

are continuous. Similarly, $\underline{P}_2$ and $\overline{P}_2$ are also continuous. From Lemma 4.4, P_1 and P_2 are two continuous interval-vector-valued functions. Therefore, this game satisfies condition (2) of Theorem 5.3.

Finally, for each $x_2 \in [25, 55]$, $\underline{p}_{11}$, $\overline{p}_{11}$, $\underline{p}_{12}$, and $\overline{p}_{12}$ are non-increasing in $[37.5 + 0.25x_2, 60]$. From Remark 4.8, $x_1 \mapsto P_1(x_1, x_2)$ is generalized $\mathcal{I}(\mathbb{R}^2_+)$-quasi-concave. Similarly, for each $x_1 \in [30, 60]$, $x_2 \mapsto P_2(x_1, x_2)$ is generalized $\mathcal{I}(\mathbb{R}^2_+)$-quasi-concave on $[30 + 0.18x_1, 55]$.

When Player 1 chooses a strategy within the interval $[43.750, 47.000]$, Player 2's feasible strategy set is constrained to the interval $[30.375, 38.700]$. Therefore, the last condition of Theorem 5.3 holds. According to Theorem 5.3, it is known that there exists at least one weak Pareto–Nash equilibrium in this game. Through calculation, we obtain that the two weak Pareto–Nash equilibria of this game are $(43.750, 30.375)$ and $(43.750, 34.375)$.

In fact, for $u_1 \in [43.750, 47.000]$ and $u_2 \in [30.375, 38.700]$, we arrive at the following consequences

$$p_{11}(u_1, 30.375) \ominus p_{11}(43.750, 30.375) = [-2(u_1 - 43.750), -2(u_1 - 43.750)] \notin \text{int } \mathcal{I}(\mathbb{R}_+),$$
$$p_{21}(43.750, u_2) \ominus p_{21}(43.750, 30.375) = [-2.5(u_2 - 30.375), -2.5(u_2 - 30.375)] \notin \text{int } \mathcal{I}(\mathbb{R}_+),$$
$$p_{12}(u_1, 34.375) \ominus p_{12}(43.750, 34.375) = (u_1 - 43.750)(150 - 2(u_1 + 43.750) + 34.375\gamma) \notin \text{int } \mathcal{I}(\mathbb{R}_+),$$
$$p_{22}(43.750, u_2) \ominus p_{22}(43.750, 34.375) = (u_2 - 34.375)(150 - 2.5(u_2 + 34.375) + 43.750\delta) \notin \text{int } \mathcal{I}(\mathbb{R}_+).$$

Therefore,

$$P_1(u_1, 30.375) \ominus P_1(43.750, 30.375) \notin \text{int } \mathcal{I}(\mathbb{R}^2_+),$$
$$P_2(43.750, u_2) \ominus P_2(43.750, 30.375) \notin \text{int } \mathcal{I}(\mathbb{R}^2_+),$$
$$P_1(u_1, 34.375) \ominus P_1(43.750, 34.375) \notin \text{int } \mathcal{I}(\mathbb{R}^2_+),$$
$$P_2(43.750, u_2) \ominus P_2(43.750, 34.375) \notin \text{int } \mathcal{I}(\mathbb{R}^2_+).$$

That is to say, $(43.750, 30.375)$ and $(43.750, 34.375)$ are two weak Pareto–Nash equilibria of $(X_i, \widetilde{G}_i, g_i, P_i)_{i\in\{1,2\}}$. □.

When firm 1 chooses a price within the range of $[43.750, 47.000]$, the feasible strategy set for firm 2 is $[30.375, 38.700]$. The price pair $(43.750, 30.375)$ constitutes a weak Pareto–Nash equilibrium when both firms prioritize sales volume as their primary objective; conversely, the pair $(43.750, 34.375)$ forms a weak Pareto–Nash equilibrium when both firms aim to maximize profit.

7. Conclusions

In this article, we investigate the generalized interval-valued multiobjective game with fuzzy constraint mappings (GIMGFCM). First, with the help of interval support functions, we discover a closed, convex, and pointed cone in the space $C(\mathbb{S}^{d-1})$. Second, this cone is of significance because it not only generates a partial order in $\mathcal{I}(\mathbb{R}^d)$ but also the semi-continuity and generalized $\mathcal{I}(\mathbb{R}^d_+)$-quasi-concavity of interval-vector-valued functions. Third, combining Fan–Glicksberge's fixed point theorem with this result leads to the existence theorem of weak Pareto–Nash equilibria for GIMGFCMs (Theorem 5.3). Finally, we research several special cases of the GIMGFCMs. By applying Theorem 5.3, we establish the existence theorems of equilibria for three types of games and further construct the relationship diagrams among these games. Finally, we demonstrate the specific application of the relevant theories in the price competition among heterogeneous markets through examples.

The GIMGFCM model proposed in this paper establishes a realistic analytical framework by incorporating fuzzy constraints, interval uncertainty, and multi-objective game mechanisms, making it well-suited for investigating game-theoretic problems in complex real-world scenarios. In subsequent work, we will pursue further exploration along two directions: first, advancing the practical application of this model in real-world complex settings; and second, leveraging insights from Ky Fan's inequality to relax the continuity and generalized $\mathcal{I}(\mathbb{R}^d_+)$-quasi-concavity requirements of the payoff function in Theorem 5.3. By introducing weaker and more realistic assumptions, we aim to extend the theory of equilibrium existence and to develop corresponding numerical methods for computing equilibria.

Author contributions

Wen Li: writing–editing & review; Du Zou: methodology & funding acquisition; Deyi Li: review & supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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Appendix

Appendix A

Proof of Lemma 3.1. If $\boldsymbol{I} = \mathbf{0}$ ($\mathbf{0} = ([0,0], \ldots, [0,0])$), the lemma obviously holds. In the subsequent discussion, we assume that $\boldsymbol{I} \neq \mathbf{0}$. Let

$$u = (u_1, \ldots, u_d) \in \mathbb{S}^{d-1},$$

and let $I_j = [\underline{a}_j, \overline{a}_j]$ for each j. The box $I_1 \times \cdots \times I_d$ consists of the points

$$\left((1-t_1)\underline{a}_1 + t_1\overline{a}_1, \ldots, (1-t_d)\underline{a}_d + t_d\overline{a}_d\right),$$

with $(t_1, \ldots, t_d) \in [0,1]^d$. So, the definition of $h_{\boldsymbol{I}}$ implies

$$h_{\boldsymbol{I}}(u) = \max_{(t_1,\cdots t_d)\in[0,1]^d} \sum_{j=1}^{d} [(1-t_j)\underline{a}_j + t_j\overline{a}_j]u_j = \sum_{j=1}^{d} \max\{\underline{a}_j u_j, \overline{a}_j u_j\}.$$

Furthermore, we have

$$|h_{\boldsymbol{I}}(u)| \leq \sum_{j=1}^{d} \max\{|\underline{a}_j|, |\overline{a}_j|\}|u_j| \leq \left(\sum_{j=1}^{d} \|h_{I_j}\|_{\max}^2\right)^{\frac{1}{2}}.$$

And since

$$\|h_{\boldsymbol{I}}\|_{\max} = \max_{u\in\mathbb{S}^{d-1}} |h_{\boldsymbol{I}}(u)|,$$

we deduce that

$$\|h_{\boldsymbol{I}}\|_{\max} \leq \left(\sum_{j=1}^{d} \|h_{I_j}\|_{\max}^2\right)^{\frac{1}{2}}. \tag{A.1}$$

On the other hand, for each j, at least one, say b_j, of the numbers $\underline{a}_j$ and $\overline{a}_j$ satisfies the equation

$$|b_j| = \max\{|\underline{a}_j|, |\overline{a}_j|\}.$$

The point $b = (b_1, \ldots, b_d)$ is nonzero and belongs to the box $I_1 \times \cdots \times I_d$. So, we can take its normalization

$$u^* = \frac{b}{\left(\sum_{j=1}^{d} b_j^2\right)^{1/2}}$$

and immediately see that

$$u^* \cdot b = \left(\sum_{j=1}^{d} b_i^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{d} \|h_{I_j}\|_{\max}^2\right)^{\frac{1}{2}}.$$

Therefore, we deduce that

$$\|h_{\boldsymbol{I}}\|_{\max} \geq h_{\boldsymbol{I}}(u^*) \geq \left(\sum_{j=1}^{d} \|h_{I_j}\|_{\max}^2\right)^{\frac{1}{2}}.$$

Combining this with (A.1), we have

$$\|h_{\boldsymbol{I}}\|_{\max} = \left(\sum_{j=1}^{d} \|h_{I_j}\|_{\max}^2\right)^{\frac{1}{2}}.$$

This completes the proof. □

Appendix B

Proof of Lemma 3.4. If $d = 1$, then $I_1^m = [\underline{a}_1^m, \overline{a}_1^m]$ and $I_2^m = [\underline{a}_2^m, \overline{a}_2^m]$ for $m = 1, 2, \cdots$. Assume that

$$\lim_{m\to\infty} I_1^m = I_1 = [\underline{a}_1, \overline{a}_1] \quad \text{and} \quad \lim_{m\to\infty} I_2^m = I_2 = [\underline{a}_2, \overline{a}_2].$$

It suffices to prove that

$$\|(I_1^m \ominus I_2^m) \ominus (I_1 \ominus I_2)\|_{\mathcal{I}(\mathbb{R})} \to 0. \tag{B.1}$$

According to Definition 2.1, we have either

$$I_1^m \ominus I_2^m = [\underline{a}_1^m - \underline{a}_2^m, \overline{a}_1^m - \overline{a}_2^m] \quad \text{or} \quad I_1^m \ominus I_2^m = [\overline{a}_1^m - \overline{a}_2^m, \underline{a}_1^m - \underline{a}_2^m].$$

Similarly, we have either

$$I_1 \ominus I_2 = [\underline{a}_1 - \underline{a}_2, \overline{a}_1 - \overline{a}_2] \quad \text{or} \quad I_1 \ominus I_2 = [\overline{a}_1 - \overline{a}_2, \underline{a}_1 - \underline{a}_2].$$

Since $I_1^m \to I_1$, $I_2^m \to I_2$, we obtain

$$\|I_1^m \ominus I_1\|_{\mathcal{I}(\mathbb{R})} = \max\{|\underline{a}_1^m - \underline{a}_1|, |\overline{a}_1^m - \overline{a}_1|\} \to 0$$

and

$$\|I_2^m \ominus I_2\|_{\mathcal{I}(\mathbb{R})} = \max\{|\underline{a}_2^m - \underline{a}_2|, |\overline{a}_2^m - \overline{a}_2|\} \to 0,$$

i.e., $\underline{a}_1^m \to \underline{a}_1, \overline{a}_1^m \to \overline{a}_1$ and $\underline{a}_2^m \to \underline{a}_2, \overline{a}_2^m \to \overline{a}_2$.

We will verify (B.1) in two following cases.

Case 1: $I_1^m \ominus I_2^m = [\underline{a}_1^m - \underline{a}_2^m, \overline{a}_1^m - \overline{a}_2^m]$. Clearly,

$$\underline{a}_1^m - \underline{a}_2^m \leq \overline{a}_1^m - \overline{a}_2^m.$$

Taking $m \to \infty$, we get $\underline{a}_1 - \underline{a}_2 \leq \overline{a}_1 - \overline{a}_2$. So,

$$\|(I_1^m \ominus I_2^m) \ominus (I_1 \ominus I_2)\|_{\mathcal{I}(\mathbb{R})} = \max\{|\underline{a}_1^m - \underline{a}_2^m - (\underline{a}_1 - \underline{a}_2)|, |\overline{a}_1^m - \overline{a}_2^m - (\overline{a}_1 - \overline{a}_2)|\} \to 0.$$

Case 2: $I_1^m \ominus I_2^m = [\overline{a}_1^m - \overline{a}_2^m, \underline{a}_1^m - \underline{a}_2^m]$. Obviously,

$$\overline{a}_1^m - \overline{a}_2^m \leq \underline{a}_1^m - \underline{a}_2^m.$$

Taking $m \to \infty$, we get $\overline{a}_1 - \overline{a}_2 \leq \underline{a}_1 - \underline{a}_2$. Moreover,

$$\|(I_1^m \ominus I_2^m) \ominus (I_1 \ominus I_2)\|_{\mathcal{I}(\mathbb{R})} = \max\{|\overline{a}_1^m - \overline{a}_2^m - (\overline{a}_1 - \overline{a}_2)|, |\underline{a}_1^m - \underline{a}_2^m - (\underline{a}_1 - \underline{a}_2)|\} \to 0.$$

If $d > 1$, then $\boldsymbol{I}_1^m = (I_{11}^m, \ldots, I_{1d}^m)$ and $\boldsymbol{I}_2^m = (I_{21}^m, \ldots, I_{2d}^m)$, for $m = 1, 2, \cdots$. Assume that

$$\lim_{m\to\infty} \boldsymbol{I}_1^m = \boldsymbol{I} = (I_{11}, \ldots, I_{1d}) \text{ and } \lim_{m\to\infty} \boldsymbol{I}_2^m = \boldsymbol{I} = (I_{21}, \ldots, I_{2d}).$$

It suffices to prove that

$$\|(\boldsymbol{I}_1^m \ominus \boldsymbol{I}_2^m) \ominus (\boldsymbol{I}_1 \ominus \boldsymbol{I}_2)\|_{\mathcal{I}(\mathbb{R}^d)} \to 0.$$

Since $\boldsymbol{I}_1^m \to \boldsymbol{I}_1$ and $\boldsymbol{I}_2^m \to \boldsymbol{I}_2$, we have

$$\|\boldsymbol{I}_1^m \ominus \boldsymbol{I}_1\|_{\mathcal{I}(\mathbb{R}^d)} \to 0 \text{ and } \|\boldsymbol{I}_2^m \ominus \boldsymbol{I}_2\|_{\mathcal{I}(\mathbb{R}^d)} \to 0.$$

By Definition 3.2, we obtain

$$\|I_{1j}^m \ominus I_{1j}\|_{\mathcal{I}(\mathbb{R})} \to 0 \text{ and } \|I_{2j}^m \ominus I_{2j}\|_{\mathcal{I}(\mathbb{R})} \to 0$$

for every j, i.e., $I_{1j}^m \to I_{1j}$ and $I_{2j}^m \to I_{2j}$. From the case $d = 1$ it follows that

$$\|(I_{1j}^m \ominus I_{2j}^m) \ominus (I_{1j} \ominus I_{2j})\|_{\mathcal{I}(\mathbb{R})} \to 0.$$

Hence,

$$\|(\boldsymbol{I}_1^m \ominus \boldsymbol{I}_2^m) \ominus (\boldsymbol{I}_1 \ominus \boldsymbol{I}_2)\|_{\mathcal{I}(\mathbb{R}^d)} \to 0.$$

This completes the proof. □

Appendix C

Proof of Lemma 3.5. If $d = 1$, assume that

$$I^m = [\underline{a}^m, \overline{a}^m] \in \mathcal{I}(\mathbb{R}_+)$$

and $h_{I^m} \to h \in C(\{-1, 1\})$, and we can see that

$$h(x) = \begin{cases} \overline{a}, & \text{if } x = 1, \\ -\underline{a}, & \text{if } x = -1, \end{cases} \tag{C.1}$$

where $\underline{a}^m \to \underline{a}$, $\overline{a}^m \to \overline{a}$, $\overline{a} \geq \underline{a} \geq 0$. Moreover, $h = h_{[\underline{a},\overline{a}]}$, and $[\underline{a}, \overline{a}] \in \mathcal{I}(\mathbb{R}_+)$. So, $\mathcal{I}(\mathbb{R}_+)$ is closed.

For $I = [\underline{a}, \overline{a}]$, $I' = [\underline{a}', \overline{a}'] \in \mathcal{I}(\mathbb{R}_+)$, $t \in [0, 1]$, and $s > 0$, it follows that

$$sI = [s\underline{a}, s\overline{a}] \in \mathcal{I}(\mathbb{R}_+)$$

and

$$tI + (1-t)I' = [t\underline{a} + (1-t)\underline{a}', t\overline{a} + (1-t)\overline{a}'] \in \mathcal{I}(\mathbb{R}_+).$$

Hence, $\mathcal{I}(\mathbb{R}_+)$ is a convex pointed cone.

If $d > 1$, assume that

$$\boldsymbol{I}^m = ([\underline{a}_1^m, \overline{a}_1^m], \ldots, [\underline{a}_d^m, \overline{a}_d^m]) \in \mathcal{I}(\mathbb{R}_+^d)$$

and $h_{\boldsymbol{I}^m} \to h$. Since Eq (C.1) and

$$h_{\boldsymbol{I}^m}(x) = \sum_{j=1}^{d} h_{[\underline{a}_j^m, \overline{a}_j^m]}(x_j),$$

it follows that

$$h = \sum_{j=1}^{d} h_{[\underline{a}_j, \overline{a}_j]}(x_j),$$

where $\underline{a}_j^m \to \underline{a}_j$, $\overline{a}_j^m \to \overline{a}_j$, $\overline{a}_j \geq \underline{a}_j \geq 0$ for each j. This implies that $([\underline{a}_1, \overline{a}_1], \ldots [\underline{a}_d, \overline{a}_d]) \in \mathcal{I}(\mathbb{R}_+^d)$. So, $\mathcal{I}(\mathbb{R}_+^d)$ is closed.

For $t \in [0, 1]$ and $\boldsymbol{I} = (I_1, \ldots, I_d)$, $\quad \boldsymbol{I}' = (I_1', \ldots, I_d') \in \mathcal{I}(\mathbb{R}_+^d)$, due to the convexity of $\mathcal{I}(\mathbb{R}_+)$, it can be deduced that

$$t\boldsymbol{I} + (1-t)\boldsymbol{I}' = (tI_1 + (1-t)I_1', \ldots, tI_d + (1-t)I_d') \in \mathcal{I}(\mathbb{R}_+^d),$$

which shows the convexity of $\mathcal{I}(\mathbb{R}_+^d)$. For $s > 0$, since $\mathcal{I}(\mathbb{R}_+)$ is a pointed cone, then

$$s\boldsymbol{I} = (sI_1, \ldots, sI_d) \in \mathcal{I}(\mathbb{R}_+^d).$$

So, $\mathcal{I}(\mathbb{R}_+^d)$ is a convex pointed cone in $C(\mathbb{S}^{d-1})$. □

Appendix D

Proof of Lemma 4.4. We only need to prove the case where P is upper semi-continuous. From the upper semi-continuity of P, it follows that for every $x \in \mathcal{X}$ and $\epsilon > 0$, there is an open neighborhood $o(x)$ of x such that for all $x' \in o(x)$,

$$P(x') \ominus P(x) \in V_\epsilon^d - \mathcal{I}(\mathbb{R}_+^d).$$

Then, for every i and $x' \in o(x)$,

$$p_i(x') \ominus p_i(x) \in V_\epsilon^1 - \mathcal{I}(\mathbb{R}_+).$$

That is to say, we obtain the following equivalence:

$$\begin{aligned}
&[\min\{\underline{p}_i(x') - \underline{p}_i(x), \overline{p}_i(x') - \overline{p}_i(x)\}, \max\{\underline{p}_i(x') - \underline{p}_i(x), \overline{p}_i(x') - \overline{p}_i(x)\}] \subset (-\infty, \epsilon) \\
&\iff \max\{\underline{p}_i(x') - \underline{p}_i(x), \overline{p}_i(x') - \overline{p}_i(x)\} < \epsilon \\
&\iff \underline{p}_i(x') < \underline{p}_i(x) + \epsilon \text{ and } \overline{p}_i(x') < \overline{p}_i(x) + \epsilon.
\end{aligned}$$

Hence, $\underline{p}_i$ and $\overline{p}_i$ are upper semi-continuous for each i. Then $\underline{P}$ and $\overline{P}$ are $\mathbb{R}_+^d$-upper semi-continuous, and vice versa. □

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