AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(1): 2979–3007.
DOI:10.3934/math.2026119
Received: 10 December 2025
Revised: 15 January 2026
Accepted: 19 January 2026
Published: 29 January 2026

Research article

Fuzzy rough graphs via fuzzy graph ideals with applications

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Abstract: In this paper, we extend the concept of fuzzy rough sets to graphs by introducing the notion of "fuzzy graph ideals". This novel approach enables the construction of rough fuzzy digraphs, termed "fuzzy graph ideal approximation spaces", along with associated methods of formation. We define new fuzzy lower and upper graphs based on any two fuzzy binary relations on the vertex and edge sets of a directed graph thereby utilizing the concept of fuzzy graph ideals. Additionally, we explore fuzzy interior and fuzzy closure graphs within rough fuzzy graphs, and examine properties such as fuzzy graph connectedness through the application of these new operators. These developments offer valuable tools to address uncertain decision-making problems, with potential practical applications in various domains. Particularly, we provide an algorithm to solve decision-making problems regarding the identification of the best location in a department to set a mobile phone Jammer.

Keywords: fuzzy rough relations; fuzzy rough digraphs; fuzzy graph ideals; decision-making
Mathematics Subject Classification: 03E72, 68R05, 68R10

Nomenclature

Symbol	Description
Ω	Simple directed graph
$\mathfrak{Q}(\Omega)$	Set of vertices (nodes) over Ω
$\mathfrak{Z}(\Omega)$	Set of edges over Ω
$Ə, Ə_1, Ə_2, Ə_3, \ldots$	Vertices (nods) of Ω
$\alpha, \alpha_1, \alpha_2, \alpha_3, \ldots$	Edges of Ω
$Ə_i Ə_j$	Edge connects the two vertices $Ə_i$ and $Ə_j$ with $i, j \in \{i, 2, 3, ...\}$
$\mathbf{I}^{\Omega}$	Family of all fuzzy graphs within Ω
$\mathbf{I}^{\mathfrak{Q}(\circlearrowleft)}, \mathbf{I}^{\mathfrak{Z}(\circlearrowleft)}$	Family of all fuzzy subsets of $\mathfrak{Q}(\Omega)$, $\mathfrak{Z}(\Omega)$, respectively
$\beth_\Omega, \daleth_\Omega, \eta_\Omega, \theta_\Omega, \Lambda_\Omega, \eta, \eta', \eta'', \ldots$	Fuzzy subgraphs of Ω
$\mu_{\mathfrak{Q}(\Omega)}, \mu_{\mathfrak{Z}(\Omega)}$	Fuzzy vertex and fuzzy edge membership functions, respectively
$(\beth_\Omega)^c$	Complement of the fuzzy graph $\beth_\Omega$
$\Re$	Fuzzy relations on the vertices set $\mathfrak{Q}(\Omega)$ of Ω
Ξ	Fuzzy relations on the edges set $\mathfrak{Z}(\Omega)$ of Ω
$\mathfrak{L}$	Graph ideal
$[\mathfrak{Q}(\beth)]_{\Re}$ and $\mathfrak{Q}(\beth)^{\Re}$	Lower fuzzy vertex set and upper fuzzy vertex set, respectively
$[\mathfrak{Z}(\beth)]_{\Xi}$, $\mathfrak{Z}(\beth)^{\Xi}$	Lower fuzzy edges set and upper fuzzy edges set, respectively
$(\mathfrak{Q}(\beth))^B$ and $(\mathfrak{Z}(\beth))^B$	Boundary fuzzy regions of $\mathfrak{Q}(\beth)$ and $\mathfrak{Z}(\beth)$, respectively
$\underline{\underline{\beth_\Omega}} = ([\mathfrak{Q}(\beth)]_{\Re}, [\mathfrak{Z}(\beth)]_{\Xi})$	Lower approximation of $\beth_\Omega$
$\overline{\overline{\beth_\Omega}} = ([\mathfrak{Q}(\beth)]^{\Re}, [\mathfrak{Z}(\beth)]^{\Xi})$	Upper approximation of $\beth_\Omega$
τ_Ω	Fuzzy topology on the rough fuzzy graph IAS ($\Omega, \Re, \Xi, \mathfrak{L}$)

1. Introduction

For the first time, in 1982, Pawlak accomplished the theory of rough sets [1]. For the study of intelligent systems, a comprehensive product of knowledge is necessary for the system to operate effectively. In this context, the principles of rough sets provide valuable tools to understand the structure and behavior of such systems. These principles serve as a foundation for the development of the upper and lower approximation operators. From this perspective, mathematicians, logicians, and researchers have advanced the theory, thereby exploring its applications in various fields [2, 3]. Theories such as the closure operator and its related concepts have been integral in this development. The primary focus is to understand how topology can be applied to analyze the structure of these systems, thus emphasizing the importance of a topological approach in the study of uncertain or incomplete information.

Graph theory, introduced in [4], and general topology are two fundamental and closely interconnected branches of mathematics. Their relationship involves constructing topologies based on the vertices and edges of a graph. Numerous studies have utilized both directed and undirected graphs to develop topological structures (see [5–7]). Most of these structures have been identified within the context of simple undirected graphs, specifically focusing on their vertex sets. Relations defined on graphs act as a link between graph theory and topological concepts, enabling the generation of new types of topological structures from this relationship. As noted in [8], there exists a one-to-one correspondence between labeled topologies on n points and labeled transitive directed graphs with n points. In 2013, [9] investigated the connection between directed graphs and finite topologies. In 2013, a topology on the vertices of an undirected graph was proposed by the authors of [10]. Recently, many communications between the rough set theory and graph theory were proposed, which are useful in physics and medicine. In 2018, the authors in [11] connected an incidence topology to a vertex set of simple graphs without isolated vertices. In 2019, the authors of [12] introduced a novel topology on the set of vertices $\mathfrak{Q}(\Omega)$ of a simple graph $\Omega = (\mathfrak{Q}(\Omega), \mathfrak{Z}(\Omega))$ that contains no isolated vertices, thereby utilizing incidence topology. This topology is generated by a subbases composed of end sets that only include the endpoints of each edge. Additionally, in [13], the same authors used the graph $\Omega = (\mathfrak{Q}(\Omega), \mathfrak{Z}(\Omega))$ to define two different topologies on its set of edges $\mathfrak{Z}(\Omega)$, namely the compatible edge topology and the incompatible edge topology. To develop new topological structures, [6] introduced a relation on graphs to generate various topological forms. The field of topology has since significantly expanded, thereby incorporating many newly developed topics and concepts. To uncover meaningful properties within these complex topological problems, researchers have introduced structures such as closure spaces, proximity, ideals [14], and primals [15].

The concept of fuzzy topology was later put out by Chang [16]. One may think of the concept as a logical extension of general topology. Numerous generic topological features in fuzzy environments have since been studied and investigated by other authors in this subject. A novel fuzzy structure named "fuzzy primal" was proposed in [17] to generate a new topology named primal fuzzy topology. In [18], Zadeh's [19] fuzzy relations were used to present the first definition of a fuzzy graph. In 1975, a more complete explanation was described by the author of [8], who introduced the fuzzy graph theory by studying fuzzy relations on fuzzy sets. He developed a few relationships pertaining to the characteristics of trees, different graphs, and path graphs. The fuzzy rough set theory [20], which unifies rough sets and fuzzy sets, is a popular mathematical model to process data with uncertainty. Over the last few years, it was successfully applied to various domains including feature selection [21–23] and outlier detection [24–26] in mixed data. In the fuzzy rough set theory, the concepts of lower and upper approximations from classic Rough Sets are adapted to work within the framework of fuzzy logic, which lead to fuzzy lower and upper approximations. This integration offers three advantages:

1) It provides a flexible framework to model data with uncertainty and imprecision by representing the boundaries of a concept with fuzzy membership functions. This allows for a partial membership where elements can belong to a set to varying degrees, thus more precisely reflecting real world scenarios.
2) It facilitates the direct processing of various data types, including numerical values, categorical variables, symbols, and more, without the necessity of data type transformation, thus preserving the data intrinsic diversity for subsequent analyses.
3) It enhances reasoning and decision making in ambiguous situations, thereby enabling the classification of objects even when their attribute values do not precisely match the criteria of a specific class, thus offering a robust approach to dealing with data ambiguity.

Therefore, the fuzzy rough set theory holds significant potential to identify outliers in heterogeneous data with uncertainty and imprecision. Additionally, several results related to rough fuzzy digraphs and their applications in decision-making scenarios were discussed in [27–29]. Considering the complexity of the analyzed systems and the interdisciplinary aspects involved in the occurring phenomena, a protocol supporting the decision-makers needs to account for possible oscillations of parameters, insufficient data, or unsuitable models [30–32]. Despite these advancements, uncertainty remains a significant challenge in safety management due to factors such as discrepancies in expert judgment, data sufficiency, and semantic ambiguities in information. On the other hand, the authors of [33, 34] developed an approach that addresses decision-making in large groups using cloud models integrated into multi-granularity linguistic environments and bidirectional trust approaches in social networks. These methodologies are beneficial for problems that require consensus and detailed analysis in dynamic environments.

The primary contribution of the paper is the development of a novel framework for the fuzzy graph theory by introducing the concept of "fuzzy graph ideals". We extend the traditional rough set theory to the context of directed graphs (digraphs) through fuzzy relations, thus leading to the construction of "fuzzy graph ideal approximation spaces". We propose new fuzzy lower and upper approximations based on any two fuzzy relations on the vertex and edge sets, and establish their properties. In this paper, following the introduction, we go over the definitions and conclusions required to fully understand the information presented in this work. After that, in Section 3, we define the notion of the "fuzzy graph ideal" on a given graph. We combine the fuzzy graph ideal with the concept of fuzzy rough sets on digraphs to produce fuzzy graph ideal approximation spaces. Based on any two fuzzy binary relations defined on the digraph, we produce two new styles of fuzzy graph ideal approximation spaces to reduce the fuzzy boundary region and increase the fuzzy accuracy degree. Additionally, key results for both types of fuzzy graph ideal approximations are established, and the relationships among the current fuzzy approximations are derived. Moreover, comparisons between the present fuzzy graph ideal approximations in [35,36] and the preceding ones in [37–39] are presented and shown to be more general. Closure spaces by relations was defined in [40], and primal topological spaces over a graph is presented in [41]. Additionally, fuzzy interior and fuzzy closure graphs of a rough fuzzy graph are discussed. In Section 4, we discuss fuzzy graph connectedness as a sample application of the new fuzzy graph operators. In Section 5, we explore applications of fuzzy graph ideal approximation spaces and introduce algorithms designed to address decision-making problems, such as identifying the optimal location within a department to install a mobile phone jammer. In Section 6, we summarize the main findings and contributions; highlight the increasing impact of topological spaces in both crisp and fuzzy graph theories; and emphasize their novel development of the "graph ideal" concept within the fuzzy graph theory. The contributions of this paper include the following:

1) **Introduction of fuzzy graph ideal:** The paper introduces and formulates the concept of a "fuzzy graph ideal" as a significant advancement in the fuzzy graph theory, thus expanding its framework and applications.
2) **Development of fuzzy graph ideal approximation spaces:** The authors extend the traditional rough set theory to directed graphs (digraphs) using fuzzy relations and the new notion of the fuzzy graph ideal, thus leading to the construction of "fuzzy graph ideal approximation spaces" (fuzzy graph IASs).
3) **Proposal of new fuzzy approximations:** The paper proposes two new styles of fuzzy lower and upper approximations on digraphs. These approximations are based on any two fuzzy relations defined on the vertex and edge sets of the digraph, in combination with the fuzzy graph ideal.
4) **Establishment of properties and generality:** Key results and properties of both types of fuzzy graph ideal approximations are established. Furthermore, comparisons are made, which show that the present fuzzy graph ideal approximations are more general than previous approaches found in the literature.

5) **Exploration of related graph concepts:** The paper discusses fuzzy interior and fuzzy closure graphs of a rough fuzzy graph, and explores fuzzy graph connectedness as a sample application of the new fuzzy graph operators.
6) **Practical application and algorithm:** The study explores applications of fuzzy graph IASs, thereby providing an algorithm to solve uncertain decision-making problems, specifically exemplified by identifying the optimal location within a department to set a mobile phone jammer.

2. Preliminaries

Throughout this study, we focus on simple directed graphs that may or may not contain loops. The symbol Ω denotes a graph. For brevity, the term "simple undirected graph with or without loops" will be referred to as "graph". Let Ω represent a graph (domain), $\mathbf{I}$ denote the unit interval $[0,1]$, and $\mathbf{I}^{\Omega}$ be the collection of all fuzzy graphs within Ω. The complement of a fuzzy subgraph $\daleth_{\Omega}$ is $(\daleth_{\Omega})^{c}$. If there is no confusion, then we write the complement as $\overline{\daleth_{\Omega}}$. A fuzzy graph approximation space ($\Omega, \Re, \Xi$) combined with a fuzzy graph ideal is called a fuzzy graph ideal approximation space (fuzzy graph IAS, for short) and denoted by ($\Omega, \Re, \Xi, \mathfrak{L}$). All undefined terminologies used in the manuscript can be referenced in [8, 16, 41].

A graph Ω is represented as the pair $(\mathfrak{Q}(\Omega), \mathfrak{Z}(\Omega))$, where $\mathfrak{Q}(\Omega)$ is a nonempty finite or infinite set, and $\mathfrak{Z}(\Omega)$ is a set of unordered pairs of members of $\mathfrak{Q}(\Omega)$. The sets $\mathfrak{Q}(\Omega) = \mathfrak{Q}$ and $\mathfrak{Z}(\Omega) = \mathfrak{Z}$ are the vertex and edge sets of Ω, respectively. The pair $\Omega = (\emptyset, \emptyset)$ stands for the empty graph. A graph Ω with no loops or numerous edges is called a simple graph. A pair $\mu_{\Omega} = (\mathfrak{Q}(\mu), \mathfrak{Z}(\mu)) = (\mu_{\mathfrak{Q}(\Omega)}, \mu_{\mathfrak{Z}(\Omega)}) \in \mathbf{I}^{\Omega}$ is said to be a fuzzy graph over the graph Ω. Here, $\mu_{\mathfrak{Q}(\Omega)}$ and $\mu_{\mathfrak{Z}(\Omega)}$ are called fuzzy vertex and fuzzy edge membership functions, respectively, of the fuzzy graph $(\mu_{\mathfrak{Q}(\Omega)}, \mu_{\mathfrak{Z}(\Omega)})$, respectively. In the operator $\mu_{\mathfrak{Q}(\Omega)} : \mathfrak{Q}(\Omega) \longrightarrow [0,1]$, the value $\mu_{\mathfrak{Q}(\Omega)}(\eth)$ is called the degree of the membership of a vertex $\eth$ in $\mu_{\mathfrak{Q}(\Omega)}$ for each $\eth \in \mathfrak{Q}(\Omega)$. Again, in the operator $\mu_{\mathfrak{Z}(\Omega)} : \mathfrak{Z}(\Omega) \longrightarrow [0,1]$, the value $\mu_{\mathfrak{Z}(\Omega)}(\alpha)$ is called the degree of the membership of an edge α in $\mu_{\mathfrak{Z}(\Omega)}$ for each $\alpha \in \mathfrak{Z}(\Omega)$. Both the members $\mu_{\mathfrak{Q}(\Omega)}(\eth)$ and $\mu_{\mathfrak{Z}(\Omega)}(\alpha)$ lie between 0 and 1. Therefore, in a fuzzy graph, there are three types of vertices: first, vertices with a membership value of 0 (indicating that the node never appears in the graph); second, vertices with a membership value of 1 (indicating that the node is definitely present); and third, vertices with membership values between 0 and 1 (representing degrees of partial presence). Typically, the edges of a fuzzy graph are depicted using solid lines, except for those edges whose membership values are 0. Consequently, only edges with positive membership values are considered in the graph. A fuzzy graph $(\mu_{\mathfrak{Q}(\Omega)}, \mu_{\mathfrak{Z}(\Omega)}) \in \mathbf{I}^{\Omega}$ over the graph Ω is called a strong fuzzy graph if $\mu_{\mathfrak{Z}(\Omega)}(\alpha) = \mu_{\mathfrak{Q}(\Omega)}(\eth_1) \wedge \mu_{\mathfrak{Q}(\Omega)}(\eth_2)$, α connects $\eth_1$ to $\eth_2$, for all $\alpha \in \mathfrak{Z}(\Omega)$. Let $\mu^1_{\Omega} = (\mu^1_{\mathfrak{Q}(\Omega)}, \mu^1_{\mathfrak{Z}(\Omega)}), \mu^2_{\Omega} = (\mu^2_{\mathfrak{Q}(\Omega)}, \mu^2_{\mathfrak{Z}(\Omega)}), \in \mathbf{I}^{\Omega}$ be two fuzzy graphs over the graph Ω. Then, $(\mu^1_{\mathfrak{Q}(\Omega)}, \mu^2_{\mathfrak{Z}(\Omega)})$ is called a fuzzy subgraph of the fuzzy graph $(\mu^2_{\mathfrak{Q}(\Omega)}, \mu^2_{\mathfrak{Z}(\Omega)})$ if $\mu^1_{\mathfrak{Q}(\Omega)}(\eth) \leq \mu^2_{\mathfrak{Q}(\Omega)}(\eth)$ and $\mu^1_{\mathfrak{Z}(\Omega)}(\alpha) \leq \mu^2_{\mathfrak{Z}(\Omega)}(\alpha)$. The union of $(\mu^1_{\mathfrak{Q}(\Omega)}, \mu^1_{\mathfrak{Z}(\Omega)})$ and $(\mu^2_{\mathfrak{Q}(\Omega)}, \mu^2_{\mathfrak{Z}(\Omega)})$ is another fuzzy graph $(\mu^3_{\mathfrak{Q}(\Omega)}, \mu^3_{\mathfrak{Z}(\Omega)})$ over the graph Ω, where $\mu^3_{\mathfrak{Q}(\Omega)} = \mu^1_{\mathfrak{Q}(\Omega)} \vee \mu^2_{\mathfrak{Q}(\Omega)}$ and $\mu^3_{\mathfrak{Z}(\Omega)} = \mu^1_{\mathfrak{Z}(\Omega)} \vee \mu^2_{\mathfrak{Z}(\Omega)}$. Additionally, the intersection is given by $\mu^3_{\mathfrak{Q}(\Omega)} = \mu^1_{\mathfrak{Q}(\Omega)} \wedge \mu^2_{\mathfrak{Q}(\Omega)}$ and $\mu^3_{\mathfrak{Z}(\Omega)} = \mu^1_{\mathfrak{Z}(\Omega)} \wedge \mu^2_{\mathfrak{Z}(\Omega)}$.

3. Fuzzy rough digraphs with fuzzy graph ideals

This section focuses on the development of fuzzy graph ideal approximation spaces and their properties. It introduces the concept of a fuzzy graph ideal within the context of the fuzzy graph theory and establishes how these ideals can be used to define new types of fuzzy approximations. The section elaborates on the construction of fuzzy lower and upper approximation operators based on fuzzy relations and the newly defined fuzzy graph ideals. Additionally, it discusses the relationships between these approximations and compares them with previous approaches in the literature, thus highlighting their generality and flexibility. Additionally, the section explores the concepts of fuzzy interior and fuzzy closure graphs of rough fuzzy graphs, thereby demonstrating how these tools can be applied to analyze the structure and properties of fuzzy graphs.

Definition 3.1. *Consider a crisp graph $\Omega = (\mathfrak{Q}(\Omega), \mathfrak{Z}(\Omega)$. The symbol $\underline{0}_\Omega = (\underline{0}_{\mathfrak{Q}(\Omega)}, \underline{0}_{\mathfrak{Z}(\Omega)}) \in \mathbf{I}^\Omega$ refers to the empty fuzzy graph over the graph Ω. In $\underline{0}_\Omega$, each vertex and edge have a degree of membership 0. Additionally, the symbol $\underline{1}_\Omega = (\underline{1}_{\mathfrak{Q}(\Omega)}, \underline{1}_{\mathfrak{Z}(\Omega)}) \in \mathbf{I}^\Omega$ refers to the full fuzzy graph over the graph Ω. In $\underline{1}_\Omega$, each vertex and edge have a degree of membership 1. Note that the crisp graph Ω is the full fuzzy graph.*

Definition 3.2. *Consider a fuzzy graph $\mu_\Omega = (\mu_{\mathfrak{Q}(\Omega)}, \mu_{\mathfrak{Z}(\Omega)}) \in \mathbf{I}^\Omega$ over a graph Ω. We define the complement of μ_Ω as $\mu^c_\Omega = (\overline{\mu}_{\mathfrak{Q}(\Omega)}, \overline{\mu}_{\mathfrak{Z}(\Omega)}) \in \mathbf{I}^\Omega$, where $\overline{\mu}_{\mathfrak{Q}(\Omega)}(\Game) = 1 - \mu_{\mathfrak{Q}(\Omega)}$ for every vertex $\Game$ in $\mathfrak{Q}(\Omega)$ and $\overline{\mu}_{\mathfrak{Z}(\Omega)}(\alpha) = \min\left\{\overline{\mu}_{\mathfrak{Z}(\Omega)}(\alpha), \overline{\mu}_{\mathfrak{Z}(\Omega)}(\alpha)\right\}$ for every edge α in $\mathfrak{Z}(\Omega)$. Note that, the complement of any fuzzy graph is a strong fuzzy graph.*

Let us define the fuzzy difference between two fuzzy graphs as follows:

$$\left(\beth_\Omega \overline{\wedge} \daleth_\Omega\right) = \begin{cases} \underline{0}_\Omega, & \text{if } \beth_\Omega \le \daleth_\Omega; \\ \beth_\Omega \wedge \daleth^c_\Omega, & \text{otherwise.} \end{cases}$$

Definition 3.3. *Consider two fuzzy subgraphs $\beth_\Omega$ and $\daleth_\Omega$ over a graph Ω. A subfamily $\mathfrak{L}$ of $\mathbf{I}^\Omega$ is said to be a fuzzy graph ideal on the graph Ω if it satisfies the following conditions:*

(1) $\underline{0}_\Omega \in \mathfrak{L}$;
(2) If $\daleth_\Omega \in \mathfrak{L}$ and $\beth_\Omega \subseteq \daleth_\Omega$, then $\beth_\Omega \in \mathfrak{L}$; and
(3) If $\beth_\Omega \in \mathfrak{L}$ and $\daleth_\Omega \in \mathfrak{L}$, then, $\beth_\Omega \cup \daleth_\Omega \in \mathfrak{L}$.

We usually consider the proper fuzzy graph ideals, that is, $\underline{1}_\Omega \notin \mathfrak{L}$. Denote a fuzzy graph ideal $\mathfrak{L}_0$ for the fuzzy graph ideal only containing $\underline{0}_\Omega$, that is, $\mathfrak{L}_0 = \{((\underline{0}_{\mathfrak{Q}(\Omega)}, \underline{0}_{\mathfrak{Z}(\Omega)}))\}$.

Here, we will define rough fuzzy sets related to vertex and edge sets of any graph Ω in a fuzzy graph approximation space ($\Omega, \mathfrak{R}, \Xi$) in a generalized form.

Definition 3.4. *Let $\mathfrak{Q}(\Omega)$ be the vertex set of a finite graph Ω, $\mathfrak{R}$ be a fuzzy relation on $\mathfrak{Q}(\Omega)$. Then, for any $\Game \in \mathfrak{Q}(\Omega)$, define the fuzzy sets $\Game\mathfrak{R}, \mathfrak{R}\Game \in \mathbf{I}^{\mathfrak{Q}(\Omega)}$ as follows:*

$$\Game\mathfrak{R}(\Game') = \mathfrak{R}(\Game, \Game') \quad \text{and} \quad \mathfrak{R}\Game(\Game') = \mathfrak{R}(\Game', \Game) \forall \Game' \in \mathfrak{Q}(\Omega).$$

Define the fuzzy sets $<\eth>\Re, \Re<\eth>\in \mathbf{I}^{\mathfrak{Q}(\Omega)}$ for any vertex $\eth \in \mathfrak{Q}(\Omega)$ as follows:

$$<\eth>\Re = \bigwedge_{\eth' \in \mathfrak{Q}(\Omega):\Re(\eth',\eth)>0} \eth'\Re \text{ and } \Re<\eth> = \bigwedge_{\eth' \in \mathfrak{Q}(\Omega),\Re(\eth,\eth')>0} \Re\eth'.$$

For any vertex $\eth \in \mathfrak{Q}(\Omega)$, define the minimal neighborhood $\Re<\eth>\Re : \mathfrak{Q}(\Omega) \to \mathbf{I}$ as follows:

$$\Re<\eth>\Re = <\eth>\Re \wedge \Re<\eth>.$$

Definition 3.5. *Let $\mathfrak{Q}(\Omega)$ be the edge set of a finite graph Ω, Ξ be a fuzzy relation on $\mathfrak{Z}(\Omega)$. Then, for any edge $\alpha \in \mathfrak{Z}(\Omega)$, define the fuzzy sets $\alpha\Xi, \Xi\alpha \in \mathbf{I}^{\mathfrak{Z}(\Omega)}$ as follows:*

$$\alpha\Xi(\alpha') = \Xi(\alpha,\alpha') \quad \text{and} \quad \Xi\alpha(\alpha') = \Xi(\alpha',\alpha) \forall \alpha' \in \mathfrak{Z}(\Omega).$$

Define the fuzzy sets $<\alpha>\Xi, \Xi<\alpha>\in \mathbf{I}^{\mathfrak{Z}(\Omega)}$ for any edge $\alpha \in \mathfrak{Z}(\Omega)$ as follows:

$$<\alpha>\Xi = \bigwedge_{\alpha' \in \mathfrak{Z}(\Omega):\Xi(\alpha',\alpha)>0} \alpha'\Xi \text{ and } \Xi<\alpha> = \bigwedge_{\alpha' \in \mathfrak{Z}(\Omega),\Xi(\alpha,\alpha')>0} \Xi\alpha'.$$

For any edge $\alpha \in \mathfrak{Z}(\Omega)$, define the minimal neighborhood $\Xi<\eth>\Xi : \mathfrak{Z}(\Omega) \to \mathbf{I}$ as follows:

$$\Xi<\eth>\Xi = <\eth>\Xi \wedge \Xi<\eth>.$$

Definition 3.6. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph approximation space and $\sqsupset$ be a fuzzy subgraph of the graph Ω. For every vertex point $\eth \in \mathfrak{Q}(\Omega)$, define $(\mathfrak{Q}(\sqsupset))_*, (\mathfrak{Q}(\sqsupset))^* \in \mathbf{I}^{\mathfrak{Q}(\Omega)}$ of a fuzzy vertex set $\mathfrak{Q}(\sqsupset) \in \mathbf{I}^{\mathfrak{Q}(\Omega)}$ by the following:*

$$(\mathfrak{Q}(\sqsupset))_*(\eth) = \begin{cases} \left(\bigvee_{\eth' \in \mathfrak{Q}(\Omega)} \Re<\eth'>\Re(\eth)\right)^c & \text{if } \Re<\eth>\Re \wedge (\mathfrak{Q}(\sqsupset))^c = \mathfrak{Q}(\eta) \text{ for some } \eta \notin \mathfrak{L} \\ & \text{and } \Re<\eth>\Re \wedge \mathfrak{Q}(\sqsupset) = \mathfrak{Q}(\eta') \text{ for some } \eta' \notin \mathfrak{L} \\ 1 & \text{if } \Re<\eth>\Re \wedge (\mathfrak{Q}(\sqsupset))^c = \mathfrak{Q}(\eta'') \text{ for some } \eta'' \in \mathfrak{L} \\ 0 & \text{if } \Re<\eth>\Re \wedge (\mathfrak{Q}(\sqsupset))^c = \mathfrak{Q}(\eta''') \text{ for some } \eta''' \notin \mathfrak{L} \\ & \text{and } \Re<\eth>\Re \wedge \mathfrak{Q}(\sqsupset) = \mathfrak{Q}(\eta'''') \text{ for some } \eta'''' \in \mathfrak{L} \end{cases}$$

$$(\mathfrak{Q}(\sqsupset))^*(\eth) = \begin{cases} \bigvee_{\eth' \in \mathfrak{Q}(\Omega)} \Re<\eth'>\Re(\eth) & \text{if } \Re<\eth>\Re \wedge \mathfrak{Q}(\sqsupset) = \mathfrak{Q}(\eta) \text{ for some } \eta \notin \mathfrak{L} \\ & \text{and } \Re<\eth>\Re \wedge (\mathfrak{Q}(\sqsupset))^c = \mathfrak{Q}(\eta') \text{ for some } \eta' \notin \mathfrak{L} \\ 1 & \text{if } \Re<\eth>\Re \wedge \mathfrak{Q}(\sqsupset) = \mathfrak{Q}(\eta'') \text{ for some } \eta'' \in \mathfrak{L} \\ 0 & \text{if } \Re<\eth>\Re \wedge \mathfrak{Q}(\sqsupset) = \mathfrak{Q}(\eta''') \text{ for some } \eta''' \notin \mathfrak{L} \\ & \text{and } \Re<\eth>\Re \wedge (\mathfrak{Q}(\sqsupset))^c = \mathfrak{Q}(\eta'''') \text{ for some } \eta'''' \in \mathfrak{L} \end{cases}$$

The roughness of a fuzzy vertex set $\mathfrak{Q}(\sqsupset) \in \mathbf{I}^{\mathfrak{Q}(\Omega)}$ is defined by the following:

$$[\mathfrak{Q}(\sqsupset)]_{\Re} = \mathfrak{Q}(\sqsupset) \wedge (\mathfrak{Q}(\sqsupset))_* \text{ and } [\mathfrak{Q}(\sqsupset)]^{\Re} = \mathfrak{Q}(\sqsupset) \vee (\mathfrak{Q}(\sqsupset))^*,$$

where $[\mathfrak{Q}(\sqsupset)]_{\Re}$ is the lower fuzzy vertex set of $\mathfrak{Q}(\sqsupset)$ and $\mathfrak{Q}(\sqsupset)^{\Re}$ is the upper fuzzy vertex set of $\mathfrak{Q}(\sqsupset)$. The boundary fuzzy region of $\mathfrak{Q}(\sqsupset)$ is $(\mathfrak{Q}(\sqsupset))^B$ given by the following: $(\mathfrak{Q}(\sqsupset))^B = [\mathfrak{Q}(\sqsupset)]^{\Re} \bar{\wedge} [\mathfrak{Q}(\sqsupset)]_{\Re}$. The pair ($[\mathfrak{Q}(\sqsupset)]^{\Re}, [\mathfrak{Q}(\sqsupset)]_{\Re}$) will be called a fuzzy rough vertex set if $[\mathfrak{Q}(\sqsupset)]^{\Re} \bar{\wedge} [\mathfrak{Q}(\sqsupset)]_{\Re} \neq \underline{0}_{\mathfrak{Q}(\Omega)}$.

Definition 3.7. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph approximation space, $\beth$ be a fuzzy subgraph of the graph Ω, and ($[\mathfrak{Q}(\beth)]^{\Re}, [\mathfrak{Q}(\beth)]_{\Re}$) be a fuzzy rough vertex set on Ω. Let Ξ be a fuzzy relation on the edges set $\mathfrak{Z}(\Omega)$ of the graph Ω such that $\Xi(\eth_1\eth_2, \eth_1'\eth_2') \leq \Re(\eth_1, \eth_1') \wedge \Re(\eth_2, \eth_2')$ for all $\eth_1\eth_2, \eth_1'\eth_2' \in \mathfrak{Z}(\Omega)$. Additionally, let γ be a fuzzy subgraph of Ω such that $\gamma_{\mathfrak{Z}(\Omega)}(\eth\eth') \leq [\mathfrak{Q}(\beth)]_{\Re}(\eth) \wedge [\mathfrak{Q}(\beth)]_{\Re}(\eth')$ for all $\eth\eth' \in \mathfrak{Z}(\Omega)$. Then, for every edge point $\alpha \in \mathfrak{Z}(\Omega)$, define $(\mathfrak{Z}(\beth))_*, (\mathfrak{Z}(\beth))^* \in \mathbf{I}^{\mathfrak{Z}(\Omega)}$ of the fuzzy edge set $\mathfrak{Z}(\beth) \in \mathbf{I}^{\mathfrak{Z}(\Omega)}$ by the following:*

$$
(\mathfrak{Z}(\beth))_*(\alpha) = \begin{cases} \left(\bigvee_{\alpha' \in \mathfrak{Z}(\Omega)} \Xi < \alpha' > \Xi(\alpha)\right)^c & \text{if } \Xi < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta) \text{ for some } \eta \notin \mathfrak{L} \\ & \text{and } \Xi < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta') \text{ for some } \eta' \notin \mathfrak{L} \\ 1 & \text{if } \Xi < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta'') \text{ for some } \eta''' \in \mathfrak{L} \\ 0 & \text{if } \Xi < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta''') \text{ for some } \eta''' \notin \mathfrak{L} \\ & \text{and } \Xi < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta'''') \text{ for some } \eta'''' \in \mathfrak{L} \end{cases}
$$

$$
(\mathfrak{Z}(\beth))^*(\alpha) = \begin{cases} \bigvee_{\alpha' \in \mathfrak{Z}(\Omega)} \Xi < \alpha' > \Xi(\alpha) & \text{if } \Xi < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta^{)} \text{ for some } \eta \notin \mathfrak{L} \\ & \text{and } \Xi < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta') \text{ for some } \eta' \notin \mathfrak{L} \\ 1 & \text{if } \Xi < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta'') \text{ for some } \eta'' \in \mathfrak{L} \\ 0 & \text{if } \Xi < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta''') \text{ for some } \eta''' \notin \mathfrak{L} \\ & \text{and } \Xi < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta'''') \text{ for some } \eta'''' \in \mathfrak{L} \end{cases}
$$

The roughness of a fuzzy edge set $\mathfrak{Z}(\beth) \in \mathbf{I}^{\mathfrak{Z}(\Omega)}$ is defined by the following:

$$
[\mathfrak{Z}(\beth)]_{\Xi} = \mathfrak{Z}(\beth) \wedge (\mathfrak{Z}(\beth))_* \text{ and } [\mathfrak{Z}(\beth)]^{\Xi} = \mathfrak{Z}(\beth) \vee (\mathfrak{Z}(\beth))^*,
$$

where $[\mathfrak{Z}(\beth)]_{\Xi}$ is the lower fuzzy edges set of $\mathfrak{Z}(\beth)$, and $\mathfrak{Z}(\beth)^{\Xi}$ is the upper fuzzy edges set of $\mathfrak{Z}(\beth)$. The boundary fuzzy region of $\mathfrak{Z}(\beth)$ is $(\mathfrak{Z}(\beth))^B$ given by the following: $(\mathfrak{Z}(\beth))^B = [\mathfrak{Z}(\beth)]^{\Xi} \bar{\wedge} [\mathfrak{Z}(\beth)]_{\Xi}$. The pair ($[\mathfrak{Z}(\beth)]^{\Xi}, [\mathfrak{Z}(\beth)]_{\Xi}$) will be called a fuzzy rough edge set (fuzzy rough relation) if $[\mathfrak{Z}(\beth)]^{\Xi} \bar{\wedge} [\mathfrak{Z}(\beth)]_{\Xi} \neq \underline{0}_{\mathfrak{Z}(\Omega)}$.

Definition 3.8. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS, and $\beth$ be a fuzzy subgraph of the graph Ω. Then,*

(i) $\Re$ is any fuzzy binary relation on the vertex set $\mathfrak{Q}(\Omega)$ of Ω,
(ii) Ξ is any fuzzy binary relation on the edge set $\mathfrak{Z}(\Omega)$ of Ω,
(iii) $\beth^{\Re}_{\mathfrak{Q}(\Omega)} = ([\mathfrak{Q}(\beth)]_{\Re}, [\mathfrak{Q}(\beth)]^{\Re}$) is a rough set on $\mathfrak{Q}(\Omega)$,
(iv) $\beth^{\Xi}_{\mathfrak{Z}(\Omega)} = ([\mathfrak{Z}(\beth)]_{\Xi}, [\mathfrak{Z}(\beth)]^{\Xi}$) is a rough relation on $\mathfrak{Z}(\Omega)$, and
(v) $\underline{\beth_{\Omega}} = ([\mathfrak{Q}(\beth)]_{\Re}, [\mathfrak{Z}(\beth)]_{\Xi})$ and $\overline{\beth_{\Omega}} = ([\mathfrak{Q}(\beth)]^{\Re}, [\mathfrak{Z}(\beth)]^{\Xi})$ are fuzzy graphs, where $\underline{\beth_{\Omega}}$ represents lower approximation of $\beth_{\Omega}$ and $\overline{\beth_{\Omega}}$ represents upper approximation of $\beth_{\Omega}$ such that

$$
[\mathfrak{Z}(\beth)]_{\Xi}(\eth\eth') \leq \min\{[\mathfrak{Q}(\beth)]_{\Re}(\eth), [\mathfrak{Q}(\beth)]_{\Re}(\eth')\},
$$

$$
[\mathfrak{Z}(\beth)]^{\Xi}(\eth\eth') \leq \min\{[\mathfrak{Q}(\beth)]^{\Re}(\eth), [\mathfrak{Q}(\beth)]^{\Re}(\eth')\}, \text{ for all } \eth\eth' \in \mathfrak{Z}(\Omega).
$$

Example 3.9. *Assume that Ω be the graph $(\mathfrak{Q}(\Omega), \mathfrak{Z}(\Omega))$, where $\mathfrak{Q}(\Omega) = \{\eth_1, \eth_2, \eth_3, \eth_4, \eth_5, \eth_6\}$, and* $\mathfrak{Z}(\Omega) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$.

Now, we define a fuzzy graph $\beth_\Omega = (\beth_{\mathfrak{Q}(\Omega)}, \beth_{\mathfrak{Z}(\Omega)})$ over the crisp graph Ω. We assume the fuzzy vertex set

$$\beth_{\mathfrak{Q}(\Omega)} = \{(\eth_1, 0.9), (\eth_2, 0.92), (\eth_3, 0.94), (\eth_4, 0.96), (\eth_5, 0.98), (\eth_6, 1.0)\},$$

and the fuzzy edge set

$$\beth_{\mathfrak{Z}(\Omega)} = \{(\alpha_1, 0.8), (\alpha_2, 0.82), (\alpha_3, 0.84), (\alpha_4, 0.86), (\alpha_5, 0.88), (\alpha_6, 0.9), (\alpha_7, 0.9), (\alpha_8, 0.9)\}.$$

The fuzzy subgraph $\beth_\Omega$ is represented in Figure 1.

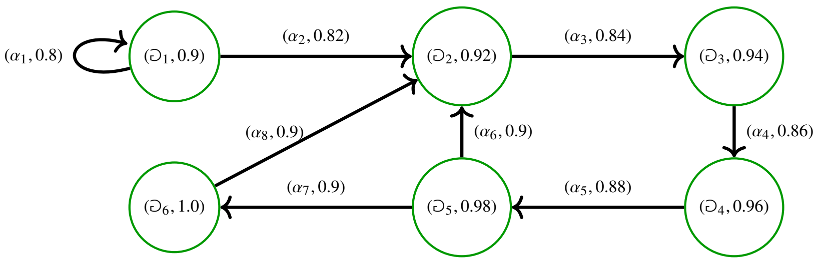


Figure 1. Representation of the fuzzy subgraph $\beth_\Omega$ given in Example 3.9.

Define a fuzzy binary relation $\mathfrak{R}$ on $\mathfrak{Q}(\Omega)$, as given in Table 1.

Table 1. Fuzzy relation $\mathfrak{R}$ in Example 3.9.

$\mathfrak{R}$	$\eth_1$	$\eth_2$	$\eth_3$	$\eth_4$	$\eth_5$	$\eth_6$
$\eth_1$	1	0.2	0.3	0.4	0.5	0.1
$\eth_2$	0.2	1	0.6	0.5	0.7	0.4
$\eth_3$	0.3	0.6	1	0.8	0.9	0.3
$\eth_4$	0.4	0.5	0.8	1	0.1	0.2
$\eth_5$	0.5	0.7	0.9	0.1	1	0.7
$\eth_6$	0.1	0.4	0.3	0.2	0.7	1

Compute the following: $\eth_1\mathfrak{R} = \{1, 0.2, 0.3, 0.4, 0.5, 0.1\}$, $\eth_2\mathfrak{R} = \{0.2, 1, 0.6, 0.5, 0.7, 0.4\}$, $\eth_3\mathfrak{R} = \{0.3, 0.6, 1, 0.8, 0.9, 0.3\}$, $\eth_4\mathfrak{R} = \{0.4, 0.5, 0.8, 1, 0.1, 0.2\}$, $\eth_5\mathfrak{R} = \{0.5, 0.7, 0.9, 0.1, 1, 0.7\}$, $\eth_6\mathfrak{R} = \{0.1, 0.4, 0.3, 0.2, 0.7, 1\}$. $\mathfrak{R}\eth_1 = \{1, 0.2, 0.3, 0.4, 0.5, 0.1\}$, $\mathfrak{R}\eth_2 = \{0.2, 1, 0.6, 0.5, 0.7, 0.4\}$, $\mathfrak{R}\eth_3 = \{0.3, 0.6, 1, 0.8, 0.9, 0.3\}$, $\mathfrak{R}\eth_4 = \{0.4, 0.5, 0.8, 1, 0.1, 0.2\}$, $\mathfrak{R}\eth_5 = \{0.5, 0.7, 0.9, 0.1, 1, 0.7\}$, and $\mathfrak{R}\eth_6 = \{0.1, 0.4, 0.3, 0.2, 0.7, 1\}$. Then, $< \eth_1 > \mathfrak{R} =< \eth_2 > \mathfrak{R} =< \eth_3 > \mathfrak{R} =< \eth_4 > \mathfrak{R} =< \eth_5 > \mathfrak{R} =< \eth_6 > \mathfrak{R} = \{0.1, 0.2, 0.3, 0.1, 0.1, 0.1\}$ and $\mathfrak{R} < \eth_1 >= \mathfrak{R} < \eth_2 >= \mathfrak{R} < \eth_3 >= \mathfrak{R} < \eth_4 >= \mathfrak{R} < \eth_5 >= \mathfrak{R} < \eth_6 >= \{0.1, 0.2, 0.3, 0.1, 0.1, 0.1\}$. Therefore, $\mathfrak{R} < \eth_1 > \mathfrak{R} = \mathfrak{R} < \eth_2 > \mathfrak{R} = \mathfrak{R} < \eth_3 > \mathfrak{R} = \mathfrak{R} < \eth_4 > \mathfrak{R} = \mathfrak{R} < \eth_5 > \mathfrak{R} = \mathfrak{R} < \eth_6 > \mathfrak{R} = \{0.1, 0.2, 0.3, 0.1, 0.1, 0.1\}$. Consider a fuzzy graph ideal $\mathfrak{L}$, where $[\mathfrak{Q}(\beth)] \in \mathfrak{L}$ iff $\mathfrak{Q}([\mathfrak{Q}(\beth)]) \leq \underline{0.09}_{\mathfrak{Q}(\Omega)}$, and the membership function of the fuzzy edge set $[\mathfrak{Q}(\beth)]_{\mathfrak{Z}(\Omega)}(\alpha)$ is given by the following: $\min\{[\mathfrak{Q}(\beth)]_{\mathfrak{Q}(\Omega)}(\eth_1), [\mathfrak{Q}(\beth)]_{\mathfrak{Q}(\Omega)}(\eth_2)\}$, where the edge α connects the vertices $\eth_1$ and $\eth_2$. The lower and upper approximations of $\beth_{\mathfrak{Q}(\Omega)}$ with respect to $\mathfrak{R}$ and $\mathfrak{L}$ are given by the following:

$$[\mathfrak{Q}(\beth)]_{\mathfrak{R}} = \{(\eth_1, 0.9), (\eth_2, 0.8), (\eth_3, 0.7), (\eth_4, 0.9), (\eth_5, 0.9), (\eth_6, 0.9)\},$$

$$[\mathfrak{Q}(\beth)]^{\mathfrak{R}} = \{(\eth_1, 0.9), (\eth_2, 0.92), (\eth_3, 0.94), (\eth_4, 0.96), (\eth_5, 0.98), (\eth_6, 1.0)\}.$$

Let $\gamma_{\mathfrak{Z}(\Omega)} = \{(\eth_1\eth_1, 0.1), (\eth_1\eth_2, 0.1), (\eth_2\eth_3, 0.2), (\eth_3\eth_4, 0.3), (\eth_4\eth_5, 0.3), (\eth_5\eth_6, 0.4), (\eth_5\eth_2, 0.2), (\eth_6\eth_2, 0.3)\}$ be a fuzzy edge set defined on $\mathfrak{Z}(\Omega)$ and Ξ be a fuzzy relation defined on $\mathfrak{Z}(\Omega)$, as given in Table 2.

Table 2. Fuzzy relation Ξ in Example 3.9.

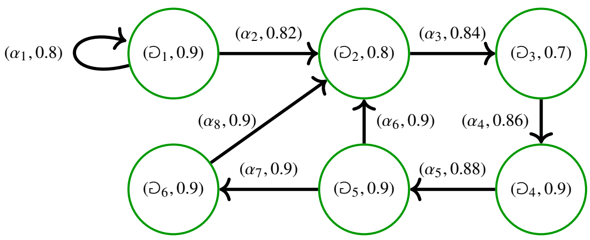
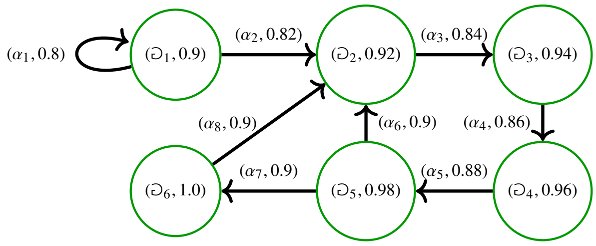
Ξ	$\eth_1\eth_1$	$\eth_1\eth_2$	$\eth_2\eth_3$	$\eth_3\eth_4$	$\eth_4\eth_5$	$\eth_5\eth_6$	$\eth_5\eth_2$	$\eth_6\eth_2$
$\eth_1\eth_1$	1	0.2	0.1	0.2	0.3	0.1	0.2	0.1
$\eth_1\eth_2$	0.2	1	0.1	0.2	0.3	0.4	0.5	0.1
$\eth_2\eth_3$	0.1	0.1	1	0.5	0.4	0.2	0.5	0.3
$\eth_3\eth_4$	0.2	0.2	0.5	1	0.1	0.1	0.4	0.2
$\eth_4\eth_5$	0.3	0.3	0.4	0.1	1	0.1	0.1	0.1
$\eth_5\eth_6$	0.1	0.4	0.2	0.1	0.1	1	0.3	0.3
$\eth_5\eth_2$	0.2	0.5	0.5	0.4	0.1	0.3	1	0.6
$\eth_6\eth_2$	0.1	0.1	0.3	0.2	0.1	0.3	0.6	1

Compute the following: $\eth_1\eth_1\Xi = \{1, 0.2, 0.1, 0.2, 0.3, 0.1, 0.2, 0.1\}$, $\eth_1\eth_2\Xi = \{0.2, 1, 0.1, 0.2, 0.3, 0.4, 0.5, 0.1\}$, $\eth_2\eth_3\Xi = \{0.1, 0.1, 1, 0.5, 0.4, 0.2, 0.5, 0.3\}$, $\eth_3\eth_4\Xi = \{0.2, 0.2, 0.5, 1, 0.1, 0.1, 0.4, 0.2\}$, $\eth_4\eth_5\Xi = \{0.3, 0.3, 0.4, 0.1, 1, 0.1, 0.1, 0.1\}$, $\eth_5\eth_6\Xi = \{0.1, 0.4, 0.2, 0.1, 0.1, 1, 0.3, 0.3\}$, $\eth_5\eth_2\Xi = \{0.2, 0.5, 0.5, 0.4, 0.1, 0.3, 1, 0.6\}$, $\eth_6\eth_2\Xi = \{0.1, 0.1, 0.3, 0.2, 0.1, 0.3, 0.6, 1\}$, $\Xi\eth_1\eth_1 = \{1, 0.2, 0.1, 0.2, 0.3, 0.1, 0.2, 0.1\}$, $\Xi\eth_1\eth_2 = \{0.2, 1, 0.1, 0.2, 0.3, 0.4, 0.5, 0.1\}$, $\Xi\eth_2\eth_3 = \{0.1, 0.1, 1, 0.5, 0.4, 0.2, 0.5, 0.3\}$, $\Xi\eth_3\eth_4 = \{0.2, 0.2, 0.5, 1, 0.1, 0.1, 0.4, 0.2\}$, $\Xi\eth_4\eth_5 = \{0.3, 0.3, 0.4, 0.1, 1, 0.1, 0.1, 0.1\}$, $\Xi\eth_5\eth_6 = \{0.1, 0.4, 0.2, 0.1, 0.1, 1, 0.3, 0.3\}$, $\Xi\eth_5\eth_2 = \{0.2, 0.5, 0.5, 0.4, 0.1, 0.3, 1, 0.6\}$, and $\Xi\eth_6\eth_2 = \{0.1, 0.1, 0.3, 0.2, 0.1, 0.3, 0.6, 1\}$. Then, $< \eth_1\eth_1 > \Xi = < \eth_1\eth_2 > \Xi = < \eth_2\eth_3 > \Xi = < \eth_3\eth_4 > \Xi = < \eth_4\eth_5 > \Xi = < \eth_5\eth_6 > \Xi = < \eth_5\eth_2 > \Xi = < \eth_6\eth_2 > \Xi = \{0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}$ and $\Xi < \eth_1\eth_1 > = \Xi < \eth_1\eth_2 > = \Xi < \eth_2\eth_3 > = \Xi < \eth_3\eth_4 > = \Xi < \eth_4\eth_5 > = \Xi < \eth_5\eth_6 > = \Xi < \eth_5\eth_2 > = \Xi < \eth_6\eth_2 > = \{0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}$. Therefore, $\Xi < \eth_1\eth_1 > \Xi = \Xi < \eth_1\eth_2 > \Xi = \Xi < \eth_2\eth_3 > \Xi = \Xi < \eth_3\eth_4 > \Xi = \Xi < \eth_4\eth_5 > \Xi = \Xi < \eth_5\eth_6 > \Xi = \Xi < \eth_5\eth_2 > \Xi = \Xi < \eth_6\eth_2 > \Xi = \{0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}$.

The lower and upper approximations of $\sqsupset_{\mathfrak{Z}(\Omega)}$ with respect to Ξ and $\mathfrak{L}$ are given by the following: $[\mathfrak{Z}(\sqsupset)]_{\Xi} = \{(\eth_1\eth_1, 0.8), (\eth_1\eth_2, 0.82), (\eth_2\eth_3, 0.84), (\eth_3\eth_4, 0.86), (\eth_4\eth_5, 0.88), (\eth_5\eth_6, 0.9), (\eth_5\eth_2, 0.9), (\eth_6\eth_2, 0.9)\}$, $[\mathfrak{Z}(\sqsupset)]^{\Xi} = \{(\eth_1\eth_1, 0.8), (\eth_1\eth_2, 0.82), (\eth_2\eth_3, 0.84), (\eth_3\eth_4, 0.86), (\eth_4\eth_5, 0.88), (\eth_5\eth_6, 0.9), (\eth_5\eth_2, 0.9), (\eth_6\eth_2, 0.9)\}$.

As a result, the lower and upper approximations $\underline{\sqsupset_{\Omega}} = ([\mathfrak{Q}(\sqsupset)]_{\mathfrak{R}}, [\mathfrak{Z}(\sqsupset)]_{\Xi})$ and $\overline{\sqsupset_{\Omega}} = ([\mathfrak{Q}(\sqsupset)]^{\mathfrak{R}}, [\mathfrak{Z}(\sqsupset)]^{\Xi})$, respectively, of $\sqsupset_{\Omega}$ are shown in Table3.

Table 3. Representation of the lower and upper approximations $\underline{\sqsupset_\Omega}$ and $\overline{\sqsupset_\Omega}$ in Example 3.9.

The fuzzy lower approximation $\underline{\sqsupset_\Omega}$	The fuzzy upper approximation $\overline{\sqsupset_\Omega}$
	

Definition 3.10. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS. For every rough fuzzy vertex set $\sqsupset_{\mathfrak{Q}(\Omega)} \in \mathbf{I}^{\mathfrak{Q}(\Omega)}$, define the accuracy fuzzy set $\mu(\sqsupset_{\mathfrak{Q}(\Omega)}) \in \mathbf{I}^{\mathfrak{Q}(\Omega)}$, for all $\eth \in \sqsupset_{\mathfrak{Q}(\Omega)}$, by the following:*

$$\mu(\sqsupset_{\mathfrak{Q}(\Omega)})(\eth) = \begin{cases} 0 & \text{if } [\mathfrak{Q}(\sqsupset)]^{\Re} = \underline{1}_{\mathfrak{Q}(\Omega)} \text{ and } [\mathfrak{Q}(\sqsupset)]_{\Re} = \underline{0}_{\mathfrak{Q}(\Omega)}, \\ \left([\mathfrak{Q}(\sqsupset)]^{\Re}(\eth) - \sqsupset_{\mathfrak{Q}(\Omega)}(\eth)\right)^c \wedge (\sqsupset_{\mathfrak{Q}(\Omega)}(\eth) - [\mathfrak{Q}(\sqsupset)]_{\Re}(\eth))^c & \text{if } [\mathfrak{Q}(\sqsupset)]^{\Re} \not\leq [\mathfrak{Q}(\sqsupset)]_{\Re}, \\ 1 & \text{otherwise;} \end{cases}$$

moreover, the accuracy value of the rough fuzzy vertex set $\sqsupset_{\mathfrak{Q}(\Omega)}$ is given by $\mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Q}(\Omega)}))$.

Whenever $[\mathfrak{Q}(\sqsupset)]^{\Re}$ be so that $[\mathfrak{Q}(\sqsupset)]^{\Re} \leq [\mathfrak{Q}(\sqsupset)]_{\Re}$, we get that $\sqsupset_{\mathfrak{Q}(\Omega)} = [\mathfrak{Q}(\sqsupset)]_{\Re} = [\mathfrak{Q}(\sqsupset)]^{\Re}$; then, $(\sqsupset_{\mathfrak{Q}(\Omega)})^B = \underline{0}_{\mathfrak{Q}(\Omega)}$ and $\mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Q}(\Omega)})) = 1$. If $[\mathfrak{Q}(\sqsupset)]_{\Re} = \underline{0}_{\mathfrak{Q}(\Omega)}$ and $[\mathfrak{Q}(\sqsupset)]^{\Re} = \underline{1}_{\mathfrak{Q}(\Omega)}$, Then, $(\sqsupset_{\mathfrak{Q}(\Omega)})^B = \underline{1}_{\mathfrak{Q}(\Omega)}$ and $\mathrm{Inf}(\mu([\mathfrak{Q}(\sqsupset)])) = 0$. Otherwise, $(\sqsupset_{\mathfrak{Q}(\Omega)})^B = [\mathfrak{Q}(\sqsupset)]^{\Re} \wedge ([\mathfrak{Q}(\sqsupset)]_{\Re})^c$ and $0 < \mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Q}(\Omega)})) < 1$, that is, the largest boundary fuzzy vertex set is associated with the lowest accuracy value and the converse is true. If $\mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Q}(\Omega)})) = 1$, Then, $\sqsupset_{\mathfrak{Q}(\Omega)}$ is crisp with respect to $\Re\left([\mathfrak{Q}(\sqsupset)]_{\Re} = [\mathfrak{Q}(\sqsupset)]^{\Re}\right.$ and $\sqsupset_{\mathfrak{Q}(\Omega)}$ is precise with respect to $\Re$). If $\mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Q}(\Omega)})) = 0$, Then, $\sqsupset_{\mathfrak{Q}(\Omega)}$ is totally rough with respect to $\Re$. Moreover, if $0 < \mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Q}(\Omega)})) < 1$, Then, $\sqsupset_{\mathfrak{Q}(\Omega)}$ is rough with respect to μ.

Note that the analogue of the above definition is valid for the rough fuzzy edge sets; then, the accuracy value of the rough fuzzy subgraph $\sqsupset_\Omega$ is given by $(\mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Q}(\Omega)})), \mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Z}(\Omega)})))$.

Lemma 3.11. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS and $\sqsupset_\Omega, \daleth_\Omega \in \mathbf{I}^\Omega$. Then, the following properties hold:*

(1) $[\mathfrak{Q}(\sqsupset)]^* = (([\mathfrak{Q}(\sqsupset)]^c)_*)^c$ and $[\mathfrak{Z}(\sqsupset)]^* = (([\mathfrak{Z}(\sqsupset)]^c)_*)^c$,
(2) $[\underline{0}_{\mathfrak{Q}(\Omega)}]^* = \underline{0}_{\mathfrak{Q}(\Omega)}$ and $[\underline{0}_{\mathfrak{Z}(\Omega)}]^* = \underline{0}_{\mathfrak{Z}(\Omega)}$,
(3) $[\underline{1}_{\mathfrak{Q}(\Omega)}]^* = \underline{1}_{\mathfrak{Q}(\Omega)}$ and $[\underline{1}_{\mathfrak{Z}(\Omega)}]^* = \underline{1}_{\mathfrak{Z}(\Omega)}$,
(4) $\sqsupset_\Omega \leq \daleth_\Omega$ implies that $[\mathfrak{Q}(\sqsupset)]_* \leq [\mathfrak{Q}(\daleth)]_*$ and $[\mathfrak{Z}(\sqsupset)]_* \leq [\mathfrak{Z}(\daleth)]_*$,
(5) $\sqsupset_\Omega \leq \daleth_\Omega$ implies that $[\mathfrak{Q}(\sqsupset)]^* \leq [\mathfrak{Q}(\daleth)]^*$ and $[\mathfrak{Z}(\sqsupset)]^* \leq [\mathfrak{Z}(\daleth)]^*$,
(6) $(\sqsupset_{\mathfrak{Q}(\Omega)} \wedge \daleth_{\mathfrak{Q}(\Omega)})^* \leq [\mathfrak{Q}(\sqsupset)]^* \wedge [\mathfrak{Q}(\daleth)]^*$ and $(\sqsupset_{\mathfrak{Z}(\Omega)} \wedge \daleth_{\mathfrak{Z}(\Omega)})^* \leq [\mathfrak{Z}(\sqsupset)]^* \wedge [\mathfrak{Z}(\daleth)]^*$,
(7) $(\sqsupset_{\mathfrak{Q}(\Omega)} \vee \daleth_{\mathfrak{Q}(\Omega)})_* \geq [\sqsupset_{\mathfrak{Q}(\Omega)}]_* \vee [\daleth_{\mathfrak{Q}(\Omega)}]_*$ and $(\sqsupset_{\mathfrak{Z}(\Omega)} \vee \daleth_{\mathfrak{Z}(\Omega)})_* \geq [\sqsupset_{\mathfrak{Z}(\Omega)}]_* \vee [\daleth_{\mathfrak{Z}(\Omega)}]_*$,
(8) $(\sqsupset_{\mathfrak{Q}(\Omega)} \wedge \daleth_{\mathfrak{Q}(\Omega)})_* \leq [\sqsupset_{\mathfrak{Q}(\Omega)}]_* \wedge [\daleth_{\mathfrak{Q}(\Omega)}]_*$ and $(\sqsupset_{\mathfrak{Z}(\Omega)} \wedge \daleth_{\mathfrak{Z}(\Omega)})_* \leq [\sqsupset_{\mathfrak{Z}(\Omega)}]_* \wedge [\daleth_{\mathfrak{Z}(\Omega)}]_*$,
(9) $(\sqsupset_{\mathfrak{Q}(\Omega)} \vee \daleth_{\mathfrak{Q}(\Omega)})^* \geq [\sqsupset_{\mathfrak{Q}(\Omega)}]^* \vee [\daleth_{\mathfrak{Q}(\Omega)}]^*$, and $(\sqsupset_{\mathfrak{Z}(\Omega)} \vee \daleth_{\mathfrak{Z}(\Omega)})^* \geq [\sqsupset_{\mathfrak{Z}(\Omega)}]^* \vee [\daleth_{\mathfrak{Z}(\Omega)}]^*$,
(10) If $\sqsupset_\Omega \in \mathfrak{L}$, Then, $[\sqsupset_{\mathfrak{Q}(\Omega)}]^* = \underline{0}_{\mathfrak{Q}(\Omega)}$ and $[\sqsupset_{\mathfrak{Z}(\Omega)}]^* = \underline{0}_{\mathfrak{Z}(\Omega)}$, and
(11) If $(\sqsupset_\Omega)^c \in \mathfrak{L}$, Then, $[\sqsupset_{\mathfrak{Q}(\Omega)}]_* = \underline{1}_{\mathfrak{Q}(\Omega)}$ and $[\sqsupset_{\mathfrak{Z}(\Omega)}]_* = \underline{1}_{\mathfrak{Z}(\Omega)}$.

Proof. We prove the fuzzy vertex set, and in a similar way, we can prove the conditions required for the fuzzy edge set.

(1) It is clear that $([\beth_{\mathfrak{Q}(\Omega)}]^*)^c = ([\beth_{\mathfrak{Q}(\Omega)}]^c)_*$, $([\beth_{\mathfrak{Q}(\Omega)}]_*)^c = ([\beth_{\mathfrak{Q}(\Omega)}]^c)^*$ and thus $[\beth_{\mathfrak{Q}(\Omega)}]^* = \left(([\beth_{\mathfrak{Q}(\Omega)}]^c)_*\right)^c$ and $[\beth_{\mathfrak{Q}(\Omega)}]_* = (([\beth_{\mathfrak{Q}(\Omega)}]^c)^*)^c$.

(2) We have $(\underline{0}_{\mathfrak{Q}(\Omega)})^*(\eth) = 0 \forall \eth \in \mathfrak{Q}(\Omega)$; then, $\underline{0}^*_{\mathfrak{Q}(\Omega)} = \underline{0}_{\mathfrak{Q}(\Omega)}$.

(3) We have $(\underline{1}_{\mathfrak{Q}(\Omega)})^*(\eth) = 1 \forall \eth \in \mathfrak{Q}(\Omega)$; then, $\underline{1}^*_{\mathfrak{Q}(\Omega)} = \underline{1}_{\mathfrak{Q}(\Omega)}$.

(4) It is proven from the definition of the fuzzy graph ideal and Definition 3.6.

(5) It is proven from the definition of the fuzzy graph ideal and Definition 3.6.

(6) We get it directly using the result in point (4).

(7) We get it directly using the result in point (5).

(8) We directly get that $[\beth_{\mathfrak{Q}(\Omega)}]_* \wedge [\daleth_{\mathfrak{Q}(\Omega)}]_* \geq ([\beth_{\mathfrak{Q}(\Omega)}] \wedge [\beth_{\mathfrak{Q}(\Omega)}])_*$.

(9) We directly get that $([\beth_{\mathfrak{Q}(\Omega)}] \vee [\daleth_{\mathfrak{Q}(\Omega)}])^* \geq [\beth_{\mathfrak{Q}(\Omega)}]^* \vee [\daleth_{\mathfrak{Q}(\Omega)}]^*$.

(10) Since $\beth_\Omega \in \mathfrak{L}$ implies that $\Re < \eth > \Re \wedge [\beth_{\mathfrak{Q}(\Omega)}] = \mathfrak{Q}([\mathfrak{Q}(\beth)])$ for some $[\mathfrak{Q}(\beth)] \in \mathfrak{L}$, then $[\beth_{\mathfrak{Q}(\Omega)}]^*(\Omega) = 0 \forall \eth \in \Omega$. Hence, $[\beth_{\mathfrak{Q}(\Omega)}]^* = \underline{0}_{\mathfrak{Q}(\Omega)}$.

(11) Since $(\beth_\Omega)^c \in \mathfrak{L}$ implies that $\Re < \eth > \Re \wedge [\beth_{\mathfrak{Q}(\Omega)}]^c = \mathfrak{Q}([\mathfrak{Q}(\beth)])$ for some $[\mathfrak{Q}(\beth)] \in \mathfrak{L}$, then $[\beth_{\mathfrak{Q}(\Omega)}]_*(\Omega) = 1 \forall \eth \in \Omega$. Hence, $[\beth_{\mathfrak{Q}(\Omega)}]_* = \underline{1}_{\mathfrak{Q}(\Omega)}$.

□

Note that: if we have the trivial fuzzy graph ideal $\mathfrak{L} = \mathbf{I}^\Omega$, Then, $[\beth_{\mathfrak{Q}(\Omega)}]_* = \underline{1}_{\mathfrak{Q}(\Omega)}$ and $[\beth_{\mathfrak{Z}(\Omega)}]_* = \underline{1}_{\mathfrak{Z}(\Omega)}$. Additionally, $[\beth_{\mathfrak{Q}(\Omega)}]^* = \underline{0}_{\mathfrak{Q}(\Omega)}$ and $[\beth_{\mathfrak{Z}(\Omega)}]^* = \underline{0}_{\mathfrak{Z}(\Omega)}$. As a result, $\underline{\beth_\Omega} = ([\mathfrak{Q}(\beth)]_\Re, [\mathfrak{Z}(\beth)]_\Xi) = \overline{\beth_\Omega} = ([\mathfrak{Q}(\beth)]^\Re, [\mathfrak{Z}(\beth)]^\Xi)$; therefore, any fuzzy subgraph has the accuracy value $(\mathrm{Inf}(\mu(\beth_{\mathfrak{Q}(\Omega)})), \mathrm{Inf}(\mu(\beth_{\mathfrak{Z}(\Omega)}))) = (1, 1)$.

Remark 3.12. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS and $\beth_\Omega, \daleth_\Omega \in \mathbf{I}^\Omega$. Then, the following examples show that in general:*

(1) $\beth_{\mathfrak{Q}(\Omega)} \nleq [\beth_{\mathfrak{Q}(\Omega)}]_* \nleq [\beth_{\mathfrak{Q}(\Omega)}]^*$, $\beth_{\mathfrak{Q}(\Omega)} \ngeq [\beth_{\mathfrak{Q}(\Omega)}]_* \ngeq [\beth_{\mathfrak{Q}(\Omega)}]^*$ and $\beth_{\mathfrak{Z}(\Omega)} \nleq [\beth_{\mathfrak{Z}(\Omega)}]_* \nleq [\beth_{\mathfrak{Z}(\Omega)}]^*$, $\beth_{\mathfrak{Z}(\Omega)} \ngeq [\beth_{\mathfrak{Z}(\Omega)}]_* \ngeq [\beth_{\mathfrak{Z}(\Omega)}]^*$;

(2) $(\beth_{\mathfrak{Q}(\Omega)} \wedge \daleth_{\mathfrak{Q}(\Omega)})^* \ngeq \beth^*_{\mathfrak{Q}(\Omega)} \wedge \daleth^*_{\mathfrak{Q}(\Omega)}$ and $(\beth_{\mathfrak{Z}(\Omega)} \wedge \daleth_{\mathfrak{Z}(\Omega)})^* \ngeq \beth^*_{\mathfrak{Z}(\Omega)} \wedge \daleth^*_{\mathfrak{Z}(\Omega)}$;

(3) $(\beth_{\mathfrak{Q}(\Omega)} \wedge \daleth_{\mathfrak{Q}(\Omega)})_* \ngeq [\beth_{\mathfrak{Q}(\Omega)}]_* \wedge [\daleth_{\mathfrak{Q}(\Omega)}]_*$ and $(\beth_{\mathfrak{Z}(\Omega)} \wedge \daleth_{\mathfrak{Z}(\Omega)})_* \ngeq [\beth_{\mathfrak{Z}(\Omega)}]_* \wedge [\daleth_{\mathfrak{Z}(\Omega)}]_*$;

(4) $(\beth_{\mathfrak{Q}(\Omega)} \vee \daleth_{\mathfrak{Q}(\Omega)})_* \nleq [\beth_{\mathfrak{Q}(\Omega)}]_* \vee [\daleth_{\mathfrak{Q}(\Omega)}]_*$ and $(\beth_{\mathfrak{Z}(\Omega)} \vee \daleth_{\mathfrak{Z}(\Omega)})_* \nleq [\beth_{\mathfrak{Z}(\Omega)}]_* \vee [\daleth_{\mathfrak{Z}(\Omega)}]_*$;

(5) $(\beth_{\mathfrak{Q}(\Omega)} \vee \daleth_{\mathfrak{Q}(\Omega)})^* \nleq [\beth_{\mathfrak{Q}(\Omega)}]^* \vee [\daleth_{\mathfrak{Q}(\Omega)}]^*$ and $(\beth_{\mathfrak{Z}(\Omega)} \vee \daleth_{\mathfrak{Z}(\Omega)})^* \nleq [\beth_{\mathfrak{Z}(\Omega)}]^* \vee [\daleth_{\mathfrak{Z}(\Omega)}]^*$;

(6) $[\beth_{\mathfrak{Q}(\Omega)}]^* = \underline{0}_{\mathfrak{Q}(\Omega)} \nRightarrow \beth_\Omega \in \mathfrak{L}$ and $[\beth_{\mathfrak{Z}(\Omega)}]^* = \underline{0}_{\mathfrak{Z}(\Omega)} \nRightarrow \beth_\Omega \in \mathfrak{L}$;

(7) $[\beth_{\mathfrak{Q}(\Omega)}]_* = \underline{1}_{\mathfrak{Q}(\Omega)} \nRightarrow (\beth_\Omega)^c \in \mathfrak{L}$ and $[\beth_{\mathfrak{Z}(\Omega)}]_* = \underline{1}_{\mathfrak{Z}(\Omega)} \nRightarrow (\beth_\Omega)^c \in \mathfrak{L}$.

Example 3.13. *Assume that* Ω *be the graph* $(\mathfrak{Q}(\Omega), \mathfrak{Z}(\Omega))$, *where* $\mathfrak{Q}(\Omega) = \{\eth_1, \eth_2, \eth_3, \eth_4\}$, *and* $\mathfrak{Z}(\Omega) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. *Define two fuzzy subgraphs over* Ω *as follows:*

$$\beth_{\mathfrak{Q}(\Omega)} = \{(\eth_1, 0.1), (\eth_2, 0.8), (\eth_3, 0.4), (\eth_4, 0.6)\},$$

and the fuzzy edge set

$$\beth_{\mathfrak{Z}(\Omega)} = \{(\alpha_1, 0.1), (\alpha_2, 0.1), (\alpha_3, 0.4), (\alpha_4, 0.2)\}.$$

Additionally, define the fuzzy vertex set

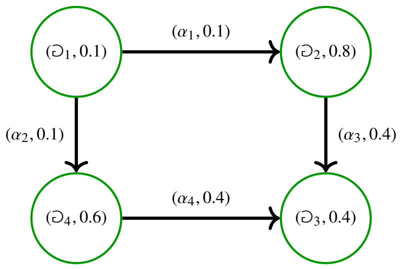
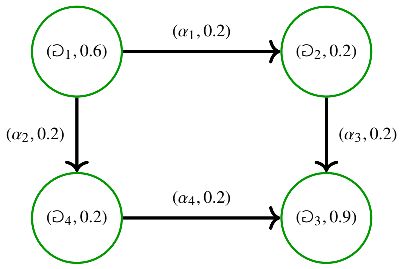
$$\daleth_{\mathfrak{Q}(\Omega)} = \{(\eth_1, 0.6), (\eth_2, 0.2), (\eth_3, 0.9), (\eth_4, 0.2)\},$$

and the fuzzy edge set

$$\daleth_{\mathfrak{Z}(\Omega)} = \{(\alpha_1, 0.2), (\alpha_2, 0.2), (\alpha_3, 0.2), (\alpha_4, 0.2)\}.$$

The two fuzzy graphs $\beth_\Omega$ and $\daleth_\Omega$ are represented in Table 4.

Table 4. Representation of the two fuzzy graphs $\beth_\Omega$ and $\daleth_\Omega$ in Example 3.13.

The fuzzy graph $\beth_\Omega$	The fuzzy graph $\daleth_\Omega$
	

Define a fuzzy binary relation $\Re$ on $\mathfrak{Q}(\Omega)$, as given in Table 5.

Table 5. Fuzzy relation $\Re$ in Example 3.13.

$\Re$	$\Game_1$	$\Game_2$	$\Game_3$	$\Game_4$
$\Game_1$	0	0.2	1	0.5
$\Game_2$	0.6	0	0.8	0.5
$\Game_3$	1	0.5	0.6	0.6
$\Game_4$	0.9	0.6	1	1

$\Game_1\Re = \{0, 0.2, 1, 0.5\}$, $\Game_2\Re = \{0.6, 0, 0.8, 0.5\}$, $\Game_3\Re = \{1, 0.5, 0.6, 0.6\}$, $\Game_4\Re = \{0.9, 0.6, 1, 1\}$, $\Re\Game_1 = \{0, 0.6, 1, 0.9\}$, $\Re\Game_2 = \{0.2, 0, 0.5, 0.6\}$, $\Re\Game_3 = \{1, 0.8, 0.6, 1\}$, and $\Re\Game_4 = \{0.5, 0.5, 0.6, 1\}$. Then, $< \Game_1 > \Re = \{0.6, 0, 0.6, 0.5\}, < \Game_2 > \Re = \{0, 0.2, 0.6, 0.5\}, < \Game_3 > \Re = \{0, 0, 0.6, 0.5\}, < \Game_4 > \Re = \{0, 0, 0.6, 0.5\}$, $\Re < \Game_1 >= \{0.2, 0, 0.5, 0.6\}, \Re < \Game_2 >= \{0, 0.5, 0.5, 0.6\}, \Re < \Game_3 >= \{0, 0, 0.5, 0.6\}$, and $\Re < \Game_4 >= \{0, 0, 0.5, 0.6\}$. Therefore, $\Re < \Game_1 > \Re = \{0.2, 0, 0.5, 0.5\}, \Re < \Game_2 > \Re = \{0, 0.2, 0.6, 0.5\}, \Re < \Game_3 > \Re = \{0, 0, 0.5, 0.5\}, \Re < \Game_4 > \Re = \{0, 0, 0.5, 0.5\}$. Consider a fuzzy graph ideal $\mathfrak{L}$, so that $[\mathfrak{Q}(\beth)] \in \mathfrak{L}$ iff $\mathfrak{Q}([\mathfrak{Q}(\beth)]) \leq \underline{0.4}_{\mathfrak{Q}(\Omega)}$, and the membership function of the fuzzy edge set $[\mathfrak{Q}(\beth)]_{\mathfrak{Z}(\Omega)}(\alpha)$ is such that

$$[\mathfrak{Q}(\beth)]_{\mathfrak{Z}(\Omega)}(\alpha) \leq [[\mathfrak{Q}(\beth)]_{\mathfrak{Q}(\Omega)}(\Game_1)] \wedge [[\mathfrak{Q}(\beth)]_{\mathfrak{Q}(\Omega)}(\Game_2)],$$

with the condition that $\mathfrak{Z}([\mathfrak{Q}(\beth)]) \leq \underline{0.2}_{\mathfrak{Z}(\Omega)}$, where the edge α connects the vertices $\Game_1$ and $\Game_2$. We compute $[\mathfrak{Q}(\beth)]_*$ and $[\mathfrak{Q}(\beth)]^*$ of $\beth_{\mathfrak{Q}(\Omega)}$ with respect to $\Re$ and $\mathfrak{L}$ are as follows: $[\mathfrak{Q}(\beth)]_* = \{(\Game_1, 0.8), (\Game_2, 0.8), (\Game_3, 0.4), (\Game_4, 0.5)\}$ and $[\mathfrak{Q}(\beth)]^* = \{(\Game_1, 0.2), (\Game_2, 0.2), (\Game_3, 0.6), (\Game_4, 0.5)\}$.

Similarly, $[\mathfrak{Q}(\daleth)]_*$ and $[\mathfrak{Q}(\daleth)]^*$ of $\daleth_{\mathfrak{Q}(\Omega)}$ with respect to $\Re$ and $\mathfrak{L}$ are: $[\mathfrak{Q}(\daleth)]_* = \{(\Game_1, 0.8), (\Game_2, 0.8), (\Game_3, 0.4), (\Game_4, 0.5)\}$ and $[\mathfrak{Q}(\daleth)]^* = \{(\Game_1, 0.2), (\Game_2, 0.2), (\Game_3, 0.6), (\Game_4, 0.5)\}$. Then, $[\mathfrak{Q}(\beth)]_* = [\mathfrak{Q}(\daleth)]_* = \{(\Game_1, 0.8), (\Game_2, 0.8), (\Game_3, 0.4), (\Game_4, 0.5)\}$ and $[\mathfrak{Q}(\beth)]^* = [\mathfrak{Q}(\daleth)]^* = \{(\Game_1, 0.2), (\Game_2, 0.2), (\Game_3, 0.6), (\Game_4, 0.5)\}$.

Let $\gamma_{\mathfrak{Z}(\Omega)} = \{(\Game_1\Game_2, 0.1), (\Game_1\Game_4, 0.1), (\Game_2\Game_3, 0.2), (\Game_4\Game_3, 0.3)\}$ be a fuzzy edge set defined on $\mathfrak{Z}(\Omega)$, and Ξ be a fuzzy relation defined on $\mathfrak{Z}(\Omega)$, as given in Table 6.

Table 6. Fuzzy relation Ξ in Example 3.13.

Ξ	$\eth_1\eth_2$	$\eth_1\eth_4$	$\eth_2\eth_3$	$\eth_4\eth_3$
$\eth_1\eth_2$	0	0	0.2	0.4
$\eth_1\eth_4$	0	0	0.2	0.3
$\eth_2\eth_3$	0.3	0.4	0	0.4
$\eth_4\eth_3$	0.3	0.5	0.4	0.5

$\eth_1\eth_2\Xi = \{0,0,0.2,0.4\}$, $\eth_1\eth_4\Xi = \{0,0,0.2,0.3\}$, $\eth_2\eth_3\Xi = \{0.3,0.4,0,0.4\}$, $\eth_4\eth_3\Xi = \{0.3,0.5,0.4,0.5\}$, $\Xi\eth_1\eth_2 = \{0,0,0.3,0.3\}$, $\Xi\eth_1\eth_4 = \{0,0,0.4,0.5\}$, $\Xi\eth_2\eth_3 = \{0.2,0.2,0,0.4\}$, and $\Xi\eth_4\eth_3 = \{0.4,0.3,0.4,0.5\}$. Then, $<\eth_1\eth_2>\Xi = \{0.3,0,0,0.3\}, <\eth_1\eth_4>\Xi = \{0,0.4,0,0.3\}, <\eth_2\eth_3>\Xi = \{0,0,0.2,0.3\}, <\eth_4\eth_3>\Xi = \{0,0,0,0.3\}$, $\Xi<\eth_1\eth_2>= \{0.2,0,0,0.3\}, \Xi<\eth_1\eth_4>= \{0,0.2,0,0.3\}, \Xi<\eth_2\eth_3>= \{0,0,0.3,0.3\}$, and $\Xi<\eth_4\eth_3>= \{0,0,0,0.3\}$. Therefore, $\Xi<\eth_1\eth_2>\Xi = \{0.2,0,0,0.3\}, \Xi<\eth_1\eth_4>\Xi = \{0,0.2,0,0.3\}, \Xi<\eth_2\eth_3>\Xi = \{0,0,0.2,0.3\}, \Xi<\eth_4\eth_3>\Xi = \{0,0,0,0.3\}$. We compute $[\mathfrak{Z}(\beth)]_*$ and $[\mathfrak{Z}(\beth)]^*$ of $\beth_{\mathfrak{Z}(\Omega)}$ with respect to Ξ and $\mathfrak{L}$ as follows:

$[\mathfrak{Z}(\beth)]_* = \{(\eth_1,0.8),(\eth_2,0.8),(\eth_3,0.8),(\eth_4,0.7)\}$ and $[\mathfrak{Z}(\beth)]^* = \{(\eth_1,0.2),(\eth_2,0.2),(\eth_3,0.2),(\eth_4,0.3)\}$.

Similarly, $[\mathfrak{Z}(\daleth)]_*$ and $[\mathfrak{Z}(\daleth)]^*$ of $\daleth_{\mathfrak{Q}(\Omega)}$ with respect to Ξ and $\mathfrak{L}$ are as follows:

$[\mathfrak{Z}(\daleth)]_* = \{(\eth_1,0.8),(\eth_2,0.8),(\eth_3,0.8),(\eth_4,0.7)\}$ and $[\mathfrak{Z}(\daleth)]^* = \{(\eth_1,0.2),(\eth_2,0.2),(\eth_3,0.2),(\eth_4,0.3)\}$.

Then, $[\mathfrak{Z}(\beth)]_* = [\mathfrak{Z}(\daleth)]_* = \{(\eth_1,0.8),(\eth_2,0.8),(\eth_3,0.8),(\eth_4,0.7)\}$ and $[\mathfrak{Z}(\beth)]^* = [\mathfrak{Z}(\daleth)]^* = \{(\eth_1,0.2),(\eth_2,0.2),(\eth_3,0.2),(\eth_4,0.3)\}$.

Now, we have the following:

(1) We have $\beth_{\mathfrak{Q}(\Omega)} = \{(\alpha_1,0.1),(\alpha_2,0.8),(\alpha_3,0.4),(\alpha_4,0.6)\}$, $[\mathfrak{Q}(\beth)]_* = \{(\eth_1,0.8),(\eth_2,0.8),(\eth_3,0.4),(\eth_4,0.5)\}$ and $[\mathfrak{Q}(\beth)]^* = \{(\eth_1,0.2),(\eth_2,0.2),(\eth_3,0.6),(\eth_4,0.5)\}$. Hence, $\beth_{\mathfrak{Q}(\Omega)} \nleq [\beth_{\mathfrak{Q}(\Omega)}]_* \nleq [\beth_{\mathfrak{Q}(\Omega)}]^*$ and $\beth_{\mathfrak{Q}(\Omega)} \ngeq [\beth_{\mathfrak{Q}(\Omega)}]_* \ngeq [\beth_{\mathfrak{Q}(\Omega)}]^*$. Furthermore, $\beth_{\mathfrak{Z}(\Omega)} = \{(\alpha_1,0.1),(\alpha_2,0.1),(\alpha_3,0.4),(\alpha_4,0.2)\}$, $[\mathfrak{Z}(\beth)]_* = \{(\eth_1,0.8),(\eth_2,0.8),(\eth_3,0.8),(\eth_4,0.7)\}$ and $[\mathfrak{Z}(\beth)]^* = \{(\eth_1,0.2),(\eth_2,0.2),(\eth_3,0.2),(\eth_4,0.3)\}$. Hence, $\beth_{\mathfrak{Z}(\Omega)} \nleq [\beth_{\mathfrak{Z}(\Omega)}]_* \nleq [\beth_{\mathfrak{Z}(\Omega)}]^*$ and $\beth_{\mathfrak{Z}(\Omega)} \ngeq [\beth_{\mathfrak{Z}(\Omega)}]_* \ngeq [\beth_{\mathfrak{Z}(\Omega)}]^*$.

(2) Compute $(\beth_{\mathfrak{Q}(\Omega)} \wedge \daleth_{\mathfrak{Q}(\Omega)})^* = \underline{0}_{\mathfrak{Q}(\Omega)} \ngeq \beth^*_{\mathfrak{Q}(\Omega)} \wedge \daleth^*_{\mathfrak{Q}(\Omega)} = \{0.2,0.2,0.6,0.5\}$. Furthermore, $(\beth_{\mathfrak{Z}(\Omega)} \wedge \daleth_{\mathfrak{Z}(\Omega)})^* = \underline{0}_{\mathfrak{Z}(\Omega)} \ngeq \beth^*_{\mathfrak{Z}(\Omega)} \wedge \daleth^*_{\mathfrak{Z}(\Omega)} = \{0.2,0.2,0.2,0.3\}$.

(3) Compute $(\beth_{\mathfrak{Q}(\Omega)} \wedge \daleth_{\mathfrak{Q}(\Omega)})_* = \underline{0}_{\mathfrak{Q}(\Omega)} \ngeq [\beth_{\mathfrak{Q}(\Omega)}]_* \wedge [\daleth_{\mathfrak{Q}(\Omega)}]_* = \{0.8,0.8,0.4,0.5\}$. Furthermore, $(\beth_{\mathfrak{Z}(\Omega)} \wedge \daleth_{\mathfrak{Z}(\Omega)})_* = \underline{0}_{\mathfrak{Z}(\Omega)} \ngeq [\beth_{\mathfrak{Z}(\Omega)}]_* \wedge [\daleth_{\mathfrak{Z}(\Omega)}]_* = \{0.8,0.8,0.4,0.5\}$.

(4) Compute $(\beth_{\mathfrak{Q}(\Omega)} \vee \daleth_{\mathfrak{Q}(\Omega)})_* = \underline{1}_{\mathfrak{Q}(\Omega)} \nleq [\beth_{\mathfrak{Q}(\Omega)}]_* \vee [\daleth_{\mathfrak{Q}(\Omega)}]_* = \{0.8,0.8,0.4,0.5\}$. Additionally, $(\beth_{\mathfrak{Q}(\Omega)} \vee \daleth_{\mathfrak{Q}(\Omega)})^* = \underline{1}_{\mathfrak{Q}(\Omega)} \nleq [\beth_{\mathfrak{Q}(\Omega)}]^* \vee [\daleth_{\mathfrak{Q}(\Omega)}]^* = \{0.2,0.2,0.6,0.5\}$.

(5) Consider a fuzzy graph ideal $\mathfrak{L}$, so that $[\mathfrak{Q}(\beth)] \in \mathfrak{L}$ iff $\mathfrak{Q}([\mathfrak{Q}(\beth)]) \leq \underline{0.6}_{\mathfrak{Q}(\Omega)}$, and the membership function of the fuzzy edge set $[\mathfrak{Q}(\beth)]_{\mathfrak{Z}(\Omega)}(\alpha)$ is given by the following:

$$\min\{[\mathfrak{Q}(\beth)]_{\mathfrak{Q}(\Omega)}(\eth_1), [\mathfrak{Q}(\beth)]_{\mathfrak{Q}(\Omega)}(\eth_2)\},$$

where the edge α connects the vertices $\eth_1$ and $\eth_2$. We compute $[\mathfrak{Q}(\beth)]_*$ and $[\mathfrak{Q}(\beth)]^*$ of $\beth_{\mathfrak{Q}(\Omega)}$ with respect to $\Re$ and $\mathfrak{L}$ as follows: $[\mathfrak{Q}(\beth)]_* = \{(\eth_1,1),(\eth_2,0),(\eth_3,1),(\eth_4,1)\}$ and $[\mathfrak{Q}(\beth)]^* = \{(\eth_1,0),(\eth_2,0),(\eth_3,0),(\eth_4,0)\} = \underline{0}_{\mathfrak{Q}(\Omega)}$, that is, $[\beth_{\mathfrak{Q}(\Omega)}]^* = \underline{0}_{\mathfrak{Q}(\Omega)} \nRightarrow \beth_\Omega \in \mathfrak{L}$.

Additionally, if $\gamma_{\mathfrak{Z}(\Omega)} = \{(\eth_1\eth_2, 0.1), (\eth_1\eth_4, 0.4), (\eth_2\eth_3, 0.4), (\eth_4\eth_3, 0.1)\}$, Then, $[\mathfrak{Z}(\beth)]_* = \{(\eth_1, 1), (\eth_2, 1), (\eth_3, 1), (\eth_4, 1)\}\underline{1}_{\mathfrak{Z}(\Omega)}$ and $[\mathfrak{Z}(\beth)]^* = \{(\eth_1, 0), (\eth_2, 0), (\eth_3, 0), (\eth_4, 0)\} = \underline{0}_{\mathfrak{Z}(\Omega)}$, that is, $[\beth_{\mathfrak{Z}(\Omega)}]_* = \underline{1}_{\mathfrak{Z}(\Omega)} \quad \nRightarrow (\beth_\Omega)^c \in \mathfrak{L}$ and $[\beth_{\mathfrak{Z}(\Omega)}]^* = \underline{0}_{\mathfrak{Z}(\Omega)} \quad \nRightarrow \beth_\Omega \in \mathfrak{L}$.

Lemma 3.14. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS and $\beth_\Omega, \daleth_\Omega \in \mathbf{I}^\Omega$. Then,*

(1) $[\mathfrak{Q}(\beth)]_\Re \le [\mathfrak{Q}(\beth)] \le [\mathfrak{Q}(\beth)]^\Re$ and $[\mathfrak{Z}(\beth)]_\Xi \le [\mathfrak{Z}(\beth)] \le [\mathfrak{Z}(\beth)]^\Xi$,
(2) $[\underline{0}_{\mathfrak{Q}(\Omega)}]_\Re = [\underline{0}_{\mathfrak{Q}(\Omega)}]^\Re = [\underline{0}_{\mathfrak{Q}(\Omega)}]$ and $[\underline{0}_{\mathfrak{Z}(\Omega)}]_\Xi = [\underline{0}_{\mathfrak{Z}(\Omega)}]^\Xi = [\underline{0}_{\mathfrak{Z}(\Omega)}]$,
(3) $[\underline{1}_{\mathfrak{Q}(\Omega)}]_\Re = [\underline{1}_{\mathfrak{Q}(\Omega)}]^\Re = [\underline{1}_{\mathfrak{Q}(\Omega)}]$ and $[\underline{1}_{\mathfrak{Z}(\Omega)}]_\Xi = [\underline{1}_{\mathfrak{Z}(\Omega)}]^\Xi = [\underline{1}_{\mathfrak{Z}(\Omega)}]$,
(4) $([\mathfrak{Q}(\beth)] \vee [\mathfrak{Q}(\daleth)])_\Re \ge [\mathfrak{Q}(\beth)]_\Re \vee [\mathfrak{Q}(\daleth)]_\Re$ and $([\mathfrak{Z}(\beth)] \vee [\mathfrak{Z}(\daleth)])_\Xi \ge [\mathfrak{Z}(\beth)]_\Xi \vee [\mathfrak{Z}(\daleth)]_\Xi$,
(5) $([\mathfrak{Q}(\beth)] \wedge [\mathfrak{Q}(\daleth)])^\Re \le [\mathfrak{Q}(\beth)]^\Re \wedge [\mathfrak{Q}(\daleth)]^\Re$ and $([\mathfrak{Z}(\beth)] \wedge [\mathfrak{Z}(\daleth)])^\Xi \le [\mathfrak{Z}(\beth)]^\Xi \wedge [\mathfrak{Z}(\daleth)]^\Xi$,
(6) $\beth_\Omega \le \daleth_\Omega$ implies that $[\mathfrak{Q}(\beth)]_\Re \le [\mathfrak{Q}(\daleth)]_\Re$ and $[\mathfrak{Z}(\beth)]_\Xi \le [\mathfrak{Z}(\daleth)]_\Xi$,
(7) $\beth_\Omega \le \daleth_\Omega$ implies that $[\mathfrak{Q}(\beth)]^\Re \le [\mathfrak{Q}(\daleth)]^\Re$ and $[\mathfrak{Z}(\beth)]^\Xi \le [\mathfrak{Z}(\daleth)]^\Xi$,
(8) $([\mathfrak{Q}(\beth)] \vee [\mathfrak{Q}(\daleth)])^\Re \ge [\mathfrak{Q}(\beth)]^\Re \vee [\mathfrak{Q}(\daleth)]^\Re$ and $([\mathfrak{Z}(\beth)] \vee [\mathfrak{Z}(\daleth)])^\Xi \ge [\mathfrak{Z}(\beth)]^\Xi \vee [\mathfrak{Z}(\daleth)]^\Xi$,
(9) $([\mathfrak{Q}(\beth)] \wedge [\mathfrak{Q}(\daleth)])_\Re \le [\mathfrak{Q}(\beth)]_\Re \wedge [\mathfrak{Q}(\daleth)]_\Re$ and $([\mathfrak{Z}(\beth)] \wedge [\mathfrak{Z}(\daleth)])_\Xi \le [\mathfrak{Z}(\beth)]_\Xi \wedge [\mathfrak{Z}(\daleth)]_\Xi$,
(10) $\left([\mathfrak{Q}(\beth)]^\Re\right)^c = ([\mathfrak{Q}(\beth)]^c)_\Re$ and $\left([\mathfrak{Z}(\beth)]^\Xi\right)^c = ([\mathfrak{Z}(\beth)]^c)_\Xi$,
(11) $([\mathfrak{Q}(\beth)]_\Re)^c = ([\mathfrak{Q}(\beth)]^c)^\Re$ and $([\mathfrak{Z}(\beth)]_\Xi)^c = ([\mathfrak{Z}(\beth)]^c)^\Xi$,
(12) $([\mathfrak{Q}(\beth)]_\Re)^\Re \ge [\mathfrak{Q}(\beth)]_\Re \ge ([\mathfrak{Q}(\beth)]_\Re)_\Re$ and $([\mathfrak{Z}(\beth)]_\Xi)^\Xi \ge [\mathfrak{Z}(\beth)]_\Xi \ge ([\mathfrak{Z}(\beth)]_\Xi)_\Xi$, and
(13) $\left([\mathfrak{Q}(\beth)]^\Re\right)_\Re \le [\mathfrak{Q}(\beth)]^\Re \le \left([\mathfrak{Q}(\beth)]^\Re\right)^\Re$ and $\left([\mathfrak{Z}(\beth)]^\Xi\right)_\Xi \le [\mathfrak{Z}(\beth)]^\Xi \le \left([\mathfrak{Z}(\beth)]^\Xi\right)^\Xi$.

Proof. Straightforward from Definitions 3.6, 3.7, and Lemma 3.11. □

Lemma 3.15. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS and $\beth_\Omega, \daleth_\Omega \in \mathbf{I}^\Omega$. Then,*

(1) $\underline{\beth_\Omega} \le \beth_\Omega \le \overline{\beth_\Omega}$,
(2) $\underline{0_\Omega} = \overline{0_\Omega} = 0_\Omega$ and $\underline{1_\Omega} = \overline{1_\Omega} = 1_\Omega$,
(3) $\underline{(\beth_\Omega \vee \daleth_\Omega)} \ge \underline{\beth_\Omega} \vee \underline{\daleth_\Omega}$,
(4) $\underline{(\beth_\Omega \wedge \daleth_\Omega)} \le \underline{\beth_\Omega} \wedge \underline{\daleth_\Omega}$,
(5) $\beth_\Omega \le \daleth_\Omega$ implies that $\underline{\beth_\Omega} \le \underline{\daleth_\Omega}$ and $\overline{\beth_\Omega} \le \overline{\daleth_\Omega}$,
(6) $\overline{(\beth_\Omega \vee \daleth_\Omega)} \ge \overline{\beth_\Omega} \vee \overline{\daleth_\Omega}$,
(7) $\overline{(\beth_\Omega \wedge \daleth_\Omega)} \le \overline{\beth_\Omega} \wedge \overline{\daleth_\Omega}$,
(8) $\left(\overline{\beth_\Omega}\right)^c = \underline{\left(\beth^c_\Omega\right)}$ and $\left(\underline{\beth_\Omega}\right)^c = \overline{\left(\beth^c_\Omega\right)}$,
(9) $\overline{\left(\underline{\beth_\Omega}\right)} \ge \underline{\beth_\Omega} \ge \underline{\left(\underline{\beth_\Omega}\right)}$, and
(10) $\underline{\left(\overline{\beth_\Omega}\right)} \le \overline{\beth_\Omega} \le \overline{\left(\overline{\beth_\Omega}\right)}$.

Proof. Straightforward from Definition 3.8 and Lemma 3.14. □

Remark 3.16. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS and $\beth_\Omega \in \mathbf{I}^\Omega$. Then,*

(1) As in the usual case, whenever $\Re$ and Ξ are reflexive fuzzy relations on Ω, Then, we have $[\beth_\Omega]_* \le \beth_\Omega \le [\beth_\Omega]^*$. In this case, the equality holds for both (8) and (9) in Lemma 3.11; thus, the equality holds for both (6) and (7) in Lemma 3.15.

(2) As in the usual case, if $\Re$ and Ξ are reflexive and transitive fuzzy relations, Then, $\underline{\left(\underline{\daleth_\Omega}\right)} = \left(\underline{\daleth_\Omega}\right)$ and $\overline{\left(\overline{\daleth_\Omega}\right)} = \left(\overline{\daleth_\Omega}\right)$. Then, a fuzzy topology τ_Ω on the rough fuzzy graph IAS ($\Omega, \Re, \Xi, \mathfrak{L}$) is generated by the following:

$$\tau_\Omega = \left\{\Lambda_\Omega \in \mathbf{I}^\Omega : \Lambda_\Omega = \underline{\Lambda_\Omega}\right\} \quad \text{or} \quad \tau_\Omega = \left\{\Lambda_\Omega \in \mathbf{I}^\Omega : [\Lambda_\Omega]^c = \overline{([\Lambda_\Omega]^c)}\right\}.$$

In the following, weaker definitions will be defined compared to Definitions 3.6 and 3.7.

Definition 3.17. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph approximation space and $\daleth$ be a fuzzy subgraph of the graph Ω. For every vertex point $\eth \in \mathfrak{Q}(\Omega)$, define $(\mathfrak{Q}(\daleth))_{**}, (\mathfrak{Q}(\daleth))^{**} \in \mathbf{I}^{\mathfrak{Q}(\Omega)}$ of a fuzzy vertex set $\mathfrak{Q}(\daleth) \in \mathbf{I}^{\mathfrak{Q}(\Omega)}$ by the following:*

$$(\mathfrak{Q}(\daleth))_{**}(\eth) = \begin{cases} \left(\bigvee_{\eth' \in \mathfrak{Q}(\Omega)} < \eth' > \Re(\eth)\right)^c & \text{if } < \eth > \Re \wedge (\mathfrak{Q}(\daleth))^c = \mathfrak{Q}(\eta) \text{ for some } \eta \notin \mathfrak{L} \\ & \text{and } < \eth > \Re \wedge \mathfrak{Q}(\daleth) = \mathfrak{Q}(\eta') \text{ for some } \eta' \notin \mathfrak{L} \\ 1 & \text{if } < \eth > \Re \wedge (\mathfrak{Q}(\daleth))^c = \mathfrak{Q}(\eta'') \text{ for some } \eta'' \in \mathfrak{L} \\ 0 & \text{if } < \eth > \Re \wedge (\mathfrak{Q}(\daleth))^c = \mathfrak{Q}(\eta''') \text{ for some } \eta''' \notin \mathfrak{L} \\ & \text{and } < \eth > \Re \wedge \mathfrak{Q}(\daleth) = \mathfrak{Q}(\eta'''') \text{ for some } \eta'''' \in \mathfrak{L} \end{cases}$$

$$(\mathfrak{Q}(\daleth))^{**}(\eth) = \begin{cases} \bigvee_{\eth' \in \mathfrak{Q}(\Omega)} < \eth' > \Re(\eth) & \text{if } < \eth > \Re \wedge \mathfrak{Q}(\daleth) = \mathfrak{Q}(\eta) \text{ for some } \eta \notin \mathfrak{L} \\ & \text{and } < \eth > \Re \wedge (\mathfrak{Q}(\daleth))^c = \mathfrak{Q}(\eta') \text{ for some } \eta' \notin \mathfrak{L} \\ 1 & \text{if } < \eth > \Re \wedge \mathfrak{Q}(\daleth) = \mathfrak{Q}(\eta'') \text{ for some } \eta'' \in \mathfrak{L} \\ 0 & \text{if } < \eth > \Re \wedge \mathfrak{Q}(\daleth) = \mathfrak{Q}(\eta''') \text{ for some } \eta''' \notin \mathfrak{L} \\ & \text{and } < \eth > \Re \wedge (\mathfrak{Q}(\daleth))^c = \mathfrak{Q}(\eta'''') \text{ for some } \eta'''' \in \mathfrak{L} \end{cases}$$

The roughness of a fuzzy vertex set $\mathfrak{Q}(\daleth) \in \mathbf{I}^{\mathfrak{Q}(\Omega)}$ is defined by the following:

$$[\mathfrak{Q}(\daleth)]_{\Re*} = \mathfrak{Q}(\daleth) \wedge (\mathfrak{Q}(\daleth))_{**} \text{ and } [\mathfrak{Q}(\daleth)]^{\Re*} = \mathfrak{Q}(\daleth) \vee (\mathfrak{Q}(\daleth))^{**},$$

where $[\mathfrak{Q}(\daleth)]_{\Re*}$ is the lower fuzzy vertex set of $\mathfrak{Q}(\daleth)$, and $\mathfrak{Q}(\daleth)^{\Re*}$ is the upper fuzzy vertex set of $\mathfrak{Q}(\daleth)$. The boundary fuzzy region of $\mathfrak{Q}(\daleth)$ is $(\mathfrak{Q}(\daleth))^{B*}$ given by $(\mathfrak{Q}(\daleth))^{B*} = [\mathfrak{Q}(\daleth)]^{\Re*} \bar{\wedge} [\mathfrak{Q}(\daleth)]_{\Re*}$. The pair ($[\mathfrak{Q}(\daleth)]^{\Re*}, [\mathfrak{Q}(\daleth)]_{\Re*}$) will be called a $*$-fuzzy rough vertex set if $[\mathfrak{Q}(\daleth)]^{\Re*} \bar{\wedge} [\mathfrak{Q}(\daleth)]_{\Re*} \neq \underline{0}_{\mathfrak{Q}(\Omega)}$.

Definition 3.18. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph approximation space, $\daleth$ be a fuzzy subgraph of the graph Ω, and ($[\mathfrak{Q}(\daleth)]^{\Re*}, [\mathfrak{Q}(\daleth)]_{\Re*}$) be a fuzzy rough vertex set on Ω. Let Ξ be a fuzzy relation on the edges set $\mathfrak{Z}(\Omega)$ of the graph Ω such that $\Xi(\eth_1\eth_2, \eth_1'\eth_2') \leq \Re(\eth_1, \eth_1') \wedge \Re(\eth_2, \eth_2')$ for all $\eth_1\eth_2, \eth_1'\eth_2' \in \mathfrak{Z}(\Omega)$. Additionally, let γ be a fuzzy subgraph of Ω such that $\gamma_{\mathfrak{Z}(\Omega)}(\eth\eth') \leq [\mathfrak{Q}(\daleth)]_\Re(\eth) \wedge [\mathfrak{Q}(\daleth)]_\Re(\eth')$ for all $\eth\eth' \in \mathfrak{Z}(\Omega)$. Then, for every edge point $\alpha \in \mathfrak{Z}(\Omega)$, define $(\mathfrak{Z}(\daleth))_{**}, (\mathfrak{Z}(\daleth))^{**} \in \mathbf{I}^{\mathfrak{Z}(\Omega)}$ of the fuzzy edge set $\mathfrak{Z}(\daleth) \in \mathbf{I}^{\mathfrak{Z}(\Omega)}$ by the following:*

$$(\mathfrak{Z}(\daleth))_{**}(\alpha) = \begin{cases} \left(\bigvee_{\alpha' \in \mathfrak{Z}(\Omega)} < \alpha' > \Xi(\alpha)\right)^c & \text{if } < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta) \text{ for some } \eta \notin \mathfrak{L} \\ & \text{and } < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta') \text{ for some } \eta' \notin \mathfrak{L} \\ 1 & \text{if } < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta'') \text{ for some } \eta''' \in \mathfrak{L} \\ 0 & \text{if } < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta''') \text{ for some } \eta''' \notin \mathfrak{L} \\ & \text{and } < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta'''') \text{ for some } \eta'''' \in \mathfrak{L} \end{cases}$$

$$(\mathfrak{Z}(\sqsupset))^{**}(\alpha) = \begin{cases} \bigvee_{\alpha' \in \mathfrak{Z}(\Omega)} < \alpha' > \Xi(\alpha) & \text{if } < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta^{)} \text{ for some } \eta \notin \mathfrak{L} \\ & \text{and } < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta') \text{ for some } \eta' \notin \mathfrak{L} \\ 1 & \text{if } < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta'') \text{ for some } \eta'' \in \mathfrak{L} \\ 0 & \text{if } < \alpha > \Xi \wedge \mathfrak{Z}(\gamma) = \mathfrak{Z}(\eta''') \text{ for some } \eta''' \notin \mathfrak{L} \\ & \text{and } < \alpha > \Xi \wedge (\mathfrak{Z}(\gamma))^c = \mathfrak{Z}(\eta'''') \text{ for some } \eta'''' \in \mathfrak{L} \end{cases}$$

The roughness of a fuzzy edge set $\mathfrak{Z}(\sqsupset) \in \mathbf{I}^{\mathfrak{Z}(\Omega)}$ is defined by the following:

$$[\mathfrak{Z}(\sqsupset)]_{\Xi*} = \mathfrak{Z}(\sqsupset) \wedge (\mathfrak{Z}(\sqsupset))_{**} \text{ and } [\mathfrak{Z}(\sqsupset)]^{\Xi*} = \mathfrak{Z}(\sqsupset) \vee (\mathfrak{Z}(\sqsupset))^{**},$$

where $[\mathfrak{Z}(\sqsupset)]_{\Xi*}$ is the lower fuzzy edges set of $\mathfrak{Z}(\sqsupset)$, and $\mathfrak{Z}(\sqsupset)^{\Xi*}$ is the upper fuzzy edges set of $\mathfrak{Z}(\sqsupset)$. The boundary fuzzy region of $\mathfrak{Z}(\sqsupset)$ is $(\mathfrak{Z}(\sqsupset))^{B*}$ given by the following: $(\mathfrak{Z}(\sqsupset))^{B*} = [\mathfrak{Z}(\sqsupset)]^{\Xi*}\bar{\wedge}[\mathfrak{Z}(\sqsupset)]_{\Xi*}$. The pair ($[\mathfrak{Z}(\sqsupset)]^{\Xi*}, [\mathfrak{Z}(\sqsupset)]_{\Xi*}$) will be called a $*$-fuzzy rough edge set (fuzzy rough relation) if $[\mathfrak{Z}(\sqsupset)]^{\Xi*}\bar{\wedge}[\mathfrak{Z}(\sqsupset)]_{\Xi*} \neq \underline{0}_{\mathfrak{Z}(\Omega)}$.

All the results given in this section are satisfied exactly, where the only main difference comes from the fact that $\Re < \partial > \Re \leq< \partial > \Re$ and $\Xi < \partial\partial' > \Xi \leq< \partial\partial' > \Xi$ for all vertices and edges of the graph Ω.

Remark 3.19. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS and $\sqsupset_\Omega \in \mathbf{I}^\Omega$. Then,*

(1) $[\sqsupset_{\mathfrak{Q}(\Omega)}]_{**} \leq [\sqsupset_{\mathfrak{Q}(\Omega)}]_*$ and $[\sqsupset_{\mathfrak{Z}(\Omega)}]_{**} \leq [\sqsupset_{\mathfrak{Z}(\Omega)}]_*$, and
(2) $[\sqsupset_{\mathfrak{Q}(\Omega)}]^* \leq [\sqsupset_{\mathfrak{Q}(\Omega)}]^{**}$ and $[\sqsupset_{\mathfrak{Z}(\Omega)}]^* \leq [\sqsupset_{\mathfrak{Z}(\Omega)}]^{**}$.

Example 3.20. *From Example 3.13, in case of $\mathfrak{L}$, $[\mathfrak{Q}(\sqsupset)] \in \mathfrak{L}$ iff $\mathfrak{Q}([\mathfrak{Q}(\sqsupset)]) \leq \underline{0.4}_{\mathfrak{Q}(\Omega)}$, and the membership function of the fuzzy edge set $[\mathfrak{Q}(\sqsupset)]_{\mathfrak{Z}(\Omega)}(\alpha)$ is such that*

$$[\mathfrak{Q}(\sqsupset)]_{\mathfrak{Z}(\Omega)}(\alpha) \leq [[\mathfrak{Q}(\sqsupset)]_{\mathfrak{Q}(\Omega)}(\partial_1)] \wedge [[\mathfrak{Q}(\sqsupset)]_{\mathfrak{Q}(\Omega)}(\partial_2)],$$

with the condition that $\mathfrak{Z}([\mathfrak{Q}(\sqsupset)]) \leq \underline{0.2}_{\mathfrak{Z}(\Omega)}$, where the edge α connects the vertices ∂_1 and ∂_2. We get $[\mathfrak{Q}(\sqsupset)]_* = \{(\partial_1, 0.8), (\partial_2, 0.8), (\partial_3, 0.4), (\partial_4, 0.5)\}$ and $[\mathfrak{Q}(\sqsupset)]^* = \{(\partial_1, 0.2), (\partial_2, 0.2), (\partial_3, 0.6), (\partial_4, 0.5)\}$. Then, $[\mathfrak{Q}(\sqsupset)]_\Re = \{(\partial_1, 0.1), (\partial_2, 0.8), (\partial_3, 0.4), (\partial_4, 0.5)\}$ and $[\mathfrak{Q}(\sqsupset)]^\Re = \{(\partial_1, 0.2), (\partial_2, 0.8), (\partial_3, 0.6), (\partial_4, 0.6)\}$. Additionally, $[\mathfrak{Z}(\sqsupset)]_* = \{(\alpha_1, 0.8), (\alpha_2, 0.8), (\alpha_3, 0.8), (\alpha_4, 0.7)\}$ and $[\mathfrak{Z}(\sqsupset)]^* = \{(\alpha_1, 0.2), (\alpha_2, 0.2), (\alpha_3, 0.2), (\alpha_4, 0.3)\}$. Then, $[\mathfrak{Z}(\sqsupset)]_\Xi = \{(\alpha_1, 0.1), (\alpha_2, 0.1), (\alpha_3, 0.4), (\alpha_4, 0.2)\}$ and $[\mathfrak{Z}(\sqsupset)]^\Xi = \{(\alpha_1, 0.2), (\alpha_2, 0.2), (\alpha_3, 0.4), (\alpha_4, 0.3)\}$, that is, $(\sqsupset_{\mathfrak{Q}(\Omega)})^B = \{0.2, 0.2, 0.6, 0.5\}$ and $(\sqsupset_{\mathfrak{Z}(\Omega)})^B = \{0.2, 0.2, 0.2, 0.3\}$. Moreover, $\mu(\sqsupset_{\mathfrak{Q}(\Omega)}) = \{0.9, 1, 0.8, 0.9\}$ and $\mu(\sqsupset_{\mathfrak{Z}(\Omega)}) = \{0.9, 0.9, 1, 0.9\}$; then, the accuracy value of $\sqsupset_\Omega$ is given by $(\mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Q}(\Omega)})), \mathrm{Inf}(\mu(\sqsupset_{\mathfrak{Z}(\Omega)}))) = (0.8, 0.9)$. If we use Definitions 3.17 and 3.18, then we get that $[\mathfrak{Q}(\sqsupset)]_{**} = \{(\partial_1, 0.4), (\partial_2, 0.8), (\partial_3, 0.4), (\partial_4, 0.5)\}$ and $[\mathfrak{Q}(\sqsupset)]^{**} = \{(\partial_1, 0.6), (\partial_2, 0.2), (\partial_3, 0.6), (\partial_4, 0.5)\}$. Then, $[\mathfrak{Q}(\sqsupset)]_{\Re*} = \{(\partial_1, 0.1), (\partial_2, 0.8), (\partial_3, 0.4), (\partial_4, 0.5)\}$ and $[\mathfrak{Q}(\sqsupset)]^{\Re*} = \{(\partial_1, 0.6), (\partial_2, 0.8), (\partial_3, 0.6), (\partial_4, 0.6)\}$. Additionally,

$[\mathfrak{Z}(\beth)]_{**} = \{(\alpha_1, 0.7), (\alpha_2, 0.6), (\alpha_3, 0.8), (\alpha_4, 0.7)\}$ and $[\mathfrak{Z}(\beth)]^{**} = \{(\alpha_1, 0.3), (\alpha_2, 0.4), (\alpha_3, 0.2), (\alpha_4, 0.3)\}$. Then,

$[\mathfrak{Z}(\beth)]_{\Xi*} = \{(\alpha_1, 0.1), (\alpha_2, 0.1), (\alpha_3, 0.4), (\alpha_4, 0.2)\}$ and $[\mathfrak{Z}(\beth)]^{\Xi*} = \{(\alpha_1, 0.3), (\alpha_2, 0.4), (\alpha_3, 0.4), (\alpha_4, 0.3)\}$, that is, $(\beth_{\mathfrak{Q}(\Omega)})^{B*} = \{0.6, 0.2, 0.6, 0.5\}$ and $(\beth_{\mathfrak{Z}(\Omega)})^{B*} = \{0.3, 0.4, 0.4, 0.3\}$. Moreover, $\mu(\beth_{\mathfrak{Q}(\Omega)}) = \{0.9, 1, 0.8, 0.9\}$ and $\mu(\beth_{\mathfrak{Z}(\Omega)}) = \{0.5, 1, 0.8, 0.9\}$; then, the accuracy value of $\beth_\Omega$ is given by $(\text{Inf}(\mu(\beth_{\mathfrak{Q}(\Omega)})), \text{Inf}(\mu(\beth_{\mathfrak{Z}(\Omega)}))) = (0.5, 0.7)$. Hence, the boundary and the accuracy value of Definitions 3.6 and 3.7 are better than those of Definitions 3.17 and 3.18.

Remark 3.21. *Let ($\Omega, \mathfrak{R}, \Xi, \mathfrak{L}$) be a fuzzy graph IAS. It should be noted from Remark 3.19 that Definitions 3.6 and 3.7 decrease the upper approximation and increase the lower approximation. This new approach is more general than many previous approaches. These are special cases (in Figure 2):*

(1) If we have equivalence fuzzy relations $\mathfrak{R}$, Ξ, $\mathfrak{L} = \mathfrak{L}_0$, and the graph Ω has no edges. Then, Definition 3.6 will be the fuzzification of the main definition given by Pawlak [1].
(2) If we have symmetric fuzzy relations $\mathfrak{R}$, Ξ,$\mathfrak{L} = \mathfrak{L}_0$, and the graph Ω has no edges. Then, Definition 3.6 will be the fuzzification of the definition given by Allam [40].
(3) If we have reflexive and symmetric fuzzy relations $\mathfrak{R}$, Ξ, and the graph Ω has no edges. Then, Definition 3.6 will be the fuzzification of the definition given by Kandil [37].
(4) If the graph Ω has no edges. Then, Definition 3.6 will be the fuzzification of the definition given by Kandil [38].
(5) If $\mathfrak{L} = \mathfrak{L}_0$. Then, Definition 3.6 will be the fuzzification of the definition given by El-Atik [36].
(6) If the graph Ω has no edges. Then, Definition 3.6 will coincides with the definition given by Ibedou [39].
(7) If we have equivalence relations $\mathfrak{R}$, Ξ, and $\mathfrak{L} = \mathfrak{L}_0$. Then, Definitions 3.6 and 3.7 will coincide with the definition given by Akram [27].
(8) If we have tolerance relations $\mathfrak{R}$, Ξ(that is $\mathfrak{R}$, Ξ are reflexive and symmetric), and $\mathfrak{L} = \mathfrak{L}_0$. Then, Definitions 3.6 and 3.7 will coincide with the definition given by Akram [28].

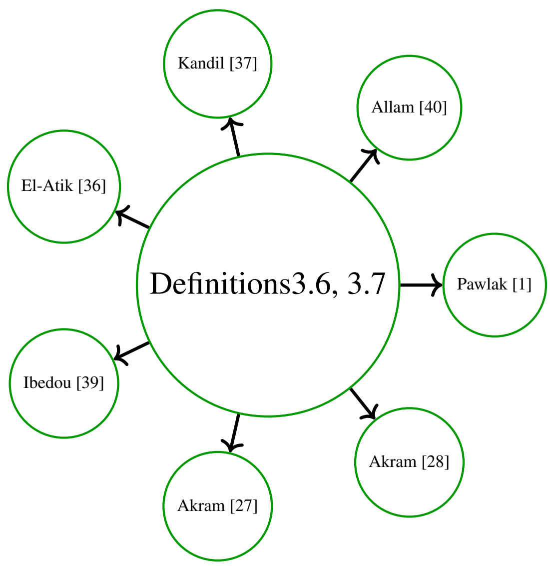


Figure 2. Implications of Remark 3.21.

As an application of these fuzzy graph IASs, we discuss fuzzy connected graphs using fuzzy closure operators where $\Re$ and Ξ are reflexive and transitive fuzzy relations.

4. Connectedness in fuzzy graph IASs

This section emphasizes that the concept of connectedness can be effectively characterized within the framework of fuzzy graph ideals, thus providing a basis to analyze the structure and connectivity properties of fuzzy graphs in this approximation space.

Definition 4.1. *Let* ($\Omega, \Re, \Xi, \mathfrak{L}$) *be a fuzzy graph IAS,* $\beth_\Omega$ *and* $\daleth_\Omega$ *be fuzzy subgraphs of the graph* Ω*. Then,*

(1) The fuzzy subgraphs $\beth_\Omega, \daleth_\Omega \in \mathbf{I}^\Omega$ are called fuzzy graph separated if $\overline{\beth_\Omega} \wedge \daleth_\Omega = \beth_\Omega \wedge \overline{\daleth_\Omega} = \underline{0}_\Omega$.

(2) A fuzzy subgraph $\Omega' \in \mathbf{I}^\Omega$ is called a fuzzy graph disconnected subgraph if there exist fuzzy graph separated subgraphs $\beth_\Omega, \daleth_\Omega \in \mathbf{I}^\Omega$, such that $\beth_\Omega \vee \daleth_\Omega = \Omega'$. A fuzzy subgraph Ω' is called a fuzzy graph connected if it is not fuzzy graph disconnected. In other words, if there are no fuzzy graph separated subgraphs $\beth_\Omega, \daleth_\Omega$ except $\beth_\Omega = \underline{0}_\Omega$, or $\daleth_\Omega = \underline{0}_\Omega$.

(3) ($\Omega, \Re, \Xi, \mathfrak{L}$) is called a fuzzy graph disconnected space if there exist fuzzy graph separated subgraphs $\beth_\Omega, \daleth_\Omega \in \mathbf{I}^\Omega$, such that $\beth_\Omega \vee \daleth_\Omega = \underline{1}_\Omega$. A fuzzy graph IAS ($\Omega, \Re, \Xi, \mathfrak{L}$) is a called a fuzzy graph connected if it is not fuzzy graph disconnected.

Example 4.2. *Assume that* Ω *be the graph* $(\mathfrak{Q}(\Omega), \mathfrak{Z}(\Omega))$*, where* $\mathfrak{Q}(\Omega) = \{Ә_1, Ә_2, Ә_3, Ә_4\}$*, and* $\mathfrak{Z}(\Omega) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$*. Define a fuzzy subgraph* θ_Ω *over* Ω *as follows:*

$$\theta_{\mathfrak{Q}(\Omega)} = \{(Ә_1, 0.3), (Ә_2, 0.3), (Ә_3, 0.2), (Ә_4, 0.2)\},$$

and the fuzzy edge set

$$\theta_{\mathfrak{Z}(\Omega)} = \{(\alpha_1, 0), (\alpha_2, 0.2), (\alpha_3, 0.1), (\alpha_4, 0.1)\}.$$

The fuzzy subgraph θ_Ω is represented in Figure 3.

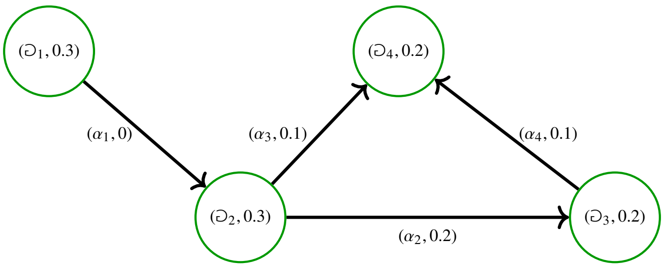


Figure 3. Representation of the fuzzy subgraph $\beth_\Omega$ given in Example 4.2.

Define a reflexive and transitive fuzzy relation $\Re$ on $\mathfrak{Q}(\Omega)$, as given in Table 7.

Table 7. Fuzzy relation $\Re$ in Example 4.2.

$\Re$	$Ә_1$	$Ә_2$	$Ә_3$	$Ә_4$
$Ә_1$	1	0	0.1	0
$Ә_2$	0	1	0.1	0
$Ә_3$	0	0	0.1	0
$Ә_4$	0	0.1	0.1	1

$< \Game_1 > \Re = \{1, 0, 0.1, 0\}, < \Game_2 > \Re = \{0, 1, 0.1, 0\}, < \Game_3 > \Re = \{0, 0, 0.1, 0\}, < \Game_4 > \Re = \{0, 0, 0.1, 1\}$, $\Re < \Game_1 >= \{0.1, 0, 0, 0\}, \Re < \Game_2 >= \{0, 0.1, 0, 0\}, \Re < \Game_3 >= \{0.1, 0.1, 1, 0.1\}$, and $\Re < \Game_4 >= \{0, 0, 0, 0.1\}$. Therefore, $\Re < \Game_1 > \Re = \{0.1, 0, 0, 0\}, \Re < \Game_2 > \Re = \{0, 0.1, 0, 0\}, \Re < \Game_3 > \Re = \{0, 0, 0.1, 0\}$, and $\Re < \Game_4 > \Re = \{0, 0, 0, 0.1\}$. Consider a fuzzy graph ideal $\mathfrak{L}$, so that $[\mathfrak{Q}(\Lambda)] \in \mathfrak{L}$ iff $\mathfrak{Q}([\mathfrak{Q}(\Lambda)]) \leq \underline{0.3}_{\mathfrak{Q}(\Omega)}$, and the membership function of the fuzzy edge set $[\mathfrak{Q}(\Lambda)]_{\mathfrak{Z}(\Omega)}(\alpha)$ is given by the following:

$$\min\{[\mathfrak{Q}(\Lambda)]_{\mathfrak{Q}(\Omega)}(\Game_1), [\mathfrak{Q}(\Lambda)]_{\mathfrak{Q}(\Omega)}(\Game_2)\},$$

where the edge α connects the vertices $\Game_1$ and $\Game_2$. Then, for any $\beth_\Omega \in \mathbf{I}^\Omega$ we compute $[\mathfrak{Q}(\beth)]_*$ and $[\mathfrak{Q}(\beth)]^*$ of $\beth_{\mathfrak{Q}(\Omega)}$ with respect to $\Re$ and $\mathfrak{L}$ as follows: $[\mathfrak{Q}(\beth)]_* = \{(\Game_1, 1), (\Game_2, 1), (\Game_3, 1), (\Game_4, 1)\} = \underline{1}_{\mathfrak{Q}(\Omega)}$, and $[\mathfrak{Q}(\beth)]^* = \{(\Game_1, 0), (\Game_2, 0), (\Game_3, 0), (\Game_4, 0)\} = \underline{0}_{\mathfrak{Q}(\Omega)}$, that is, $[\mathfrak{Q}(\beth)]_\Re = [\mathfrak{Q}(\beth)] = [\mathfrak{Q}(\beth)]^\Re$ for any $\beth_\Omega \in \mathbf{I}^\Omega$. Let $\gamma_{\mathfrak{Z}(\Omega)} = \{(\Game_1\Game_2, 0.1), (\Game_2\Game_3, 0.1), (\Game_2\Game_4, 0.1), (\Game_3\Game_4, 0.1)\}$ be a fuzzy edge set defined on $\mathfrak{Z}(\Omega)$, and Ξ be a fuzzy relation defined on $\mathfrak{Z}(\Omega)$, as given in Table 8.

Table 8. Fuzzy relation Ξ in Example 4.2.

Ξ	$\Game_1\Game_2$	$\Game_2\Game_3$	$\Game_2\Game_4$	$\Game_3\Game_4$
$\Game_1\Game_2$	1	0	0	0
$\Game_2\Game_3$	0	0.1	0	0
$\Game_2\Game_4$	0	0.1	1	0.1
$\Game_3\Game_4$	0	0	0	0.1

$\Game_1\Game_2\Xi = \{1, 0, 0, 0\}$, $\Game_2\Game_3\Xi = \{0, 0.1, 0, 0\}$, $\Game_2\Game_4\Xi = \{0, 0.1, 1, 0.1\}$, $\Game_3\Game_4\Xi = \{0, 0, 0, 0.1\}$, $\Xi\Game_1\Game_2 = \{1, 0, 0, 0\}$, $\Xi\Game_2\Game_3 = \{0, 0.1, 0.1, 0\}$, $\Xi\Game_2\Game_4 = \{0, 0, 1, 0\}$, and $\Xi\Game_3\Game_4 = \{0, 0, 0.1, 0.1\}$. Then, $< \Game_1\Game_2 > \Xi = \{1, 0, 0, 0\}, < \Game_2\Game_3 > \Xi = \{0, 0.1, 0, 0\}, < \Game_2\Game_4 > \Xi = \{0, 0, 1, 0\}, < \Game_3\Game_4 > \Xi = \{0, 0, 0, 0.1\}$, $\Xi < \Game_1\Game_2 >= \{1, 0, 0, 0\}, \Xi < \Game_2\Game_3 >= \{0, 0.1, 0, 0\}, \Xi < \Game_2\Game_4 >= \{0, 0, 0.1, 0\}$, and $\Xi < \Game_3\Game_4 >= \{0, 0, 0, 0.1\}$. Therefore, $\Xi < \Game_1\Game_2 > \Xi = \{1, 0, 0, 0\}, \Xi < \Game_2\Game_3 > \Xi = \{0, 0.1, 0, 0\}, \Xi < \Game_2\Game_4 > \Xi = \{0, 0, 0.1, 0\}$, and $\Xi < \Game_3\Game_4 > \Xi = \{0, 0, 0, 0.1\}$. Then, for any $\beth_\Omega \in \mathbf{I}^\Omega$, we compute $[\mathfrak{Z}(\beth)]_*$ and $[\mathfrak{Z}(\beth)]^*$ of $\beth_{\mathfrak{Z}(\Omega)}$ with respect to Ξ and $\mathfrak{L}$ as follows: $[\mathfrak{Z}(\beth)]_* = \{(\Game_1\Game_2, 1), (\Game_2\Game_3, 1), (\Game_2\Game_4, 1), (\Game_3\Game_4, 1)\} = \underline{1}_{\mathfrak{Z}(\Omega)}$, and $[\mathfrak{Z}(\beth)]^* = \{(\Game_1\Game_2, 0), (\Game_2\Game_3, 0), (\Game_2\Game_3, 0), (\Game_3\Game_4, 0)\} = \underline{0}_{\mathfrak{Z}(\Omega)}$.

Specifically, $[\mathfrak{Z}(\beth)]_\Xi = [\mathfrak{Z}(\beth)] = [\mathfrak{Z}(\beth)]^\Xi$ for any $\beth_\Omega \in \mathbf{I}^\Omega$. As a result, $\underline{\beth_\Omega} = \beth_\Omega = \overline{\beth_\Omega}$ for any $\beth_\Omega \in \mathbf{I}^\Omega$. Hence, we can find $\beth_\Omega = (\{0, 0.3, 0.2, 0.2\}, \{0, 0.2, 0.1, 0.1\}), \daleth_\Omega = (\{0.3, 0, 0, 0\}, \{0, 0, 0, 0\})$ so that the fuzzy subgraph $(\beth_\Omega \vee \daleth_\Omega) = \theta_\Omega = (\{0.3, 0.3, 0.2, 0.2\}, \{0, 0.2, 0.1, 0.1\})$ is a fuzzy subgraph for which $\overline{\beth_\Omega} \wedge \daleth_\Omega = \beth_\Omega \wedge \overline{\daleth_\Omega} = \beth_\Omega \wedge \daleth_\Omega = \underline{0}_\Omega$. Thus, $\theta_\Omega = (\{0.3, 0.3, 0.2, 0.2\}, \{0, 0.2, 0.1, 0.1\})$ is fuzzy graph disconnected subgraph. The choice of the fuzzy relations $\Re$, Ξ and the graph ideal $\mathfrak{L}$ played the main role of being $(\beth_\Omega)^* = ([\mathfrak{Q}(\beth)]^*, [\mathfrak{Z}(\beth)]^*) = \underline{0}_\Omega$, that is, $\beth_\Omega = \overline{\beth_\Omega}$ for any $\beth_\Omega \in \mathbf{I}^\Omega$. Therefore, we could find a pair of fuzzy graph separated subgraphs as shown above, and thus we found a fuzzy subgraph which is fuzzy graph disconnected. Furthermore, if a fuzzy graph contains more than one component, Then, it's fuzzy graph disconnected.

Proposition 4.3. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS, $\beth_\Omega$ and $\daleth_\Omega$ be fuzzy subgraphs of the graph Ω. Then, the following are equivalent:*

(1) ($\Omega, \Re, \Xi, \mathfrak{L}$) is fuzzy graph connected;

(2) $\beth_\Omega \wedge \daleth_\Omega = \underline{0}_\Omega, \underline{\beth_\Omega} = \beth_\Omega, \underline{\daleth_\Omega} = \daleth_\Omega$, and $\beth_\Omega \vee \daleth_\Omega = \underline{1}_\Omega$ imply $\beth_\Omega = \underline{0}_\Omega$ or $\daleth_\Omega = \underline{0}_\Omega$;

(3) $\beth_\Omega \wedge \daleth_\Omega = \underline{0}_\Omega, \overline{\beth_\Omega} = \beth_\Omega, \overline{\daleth_\Omega} = \daleth_\Omega$, and $\beth_\Omega \vee \daleth_\Omega = \underline{1}_\Omega$ imply $\beth_\Omega = \underline{0}_\Omega$ or $\daleth_\Omega = \underline{0}_\Omega$.

Proof. (1) $\Rightarrow$ (2): Let $\beth_\Omega, \daleth_\Omega \in \mathbf{I}^\Omega$ with $\underline{\beth_\Omega} = \beth_\Omega, \underline{\daleth_\Omega} = \daleth_\Omega$ such that $\beth_\Omega \wedge \daleth_\Omega = \underline{0}_\Omega$ and $\beth_\Omega \vee \daleth_\Omega = \underline{1}_\Omega$. Then, $\overline{\beth_\Omega} = \overline{(\daleth_\Omega)^c} = (\daleth)\Omega)^c = \beth_\Omega, \overline{\daleth_\Omega} = \overline{(\beth_\Omega)^c} = \mu^c = \daleth_\Omega$. Hence, $\overline{\beth_\Omega} \wedge \daleth_\Omega = \beth_\Omega \wedge \overline{\daleth_\Omega} = \beth_\Omega \wedge \nu = \underline{0}_\Omega$, that is, $\beth_\Omega, \daleth_\Omega$ are fuzzy graph separated subgraphs so that $\beth_\Omega \vee \daleth_\Omega = \underline{1}_\Omega$. However, ($\Omega, \Re, \Xi, \mathfrak{L}$) is a fuzzy graph connected, which implies that $\beth_\Omega = \underline{0}_\Omega$ or $\daleth_\Omega = \underline{0}_\Omega$.
(2) $\Rightarrow$ (3) : (3) $\Rightarrow$ (1) : Clear.

□

Proposition 4.4. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS, and $\beth_\Omega$ be a fuzzy subgraph of the graph Ω. Then, the following are equivalent:*

(1) $\beth_\Omega$ is fuzzy graph connected subgraph;
(2) If $\daleth_\Omega, \eta_\Omega$ are fuzzy graph separated subgraphs with $\beth_\Omega \leq (\daleth_\Omega \vee \eta_\Omega)$, then $\beth_\Omega \wedge \daleth_\Omega = \underline{0}_\Omega$ or $\beth_\Omega \wedge \eta_\Omega = \underline{0}_\Omega$;
(3) If $\daleth_\Omega, \eta_\Omega$ are fuzzy graph separated subgraphs with $\beth_\Omega \leq (\daleth_\Omega \vee \eta_\Omega)$, then $\beth_\Omega \leq \daleth_\Omega$ or $\beth_\Omega \leq \eta_\Omega$.

Proof. (1) $\Rightarrow$ (2): Let $\daleth_\Omega, \eta_\Omega$ be fuzzy graph separated subgraphs with $\beth_\Omega \leq (\daleth_\Omega \vee \eta_\Omega)$, that is, $\overline{\daleth_\Omega} \wedge \eta_\Omega = \overline{\eta_\Omega} \wedge \daleth_\Omega = \underline{0}_\Omega$ so that $\beth_\Omega \leq (\daleth_\Omega \vee \eta_\Omega)$, since

$$\overline{(\beth_\Omega \wedge \daleth_\Omega)} \wedge (\beth_\Omega \wedge \eta_\Omega) = \overline{\beth_\Omega} \wedge \overline{\daleth_\Omega} \wedge (\beth_\Omega \wedge \eta_\Omega) = \overline{\beth_\Omega} \wedge \beth_\Omega \wedge \overline{\daleth_\Omega} \wedge \eta_\Omega = \beth_\Omega \wedge \underline{0}_\Omega = \underline{0}_\Omega.$$
$$\overline{(\beth_\Omega \wedge \eta_\Omega)} \wedge (\beth_\Omega \wedge \daleth_\Omega) = \overline{\beth_\Omega} \wedge \overline{\eta_\Omega} \wedge (\beth_\Omega \wedge \daleth_\Omega) = \overline{\beth_\Omega} \wedge \beth_\Omega \wedge \overline{\eta_\Omega} \wedge \daleth_\Omega = \beth_\Omega \wedge \underline{0}_\Omega = \underline{0}_\Omega.$$

Then, $(\beth_\Omega \wedge \daleth_\Omega)$ and $(\beth_\Omega \wedge \eta_\Omega)$ are fuzzy graph separated subgraphs with $\beth_\Omega = (\beth_\Omega \wedge \daleth_\Omega) \vee (\beth_\Omega \wedge \eta_\Omega)$. However, $\beth_\Omega$ is a fuzzy graph connected, which means that $\beth_\Omega \wedge \daleth_\Omega = \underline{0}_\Omega$ or $\beth_\Omega \wedge \eta_\Omega = \underline{0}_\Omega$.
(2) $\Rightarrow$ (3): If $\beth_\Omega \wedge \daleth_\Omega = \underline{0}_\Omega$, then $\beth_\Omega \leq (\daleth_\Omega \vee \eta_\Omega)$ means that $\beth_\Omega = \beth_\Omega \wedge (\daleth_\Omega \vee \eta_\Omega) = (\beth_\Omega \wedge \daleth_\Omega) \vee (\beth_\Omega \wedge \eta_\Omega) = \beth_\Omega \wedge \eta_\Omega$, and thus $\beth_\Omega \leq \eta_\Omega$. Additionally, if $\beth_\Omega \wedge \eta_\Omega = \underline{0}_\Omega$, Then, $\beth_\Omega \leq \daleth_\Omega$.
(3) $\Rightarrow$ (1): Let $\daleth_\Omega, \eta_\Omega$ be fuzzy graph separated subgraphs so that $\beth_\Omega = \daleth_\Omega \vee \eta_\Omega$. Then, $\beth_\Omega \leq \daleth_\Omega$ or $\beth_\Omega \leq \eta_\Omega$. If $\beth_\Omega \leq \daleth_\Omega$, then

$$\eta_\Omega = (\daleth_\Omega \vee \eta_\Omega) \wedge \eta_\Omega = \beth_\Omega \wedge \eta_\Omega \leq \daleth_\Omega \wedge \eta_\Omega \leq \overline{\daleth_\Omega} \wedge \eta_\Omega = \underline{0}_\Omega.$$

Additionally, if $\beth_\Omega \leq \eta_\Omega$, then $\daleth_\Omega = (\daleth_\Omega \vee \eta_\Omega) \wedge \daleth_\Omega = \beth_\Omega \wedge \daleth_\Omega \leq \eta_\Omega \wedge \daleth_\Omega \leq \overline{\eta_\Omega} \wedge \daleth_\Omega = \underline{0}_\Omega$. Hence, $\beth_\Omega$ is a fuzzy graph connected subgraph.

□

Definition 4.5. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS, and $\beth_\Omega$ be a fuzzy subgraph of the graph Ω. Then, $\beth_\Omega$ is called a fuzzy graph component if $\beth_\Omega$ is a maximal fuzzy graph connected subgraph in Ω, that is, if $\daleth_\Omega \geq \beth_\Omega$ and $\beth_\Omega$ is fuzzy graph connected subgraph, Then, $\beth_\Omega = \daleth_\Omega$.*

Proposition 4.6. *Let ($\Omega, \Re, \Xi, \mathfrak{L}$) be a fuzzy graph IAS, and $\beth_\Omega$ be a fuzzy subgraph of the graph Ω. If $\beth_\Omega \neq \underline{0}_\Omega$ is a fuzzy graph connected subgraph in ($\Omega, \Re, \Xi, \mathfrak{L}$) and $\beth_\Omega \leq \daleth_\Omega \leq \overline{\beth_\Omega}$, then $\daleth_\Omega$ is a fuzzy graph connected.*

Proof. Let $\theta_\Omega, \eta_\Omega$ be fuzzy graph separated subgraphs in $\mathbf{I}^\Omega$ such that $\daleth_\Omega = \theta_\Omega \vee \eta_\Omega$, that is, $\overline{\theta_\Omega} \wedge \eta_\Omega = \overline{\eta_\Omega} \wedge \theta_\Omega = \underline{0}_\Omega$. Since $\beth_\Omega \leq \daleth_\Omega$ implies that $\beth_\Omega \leq (\theta_\Omega \vee \eta_\Omega)$ and $\beth_\Omega$ is a fuzzy graph connected, Then, from (3) in Proposition 4.4, we have $\beth_\Omega \leq \theta_\Omega$ or $\beth_\Omega \leq \eta_\Omega$. From $\daleth_\Omega \leq \overline{\beth_\Omega}$, if $\beth_\Omega \leq \theta_\Omega$, then

$$\eta_\Omega = (\theta_\Omega \vee \eta_\Omega) \wedge \eta_\Omega = \daleth_\Omega \wedge \eta_\Omega \leq \overline{\beth_\Omega} \wedge \eta_\Omega \leq \overline{\theta_\Omega} \wedge \eta_\Omega = \underline{0}_\Omega.$$

If $\beth_\Omega \le \eta_\Omega$, then

$$\theta_\Omega = (\theta_\Omega \vee \eta_\Omega) \wedge \theta_\Omega = \daleth_\Omega \wedge \theta_\Omega \le \overline{\beth_\Omega} \wedge \theta_\Omega \le \overline{\eta_\Omega} \wedge \theta_\Omega = \underline{0}_\Omega.$$

Hence, $\daleth_\Omega$ is a fuzzy graph connected. □

5. Applications

Decision-making is essential in our daily lives. Many uncertain systems exist and decision making under uncertainty or the choice in uncertain environment is the central subject in many of the disciplines that are alloyed in the management curriculum. Decision making involves recognizing a problem, formulating options, assessing them, and picking the best solution. This section presents a method to make decisions in uncertain systems using fuzzy rough information on graphs via graph ideals. This technique provides an in-depth analysis of the problem by using lower and upper approximations for unknown data.

Example 5.1. *Identification of best location in a department to set a mobile phone Jammer:*

For instance, a director of an institute may want to install mobile phone jammer across multiple departments, thus ensuring that each department is influenced by at least one jammer. To minimize the cost associated with deploying high-quality and powerful jammers, it is necessary to determine the smallest number of jammers to be installed. We assume a network of seven departments represented as vertices: $\mathfrak{Q}(\Omega) = \{\eth_1, \eth_2, \eth_3, \eth_4, \eth_5, \eth_6, \eth_7\}$. An edge exists between two vertices if one department is within the effect of a jammer installed in the other. The set of edges is denoted by $\mathfrak{Z}(\Omega) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}\}$. Now, we define a fuzzy graph $\beth_\Omega = (\beth_{\mathfrak{Q}(\Omega)}, \beth_{\mathfrak{Z}(\Omega)})$ over the crisp graph Ω, which describes the strength of the jammer (fuzzy vertices) in each department and the strength of their relations (fuzzy edges). We assume the fuzzy vertex set

$$\beth_{\mathfrak{Q}(\Omega)} = \{(\eth_1, 0.5), (\eth_2, 0.7), (\eth_3, 0.6), (\eth_4, 0.6), (\eth_5, 0.6), (\eth_6, 0.6), (\eth_7, 0.6)\},$$

and the fuzzy edge set

$$\beth_{\mathfrak{Z}(\Omega)} = \{(\alpha_1, 0.5), (\alpha_2, 0.5), (\alpha_3, 0.5), (\alpha_4, 0.5), (\alpha_5, 0.5), (\alpha_6, 0.5), (\alpha_7, 0.4), (\alpha_8, 0.5), (\alpha_9, 0.6), (\alpha_{10}, 0.5)\}.$$

The fuzzy subgraph $\beth_\Omega$ is represented in Figure 4.

Define a fuzzy binary relation $\Re$ on $\mathfrak{Q}(\Omega)$ as given in Table 9, where $\Re(\eth_i, \eth_j)$, $(i, j \in \{1, 2, \ldots, 7\})$ represents the relationship of comparison between the strength of the jammer in $\eth_i$ and the strength of the jammer in $\eth_j$.

Table 9. Fuzzy relation $\Re$ in Example 5.1.

$\Re$	$\eth_1$	$\eth_2$	$\eth_3$	$\eth_4$	$\eth_5$	$\eth_6$	$\eth_7$
$\eth_1$	1	0.4	0.5	0.5	0.3	0.5	0.6
$\eth_2$	0.4	1	0.4	0.5	0.6	0.6	0.6
$\eth_3$	0.5	0.4	1	0.7	0.6	0.6	0.5
$\eth_4$	0.5	0.5	0.7	1	0.4	0.6	0.6
$\eth_5$	0.3	0.6	0.6	0.4	1	0.9	0.2
$\eth_6$	0.5	0.6	0.6	0.6	0.9	1	0.6
$\eth_7$	0.6	0.6	0.5	0.6	0.2	0.6	1

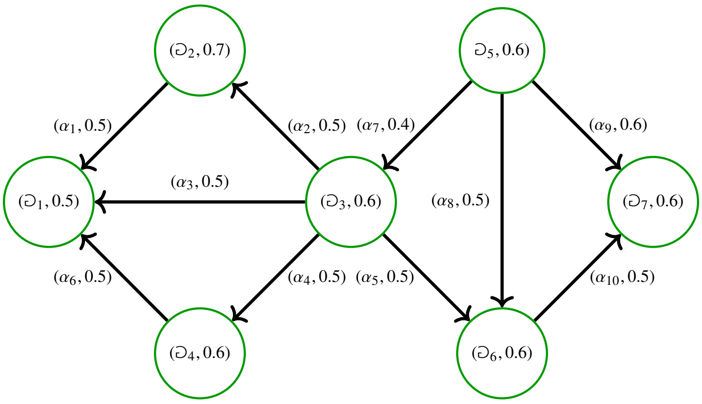


Figure 4. Representation of the fuzzy subgraph $\beth_\Omega$ given in Example 5.1.

$\Re < \eth_1 > \Re = \Re < \eth_2 > \Re = \Re < \eth_3 > \Re = \Re < \eth_4 > \Re = \Re < \eth_5 > \Re = \Re < \eth_6 > \Re = \Re < \eth_7 > \Re = \{0.3, 0.4, 0.4, 0.4, 0.2, 0.5, 0.2\}$. Consider a fuzzy graph ideal $\mathfrak{L}$ so that $[\mathfrak{Q}(\beth)] \in \mathfrak{L}$ iff $\mathfrak{Q}([\mathfrak{Q}(\beth)]) \leq \underline{0.1}_{\mathfrak{Q}(\Omega)}$, and the membership function of the fuzzy edge set $[\mathfrak{Q}(\beth)]_{\mathfrak{Z}(\Omega)}(\alpha)$ is given by the following:

$$\min\{[\mathfrak{Q}(\beth)]_{\mathfrak{Q}(\Omega)}(\eth_1), [\mathfrak{Q}(\beth)]_{\mathfrak{Q}(\Omega)}(\eth_2)\},$$

where the edge α connects the vertices $\eth_1$ and $\eth_2$. The lower and upper approximations of $\beth_{\mathfrak{Q}(\Omega)}$ with respect to $\Re$ and $\mathfrak{L}$ are given by the following:

$$[\mathfrak{Q}(\beth)]_{\Re} = \{(\eth_1, 0.5), (\eth_2, 0.6), (\eth_3, 0.6), (\eth_4, 0.6), (\eth_5, 0.6), (\eth_6, 0.5), (\eth_7, 0.6)\},$$

$$[\mathfrak{Q}(\beth)]^{\Re} = \{(\eth_1, 0.5), (\eth_2, 0.7), (\eth_3, 0.6), (\eth_4, 0.6), (\eth_5, 0.6), (\eth_6, 0.6), (\eth_7, 0.6)\}.$$

Let $\gamma_{\mathfrak{Z}(\Omega)} = \{(\eth_2\eth_1, 0.5), (\eth_3\eth_2, 0.5), (\eth_3\eth_1, 0.5), (\eth_3\eth_4, 0.5), (\eth_3\eth_6, 0.5), (\eth_4\eth_1, 0.5), (\eth_5\eth_3, 0.4), (\eth_5\eth_6, 0.5), (\eth_5\eth_7, 0.6), (\eth_6\eth_7, 0.5)\}$ be a fuzzy edge set defined on $\mathfrak{Z}(\Omega)$, and $\gamma_{\mathfrak{Z}(\Omega)}(\eth_i\eth_j)$ describes the degree of interference created by the jammers of $\eth_i$ at the same frequency range that is used by cell phones in the surroundings of $\eth_j$. Let Ξ be a fuzzy relation defined on $\mathfrak{Z}(\Omega)$, as given in Table 10.

Table 10. Fuzzy relation Ξ in Example 5.1.

Ξ	$\eth_2\eth_1$	$\eth_3\eth_2$	$\eth_3\eth_1$	$\eth_3\eth_4$	$\eth_3\eth_6$	$\eth_4\eth_1$	$\eth_5\eth_3$	$\eth_5\eth_6$	$\eth_5\eth_7$	$\eth_6\eth_7$
$\eth_2\eth_1$	1	0.3	0.3	0.4	0.4	0.5	0.5	0.4	0.2	0.2
$\eth_3\eth_2$	0.3	1	0.3	0.4	0.5	0.3	0.4	0.5	0.6	0.3
$\eth_3\eth_1$	0.3	0.3	1	0.5	0.5	0.6	0.5	0.5	0.6	0.2
$\eth_3\eth_4$	0.4	0.4	0.5	1	0.5	0.5	0.4	0.5	0.6	0.3
$\eth_3\eth_6$	0.4	0.5	0.5	0.5	1	0.5	0.6	0.5	0.2	0.2
$\eth_4\eth_1$	0.5	0.3	0.6	0.5	0.5	1	0.3	0.3	0.2	0.2
$\eth_5\eth_3$	0.5	0.4	0.5	0.4	0.6	0.3	1	0.5	0.4	0.4
$\eth_5\eth_6$	0.4	0.5	0.5	0.5	0.5	0.3	0.5	1	0.6	0.2
$\eth_5\eth_7$	0.2	0.6	0.6	0.6	0.2	0.2	0.4	0.6	1	0.4
$\eth_6\eth_7$	0.2	0.3	0.2	0.3	0.2	0.2	0.4	0.2	0.4	1

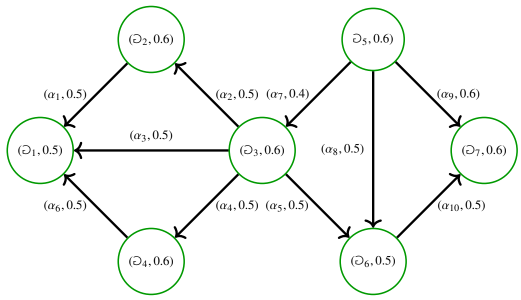
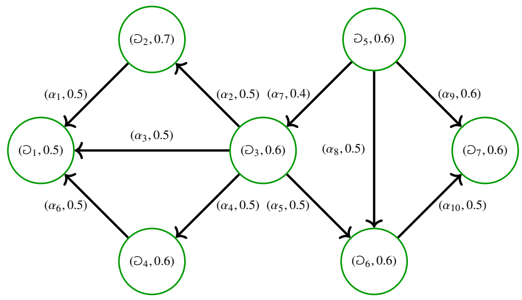
$\Xi < \eth_2\eth_1 > \Xi = \Xi < \eth_3\eth_2 > \Xi = \Xi < \eth_3\eth_1 > \Xi = \Xi < \eth_3\eth_4 > \Xi = \Xi < \eth_3\eth_6 > \Xi = \Xi < \eth_4\eth_1 > \Xi = \Xi < \eth_5\eth_3 > \Xi = \Xi < \eth_5\eth_6 > \Xi = \Xi < \eth_5\eth_7 > \Xi = \Xi < \eth_6\eth_7 > \Xi = \{0.2, 0.3, 0.2, 0.3, 0.2, 0.2, 0.3, 0.2, 0.2, 0.2\}$. The lower and upper approximations of $\beth_{\mathfrak{Z}(\Omega)}$ with respect to Ξ and $\mathfrak{L}$ are given by the following:

$[\mathfrak{Z}(\sqsupset)]_{\Xi} = \{(\eth_2\eth_1, 0.5), (\eth_3\eth_2, 0.5), (\eth_3\eth_1, 0.5), (\eth_3\eth_4, 0.5), (\eth_3\eth_6, 0.5), (\eth_4\eth_1, 0.5), (\eth_5\eth_3, 0.4), (\eth_5\eth_6, 0.5), (\eth_5\eth_7, 0.6), (\eth_6\eth_7, 0.5)\}$,

$[\mathfrak{Z}(\sqsupset)]^{\Xi} = \{(\eth_2\eth_1, 0.5), (\eth_3\eth_2, 0.5), (\eth_3\eth_1, 0.5), (\eth_3\eth_4, 0.5), (\eth_3\eth_6, 0.5), (\eth_4\eth_1, 0.5), (\eth_5\eth_3, 0.4), (\eth_5\eth_6, 0.5), (\eth_5\eth_7, 0.6), (\eth_6\eth_7, 0.5)\}$.

The problem can be represented by fuzzy rough digraphs, as shown in Table 11.

Table 11. Representation of the lower and upper approximations $\underline{\sqsupset_\Omega}$ and $\overline{\sqsupset_\Omega}$ in Example 5.1.

Fuzzy lower $\underline{\sqsupset_\Omega} = ([\mathfrak{Q}(\sqsupset)]_{\Re}, [\mathfrak{Z}(\sqsupset)]_{\Xi})$	Fuzzy upper $\overline{\sqsupset_\Omega} = \left([\mathfrak{Q}(\sqsupset)]^{\Re}, [\mathfrak{Z}(\sqsupset)]^{\Xi}\right)$
	

Apply the following formulae:

$$\left([\mathfrak{Q}(\sqsupset)]_{\Re} \oplus [\mathfrak{Q}(\sqsupset)]^{\Re}\right)(\eth_i) = [\mathfrak{Q}(\sqsupset)]_{\Re}(\eth_i) + [\mathfrak{Q}(\sqsupset)]^{\Re}(\eth_i) - \left([\mathfrak{Q}(\sqsupset)]_{\Re}(\eth_i) * [\mathfrak{Q}(\sqsupset)]^{\Re}(\eth_i)\right), \text{ and}$$

$$\left([\mathfrak{Z}(\sqsupset)]_{\Xi} \oplus [\mathfrak{Z}(\sqsupset)]^{\Xi}\right)(\eth_i\eth_j) = [\mathfrak{Z}(\sqsupset)]_{\Xi}(\eth_i\eth_j) + [\mathfrak{Z}(\sqsupset)]^{\Xi}(\eth_i\eth_j) - \left([\mathfrak{Z}(\sqsupset)]_{\Xi}(\eth_i\eth_j) * [\mathfrak{Z}(\sqsupset)]^{\Xi}(\eth_i\eth_j)\right).$$

This implies that $[\mathfrak{Q}(\sqsupset)]_{\Re} \oplus [\mathfrak{Q}(\sqsupset)]^{\Re} = \{(\eth_1, 0.75), (\eth_2, 0.88), (\eth_3, 0.84), (\eth_4, 0.84), (\eth_5, 0.84), (\eth_6, 0.8), (\eth_7, 0.84)\}$ and $[\mathfrak{Z}(\sqsupset)]_{\Xi} \oplus [\mathfrak{Z}(\sqsupset)]^{\Xi} = \{(\eth_2\eth_1, 0.75), (\eth_3\eth_2, 0.75), (\eth_3\eth_1, 0.75), (\eth_3\eth_4, 0.75), (\eth_3\eth_6, 0.75), (\eth_4\eth_1, 0.75), (\eth_5\eth_3, 0.64), (\eth_5\eth_6, 0.75), (\eth_5\eth_7, 0.84), (\eth_6\eth_7, 0.75)\}$.

Thus, a fuzzy digraph is depicted in Figure 5.

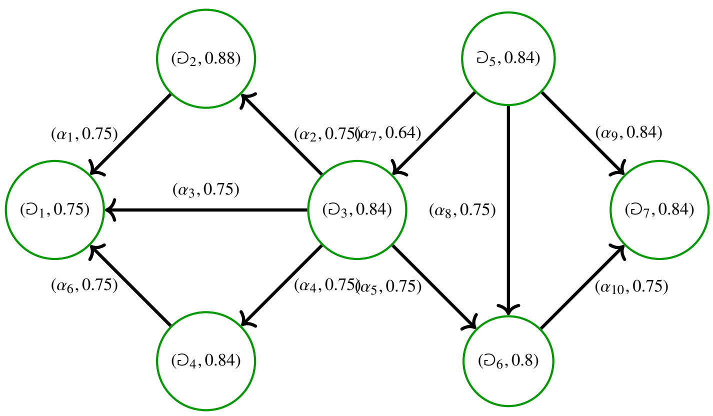


Figure 5. Fuzzy digraph $(\underline{\sqsupset_\Omega} \oplus \overline{\sqsupset_\Omega}) = \left([\mathfrak{Q}(\sqsupset)]_{\Re} \oplus [\mathfrak{Q}(\sqsupset)]^{\Re}, [\mathfrak{Z}(\sqsupset)]_{\Xi} \oplus [\mathfrak{Z}(\sqsupset)]^{\Xi}\right)$ defined in Example 5.1.

The final step involves identifying the minimal dominating vertex set of this digraph, which constitutes the solution. The minimal dominating set is $\{\eth_3, \eth_5\}$. Therefore, by installing a jammer at these two locations, the overall cost can be minimized. The process to determine a minimal dominating vertex set is detailed and illustrated in Algorithm 3.6.

Algorithm 1 The algorithm for determining a minimal dominating vertex set for a rough fuzzy digraph

Input: A finite fuzzy digraph $\beth_\Omega = (\mathfrak{Q}_\Omega, \mathfrak{Z}_\Omega)$, fuzzu graph ideal $\mathfrak{L}$ and two finite fuzzy relations $\Re$, Ξ.
Output: The fuzzy minimal dominating set D.

Input the vertex set $\mathfrak{Q}(\Omega) = \{\eth_1, \eth_2, \ldots, \eth_n\}$.
Input the fuzzy vertex set $\beth_{\mathfrak{Q}(\Omega}$ on $\mathfrak{Q}(\Omega)$.
Input the fuzzy relation $\Re$ on $\mathfrak{Q}(\Omega)$.
Input the fuzzy graph ideal $\mathfrak{L}$ on the graph Ω.
Input the edges set $\mathfrak{Z}(\Omega) = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, where $\alpha_i = \eth_j\eth_k$, for some $1 \leq j, k \leq n$.
Input the fuzzy relation Ξ on $\mathfrak{Z}(\Omega)$.
for $i = 1 : n$ **do**
  $[\mathfrak{Q}(\beth)]_\Re(\eth_i) = 1$ and $[\mathfrak{Q}(\beth)]^\Re(\eth_i) = 0$
  for $j = 1 : n$ **do**
    Compute: $(\mathfrak{Q}(\beth))_*(\eth_j)$ and $(\mathfrak{Q}(\beth))^*(\eth_j)$
    $[\mathfrak{Q}(\beth)]_\Re(\eth_i) = \min\{[\mathfrak{Q}(\beth)]_*(\eth_i), (\mathfrak{Q}(\beth))(\eth_j)\}$
    $[\mathfrak{Q}(\beth)]^\Re(\eth_i) = \max\{[\mathfrak{Q}(\beth)]^*(\eth_i), (\mathfrak{Q}(\beth))(\eth_j)\}$
  end for
end for
for $i = 1 : r$ **do**
  $[\mathfrak{Z}(\beth)]_\Xi(\alpha_i) = 1$
  $[\mathfrak{Z}(\beth)]^\Xi(\alpha_i) = 0$
  for $j = 1 : n$ **do**
    Compute: $(\mathfrak{Z}(\beth))_*(\alpha_j)$ and $(\mathfrak{Z}(\beth))^*(\alpha_j)$
    $[\mathfrak{Z}(\beth)]_\Xi(\eth_i) = \min\{[\mathfrak{Z}(\beth)]_*(\alpha_i), (\mathfrak{Q}(\beth))(\alpha_j)\}$
    $[\mathfrak{Z}(\beth)]^\Xi(\alpha_i) = \max\{[\mathfrak{Z}(\beth)]^*(\alpha_i), (\mathfrak{Z}(\beth))(\alpha_j)\}$
  end for
end for
$k = 0, D = \emptyset$
for $i = 1 : n$ **do**
  for $j = i + 1 : n$ **do**
    Compute:$\left([\mathfrak{Z}(\beth)]_\Xi \oplus [\mathfrak{Z}(\beth)]^\Xi\right)(\eth_i\eth_j)$
    if $\left([\mathfrak{Z}(\beth)]_\Xi \oplus [\mathfrak{Z}(\beth)]^\Xi\right)(\eth_i\eth_j) = \min\{\left([\mathfrak{Q}(\beth)]_\Re \oplus [\mathfrak{Q}(\beth)]^\Re\right)(\eth_i), \left([\mathfrak{Q}(\beth)]_\Re \oplus [\mathfrak{Q}(\beth)]^\Re\right)(\eth_j)\}$ **then**
      $\eth_i \in D,\ k = k + 1, \eth_k = \eth_i$
    end if
  end for
end for
Arrange $\mathfrak{Q}(\beth) \backslash D = \{\eth_{k+1}, \eth_{k+2}, \ldots, \eth_n\} = J$
for $i = 1 : k$ **do**
  $D' = D \backslash \{\eth_i\}$
  if $D'$is a dominating set **then**
    $D = D'\ J = J \cup \{\eth_i\}$
  end if
end for
if $D \cup J = \mathfrak{Q}(\beth)$ **then**
  Print: D is a minimal dominating set. **else**
  Print: There is no dominating set.
end if

Discussion

Applying a single uncertain approach to real-world situations is often challenging due to the limitations of human knowledge in analyzing complex problems. In decision-making scenarios, it is essential to account for parametric uncertainty within graphical models. For example, when selecting the best social organization, one must consider not only the norms and characteristics of each option, but also the interconnections and coordination links between them. The fuzzy rough set theory provides a mathematical framework to address this challenge. This algorithm employs fuzzy binary relations between pairs of items to generate the lower and upper approximations of the target set. In our application, we compare fuzzy rough graphs with fuzzy graphs by applying fuzzy sets and fuzzy graph ideals. As illustrated by the graphical representation of the fuzzy graph $\beth_\Omega$ in Figure 4, applying the given fuzzy information to identify locations for a mobile phone jammer does not result in a feasible dominating set, which indicates ambiguous information with no solutions. To resolve this, one must either refine the fuzzy data or establish fuzzy relations and fuzzy graph ideals to obtain a suitable fuzzy graph IAS that identifies at least one location. Therefore, the fuzzy rough set theory proves to be more effective and reliable in such decision-making problems. Consequently, we generated fuzzy rough digraphs as shown in Table 11, and from these, constructed the fuzzy rough digraph in Figure 5, which facilitated the determination of the minimal dominating set.

6. Conclusions

Researchers are progressively enhancing the role of topological spaces in both the classical graph theory and fuzzy graph theory. The fuzzy rough set theory offers methods to determine the upper and lower approximations of fuzzy sets. Traditional approaches in the literature often rely on arbitrary or equivalence relations as approximation techniques within the generalized rough set theory. In this work, we introduced a significant advancement to the fuzzy graph theory by formulating the concept of "graph ideal", thereby expanding the framework and applications of the theory. We developed a method using any two fuzzy relations on vertex set and edge set of a digraph as an approximation tool and the new notion of fuzzy graph ideal to introduce fuzzy graph IAS over digraphs. We proposed new two kinds of fuzzy lower and upper approximations on digraphs based on any two fuzzy relations and the fuzzy graph ideal. Additionally, some vital and results of both fuzzy graph ideal approximations were established. The relationships between the present fuzzy approximations were induced. Moreover, comparisons between the present fuzzy graph ideal approximations and the preceding ones [35,37,39] were presented and shown to be more general. Fuzzy interior and fuzzy closure graphs of a rough fuzzy graph were discussed. Furthermore, we discussed fuzzy graph connectedness as a sample application of the new fuzzy graph operators. Finally, we considered applications of fuzzy graph IASs. We presented fuzzy rough digraphs which utilized the fuzzy graph ideal as an enormous tool to solve uncertain decision-making problems. The method's effectiveness in resolving ambiguity relies on the successful definition and integration of these complex structures (fuzzy relations, fuzzy graph ideals, and the resulting approximation spaces). This complexity might be a practical limitation compared to simpler methods, although it was presented as a strength to deal with uncertainty. We plan to extend our research work of fuzzification to (1) Intuitionistic fuzzy rough graphs, (2) Bipolar fuzzy rough graphs, and (3) Picture fuzzy rough graphs.

Author contributions

Dali Shi: Methodology, data curation, funding; Salah Eldin Abbas: Validation, formal analiyses, investigation; Hossam M. Khiamy: Software, writing–original draft, writing–review final form; Ismail Ibedou: Conceptualization, supervision, project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of interest

The authors declare that they have no conflicts of interest.

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