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Research article

Innovative examination of noise impacts on explicit solitary wave solutions of (2+1)-dimensional stochastic Chaffee-Infante equation

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Abstract: In this paper, we explored new families of solitary wave solutions of stochastic Chaffee-Infante equation (SCIE) with Wiener process in Itô sense. SCIE is one of the most important models in mathematical physics used to describe the processes of diffusion and wave propagation. Efficient (G'/G)-expansion and its version generalized $(r + G'/G)$-expansion are applied to obtain explicit solitary wave solutions to the targeted SCIE. The strategic (G'/G)-expansion method originally converts SCIE to nonlinear ordinary differential equation (NODE) by wave transformation and then converts it to a set of nonlinear algebraic equations on the assumption of finite series-form solutions. Under the analysis of the solutions of the resulting system with Maple, a number of solitary wave solutions in the form of trigonometric, hyperbolic, and rational functions were found. To verify the presence of solitary wave solutions, such as soliton, dark, bright, kink, and anti-kink solitary wave solutions in SCIEs, several solitary wave solutions were evaluated using illustrated 3D visualizations for given parameter values under zero and nonzero noise effects. The implication of our results extends widely across various fields both in stochastic phenomena and nonlinear dynamics and has a contributions to physics and nonlinear science.

Keywords: stochastic Chaffee-Infante equation; $(\frac{G'}{G})$-expansion method; solitary waves; solitons; Wiener process
Mathematics Subject Classification: 35R11, 35R30, 47A52, 49M41, 49N45

1. Introduction

Nonlinear partial differential equations (NPDEs) are fundamental frameworks in mathematical modeling that provide a solid foundation for phenomena that regulate not just over time but also over various geographical areas. From simulating fluid flow and quantum mechanics to comprehending heat transfer and wave behavior, NPDEs are employed in a wide range of scientific and industrial domains. Significant insights into the dynamics of NPDEs may be gained from studying the solitary wave phenomenon, which depends on explicit solutions of NPDEs, particularly the estimation of the traveling wave and solitary wave solution. As a result, research on solitary wave solutions of NPDEs is expanding significantly. As a consequence, a number of useful methods have been developed to get solitary wave solutions in order to understand the underlying mechanism of these nonlinear models [1–3]. Over the past decade, there has been a growing interest in using symbolic computing algorithms to provide explicit solitary wave solutions for NPDEs that simplify complicated algebraic computations [4–6].The literature has established a number of analytical techniques for finding new solitary wave solutions, including the Riccati-Bernoulli sub-ODE method [7], the Khater method [8], the unified method [9], the Riccati modified extended simple equation approach [10, 11], the (G'/G) -expansion technique [12–14], the exp-function approach [15], the Sardar sub-equation methodology [16], the sub-equation technique [17], the Kudryashov technique [18], the Poincaré-Lighthill-Kuo technique [19], the extended direct algebraic method [20–22], the Hirota's bilinear method [23], the generalized Kudryashov auxiliary method [24], and the extended hyperbolic function method [25], among others [26–28]. As a result, complex systems that often have no analytical solutions may now be analyzed with the help of these methods [29]. Researchers and scholars in a variety of sectors must thus be knowledgeable with NPDE theory and its applications. It gives researchers the information and abilities needed to carry out practical research and develop a deeper comprehension of both artificial and natural systems [30–32].

In several scientific and technological domains, a variety of nonlinear models have been created to study various physical processes [33–35]. These models include the Maccari equation [36], the Fokas-Lenells equation [37], the Schrödinger equation [38], and a number of others [39–41]. Chaffee-Infante equations (CIEs) are modified reaction-diffusion equations that are commonly used to study pattern formation, phase transitions, forecasting, complex interaction of solitons and nonlinear wave behavior in the propagation of a nonlinear optical material. The CIE in (2 + 1)-dimensions is shown as [42]:

$$F_{tx} + (kF^3 - F_{xx} - kF)_x + aF_{yy} = 0, \tag{1.1}$$

In contrast, the CIE in (1 + 1)-dimensions is represented as:

$$F_t - F_{xx} - kF(1 - F^2) = 0, \tag{1.2}$$

where physical structures are represented by $F = F(x, y, t)$ in (1.1) and $F = F(x, t)$ in (1.2), a is the diminution parameter, k is the coefficient of diffusion rate parameter, and the nonlinear term F^3 regulates saturating effects. The coefficient k modifies the proportional ratio within the diffusion component and the nonlinear factor. The CIEs provide a useful framework for researching the important scientific phenomena of gas diffusion in an evenly built media. The CIEs [43, 44] are an established illustration of a reactive Duffing model having origins in the physical science. Similarly, (1.1) and (1.2) result in Newell-Whitehead equations at $k = 1$. The CIEs are essential frameworks for studying chemical kinetics and more complicated chemical processes in general. Robert L. Chaffee and Carlos Infante established these mathematical equations in the middle of the twentieth century [45–47]. The kinetics of certain chemical reactions are succinctly and thoroughly described by these equations.

Stochastic differential equations (SDEs) are mathematical frameworks of modeling dynamic systems with randomness, and they have been popular since the middle of the 20th century in such applications as biology, physics, finance, and engineering [48, 49]. In the 1940s, Kiyosi Itô and Andrei Kolmogorov pioneered the foundation of stochastic calculus [50], which is crucial in the treatment of uncertainty in many fields. The difference between SDEs and traditional differential equations is that the former make the system evolution less predictable [51–53]. SDEs describe the lack of predictability in different systems because of the background noise among others and economic dynamics, combining predictability and spontaneity. SDEs are based on Itô calculus, which are used to deal with stochastic differentials and integrals. SDEs have strong and weak solutions [54, 55] that characterize their behavioral characteristics. Wiener process or Brownian motion, can be used to describe random variations in many disciplines, such as engineering and economics, whereby the change in the system can be analyzed and predicted through this process based on the outcome of small random vagaries in the system under examination [56].

SDEs are essential in different areas as it is multifaceted. In the field of physics, SDEs play vital role in modeling stochastic processes, such as Brownian diffusion and motion. Their application in risk assessment and portfolio optimization is an example in economics in the Black-Scholes equation as introduced by [57]. SDEs can be used in biology to comprehend changes in populations and biochemical reactions. The Itô and Stratonovich calculus are also crucial mathematical tools that help to solve SDEs [58]. Furthermore, SDEs also affected the investigation of stochastic solitons in NPDEs, such as the SIdV equation [59], stochastic Chen-Lee-Liu equation [60], and other equations [61, 62]. Inspired by the significant work on the stochastic analysis of nonlinear NPDEs, we seek to examine the SCIE with regard to stochastic. The SCIE is stated as [63]:

$$F_{tx} + (kF^3 - F_{xx} - kF)_x + aF_{yy} - \varrho F_x S_t(t) = 0, \tag{1.3}$$

where $S(t)$ is the Brownian motion. The stochastic process $S(t)_{t\geq 0}$ is termed a Brownian motion if it meets the subsequent conditions [63]:

- $S(0) = 0$,
- $S(t)$ is a continuous function,
- $S(t_1) - S(t_0)$ is independent for $t_0 < t_1$,
- $S(t_1) - S(t_0)$ has a Gaussian distribution with mean 0 and variance $t_1 - t_0$.

In contrast to CIE, the SCIE offers a more realistic model in real-world systems of stability variation and modeling of the dynamics of noise-driven system by including random fluctuations in the form of Browning motion. The modified extended tanh method [63] and modified auxiliary equation method [64] are some of the techniques that have emerged in the search for analytic solutions for SCIE. This study aims to utilize the efficient $(\frac{G'}{G})$-expansion method and its version generalized $(r + \frac{G'}{G})$-expansion method in a stochastic setting to discover novel explicit solitary wave solutions for the SCIE. The strategic $(\frac{G'}{G})$-expansion approach uses wave transformation to first convert the SCIE into a NODE, which is then transformed into a system of nonlinear algebraic equations under the presumption of finite series-form solutions. Several explicit solitary wave solutions in the form of trigonometric, hyperbolic, and rational functions are found when the solutions of the resultant system are examined using the Maple. Several solitary wave solutions are assessed using illustrated 3D visualizations for specified parameter values under zero and nonzero noise effects in order to confirm the existence of solitary wave solutions, such as soliton, dark, bright, kink, anti-kink, and other solitary wave solutions in SCIEs. Our results contribute to advances in nonlinear science and physics and have wide-ranging ramifications in several fields, including stochastic phenomena and nonlinear dynamics.

The remaining work is arranged as follows: Section 2 presents the methodological framework for the proposed $(\frac{G'}{G})$-expansion method, Section 3 uses the suggested approach to generate novel families of explicit solitary wave solutions for the intended SCIE, Section 4 discusses and illustrates the identified explicit solitary wave solutions, Section 5 provides a brief summary of the results, and the appendix provides detail about terms-by-term transformation of model into NODE.

2. The working methodology

The operations of the $(\frac{G'}{G})$-expansion technique and its variant generalized $(r+\frac{G'}{G})$-expansion method are explained in this section of our study, with a focus on solving the following generic NPDE [12,65]:

$$M_1(F, F_t, F_{t_1}, F_{t_2}, zF_{t_1}, \ldots) = 0, \tag{2.1}$$

where $F = F(t, t_1, t_2, t_3, \ldots, t_l)$. The following approach is used to solve Eq (2.1):

A. We begin with the application of a wave transformation $F(t, t_1, t_2, t_3, \ldots, t_l) = \zeta(\theta)$, where θ can be defined in several ways. The following nonlinear ordinary differential equation is produced by the application of this transformation to Eq (2.1):

$$M_2(\zeta, \zeta'\zeta, \zeta', \ldots) = 0, \tag{2.2}$$

where $\zeta' = \frac{d\zeta}{d\theta}$. Sometimes, in order to coerce NODE to adhere to the homogeneous balancing principle, the integration of Eq (2.2) is required.

B. Next, we assume the following series form solution for (2.2):

a. In the simple $(\frac{G'}{G})$-expansion method, we suppose the following solution:

$$\zeta(\theta) = \sum_{i=-\sigma}^{\sigma} S_i (\frac{G'(\theta)}{G(\theta)})^i, \tag{2.3}$$

b. In the generalized $(r + \frac{G'}{G})$-expansion method, we suppose the following solution:

$$\zeta(\theta) = \sum_{i=-\sigma}^{\sigma} S_i (r + \frac{G'(\theta)}{G(\theta)})^i, \quad r \in R, \tag{2.4}$$

where $S_i's(i = -\sigma...\sigma)$ will be calculated later. By homogenously balancing the nonlinearity and highest derivative term in Eq (2.2), the positive integer σ called balance number may be found. The following mathematical formulae are applied in order to determine exact values of the balance number σ:

$$D(\frac{d^\gamma \zeta}{d\theta^\gamma}) = \sigma + \gamma, \text{ and } D(\zeta^\rho (\frac{d^\gamma \zeta}{d\theta^\gamma})^m) = \sigma\rho + m(\gamma + \sigma), \tag{2.5}$$

where D means the $\zeta(\theta)$'s degree and γ, ρ, and m belongs to positive integers. Furthermore, the function $G(\theta)$ observed in Eqs (2.3) and (2.4) is the solution to the subsequent auxiliary equation:

$$G''(\theta) + \psi G'(\theta) + \tau G(\theta) = 0, \tag{2.6}$$

where ψ, τ are constants. Moreover, $G'(\theta)$ denotes derivative of $G(\theta)$. Additionally, using the general solution of Eq (2.6), we have [65]

$$\left(\frac{G'(\theta)}{G(\theta)}\right)=\begin{cases}\frac{1}{2}\,\frac{\sqrt{\Omega}\left(\chi_1\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}{\chi_1\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)}-\frac{1}{2}\psi, & \Omega<0,\\ \frac{1}{2}\,\frac{\sqrt{-\Omega}\left(-\chi_1\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}{\chi_1\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)}-\frac{1}{2}\psi, & \Omega>0,\\ \frac{\chi_2}{\chi_1+\chi_2\theta}-\frac{1}{2}\psi, & \Omega=0,\end{cases} \tag{2.7}$$

where $\Omega=\psi^2-4\tau$ and χ_1 and χ_2 in (2.7) are arbitrary constants.

C. Next, we insert Eqs (2.3) or (2.4) into (2.2) and gather any terms that have equal powers of $(\frac{G'(\theta)}{G(\theta)})$.

D. A set of nonlinear algebraic equations in $S_i(i=-\sigma,...,\sigma)$, ψ, τ, and other necessary parameters result from the obtained polynomial $(\frac{G'(\theta)}{G(\theta)})^i$ having all of its coefficients equal to zero.

E. The unknown parameters are discovered by using Maple to solve the resultant system.

F. Families of solitary wave solutions for Eq (2.1) are then created by substituting the estimated values from step **E** in either (2.3) or (2.4).

3. Results

The model's governing equation and explicit solitary wave solutions using the suggested approach are demonstrated in this section. We apply the following transformation to Eq (1.3) in order to determine the governing NODE:

$$F(t,x,y)=\zeta(\theta)\mathrm{e}^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}},\theta=x+y-ct. \tag{3.1}$$

Consequently, we obtain

$$\begin{aligned}&-c\zeta''(\theta)\mathrm{e}^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}}+\left(\varrho S_t(t)\zeta'(\theta)\mathrm{e}^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}}-\frac{\varrho^2}{2}\zeta'(\theta)\mathrm{e}^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}}\right)-\zeta'''(\theta)\mathrm{e}^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}}\\&+3k\zeta^2(\theta)(\mathrm{e}^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}})^3\zeta'(\theta)-k\zeta'(\theta)\mathrm{e}^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}}+a\zeta''(\theta)\mathrm{e}^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}}-\varrho\zeta'(\theta)\mathrm{e}^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}}S_t(t)=0,\end{aligned} \tag{3.2}$$

where the total derivatives of $\zeta(\theta)$ with regard to θ are shown by primes. We acquire a NODE after taking into account both sides' expectations. The resultant NODE is further integrated with respect to θ with zero constant of integration. As a result, we obtain the following governing NODE:

$$k\left(\zeta\left(\theta\right)\right)^3-\left(k+\frac{\varrho^2}{2}\right)\zeta\left(\theta\right)-\zeta''\left(\theta\right)+\left(a-c\right)\zeta'\left(\theta\right)=0. \tag{3.3}$$

Using the formulae in Eq (2.5), we achieve $\sigma=1$ by demonstrating the homogeneous balancing principle between $\zeta''(\theta)$ and $\zeta^3(\theta)$ given in (3.3).

3.1. Simple $(\frac{G'}{G})$-expansion method

To generate solitary wave solutions for Eq (1.3), we first address (3.3) using the simple $(\frac{G'}{G})$-expansion approach. This approach suggests the following closed form solution for (3.3) when we replace σ in (2.3) by 1:

$$\zeta(\theta)=\sum_{i=-1}^{1}S_i(\frac{G'(\theta)}{G(\theta)})^i. \tag{3.4}$$

An expression in $(\frac{G'(\theta)}{G(\theta)})$ is obtained by putting (3.4) in (3.3) with the help of Eq (2.6) and collecting all the terms with the same power of $(\frac{G'(\theta)}{G(\theta)})$. An algebraic system of equations is produced when all of the expression's coefficients are set to zero. The following four cases of solutions for the pertinent parameters are produced by using Maple to solve the resulting problem:

Case 1.1.

$$S_0 = -\frac{\sqrt{\Omega}+\psi}{\sqrt{-\varrho^2+4\Omega}}, S_1 = 0, S_{-1} = 2\frac{\tau}{\sqrt{-\varrho^2+4\Omega}}, c = a + 3\sqrt{\Omega}, k = 2\Omega - \frac{1}{2}\varrho^2, a = a. \tag{3.5}$$

Case 1.2.

$$S_0 = \frac{\sqrt{\Omega}-\psi}{\sqrt{-\varrho^2+4\Omega}}, S_1 = 2\frac{1}{\sqrt{-\varrho^2+4\Omega}}, S_{-1} = 0, c = a + 3\sqrt{\Omega}, k = 2\Omega - \frac{1}{2}\varrho^2, a = a. \tag{3.6}$$

Case 1.3.

$$S_0 = -\frac{\psi}{\sqrt{-\varrho^2+\Omega}}, S_1 = 0, S_{-1} = 2\frac{\tau}{\sqrt{-\varrho^2+\Omega}}, c = a, k = \frac{1}{2}\Omega - \frac{1}{2}\varrho^2, a = a. \tag{3.7}$$

Case 1.4.

$$S_0 = -\frac{\psi}{\sqrt{-\varrho^2+\Omega}}, S_1 = 2\frac{1}{\sqrt{-\varrho^2+\Omega}}, S_{-1} = 0, c = a, k = \frac{1}{2}\Omega - \frac{1}{2}\varrho^2, a = a. \tag{3.8}$$

Where $\Omega = \psi^2 - 4\tau$.

Assuming Case 1.1 and using Eqs (3.1) and (3.4) with the corresponding solutions of (2.6) stated in (2.7), we get the following families of solutions:

Set 1.1.1. In case of $\Omega > 0$,

(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,1,1}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(\frac{2\tau}{\sqrt{-\varrho^2+4\Omega}\left(\frac{1}{2}\frac{\sqrt{\Omega}\left(\chi_1\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}{\chi_1\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)} - \frac{1}{2}\psi\right)} - \frac{\sqrt{\Omega}+\psi}{\sqrt{-\varrho^2+4\Omega}}\right). \tag{3.9}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$

$$F_{1,1,2}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(2\frac{\tau}{\sqrt{-\varrho^2+4\Omega}\left(\frac{1}{2}\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\theta\right) - \frac{1}{2}\psi\right)} - \frac{\sqrt{\Omega}+\psi}{\sqrt{-\varrho^2+4\Omega}}\right). \tag{3.10}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,1,3}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(2\frac{\tau}{\sqrt{-\varrho^2+4\Omega}\left(\frac{1}{2}\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\theta\right) - \frac{1}{2}\psi\right)} - \frac{\sqrt{\Omega}+\psi}{\sqrt{-\varrho^2+4\Omega}}\right). \tag{3.11}$$

Set 1.1.2. In case of $\Omega < 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,1,4}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(\frac{2\tau}{\sqrt{-\varrho^2+4\Omega}\left(\frac{1}{2}\frac{\sqrt{-\Omega}\left(-\chi_1\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}{\chi_1\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)}-\frac{1}{2}\psi\right)}-\frac{\sqrt{\Omega}+\psi}{\sqrt{-\varrho^2+4\Omega}}\right). \tag{3.12}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{1,1,5}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(2\frac{\tau}{\sqrt{-\varrho^2+4\Omega}\left(\frac{1}{2}\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)-\frac{1}{2}\psi\right)}-\frac{\sqrt{\Omega}+\psi}{\sqrt{-\varrho^2+4\Omega}}\right). \tag{3.13}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,1,6}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(2\frac{\tau}{\sqrt{-\varrho^2+4\Omega}\left(-\frac{1}{2}\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)-\frac{1}{2}\psi\right)}-\frac{\sqrt{\Omega}+\psi}{\sqrt{-\varrho^2+4\Omega}}\right). \tag{3.14}$$

Set 1.1.3. In case of $\Omega = 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,1,7}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(-\frac{\psi^2\theta\chi_2+4\tau\theta\chi_2+\psi^2\chi_1+4\tau\chi_1-2\psi\chi_2}{i\varrho\left(\psi\theta\chi_2+\psi\chi_1-2\chi_2\right)}\right). \tag{3.15}$$

(ii) When $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{1,1,8}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(-\frac{\psi^2\theta+4\tau\theta-2\psi}{i\varrho\left(\psi\theta-2\right)}\right). \tag{3.16}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,1,9}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(-\frac{\psi^2+4\tau}{i\varrho\psi}\right). \tag{3.17}$$

In the above solutions, $\theta = x + y - (a + 3\sqrt{\Omega})t$.

Assuming Case 1.2 and using Eqs (3.1) and (3.4) with the corresponding solutions of (2.6) stated in (2.7), we get the following families of solutions:

Set 1.2.1. In case of $\Omega > 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,2,1}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(\frac{\sqrt{\Omega}-\psi}{\sqrt{-\varrho^2+4\Omega}}+\frac{\left(\frac{\sqrt{\Omega}\left(\chi_1\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}{\chi_1\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)}-\psi\right)}{\sqrt{-\varrho^2+4\Omega}}\right). \tag{3.18}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$

$$F_{1,2,2}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(\frac{\sqrt{\Omega}-\psi}{\sqrt{-\varrho^2+4\Omega}}+\frac{\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\theta\right)-\frac{1}{2}\psi}{\sqrt{-\varrho^2+4\Omega}}\right). \tag{3.19}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,2,3}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(\frac{\sqrt{\Omega} - \psi}{\sqrt{-\varrho^2 + 4\,\Omega}} + \frac{\sqrt{\Omega}\tanh\left(\frac{1}{2}\ \sqrt{\Omega}\theta\right) - \frac{1}{2}\,\psi}{\sqrt{-\varrho^2 + 4\,\Omega}}\right). \tag{3.20}$$

Set 1.2.2. In case of $\Omega < 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,2,4}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(\frac{\sqrt{\Omega} - \psi}{\sqrt{-\varrho^2 + 4\,\Omega}} + \frac{\left(\frac{\sqrt{-\Omega}\left(-\chi_1 \sin\left(\frac{1}{2}\ \sqrt{-\Omega}\theta\right) + \chi_2 \cos\left(\frac{1}{2}\ \sqrt{-\Omega}\theta\right)\right)}{\chi_1 \cos\left(\frac{1}{2}\ \sqrt{-\Omega}\theta\right) + \chi_2 \sin\left(\frac{1}{2}\ \sqrt{-\Omega}\theta\right)} - \psi\right)}{\sqrt{-\varrho^2 + 4\,\Omega}}\right). \tag{3.21}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{1,2,5}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(\frac{\sqrt{\Omega} - \psi}{\sqrt{-\varrho^2 + 4\,\Omega}} + \frac{\sqrt{-\Omega}\cot\left(\frac{1}{2}\ \sqrt{-\Omega}\theta\right) - \frac{1}{2}\,\psi}{\sqrt{-\varrho^2 + 4\,\Omega}}\right). \tag{3.22}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,2,6}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(\frac{\sqrt{\Omega} - \psi}{\sqrt{-\varrho^2 + 4\,\Omega}} + \frac{-\ \sqrt{-\Omega}\tan\left(\frac{1}{2}\ \sqrt{-\Omega}\theta\right) - \frac{1}{2}\,\psi}{\sqrt{-\varrho^2 + 4\,\Omega}}\right). \tag{3.23}$$

Set 1.2.3. In case of $\Omega = 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,2,7}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(-2\,\frac{\psi\,\theta\,\chi_2 + \psi\,\chi_1 - \chi_2}{(\theta\,\chi_2 + \chi_1)\,i\varrho}\right). \tag{3.24}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{1,2,8}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(-2\,\frac{\psi\,\theta - 1}{\theta\,i\varrho}\right). \tag{3.25}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,2,9}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(-2\,\frac{\psi}{i\varrho}\right). \tag{3.26}$$

In the above solutions, $\theta = x + y - (a + 3\ \sqrt{\Omega})t$.

Assuming Case 1.3 and using Eqs (3.1) and (3.4) with the corresponding solutions of (2.6) stated in (2.7), we get the following families of solutions:
Set 1.3.1. In case of $\Omega > 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,3,1}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}}\left(2\,\tau\frac{1}{\sqrt{-\varrho^2 + \Omega}\left(\frac{1}{2}\ \frac{\sqrt{\Omega}\left(\chi_1 \sinh\left(\frac{1}{2}\ \sqrt{\Omega}\theta\right) + \chi_2 \cosh\left(\frac{1}{2}\ \sqrt{\Omega}\theta\right)\right)}{\chi_1 \cosh\left(\frac{1}{2}\ \sqrt{\Omega}\theta\right) + \chi_2 \sinh\left(\frac{1}{2}\ \sqrt{\Omega}\theta\right)} - \frac{1}{2}\,\psi\right)} - \frac{\psi}{\sqrt{-\varrho^2 + \Omega}}\right). \tag{3.27}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$

$$F_{1,3,2}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(2\,\frac{\tau}{\sqrt{-\varrho^2+\Omega}\left(\frac{1}{2}\,\sqrt{\Omega}\coth\left(\frac{1}{2}\,\sqrt{\Omega}\theta\right)-\frac{1}{2}\,\psi\right)}-\frac{\psi}{\sqrt{-\varrho^2+\Omega}}\Big). \tag{3.28}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,3,3}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(2\,\frac{\tau}{\sqrt{-\varrho^2+\Omega}\left(\frac{1}{2}\,\sqrt{\Omega}\tanh\left(\frac{1}{2}\,\sqrt{\Omega}\theta\right)-\frac{1}{2}\,\psi\right)}-\frac{\psi}{\sqrt{-\varrho^2+\Omega}}\Big). \tag{3.29}$$

Set 1.3.2. In case of $\Omega < 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,3,4}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(2\,\tau\frac{1}{\sqrt{-\varrho^2+\Omega}\left(\frac{1}{2}\,\frac{\sqrt{-\Omega}\left(-\chi_1\sin\left(\frac{1}{2}\,\sqrt{-\Omega}\theta\right)+\chi_2\cos\left(\frac{1}{2}\,\sqrt{-\Omega}\theta\right)\right)}{\chi_1\cos\left(\frac{1}{2}\,\sqrt{-\Omega}\theta\right)+\chi_2\sin\left(\frac{1}{2}\,\sqrt{-\Omega}\theta\right)}-\frac{1}{2}\,\psi\right)}-\frac{\psi}{\sqrt{-\varrho^2+\Omega}}\Big). \tag{3.30}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{1,3,5}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(2\,\frac{\tau}{\sqrt{-\varrho^2+\Omega}\left(\frac{1}{2}\,\sqrt{-\Omega}\cot\left(\frac{1}{2}\,\sqrt{-\Omega}\theta\right)-\frac{1}{2}\,\psi\right)}-\frac{\psi}{\sqrt{-\varrho^2+\Omega}}\Big). \tag{3.31}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,3,6}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(2\,\frac{\tau}{\sqrt{-\varrho^2+\Omega}\left(-\frac{1}{2}\,\sqrt{-\Omega}\tan\left(\frac{1}{2}\,\sqrt{-\Omega}\theta\right)-\frac{1}{2}\,\psi\right)}-\frac{\psi}{\sqrt{-\varrho^2+\Omega}}\Big). \tag{3.32}$$

Set 1.3.3. In case of $\Omega = 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,3,7}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(-\frac{\psi^2\theta\chi_2+4\,\tau\,\theta\chi_2+\psi^2\chi_1+4\,\tau\chi_1-2\,\psi\chi_2}{i\varrho\,(\psi\,\theta\chi_2+\psi\chi_1-2\,\chi_2)}\Big). \tag{3.33}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{1,3,8}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(-\frac{\psi^2\theta+4\,\tau\,\theta-2\,\psi}{i\varrho\,(\psi\,\theta-2)}\Big). \tag{3.34}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,3,9}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(-\frac{\psi^2+4\,\tau}{i\varrho\psi}\Big). \tag{3.35}$$

In the above solutions, $\theta = x + y - at$.

Assuming Case 1.4 and using Eqs (3.1) and (3.4) with the corresponding solutions of (2.6) stated in (2.7), we get the following families of solutions:

Set 1.4.1. In case of $\Omega > 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,4,1}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(-\frac{\psi}{\sqrt{-\varrho^2+\Omega}} + \frac{\left(\frac{\sqrt{\Omega}\left(\chi_1 \sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right) + \chi_2 \cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}{\chi_1 \cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right) + \chi_2 \sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)} - \psi \right)}{\sqrt{-\varrho^2+\Omega}} \right). \tag{3.36}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$

$$F_{1,4,2}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(-\frac{\psi}{\sqrt{-\varrho^2+\Omega}} + \frac{\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\theta\right) - \frac{1}{2}\psi}{\sqrt{-\varrho^2+\Omega}} \right). \tag{3.37}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,4,3}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(-\frac{\psi}{\sqrt{-\varrho^2+\Omega}} + \frac{\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\theta\right) - \frac{1}{2}\psi}{\sqrt{-\varrho^2+\Omega}} \right). \tag{3.38}$$

Set 1.4.2. In case of $\Omega < 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,4,4}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(-\frac{\psi}{\sqrt{-\varrho^2+\Omega}} + \frac{\left(\frac{\sqrt{-\Omega}\left(-\chi_1 \sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right) + \chi_2 \cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}{\chi_1 \cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right) + \chi_2 \sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)} - \psi \right)}{\sqrt{-\varrho^2+\Omega}} \right). \tag{3.39}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{1,4,5}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(-\frac{\psi}{\sqrt{-\varrho^2+\Omega}} + \frac{\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\theta\right) - \frac{1}{2}\psi}{\sqrt{-\varrho^2+\Omega}} \right). \tag{3.40}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,4,6}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(-\frac{\psi}{\sqrt{-\varrho^2+\Omega}} + \frac{-\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\theta\right) - \frac{1}{2}\psi}{\sqrt{-\varrho^2+\Omega}} \right). \tag{3.41}$$

Set 1.4.3. In case of $\Omega = 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{1,4,7}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(-2\,\frac{\psi\,\theta\chi_2 + \psi\chi_1 - \chi_2}{(\theta\chi_2 + \chi_1)\,i\varrho} \right). \tag{3.42}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{1,4,8}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(-2\,\frac{\psi\,\theta - 1}{\theta\,i\varrho} \right). \tag{3.43}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{1,4,9}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(-2\,\frac{\psi}{i\varrho} \right). \tag{3.44}$$

In the above solutions, $\theta = x + y - at$.

3.2. Generalized $(r + \frac{G'}{G})$-expansion method

To generate solitary wave solutions for Eq (1.3), we now address (3.3) using the generalized $(r+\frac{G'}{G})$-expansion approach. This approach suggests the following closed form solution for (3.3) when we replace σ in (2.4) by 1:

$$\zeta(\theta) = \sum_{i=-1}^{1} S_i(r + \frac{G'(\theta)}{G(\theta)})^i. \tag{3.45}$$

An expression in $(\frac{G'(\theta)}{G(\theta)})$ is obtained by putting (3.45) in (3.3) with the help of Eq (2.6) and collecting all the terms with the same power of $(\frac{G'(\theta)}{G(\theta)})$. An algebraic system of equations is produced when all of the expression's coefficients are set to zero. The following four cases of solutions for the pertinent parameters are produced by using Maple to solve the resulting problem:

Case 2.1.

$$S_0 = \sqrt{\frac{\Omega}{-\varrho^2 + 4\,\Omega}}, S_1 = 2\,\frac{1}{\sqrt{-\varrho^2 + 4\,\Omega}}, S_{-1} = 0, c = c, k = 2\,\Omega - \frac{1}{2}\,\varrho^2, a = c + 3\,\sqrt{\Omega}, r = \frac{1}{2}\,\psi. \tag{3.46}$$

Case 2.2.

$$S_0 = 0, S_1 = 0, S_{-1} = -\frac{1}{2}\,\frac{\Omega}{\sqrt{-\varrho^2 + \Omega}}, a = c, c = c, k = \frac{1}{2}\,\Omega - \frac{1}{2}\,\varrho^2, r = \frac{1}{2}\,\psi. \tag{3.47}$$

Case 2.3.

$$S_0 = 0, S_1 = 2\,\sqrt{(-\varrho^2 - 2\,\Omega)^{-1}}, S_{-1} = -\frac{1}{2}\,\frac{\Omega}{\sqrt{-\varrho^2 - 2\,\Omega}}, a = c, c = c, k = -\Omega - \frac{1}{2}\,\varrho^2, r = \frac{1}{2}\,\psi. \tag{3.48}$$

Case 2.4.

$$\begin{aligned} &S_0 = 2\,\sqrt{\frac{\Omega}{-\varrho^2 + 16\,\Omega}}, S_1 = 2\,\frac{1}{\sqrt{-\varrho^2 + 16\,\Omega}}, S_{-1} = \frac{1}{2}\,\frac{\Omega}{\sqrt{-\varrho^2 + 16\,\Omega}},\\ &c = c, k = 8\,\Omega - \frac{\varrho^2}{2}, a = c + 6\,\sqrt{\Omega}, r = \frac{\psi}{2}\,. \end{aligned} \tag{3.49}$$

Where $\Omega = \psi^2 - 4\tau$.

Assuming Case. 2.1 and using Eqs (3.1) and (3.45) with the corresponding solutions of (2.6) stated in (2.7), we get the following families of solutions:

Set 2.1.1. In case of $\Omega > 0$,

(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,1,1}(t, x, y) = \mathrm{e}^{\varrho \mathrm{S(t)} - \frac{\varrho^2 \mathrm{t}}{2}}\left(\sqrt{\frac{\Omega}{-\varrho^2 + 4\,\Omega}} + \frac{\sqrt{\Omega}\left(\chi_1 \sinh\left(\frac{1}{2}\,\sqrt{\Omega}\theta\right) + \chi_2 \cosh\left(\frac{1}{2}\,\sqrt{\Omega}\theta\right)\right)}{\sqrt{-\varrho^2 + 4\,\Omega}\left(\chi_1 \cosh\left(\frac{1}{2}\,\sqrt{\Omega}\theta\right) + \chi_2 \sinh\left(\frac{1}{2}\,\sqrt{\Omega}\theta\right)\right)}\right). \tag{3.50}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$

$$F_{2,1,2}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(\sqrt{\frac{\Omega}{-\varrho^2 + 4\Omega}} + \frac{\sqrt{\Omega} \coth\left(\frac{1}{2}\sqrt{\Omega}\theta\right)}{\sqrt{-\varrho^2 + 4\Omega}} \right). \tag{3.51}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{2,1,3}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(\sqrt{\frac{\Omega}{-\varrho^2 + 4\Omega}} + \frac{\sqrt{\Omega} \tanh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)}{\sqrt{-\varrho^2 + 4\Omega}} \right). \tag{3.52}$$

Set 2.1.2. In case of $\Omega < 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,1,4}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(\sqrt{\frac{\Omega}{-\varrho^2 + 4\Omega}} + \frac{\sqrt{-\Omega}\left(-\chi_1 \sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right) + \chi_2 \cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}{\sqrt{-\varrho^2 + 4\Omega}\left(\chi_1 \cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right) + \chi_2 \sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)} \right). \tag{3.53}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{2,1,5}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(\sqrt{\frac{\Omega}{-\varrho^2 + 4\Omega}} + \frac{\sqrt{-\Omega} \cot\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)}{\sqrt{-\varrho^2 + 4\Omega}} \right). \tag{3.54}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{2,1,6}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(\sqrt{\frac{\Omega}{-\varrho^2 + 4\Omega}} - \frac{\sqrt{-\Omega} \tan\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)}{\sqrt{-\varrho^2 + 4\Omega}} \right). \tag{3.55}$$

Set 2.1.3. In case of $\Omega = 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,1,7}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(2 \frac{\chi_2}{(\theta\chi_2 + \chi_1)\, i\varrho} \right). \tag{3.56}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{2,1,8}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(2 \frac{1}{\theta\, i\varrho} \right). \tag{3.57}$$

In the above solutions, $\theta = x + y - ct$.

Assuming Case 2.2 and using Eqs (3.1) and (3.45) with the corresponding solutions of (2.6) stated in (2.7), we get the following families of solutions:
Set 2.2.1. In case of $\Omega > 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,2,1}(t,x,y) = e^{\varrho S(t) - \frac{\varrho^2 t}{2}} \left(- \frac{\sqrt{\Omega}\left(\chi_1 \cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right) + \chi_2 \sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}{\sqrt{-\varrho^2 + \Omega}\left(\chi_1 \sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right) + \chi_2 \cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)} \right). \tag{3.58}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$

$$F_{2,2,2}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(-\frac{\sqrt{\Omega}}{\sqrt{-\varrho^2+\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\theta\right)}\Big). \tag{3.59}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{2,2,3}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(-\frac{\sqrt{\Omega}}{\sqrt{-\varrho^2+\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)}\Big). \tag{3.60}$$

Set 2.2.2. In case of $\Omega < 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,2,4}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(-\frac{\Omega\left(\chi_1\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}{\sqrt{-\varrho^2+\Omega}\sqrt{-\Omega}\left(-\chi_1\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}\Big). \tag{3.61}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{2,2,5}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(-\frac{\Omega}{\sqrt{-\varrho^2+\Omega}\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)}\Big). \tag{3.62}$$

(iii) In case of $\chi_1 \neq 0, \chi_2 = 0$:

$$F_{2,2,6}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(\frac{\Omega}{\sqrt{-\varrho^2+\Omega}\sqrt{-\Omega}\tan\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)}\Big). \tag{3.63}$$

In the above solutions, $\theta = x + y - ct$.

Assuming Case. 2.3 and using Eqs (3.1) and (3.45) with the corresponding solutions of (2.6) stated in (2.7), we get the following families of solutions:
Set 2.3.1. In case of $\Omega > 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$\begin{aligned} F_{2,3,1}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(&\frac{-\sqrt{\Omega}\left(\chi_1\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}{\sqrt{-\varrho^2-2\Omega}\left(\chi_1\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)} \\ &+\frac{\sqrt{\frac{1}{(-\varrho^2-2\Omega)}}\sqrt{\Omega}\left(\chi_1\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}{\chi_1\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)}\Big). \end{aligned} \tag{3.64}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$

$$F_{2,3,2}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Big(-\frac{\sqrt{\Omega}}{\sqrt{-\varrho^2-2\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\theta\right)}+\sqrt{\frac{1}{(-\varrho^2-2\Omega)}}\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\Big). \tag{3.65}$$

Set 2.3.2. In case of $\Omega < 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,3,3}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Bigg(\frac{-\Omega\left(\chi_1\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}{\sqrt{-\varrho^2-2\Omega}\sqrt{-\Omega}\left(-\chi_1\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)} + \frac{\sqrt{\frac{1}{(-\varrho^2-2\Omega)}}\sqrt{-\Omega}\left(-\chi_1\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}{\chi_1\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)}\Bigg). \tag{3.66}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{2,3,4}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(-\frac{\Omega}{\sqrt{-\varrho^2-2\Omega}\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)} + \sqrt{\frac{1}{(-\varrho^2-2\Omega)}}\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right). \tag{3.67}$$

Set 2.3.3. In case of $\Omega = 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,3,5}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(\frac{2i\chi_2}{\varrho(\theta\chi_2+\chi_1)}\right). \tag{3.68}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{2,3,6}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(\frac{2i}{\varrho\theta}\right). \tag{3.69}$$

In the above solutions, $\theta = x + y - ct$.

Assuming Case. 2.4 and using Eqs (3.1) and (3.45) with the corresponding solutions of (2.6) stated in (2.7), we get the following families of solutions:
Set 2.4.1. In case of $\Omega > 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,4,1}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Bigg(\frac{\sqrt{\Omega}\left(\chi_1\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}{\sqrt{-\varrho^2+16\Omega}\left(\chi_1\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)} + 2\sqrt{\frac{\Omega}{-\varrho^2+16\Omega}} + \frac{\sqrt{\Omega}\left(\chi_1\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}{\sqrt{-\varrho^2+16\Omega}\left(\chi_1\cosh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)+\chi_2\sinh\left(\frac{1}{2}\sqrt{\Omega}\theta\right)\right)}\Bigg). \tag{3.70}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$

$$F_{2,4,2}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(\frac{\sqrt{\Omega}}{\sqrt{-\varrho^2+16\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\theta\right)} + 2\sqrt{\frac{\Omega}{-\varrho^2+16\Omega}} + \frac{\sqrt{\Omega}\coth\left(\frac{1}{2}\sqrt{\Omega}\theta\right)}{\sqrt{-\varrho^2+16\Omega}}\right). \tag{3.71}$$

Set 2.4.2. In case of $\Omega < 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,4,3}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Bigg(\frac{\Omega\left(\chi_1\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}{\sqrt{-\varrho^2+16\,\Omega}\sqrt{-\Omega}\left(-\chi_1\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)} + 2\sqrt{\frac{\Omega}{-\varrho^2+16\,\Omega}} + \frac{\sqrt{-\Omega}\left(-\chi_1\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}{\sqrt{-\varrho^2+16\,\Omega}\left(\chi_1\cos\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)+\chi_2\sin\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)\right)}\Bigg). \tag{3.72}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{2,4,4}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\Bigg(\frac{\Omega}{\sqrt{-\varrho^2+16\,\Omega}\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)} + 2\sqrt{\frac{\Omega}{-\varrho^2+16\,\Omega}} + \frac{\sqrt{-\Omega}\cot\left(\frac{1}{2}\sqrt{-\Omega}\theta\right)}{\sqrt{-\varrho^2+16\,\Omega}}\Bigg). \tag{3.73}$$

Set 2.4.3. In case of $\Omega = 0$,
(i) In case of $\chi_1 \neq 0, \chi_2 \neq 0$:

$$F_{2,4,5}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(\frac{2\chi_2}{i\varrho\left(\theta\chi_2+\chi_1\right)}\right). \tag{3.74}$$

(ii) In case of $\chi_1 = 0, \chi_2 \neq 0$:

$$F_{2,4,6}(t,x,y) = e^{\varrho S(t)-\frac{\varrho^2 t}{2}}\left(\frac{2}{i\varrho\theta}\right). \tag{3.75}$$

In the above solutions, $\theta = x + y - ct$.

4. Graphical analysis

The visual behaviors of the numerous solitary waves discovered in the model under investigation are shown in the current section of the research. We were able to extract and display wave patterns for the explicit solitary waves in three-dimensional representations by using the recommended method. Comprehending the behavior of connected physical processes requires an understanding of these solitary wave structures. A solitary wave is a non-dispersion and localized waveform that maintains its speed and shape while propagating. In the setting of the SCIE, the solitary waves result from the action of the nonlinear term that promotes the formation of localized structures and the diffusion term that prevents the localized structures from spreading. The profiles of the wave may be more complicated when the stochastic effects are considered, showing changes in the amplitude, shape, or stability. On the other hand, a powerful, self-propagating wave forms that keeps its form and speed as it travels across a nonlinear medium is known as a conventional soliton, which is a special type of solitary waves. When the solutions of the resultant systems are analyzed using the Maple, several explicit solutions exhibiting kink, anti-kink, dark, soliton, and bright solitary wave solutions in the form of rational, trigonometric, and hyperbolic functions are produced. We provide adaptive perspectives on the dynamics of the stochastic processes dampening the solitary wave forms in the underlying framework of the SCIE by integrating the Wiener process in the sinusoidal eigen-expansion sense. 3D graphs (Figures 1–8) are used to graphically depict the dynamics of the generated solitary wave solutions with and without Brownian motion. Waveforms composed of both bright and dark solitary waves are known as bright-dark solitary waves. A dark solitary wave has a drop in amplitude, whereas a bright solitary wave has a peak in amplitude. This mix structure is the consequence of the combined influence of cubic response dynamics and fractional diffusion. A kink solitary wave is a solitary wave that exhibits a monotonic transition between two distinct asymptotic states due to fractional diffusion transportation, which permits a smooth unstable connection between the two stationary equilibria of the reaction term. Finally, an anti-kink is a kink solitary wave that travels in opposite to the direction of kink solitary wave. Additionally, the solitary waves remain smooth when the noise is zero; when the noise is nonzero, the solitary waves show minor amplitude fluctuations without changing the fundamental structure. In other words, the noise term only contributes perturbative aberrations rather than causing the wave to become unstable. These behaviors show that the solitary wave solutions are stochastically stable, which means that even in the presence of random disturbances, the intrinsic wave pattern endures even though the tiniest details change according to the noise level.

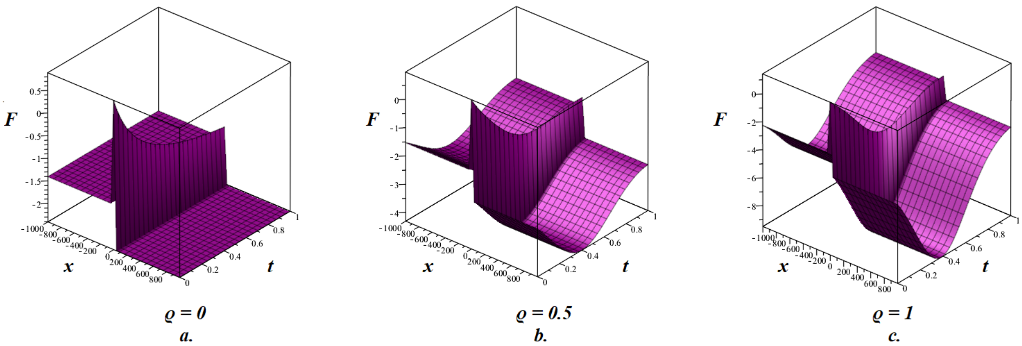

Figure 1. The depiction of kink solution $F_{1,1,1}(t, x, y)$ given in (3.9) under zero and nonzero noises ϱ at $\psi = 0.9, \tau = 0.1, y = 1, a = 0.001, \chi_1 = 1, \chi_2 = 2$.

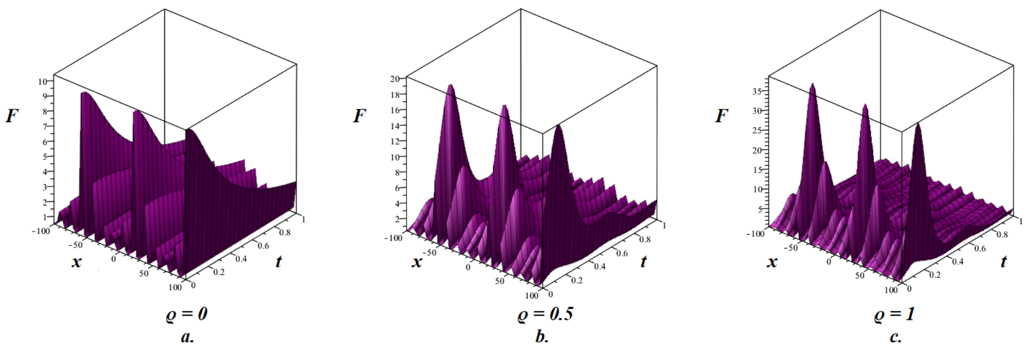

Figure 2. The depiction of bright solution $F_{1,2,4}(t, x, y)$ given in (3.21) under zero and nonzero noises ϱ at $\psi = 0.05, \tau = 0.04, y = 5, a = 0.003, \chi_1 = 2, \chi_2 = 1$.

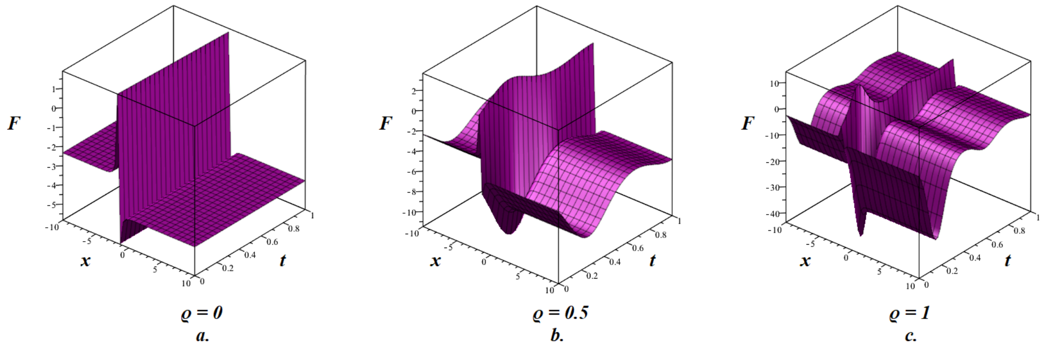

Figure 3. The depiction of kink solution $F_{1,3,2}(t, x, y)$ given in (3.28) under zero and nonzero noises ϱ at $\psi = 5, \tau = 4, y = 2, a = 0.005$.

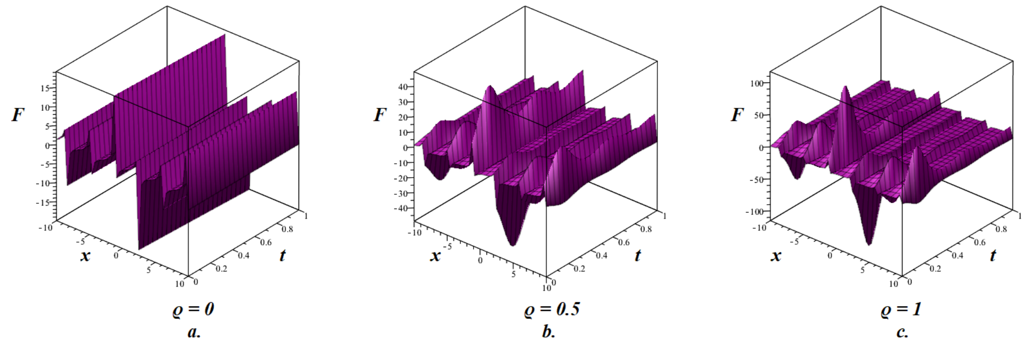

Figure 4. The depiction of bright-dark type solution $F_{1,4,6}(t, x, y)$ given in (3.41) under zero and nonzero noises ϱ at $\psi = 1, \tau = 1, y = 3, a = 0.004$.

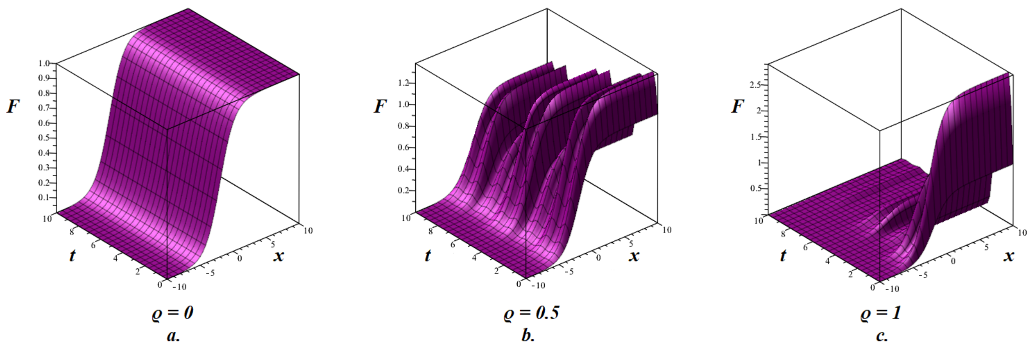

Figure 5. The depiction of anti-kink solution $F_{2,1,3}(t, x, y)$ given in (3.52) under zero and nonzero noises ϱ at $\psi = 0.95, \tau = 0.01, y = 2, c = 0.001$.

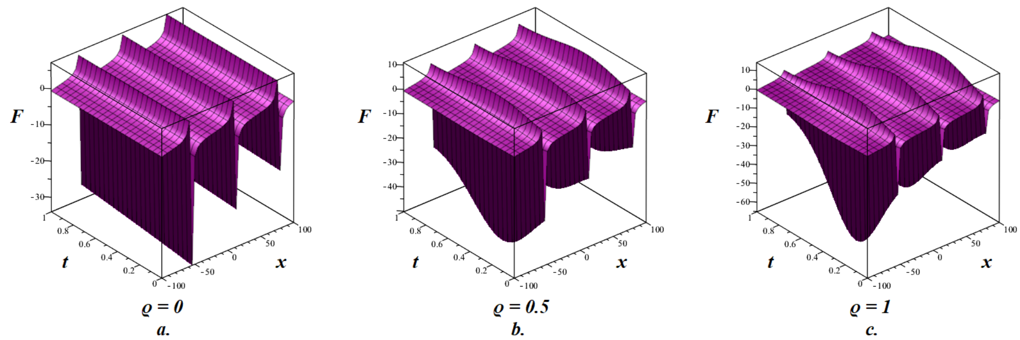

Figure 6. The depiction of periodic solution $F_{2,2,5}(t, x, y)$ given in (3.62) under zero and nonzero noises ϱ at $\psi = 0.05, \tau = 0.5, y = 3, c = 0.0055$.

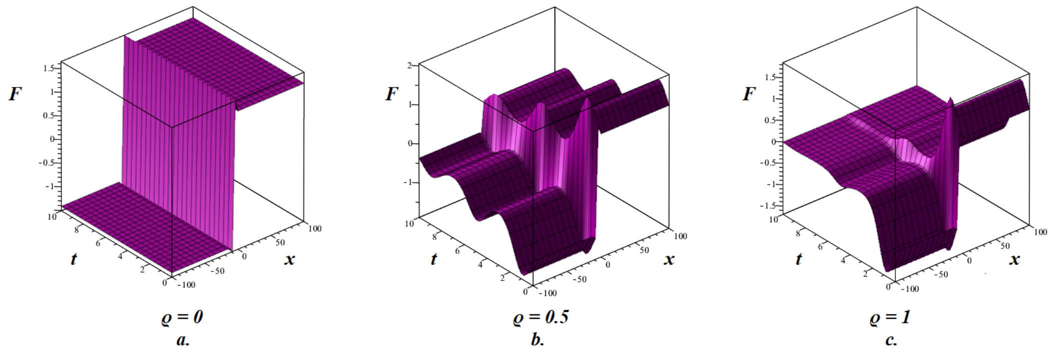

Figure 7. The depiction of anti-kink solution $F_{2,3,1}(t, x, y)$ given in (3.64) under zero and nonzero noises ϱ at $\psi = 0.75, \tau = 0.01, y = 5, c = 0.0015, \chi_1 = 3, \chi_2 = 1$.

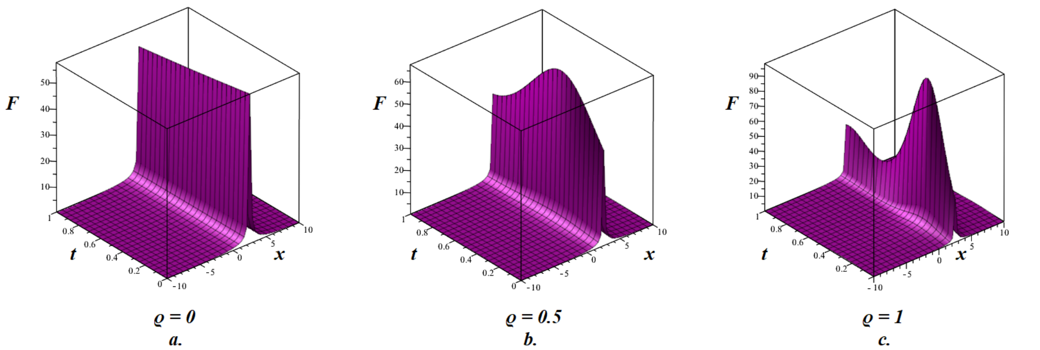

Figure 8. The depiction of bright solution $F_{2,4,3}(t, x, y)$ given in (3.72) under zero and nonzero noises ϱ at $\psi = 0.005, \tau = 0.005, y = 4, c = 0.0085, \chi_1 = 2, \chi_2 = 1$.

5. Conclusions

In conclusion, this exploration established and analyzed new classes of solitary wave solutions for SCIE with Wiener process. The effective (G'/G)-expansion method with its version generalized

$(r + G'/G)$-expansion method are used to get explicit solitary wave solutions of the desired SCIE. The strategic (G'/G) -expansion strategy employed wave transformation to convert the SCIE in to a NODE, which was then transformed into a system of nonlinear algebraic equations assuming finite series-form solutions. On analyzing the solutions of the resultant system with the Maple, several solitary wave solutions in the trigonometric, hyperbolic, and rational form were discovered. Several solitary wave solutions were assessed using illustrated 3D visualizations for specified parameter values under zero and nonzero noise effects in order to confirm the existence of solitary wave solutions, such as dark, bright, kink, anti-kink, and other solitary wave solutions in SCIEs. Our findings can be applied to further development of nonlinear science and physics and can find extensive implications across a variety of disciplines, including stochastic processes and nonlinear dynamics.

The study also indicated that the method of the (G'/G)-expansion is simple and efficient approach that gives a large number of solitary wave solutions that may be applied to diverse NPDEs in the mathematical sciences. The suggested simple algebraic ansatz avoids the use of complex processes, such as linearization, perturbation, and other processes involved in the transformation that may be required in the other techniques. Furthermore, the approaches' shortcomings lie in the failure of homogeneous balancing principle, which is necessary to construct solitary wave solutions by balancing nonlinearity with dispersion. Moreover, the $(r + G'/G)$-expansion method is the generalization of (G'/G)-expansion method. Thus by setting $r = 0$, the results of (G'/G)-expansion method can be achieved from the results of $(r + G'/G)$-expansion method. However, we attain new solutions by $(r + G'/G)$-expansion as opposed to (G'/G)-expansion method when S_{-1} in (55) is nonzero and may get the same solutions if $S_{-1} = 0$. Consequently, the solutions of $(r + G'/G)$-expansion method obtained under the Case 2.1 can be achieved by (G'/G)-expansion method, but the solutions under Cases 2.2–2.4 are novel and cannot be acquired by (G'/G)-expansion method unless $r = 0$. It is also notable that the condition $\Omega < 0$ (occur in a few families of solutions) causes negative radicands in the analytical manifestations in some instances, introducing the imaginary unit ι, which makes the solitary wave solutions complex valued. Complex valued solitary wave solutions are mathematically admissible and physically significant in stochastic and nonlinear dynamical systems, where they can be interpreted as phase-modulated, oscillatory, or internal-mode wave structures as opposed to real-valued amplitude profiles.

The future objective of the research is to study the given model with some noise effects in the sense of a fractional derivatives.

Author contributions

Both authors contributed equally in preparing the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. S. A. El-Tantawy, H. A. Alyousef, R. T. Matoog, R. Shah, On the optical soliton solutions to the fractional complex structured (1+ 1)-dimensional perturbed Gerdjikov-Ivanov equation, *Phys. Scr.*, **99** (2024), 035249. https://doi.org/10.1088/1402-4896/ad241b
2. S. Noor, A. S. Alshehry, A. Shafee, R. Shah, Families of propagating soliton solutions for (3+ 1)-fractional Wazwaz-BenjaminBona-Mahony equation through a novel modification of modified extended direct algebraic method, *Phys. Scr.*, **99** (2024), 045230. https://doi.org/ 10.1088/1402-4896/ad23b0
3. M. M. Al-Sawalha, S. Noor, R. Shah, H. Yasmin, Unveiling solitary and twinning kink solitons in (2+ 1)-dimensional modified Zakharov-Kuznetsov equation arising in electronics, *Sci. Rep.*, **15** (2025), 42126. https://doi.org/10.1038/s41598-025-26017-w
4. N. Iqbal, W. W. Mohammed, A. E. Hamza, S. Hussain, Y. Jawarneh, R. Shah, Fractals and chaotic solitons phenomena in conformable coupled Higgs syste, *Discr. Dyn. Nature Soc.*, **2025** (2025), 8384630. https://doi.org/10.1155/ddns/8384630
5. M. M. Al-Sawalha, S. Noor, R. Shah, H. Yasmin, Chaotic analysis, hopf bifurcation and collision of optical periodic solitons in (2+ 1)-dimensional degenerated Biswas–Milovic equation with Kerr law of nonlinearity, *Sci. Rep.*, **15** (2025), 44322. https://doi.org/10.1038/s41598-025-30009-1
6. W. Alhejaili, R. Shah, A. H. Salas, S. Raut, S. Roy, A. Roy, et al., Unearthing the existence of intermode soliton-like solutions within integrable quintic Kundu-Eckhaus equation, *Rend. Fis. Acc. Lincei*, **36** (2025), 233–255. https://doi.org/ 10.1007/s12210-024-01288-z
7. M. A. Abdelrahman, A note on Riccati-Bernoulli sub-ODE method combined with complex transform method applied to fractional differential equations, *Nonlinear Eng.*, **7** (2018), 279–285. https://doi.org/10.1515/nleng-2017-0145
8. A. H. Abdel-Aty, New analytical solutions of Wick-Type stochastic schamel KdV equation via modified Khater method, *J. Inf. Sci. Eng.*, **36** (2020), 1279–1291. https://doi.org/10.6688/JISE.20201136(6).0010
9. M. S. Ullah, A. Abdeljabbar, H. O. Roshid, M. Z. Ali, Application of the unified method to solve the Biswas-Arshed model, *Res. Phys.*, **42** (2022), 105946. https://doi.org/10.1016/j.rinp.2022.105946
10. Y. Xiao, S. Barak, M. Hleili, K. Shah, Exploring the dynamical behaviour of optical solitons in integrable kairat-II and kairat-X equations, *Phys. Scr.*, **99** (2024), 095261. https://doi.org/10.1088/1402-4896/ad6e34

11. Z. Yan, J. Li, S. Barak, S. Haque, N. Mlaiki, Delving into quasi-periodic type optical solitons in fully nonlinear complex structured perturbed Gerdjikov-Ivanov equation, *Sci. Rep.*, **15** (2025), 8818. https://doi.org/10.1038/s41598-025-91978-x

12. H. Khan, S. Barak, P. Kumam, M. Arif, Analytical solutions of fractional Klein-Gordon and gas dynamics equations, via the $(\frac{G'}{G})$-expansion method, *Symmetry*, **11** (2019), 566. https://doi.org/10.3390/sym11040566

13. J. Zhang, X. Wei, Y. Lu, A generalized $(\frac{G'}{G})$-expansion method and its applications. *Phys. Lett. A*, **372** (2008), 3653–3658. https://doi.org/10.1016/j.physleta.2008.02.027

14. E. M. E. Zayed, K. A. Gepreel, The $(\frac{G'}{G})$-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics, *J. Math. Phys.*, **50** (2009), 013502. https://doi.org/10.1063/1.3033750

15. W. X. Ma, T. Huang, Y. Zhang, A multiple exp-function method for nonlinear differential equations and its application, *Phys. Scr.*, **82** (2010), 065003. https://doi.org/10.1088/0031-8949/82/06/065003

16. T. Muhammad, A. A. Hamoud, H. Emadifar, F. K. Hamasalh, H. Azizi, M. Khademi, Traveling wave solutions to the Boussinesq equation via Sardar sub-equation technique. *AIMS Math.*, **7** (2022), 11134–11149. https://doi.org/10.3934/math.2022623

17. A. Bekir, E. Aksoy, A. C. Cevikel, Exact solutions of nonlinear time fractional partial differential equations by sub-equation method, *Math. Meth. Appl. Sci.*, **38** (2015), 2779–2784. https://doi.org/10.1002/mma.3260

18. P. N. Ryabov, D. I. Sinelshchikov, M. B. Kochanov, Application of the Kudryashov method for finding exact solutions of the high order nonlinear evolution equations, *Appl. Math. Comput.*, **218** (2011), 3965–3972. https://doi.org/10.1016/j.amc.2011.09.027

19. B. K. Shivamoggi, Poincaré-lighthill-kuo method and non-linear hyperbolic waves, *J. Sound Vibr.*, **54** (1977), 603–605. https://doi.org/10.1016/0022-460X(77)90618-6

20. R. Ali, D. Kumar, A. Akguel, A. Altalbe, On the periodic soliton solutions for fractional Schrödinger equations, *Fractals*, **32** (2024), 2440033. https://doi.org/10.1142/S0218348X24400334

21. R. Ali, Z. Zhang, H. Ahmad, Exploring soliton solutions in nonlinear spatiotemporal fractional quantum mechanics equations: an analytical study, *Opt. Quant. Elect.*, **56** (2024), 838. https://doi.org/10.1007/s11082-024-06370-2

22. M. Bilal, J. Iqbal, R. Ali, F. A. Awwad, E. A. A. Ismail, Exploring families of solitary wave solutions for the fractional coupled Higgs system using modified extended direct algebraic method, *Fractal Fract.*, **7** (2023), 653. https://doi.org/10.3390/fractalfract7090653

23. A. M. Wazwaz, The Hirota's bilinear method and the tanh-coth method for multiple-soliton solutions of the Sawada-Kotera-Kadomtsev-Petviashvili equation, *Appl. Math. Comput.*, **200** (2008), 160–166. https://doi.org/10.1016/j.amc.2007.11.001

24. R. Ali, A. Ahmad, M. M. Alam, Investigating bifurcating dynamics of cnoidal solitons in complex fractional Maccari system, *Eng. Comput.*, 2025, 1–24. https://doi.org/10.1108/EC-11-2024-1018

25. Y. Shang, Y. Huang, W. Yuan, The extended hyperbolic functions method and new exact solutions to the Zakharov equations, *Appl. Math. Comput.*, **200** (2008), 110–122. https://doi.org/10.1016/j.amc.2007.10.059

26. R. Ali, R. Imtiaz, M. U. D. Junjua, F. A. Awwad, E. A. Ismail, A. S. Hendy, Fractal formation and chaotic soliton phenomena in nonlinear conformable Heisenberg ferromagnetic spin chain equation, *Open Phys.*, **23** (2025), 20250166. https://doi.org/10.1515/phys-2025-0166

27. R. Ali, S. Barak, A. Altalbe, Analytical study of soliton dynamics in the realm of fractional extended shallow water wave equations, *Phys. Scr.*, **99** (2024), 065235. https://doi.org/10.1088/1402-4896/ad4784

28. R. Ali, M. M. Alam, S. Barak, Exploring chaotic behavior of optical solitons in complex structured conformable perturbed Radhakrishnan-Kundu-Lakshmanan model, *Phys. Scr.*, **99** (2024), 095209. https://doi.org/10.1088/1402-4896/ad67b1

29. K. Suwais, N. Mlaiki, S. Barak, R. Ali, Novel quasi-periodic type optical solitons and the formation of fractal structures in non-integrable nonlinear Helmholtz equations with phase portraits and chaotic analysis, *European J. Pure Appl. Math.*, **18** (2025), 6764–6764. https://doi.org/10.29020/nybg.ejpam.v18i4.6764

30. A. Barbosa, J. P. Sena, N. Kacem, N. Bouhaddi, An artificial intelligence approach to design periodic nonlinear oscillator chains under external excitation with stable damped solitons, *Mech. Syst. Signal Proc.*, **205** (2023), 110879. https://doi.org/10.1016/j.ymssp.2023.110879

31. M. V. Flamarion, E. Pelinovsky, E. Didenkulova, Soliton dynamics in random fields: The Benjamin-Ono equation framework, *Appl. Math. Model.*, **144** (2025), 116092. https://doi.org/10.1016/j.apm.2025.116092

32. M. V. Flamarion, E. Pelinovsky, D. V. Makarov, Wave fields under the influence of a random-driven force: The Burgers equation, *Phys. Lett. A*, **527** (2024), 130000. https://doi.org/10.1016/j.physleta.2024.130000

33. S. H. Dong, A new approach to the relativistic Schrödinger equation with central potential: Ansatz method, *Int. J. Theoret. Phys.*, **40** (2001), 559–567. https://doi.org/10.1023/A:1004119928867

34. Y. S. Guo, W. Li, S. H. Dong, Gaussian solitary solution for a class of logarithmic nonlinear Schrödinger equation in (1+ n) dimensions, *Results Phys.*, **44** (2023), 106187. https://doi.org/10.1016/j.rinp.2022.106187

35. M. S. Rahaman, M. N. Islam, M. S. Ullah, Bifurcation, chaos, modulation instability, and soliton analysis of the Schrödinger equation with cubic nonlinearity, *Sci. Rep.*, **15** (2025), 11689. https://doi.org/10.1038/s41598-025-96327-6

36. S. T. Demiray, Y. Pandir, H. Bulut, New solitary wave solutions of Maccari system, *Ocean Eng.*, **103** (2015), 153–159. https://doi.org/10.1016/j.oceaneng.2015.04.037

37. J. He, S. Xu, K. Porsezian, Rogue waves of the Fokas-Lenells equation, *J. Phys. Soc. Japan*, **81** (2012), 124007. https://doi.org/10.1143/JPSJ.81.124007

38. N. K. Vitanov, A. Chabchoub, N. Hoffmann, Deep-water waves: On the nonlinear Schrödinger equation and its solutions, preprint paper, 2013. https://doi.org/10.48550/arXiv.1301.0990

39. K. Wang, K. Yan, F. Shi, G. Li, X. Liu, Qualitative study of the (2+ 1)-dimensional BLMPE equation: Variational principle, Hamiltonian and diverse wave solutions, *AIMS Math.*, **10** (2025), 26168–26186. https://doi.org/10.3934/math.20251152

40. M. Almatrafi, A. Alharbi, New soliton wave solutions to a nonlinear equation arising in plasma physics, *Comput. Model. Eng. Sci.*, **137** (2023), 827. https://doi.org/10.32604/cmes.2023.027344

41. M. S. Rahaman, M. S. Ullah, M. N. Islam, Novel dynamics of the nonlinear fractional soliton neuron model with sensitivity analysis, *AIP Adv.*, **15** (2025), 085116. https://doi.org/10.1063/5.0282276

42. Z. Alsheekhhussain, T. S. Alshammari, Y. Amadou, S. Alshammari, M. Alshammari, M. M. Al-sawalha, Dynamical behavior of kink solitons in nonlinear Chaffee-Infante equations with chaotic and bifurcation analysis, *Ain Shams Eng. J.*, **16** (2025), 103438. https://doi.org/10.1016/j.asej.2025.103438

43. M. A. Akbar, N. H. M. Ali, J. Hussain, Optical soliton solutions to the (2+1)-dimensional Chaffee–Infante equation and the dimensionless form of the Zakharov equation, *Adv. Differ. Equ.*, **2019** (2019), 446. https://doi.org/10.1186/s13662-019-2377-9

44. R. Sakthivel, C. Chun, New soliton solutions of Chaffee-Infante equations using the exp-function method, *Zeitschrift Naturforschung A*, **65** (2010), 197–202. https://doi.org/10.1515/zna-2010-0307

45. N. Chafee, E. F. Infante, A bifurcation problem for a nonlinear partial differential equation of parabolic type, *Appl. Anal.*, **4** (1974), 17–37. https://doi.org/ 10.1080/00036817408839081

46. T. Caraballo, H. Crauel, J. Langa, J. Robinson, The effect of noise on the Chafee-Infante equation: A nonlinear case study, *Proc. Amer. Math. Soc.*, **135** (2007), 373–382. https://doi.org/10.1090/S0002-9939-06-08593-5

47. H. Huang, R. Huang, Asymptotic behavior of solutions for the Chafee-infante equation, *Acta Math. Sci.*, **40** (2020), 425–441. https://doi.org/10.1007/s10473-020-0209-3

48. B. Oksendal, *Stochastic Differential Equations: An Introduction with Applications*, Berlin: Springer, 2013.

49. M. Z. Baber, M. W. Yasin, C. Xu, N. Ahmed, M. S. Iqbal, Numerical and analytical study for the stochastic spatial dependent prey-predator dynamical system, *J. Comput. Nonlinear Dyn.*, **19** (2024), 101003. https://doi.org/10.1115/1.4066038

50. K. Itô, *On Stochastic Differential Equations*, New York: American Mathematical Society, 1951. https://doi.org/10.1090/memo/0004

51. M. Z. Baber, N. Ahmed, C. Xu, M. S. Iqbal, T. A. Sulaiman, A computational scheme and its comparison with optical soliton solutions for the stochastic Chen-Lee-Liu equation with sensitivity analysis, *Modern Phys. Lett. B*, **39** (2025), 2450376. https://doi.org/10.1142/S0217984924503767

52. F. Vadillo, Comparing stochastic Lotka-Volterra predator-prey models, *Appl. Math. Comput.*, **360** (2019), 181–189. https://doi.org/10.1016/j.amc.2019.05.002

53. M. T. Islam, M. A. Akter, J. F. Gómez-Aguilar, M. A. Akbar, Novel and diverse soliton constructions for nonlinear space-time fractional modified Camassa-Holm equation and Schrodinger equation, *Opt. Quant. Elect.*, **54** (2022), 227. https://doi.org/10.1007/s11082-022-03602-1

54. S. Shreve, *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*, Berlin: Springer Science Business Media, 2005.

55. C. Kuehn, Travelling waves in monostable and bistable stochastic partial differential equations, *Jahresber. Dtsch. Math. Ver.*, **122** (2020), 73–107. https://doi.org/10.1365/s13291-019-00206-9

56. N. Bruti-Liberati, *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*, Berlin: Springer, 2007. https://doi.org/10.1007/978-3-642-13694-8

57. R. D. T. Strauss, F. Effenberger, A Hitch-hiker's Guide to stochastic differential equations: solution methods for energetic particle transport in space physics and astrophysics, *Space Sci. Rev.*, **212** (2017), 151–192. https://doi.org/10.1007/s11214-017-0351-y

58. C. Bauzet, *Stochastic PDEs, Finite-Volume Schemes and Applications in Mechanics*, Aix-en-Provence: Aix-Marseille Université, 2023.

59. S. Ahmad, S. F. Aldosary, M. A. Khan, Stochastic solitons of a short-wave intermediate dispersive variable (SIdV) equation, *AIMS Math.*, **9** (2024), 10717–10733. https://doi.org/10.3934/math.2024523

60. M. Z. Baber, N. Ahmed, C. Xu, M. S. Iqbal, T. A. Sulaiman, A computational scheme and its comparison with optical soliton solutions for the stochastic Chen-Lee-Liu equation with sensitivity analysis, *Modern Phys. Lett. B*, **39** (2025), 2450376. https://doi.org/10.1142/S0217984924503767

61. P. Albayrak, M. Ozisik, A. Secer, M. Bayram, S. E. Das, Optical solitons of stochastic perturbed Radhakrishnan-Kundu-Lakshmanan model with Kerr law of self-phase-modulation, *Modern Phys. Lett. B*, **38** (2024), 2450122. https://doi.org/10.1142/S0217984924501227

62. Z. Li, E. Zhu, Optical soliton solutions of stochastic Schrödinger-Hirota equation in birefringent fibers with spatiotemporal dispersion and parabolic law nonlinearity, *J. Opt.*, **53** (2024), 1302–1308. https://doi.org/10.1007/s12596-023-01287-7

63. S. Ahmad, N. Becheikh, L. Kolsi, T. Muhammad, Z. Ahmad, M. K. Nasrat, Uncovering the stochastic dynamics of solitons of the Chaffee-Infante equation, *Sci. Rep.*, **14** (2024), 19485. https://doi.org/10.1038/s41598-024-67116-4

64. A. Kamel, H. Y. Louati, K. Aldwoah, F. Alqarni, M. Almalahi, M. Hleili, Stochastic analysis and soliton solutions of the Chaffee-Infante equation in nonlinear optical media, *Boundary Value Probl.*, **2024** (2024), 119. https://doi.org/10.1186/s13661-024-01930-7

65. R. Ali, E. Tag-eldin, A comparative analysis of generalized and extended (G'/G)-expansion methods for travelling wave solutions of fractional Maccari's system with complex structure, *Alex. Eng. J.*, **79** (2023), 508–530. https://doi.org/10.1016/j.aej.2023.08.007

A. Appendix

This section presents term-by-term transformation of the SCIE to NODE using the stochastic wave transformation presented in Eq (3.1). Using Maple and applying the proposed transformation to the terms of Eq (1.3) we get:

$$
\begin{aligned}
F_t &= -\zeta'(\theta)ce^{\varrho S(t)-\frac{1}{2}\varrho^2 t} + \zeta(\theta)\left(\varrho\frac{d}{dt}S(t) - \frac{1}{2}\varrho^2\right)e^{\varrho S(t)-\frac{1}{2}\varrho^2 t},\\
F_x &= \zeta'(\theta)e^{\varrho S(t)-\frac{1}{2}\varrho^2 t},\\
F_{tx} &= -\zeta''(\theta)ce^{\varrho S(t)-\frac{1}{2}\varrho^2 t} + \zeta'(\theta)\left(\varrho\frac{d}{dt}S(t) - \frac{1}{2}\varrho^2\right)e^{\varrho S(t)-\frac{1}{2}\varrho^2 t},\\
F_{xx} &= \zeta''(\theta)e^{\varrho S(t)-\frac{1}{2}\varrho^2 t},\\
F_{xxx} &= \zeta'''(\theta)e^{\varrho S(t)-\frac{1}{2}\varrho^2 t},\\
F_{yy} &= \zeta''(\theta)e^{\varrho S(t)-\frac{1}{2}\varrho^2 t},
\end{aligned}
\tag{A.1}
$$

which are used in the transformation of Eq (1.3) to Eq (3.2). Moreover, using the fact that the expectation $\mathbb{E}(e^{\varrho S(t)-\frac{1}{2}\varrho^2 t}) = 1$ and $\mathbb{E}(ah(\theta)) = a\mathbb{E}(h(\theta))$ where a is a constant, we get the following expected values of the obtained terms:

$$
\begin{aligned}
&\mathbb{E}(\zeta'(\theta)e^{\varrho S(t)-\frac{1}{2}\varrho^2 t}) = \zeta'(\theta),\\
&\mathbb{E}(\zeta''(\theta)e^{\varrho S(t)-\frac{1}{2}\varrho^2 t}) = \zeta''(\theta),\\
&\mathbb{E}(\zeta'''(\theta)e^{\varrho S(t)-\frac{1}{2}\varrho^2 t}) = \zeta'''(\theta),\\
&\mathbb{E}(\frac{\varrho^2}{2}\zeta'(\theta)e^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}}) = \frac{\varrho^2}{2}\zeta'(\theta),\\
&\mathbb{E}(3k\zeta^2(\theta)(e^{\varrho \mathrm{S(t)}-\frac{\varrho^2 \mathrm{t}}{2}})^3\zeta'(\theta)) = 3k\zeta^2(\theta)\zeta'(\theta),
\end{aligned}
\tag{A.2}
$$

which are used in obtaining Eq (3.3) from Eq (3.2).