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Research article

Unit graphs of group rings

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Abstract: In this paper, we studied the unit graph $\Upsilon(RG)$ of a finite group ring RG, where two vertices $x, y \in RG$ were adjacent when $x + y$ was a unit. We established algebraic criteria for connectivity, described all connected components, and determined the diameter of $\Upsilon(RG)$ in terms of the semisimple quotient $RG/J(RG)$. We classified completely when the unit graph was a cycle or a complete graph, and we determined the girth of $\Upsilon(RG)$ from the field factorization of the quotient. The results revealed a tight correspondence between the additive structure generated by units and the global geometry of the unit graph.

Keywords: unit graph; group ring; connectivity; diameter; cycle classification; girth
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1. Introduction

The structure of a group ring RG reflects a delicate interaction between its additive and multiplicative components. While the multiplicative structure, especially the unit group $U(RG)$, has been extensively developed in the classical theory of group rings (see, for example, [1–3]), the behavior of units relative to the additive group $(RG, +)$ remains much less understood. A natural combinatorial framework for exploring this interaction is the *unit graph* $\Upsilon(RG)$, whose vertices are the elements of RG and where two vertices x, y are adjacent when

$$x + y \in U(RG).$$

This construction captures how multiplicative units are distributed across additive cosets and provides a geometric perspective on the internal arithmetic of the group ring.

For any finite ring R and finite group G, the group ring RG is a finite additive abelian group. A key structural principle, implicit in the theory of lifting units modulo, the Jacobson radical (see [2]), is that

many global features of $U(RG)$ are governed by the semisimple quotient $RG/J(RG)$. By Wedderburn theory, this quotient decomposes as a finite direct product of fields

$$RG/J(RG) \cong \prod_i \mathbb{F}_{q_i},$$

enabling global properties of $\Upsilon(RG)$ to be analyzed through explicit field–theoretic data.

Several graph constructions associated with rings illustrate how algebraic properties impose strong combinatorial constraints. Zero-divisor graphs [4] and unitary Cayley graphs [5] provide prominent examples in which graph invariants encode structural information about rings. Unit graphs of commutative rings have also been studied previously (see, for instance, [6]), where adjacency is, again defined by the sum of two elements being a unit. However, the passage from commutative rings to group rings introduces substantial new features: The presence of a nontrivial group basis, the augmentation structure, and the typically much larger and more intricate unit group. As a consequence, many arguments and results from the commutative setting do not extend directly to group rings and require fundamentally different techniques.

1.1. Contribution of the article

Our aim of this paper is to provide a systematic and unified study of the unit graph $\Upsilon(RG)$ for finite group rings. Our major contributions are summarized below, with explicit references to the corresponding results in the paper:

- In Theorem 4.1, we establish a complete and purely algebraic criterion for the connectivity of $\Upsilon(RG)$, showing that the graph is connected if and only if the additive subgroup generated by the unit group equals the whole ring.
- In Theorem 4.2, we describe all connected components of $\Upsilon(RG)$ explicitly as unions of cosets of the additive subgroup $\langle U(RG)\rangle_{\text{add}}$.
- In Theorem 5.5, we give a full classification of the diameter of $\Upsilon(RG)$, proving that only the values 2, 3, and ∞ can occur, depending solely on whether RG admits quotients isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- In Theorem 6.1, we characterize exactly when the unit graph $\Upsilon(RG)$ is a simple cycle and obtain a complete list of all finite group rings for which this occurs.
- In Theorem 7.3, we determine the girth of $\Upsilon(RG)$ in terms of the finite field decomposition of the semisimple quotient $RG/J(RG)$, yielding a complete characterization of when the unit graph is triangle-free.

These results are exact and exhaustive classification theorems rather than partial bounds or example-based observations. They demonstrate that the global graph-theoretic structure of $\Upsilon(RG)$ is rigidly controlled by algebraic properties of the underlying group ring.

1.2. Organization of the paper

The paper is organized as follows: In Section 2, we recall the algebraic and graph-theoretic preliminaries needed throughout the paper. In Section 3, we establish degree formulas and basic structural properties of $\Upsilon(RG)$. Connectivity and connected components are studied in Section 4.

Section 5 is devoted to the classification of the diameter of $\Upsilon(RG)$. In Section 6, we characterize cycle graphs arising from group rings, and in Section 7, we determine the girth of $\Upsilon(RG)$. We conclude with remarks on possible directions for future research.

2. Preliminaries

Throughout the paper, we work with finite rings that contain an identity element and finite groups. All graphs considered are simple and undirected unless explicitly stated otherwise. In this section, we recall the basic algebraic and graph-theoretic notions that will be needed later; all results stated here are standard.

2.1. Rings, group rings, and units

The basic theory of group rings used throughout this paper is standard. For background on group rings, augmentation, and units, we refer to [3, Chapter 1] and [7, Chapter 1].

Let R be a finite ring with identity 1, and let G be a finite group with identity element 1_G. The group ring RG consists of all formal sums

$$x = \sum_{g \in G} r_g g, \quad r_g \in R,$$

with componentwise addition and multiplication given by convolution. As each coefficient may be chosen independently, the ring RG has

$$|RG| = |R|^{|G|}$$

elements. The additive group of RG is denoted $(RG, +)$.

The group of units of a ring A is denoted by

$$U(A) = \{u \in A : \exists v \in A \text{ such that } uv = vu = 1\}.$$

For any subset $X \subseteq A$, we write $\langle X \rangle_{\text{add}}$ for the additive subgroup generated by X.

2.2. Jacobson radical and semisimple quotients

The Jacobson radical of a ring A is denoted by $J(A)$. We recall the following well-known facts (see, for example, [2, Chapter 4] or [8, Chapter VII]):

- If A is a finite ring, then $A/J(A)$ is semisimple.
- If A is finite and commutative, then

$$A/J(A) \;\cong\; \prod_{i=1}^{t} \mathbb{F}_{q_i},$$

  a direct product of finite fields.
- The canonical homomorphism $U(A) \to U(A/J(A))$ is surjective.

In this paper, whenever we use the above decomposition of $RG/J(RG)$ into a product of finite fields, we explicitly assume that R is commutative and G is abelian, so that the group ring RG is commutative.

2.3. Augmentation in modular group rings

Let F be a finite field of characteristic p and let G be a finite p-group. The structure of the group ring FG is well understood and plays an important role in several parts of this paper.

It is a classical result that FG is a local ring in this situation; see [3, Chapter 4] or [9]. The Jacobson radical of FG is given by

$$J(FG) = \ker(\varepsilon),$$

where the *augmentation map* $\varepsilon\colon FG \to F$ is defined by

$$\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g.$$

In particular, the quotient $FG/J(FG)$ is isomorphic to F, and the unit group of FG admits the description

$$U(FG) = \{\, u \in FG : \varepsilon(u) \neq 0 \,\}.$$

That is, an element of FG is a unit if and only if its augmentation is nonzero. This characterization will be used repeatedly in our analysis of adjacency, common neighbors, and diameter bounds in the unit graph.

More generally, for a finite group ring RG, the natural homomorphism

$$U(RG) \longrightarrow U(RG/J(RG))$$

is surjective; see, for example, [2, Chapter 4] or [3, Chapter 1]. This lifting property enables many graph-theoretic features of $\Upsilon(RG)$ to be studied via the semisimple quotient $RG/J(RG)$.

2.4. Basic graph-theoretic notions

We briefly recall the graph-theoretic terminology used throughout the paper; for general background, see, for example, [10, 11].

A (simple, undirected) graph is a pair

$$\Gamma = (V, E),$$

where V is a set of *vertices* and E is a set of unordered pairs $\{x, y\}$ of *distinct* vertices, called *edges*. If $\{x, y\} \in E$, we write $x \sim_\Gamma y$ (or simply $x \sim y$) and say that x and y are *adjacent*.

For a vertex $x \in V$, the *neighborhood* of x is

$$N_\Gamma(x) = \{\, y \in V :\ y \sim_\Gamma x \,\},$$

and the *degree* of x is

$$\deg_\Gamma(x) = |N_\Gamma(x)|.$$

A *path* in Γ from x to y is a sequence of vertices

$$x = v_0,\ v_1,\ \ldots,\ v_k = y \quad \text{such that} \quad v_{i-1} \sim_\Gamma v_i \ \ (1 \le i \le k).$$

The *length* of the path is k. If there exists a path from x to y, the *distance* between them is the length of a shortest such path and is denoted $\mathrm{dist}_\Gamma(x, y)$; otherwise, we set

$$\mathrm{dist}_\Gamma(x, y) = \infty.$$

A graph is *connected* if

$$\mathrm{dist}_\Gamma(x, y) < \infty$$

for all $x, y \in V$. The *connected component* of a vertex x is the (unique) maximal connected subgraph of Γ containing x.

The *diameter* of Γ is

$$\mathrm{diam}(\Gamma) = \sup_{x,y \in V} \mathrm{dist}_\Gamma(x, y),$$

and the *eccentricity* of a vertex x is

$$\mathrm{ecc}_\Gamma(x) = \sup_{y \in V} \mathrm{dist}_\Gamma(x, y).$$

A subset $D \subseteq V$ is called a *dominating set* if every vertex of V either belongs to D or is adjacent to at least one vertex of D.

A graph is *k-regular* if

$$\deg_\Gamma(x) = k$$

for all $x \in V$. It is *semi-regular* (or *biregular*) if V can be partitioned as

$$V = A \cup B$$

such that all vertices in A have the same degree and all vertices in B have the same degree (not necessarily equal to the degree in A).

A graph is *complete* if every pair of distinct vertices is adjacent. A graph is *bipartite* if V can be partitioned into two parts

$$V = V_0 \cup V_1$$

such that no edge has both endpoints in the same part.

A *cycle* of length n is a connected graph C_n on vertices $v_1, \ldots, v_n$ with edges $v_i \sim v_{i+1}$ for

$$1 \leq i \leq n - 1$$

and $v_n \sim v_1$. The *girth* of Γ is the length of its shortest cycle; if Γ contains no cycles we set

$$\mathrm{girth}(\Gamma) = \infty.$$

A graph Γ is called *triangle-free* if its girth is at least 4, that is, Γ contains no cycles of length 3.

For a graph

$$\Gamma = (V, E),$$

the *complement graph* Γ^c has the same vertex set V, and two distinct vertices x, y are adjacent in Γ^c if and only if they are *not* adjacent in Γ. If $W \subseteq V$, the *induced subgraph* $\Gamma[W]$ is the graph with vertex set W and all edges of Γ whose endpoints both lie in W.

A *perfect matching* is a collection of disjoint edges whose endpoints cover all vertices.

2.5. Difference sets and Cayley graphs

Let A be a ring and let $X \subseteq A$ be a subset. We define the *difference set* of X by

$$X - X = \{x - y : x, y \in X\}.$$

In particular, for the unit group of a group ring, we write

$$U(RG) - U(RG) = \{u - v : u, v \in U(RG)\}.$$

Such difference sets arise naturally in the study of additive properties of unit groups and play a central role in bounding distances in unit graphs (see, for example, [10, 11]).

Now, let $(A, +)$ be a finite abelian group and let $S \subseteq A$ satisfy

$$S = -S$$

and $0 \notin S$. The *Cayley graph* $\mathrm{Cay}(A, S)$ is the graph with vertex set A and adjacency relation

$$x \sim y \quad \Longleftrightarrow \quad y - x \in S.$$

Cayley graphs provide a natural bridge between algebraic generating sets and graph-theoretic structure, and they are fundamental objects in algebraic graph theory (see [10, 11]).

In particular, when

$$2 \cdot 1_{RG} \notin U(RG),$$

the unit graph $\Upsilon(RG)$ can be viewed as a Cayley sum graph on the additive group $(RG, +)$ with respect to the generating set $U(RG)$. This viewpoint is essential in our analysis of connectivity, diameter, and cycle structure.

3. Structural properties of the unit graph $\Upsilon(RG)$

Definition 3.1. *Let R be a ring and G a group. The* unit graph *of the group ring RG, denoted $\Upsilon(RG)$, is the simple graph whose vertices are the elements of RG, and for distinct $x, y \in RG$,*

$$x \sim y \quad \Longleftrightarrow \quad x + y \in U(RG).$$

The closed unit graph*, denoted $\overline{\Upsilon}(RG)$, is obtained by adding a loop at each vertex x, satisfying $2x \in U(RG)$.*

Lemma 3.2. *Let R be a ring and G a group. If*

$$2 \cdot 1_{RG} \notin U(RG),$$

then

$$\overline{\Upsilon}(RG) = \Upsilon(RG).$$

Proof. If a loop appears at $x \in RG$, then

$$x + x = 2x \in U(RG),$$

so

$$1_{RG} = (2x)y = (2 \cdot 1_{RG})(xy)$$

for some $y \in RG$, which shows

$$2 \cdot 1_{RG} \in U(RG).$$

Thus, if $2 \cdot 1_{RG}$ is not a unit, no loops occur. □

Remark 3.3. *If*

$$\mathrm{char}(R) = 2,$$

then

$$2 \cdot 1_{RG} = 0$$

is not a unit, so the open and closed unit graphs coincide. If $2 \cdot 1_{RG}$ is *a unit, then loops appear precisely at the units of RG.*

Proposition 3.4. *Let* R_1, R_2 *be finite rings and* G_1, G_2 *finite groups, such that*

$$R_1G_1 \cong R_2G_2$$

as rings. Then

$$\Upsilon(R_1G_1) \cong \Upsilon(R_2G_2).$$

Proof. A ring isomorphism φ: $R_1G_1 \to R_2G_2$ preserves addition and sends units to units. Thus,

$$x + y \in U(R_1G_1) \iff \varphi(x) + \varphi(y) \in U(R_2G_2),$$

so φ induces a graph isomorphism between the unit graphs. □

Proposition 3.5. *Let*

$$R = \mathbb{F}_q$$

be a finite field and let G_1, G_2 *be finite groups. If*

$$\Upsilon(RG_1) \cong \Upsilon(RG_2),$$

then:

(1) $|G_1| = |G_2|$*;*
(2) The groups G_1 *and* G_2 *need not be isomorphic.*

Proof. (1) Since

$$|RG| = q^{|G|},$$

the unit graph $\Upsilon(RG)$ has exactly $q^{|G|}$ vertices. If

$$\Upsilon(RG_1) \cong \Upsilon(RG_2),$$

then the graphs have the same number of vertices. Hence,

$$q^{|G_1|} = q^{|G_2|},$$

which implies

$$|G_1| = |G_2|.$$

(2) We show that isomorphic unit graphs do not determine the underlying group. Let

$$R = \mathbb{F}_2$$

and let $G \in \{C_4, V_4\}$. Both groups C_4 and V_4 are 2-groups of order 4. An element of the group ring $\mathbb{F}_2G$ can be written in the form

$$x = \sum_{g \in G} a_g g, \quad a_g \in \mathbb{F}_2.$$

Let

$$\varepsilon : \mathbb{F}_2G \longrightarrow \mathbb{F}_2$$

denote the augmentation map, defined by

$$\varepsilon(x) = \sum_{g \in G} a_g.$$

Since G is a 2-group and

$$\text{char}(\mathbb{F}_2) = 2,$$

the group ring $\mathbb{F}_2G$ is a local ring with Jacobson radical

$$J(\mathbb{F}_2G) = \ker(\varepsilon),$$

Moreover,

$$\mathbb{F}_2G/J(\mathbb{F}_2G) \cong \mathbb{F}_2.$$

Hence, an element $x \in \mathbb{F}_2G$ is a unit if and only if its image in the residue field is nonzero, that is,

$$x \in U(\mathbb{F}_2G) \Longleftrightarrow \varepsilon(x) = 1.$$

The vertex set of the unit graph $\Upsilon(\mathbb{F}_2G)$ is the additive group $(\mathbb{F}_2G, +)$, which has

$$2^{|G|} = 16$$

elements, and since an element

$$x \in \mathbb{F}_2G$$

is a unit if and only if

$$\varepsilon(x) = 1,$$

then the vertex set splits into two equal parts

$$A_0 = \{x \in \mathbb{F}_2G : \varepsilon(x) = 0\}, \quad A_1 = \{x \in \mathbb{F}_2G : \varepsilon(x) = 1\},$$

and each is of cardinality 8.

By definition of the unit graph, two vertices $x, y \in \mathbb{F}_2 G$ are adjacent if and only if

$$x + y \in U(\mathbb{F}_2 G),$$

which is equivalent to

$$\varepsilon(x + y) = 1.$$

Since

$$\varepsilon(x + y) = \varepsilon(x) + \varepsilon(y)$$

and the characteristic is 2, this occurs precisely when

$$\varepsilon(x) \neq \varepsilon(y).$$

Therefore:

- Every vertex in A_0 is adjacent to all vertices in A_1;
- There are no edges inside A_0 or inside A_1.

Hence, the unit graph $\Upsilon(\mathbb{F}_2 G)$ is the complete bipartite graph

$$\Upsilon(\mathbb{F}_2 G) \cong K_{8,8}.$$

This description depends only on the augmentation map and not on the internal group structure of G.

Consequently,

$$\Upsilon(\mathbb{F}_2 C_4) \cong K_{8,8} \cong \Upsilon(\mathbb{F}_2 V_4),$$

even though

$$C_4 \ncong V_4.$$

This completes the proof. □

Proposition 3.6. *Let R be a finite ring with identity and G a finite group. Then for every vertex $x \in RG$, the degree of x in the unit graph $\Upsilon(RG)$ is*

$$\deg(x) = \begin{cases} |U(RG)| - 1, & \text{if } 2x \in U(RG), \\ |U(RG)|, & \text{otherwise.} \end{cases}$$

Proof. Fix $x \in RG$ and define

$$A_x := \{-x + u \mid u \in U(RG)\}.$$

Then A_x is the set of all vertices $y \in RG$, such that

$$x + y \in U(RG).$$

The map

$$U(RG) \longrightarrow A_x, \qquad u \mapsto -x + u,$$

is a bijection, and hence

$$|A_x| = |U(RG)|.$$

If $2x \in U(RG)$, then $x \in A_x$, so x must be excluded when counting neighbors. In this case,

$$\deg(x) = |A_x| - 1 = |U(RG)| - 1.$$

Otherwise, $x \notin A_x$ and

$$\deg(x) = |A_x| = |U(RG)|.$$

This completes the proof. □

Corollary 3.7. *Let R and G be as above.*

(1) If $2 \cdot 1_{RG} \notin U(RG)$, then $\Upsilon(RG)$ is a $|U(RG)|$-regular graph.
(2) If $2 \cdot 1_{RG} \in U(RG)$, then $\Upsilon(RG)$ has exactly two degree classes:

$$\deg(x) = \begin{cases} |U(RG)| - 1, & x \in U(RG), \\ |U(RG)|, & x \notin U(RG). \end{cases}$$

Thus, $\Upsilon(RG)$ is semi-regular.

Proposition 3.8. *Let R be a finite ring and G a finite group. If*

$$\mathrm{char}(R) = 2,$$

then the unit graph $\Upsilon(RG)$ is vertex-transitive under the action of the additive group of RG.

Proof. For each $a \in RG$, define the translation map

$$\psi_a(x) = x + a.$$

Since addition in RG is a group operation, ψ_a is a bijection. If $x \sim y$, then

$$x + y \in U(RG)$$

and

$$\psi_a(x) + \psi_a(y) = (x + a) + (y + a) = x + y + 2a = x + y,$$

because

$$\mathrm{char}(R) = 2.$$

Thus, ψ_a preserves adjacency and is a graph automorphism. Translations act transitively on the vertex set, so $\Upsilon(RG)$ is vertex-transitive. □

Remark 3.9. *When*

$$\mathrm{char}(R) \neq 2,$$

translation need not preserve adjacency, and vertex-transitivity generally fails.

Theorem 3.10. *Let R be a finite ring with identity and G a finite group. Set*

$$n = |RG| \ \text{ and } \ u = |U(RG)|.$$

Then

$$|E(\Upsilon(RG))| = \frac{1}{2}\sum_{x\in RG}\begin{cases} u-1, & \text{if } 2x \in U(RG), \\ u, & \text{otherwise.}\end{cases}$$

In particular:

(1) If $2\cdot 1_{RG} \notin U(RG)$, then

$$|E(\Upsilon(RG))| = \frac{1}{2}\,n\,u.$$

(2) If $2\cdot 1_{RG} \in U(RG)$, then

$$|E(\Upsilon(RG))| = \frac{1}{2}(u(u-1) + (n-u)u).$$

Proof. By Proposition 3.6,

$$\deg(x) = \begin{cases} u-1, & 2x \in U(RG), \\ u, & \text{otherwise.}\end{cases}$$

The handshake lemma yields

$$2|E(\Upsilon(RG))| = \sum_{x\in RG}\deg(x),$$

giving the stated formula.

If

$$2\cdot 1_{RG} \notin U(RG),$$

then $2x \notin U(RG)$ for all x, so every vertex has degree u and

$$2|E| = nu.$$

If

$$2\cdot 1_{RG} \in U(RG),$$

then $2x \in U(RG)$ if $x \in U(RG)$, hence

$$2|E| = \sum_{x\in U(RG)}(u-1) + \sum_{x\notin U(RG)} u = u(u-1) + (n-u)u.$$

Dividing by 2 yields the result. □

Definition 3.11. *Let R be a finite ring and G a finite group. The* complement *of the unit graph $\Upsilon(RG)$, denoted $\Upsilon^c(RG)$, is the simple graph on the vertex set RG in which distinct vertices $x, y \in RG$ are adjacent exactly when they are not adjacent in $\Upsilon(RG)$; that is,*

$$x \sim_{\Upsilon^c} y \iff x \neq y \ \text{ and } \ x + y \notin U(RG).$$

Proposition 3.12. *Let R be a finite ring and G a finite group. For each vertex $x \in RG$,*

$$\deg_{\Upsilon^c}(x) = (|RG| - 1) - \deg_{\Upsilon}(x).$$

Consequently:

(1) If

$$2 \cdot 1_{RG} \notin U(RG),$$

then $\Upsilon^c(RG)$ is $(|RG| - 1 - |U(RG)|)$-regular.

(2) If

$$2 \cdot 1_{RG} \in U(RG),$$

then $\Upsilon^c(RG)$ is semi-regular with degree classes

$$\deg_{\Upsilon^c}(x) = \begin{cases} |RG| - |U(RG)|, & x \in U(RG), \\ |RG| - 1 - |U(RG)|, & x \notin U(RG). \end{cases}$$

Proof. Every vertex has at most $|RG| - 1$ possible neighbors. Exactly $\deg_{\Upsilon}(x)$ of these are adjacent in $\Upsilon(RG)$, and the remaining ones form its neighbors in the complement. Thus,

$$\deg_{\Upsilon^c}(x) = (|RG| - 1) - \deg_{\Upsilon}(x).$$

Substituting the degree values from Proposition 3.6 yields the two cases of the statement. □

setup 3.13. *Let F be a finite field of characteristic p, and let G be a finite p-group. Write*

$$q = |F|.$$

Then FG is a local ring with Jacobson radical

$$J(FG) = \ker(\varepsilon),$$

where ε: $FG \to F$ is the augmentation map, and

$$FG/J(FG) \cong F.$$

Thus, $u \in FG$ is a unit if and only if $\varepsilon(u) \neq 0$.

Proposition 3.14. *Let F be a finite field of order $q > 2$ and G a finite p-group. Then in the closed unit graph $\overline{\Upsilon}(FG)$, every two vertices $x, y \in FG$ have a common neighbor:*

$$N_{\overline{\Upsilon}(FG)}(x) \cap N_{\overline{\Upsilon}(FG)}(y) \neq \emptyset.$$

Proof. Since

$$FG/J(FG) \cong F,$$

an element v is a unit precisely when $\varepsilon(v) \neq 0$. We seek $z \in FG$ such that $x + z$ and $y + z$ are units, i.e.,

$$\varepsilon(x) + \varepsilon(z) \neq 0, \qquad \varepsilon(y) + \varepsilon(z) \neq 0.$$

Equivalently, $\varepsilon(z)$ must avoid the values $-\varepsilon(x)$ and $-\varepsilon(y)$. Because $q > 2$, we may choose $t \in F$ distinct from both. Select $z \in FG$ with

$$\varepsilon(z) = t \ \ (e.g., \quad z = t \cdot 1).$$

Then

$$\varepsilon(x + z) \neq 0 \ \text{ and } \ \varepsilon(y + z) \neq 0,$$

so $x + z$ and $y + z$ are units. Thus, z is a common neighbor in $\overline{\Upsilon}(FG)$. □

4. Connectivity criterion of the unit graph

We now establish a fundamental criterion describing when the unit graph $\Upsilon(RG)$ is connected in terms of the additive subgroup generated by its units.

Theorem 4.1. *Let R be a ring and G a group. Then the unit graph $\Upsilon(RG)$ is connected if and only if the additive subgroup of RG generated by its units equals the whole group ring:*

$$\langle U(RG)\rangle_{\text{add}} = RG.$$

Proof. Assume first that $\Upsilon(RG)$ is connected. Let $a \in RG$. Since $\Upsilon(RG)$ is connected, there exists a path

$$0 = x_0 \sim x_1 \sim \cdots \sim x_n = a,$$

and each edge satisfies $x_{i-1} + x_i \in U(RG)$. Setting

$$u_i = x_{i-1} + x_i,$$

we obtain an alternating sum expression

$$a = u_n - u_{n-1} + \cdots \pm u_1,$$

so a lies in the additive subgroup generated by the units.

Conversely, assume

$$\langle U(RG)\rangle_{\text{add}} = RG.$$

Write

$$a = s_1 - s_2 + \cdots \pm s_m$$

with $s_j \in U(RG)$. Define

$$y_0 = 0 \text{ and } y_j = s_j - y_{j-1}.$$

Then

$$y_{j-1} + y_j = s_j \in U(RG),$$

giving

$$0 = y_0 \sim y_1 \sim \cdots \sim y_m = a,$$

so $\Upsilon(RG)$ is connected. □

Theorem 4.2. *Let R be a ring and G a group. Write*

$$A = (RG, +)$$

and let

$$H = \langle U(RG)\rangle_{\text{add}} \leq A.$$

For each vertex $x \in RG$, denote by $\text{Comp}(x)$ *the connected component of x in $\Upsilon(RG)$. Then*

$$\text{Comp}(x) = (x + H) \cup (-x + H).$$

In particular, if

$$\operatorname{char}(RG) = 2,$$

then

$$-x = x \quad \textit{and} \quad \operatorname{Comp}(x) = x + H$$

for all x. Thus $\Upsilon(RG)$ *is connected if and only if*

$$H = RG.$$

Proof. Each edge $v \sim w$ satisfies $w - v \in U(RG)$. Following a path from 0 to x shows that any vertex reachable from 0 lies in $H \cup (-H)$, giving

$$\operatorname{Comp}(0) \subseteq H \cup (-H).$$

Translation by x yields

$$\operatorname{Comp}(x) \subseteq (x + H) \cup (-x + H).$$

Conversely, any $h \in H$ is an alternating sum of units, providing a path from 0 to h and from 0 to $-h$. Translation by x gives paths from x to $x + h$ and x to $-x + h$, proving equality. □

Corollary 4.3. *Let*

$$A = (RG, +), \quad H = \langle U(RG) \rangle_{\text{add}}, \quad \textit{and} \quad Q = A/H.$$

Let $\iota\colon Q \to Q$ *be the involution*

$$\iota(x + H) = -x + H.$$

Then the connected components of $\Upsilon(RG)$ *are the* ι*-orbits in* Q*. Hence*

$$\#\text{components} = \frac{1}{2}\Big([A : H] + \big|\{\, c \in Q : 2c = 0 \,\}\big|\Big).$$

In particular, if

$$\operatorname{char}(RG) = 2,$$

then

$$\#\text{components} = [A : H].$$

Proof. By Theorem 4.2, each connected component is

$$(x + H) \cup (-x + H),$$

corresponding to the ι-orbit of $x + H$ in Q. Fixed points of ι are those cosets c with

$$2c = 0.$$

Counting orbits yields the stated formula. □

5. Diameter of the unit graph $\Upsilon(RG)$

In this section, we determine the diameter of the unit graph $\Upsilon(RG)$. Our strategy proceeds through successive structural reductions: First to quotients, then to the Jacobson radical, and finally to the semisimple decomposition into finite fields. This yields a complete classification of the diameter in terms of the elementary quotient structure of RG.

Throughout, dist denotes the graph distance in $\Upsilon(RG)$.

Lemma 5.1. *Let φ: $RG \twoheadrightarrow S$ be a surjective ring homomorphism. Then the induced map*

$$\varphi : \Upsilon(RG) \longrightarrow \Upsilon(S)$$

is a graph homomorphism and satisfies

$$\mathrm{dist}_{\Upsilon(S)}(\varphi(x), \varphi(y)) \ \leq \ \mathrm{dist}_{\Upsilon(RG)}(x, y) \quad (x, y \in RG).$$

Proof. If $x \sim y$ in $\Upsilon(RG)$, then

$$x + y \in U(RG),$$

hence

$$\varphi(x) + \varphi(y) = \varphi(x + y) \in U(S),$$

so $\varphi(x) \sim \varphi(y)$. Since graph homomorphisms do not increase distances, the inequality follows. □

Lemma 5.2. *For $x, y \in RG$ the following are equivalent:*

$$\mathrm{dist}(x, y) \leq 2 \iff x - y \in U(RG) - U(RG).$$

Proof. If $x \sim z \sim y$, then there exist units $u, v \in U(RG)$, such that

$$x + z = u \ \text{ and } \ y + z = v,$$

giving

$$x - y = u - v.$$

Conversely, if

$$x - y = u - v$$

with $u, v \in U(RG)$, set

$$z = u - x.$$

Then

$$x + z = u \ \text{ and } \ y + z = v,$$

so $x \sim z \sim y$. □

Lemma 5.3. *Let*

$$J = J(RG)$$

be the Jacobson radical. Then

$$RG = U(RG) - U(RG) \quad \iff \quad RG/J = U(RG/J) - U(RG/J).$$

Moreover, the natural map $U(RG) \twoheadrightarrow U(RG/J)$ is surjective.

Proof. Surjectivity is classical: Every unit in RG/J lifts to a unit in RG.

If

$$RG = U(RG) - U(RG),$$

applying the quotient map gives

$$RG/J = U(RG/J) - U(RG/J).$$

Conversely, assume

$$x + J = u' - v'$$

with $u', v' \in U(RG/J)$. Lift u', v' to units $u, v \in U(RG)$. Then

$$x = u - v + j$$

for some $j \in J$, but $J \subseteq U(RG)$, so

$$x = (u + j) - v$$

is a difference of units. □

Lemma 5.4. *Let*

$$S = \prod_{i=1}^{t} \mathbb{F}_{q_i}.$$

Then:

$$U(S) - U(S) = S \iff \textit{no factor } \mathbb{F}_2 \textit{ occurs}.$$

If

$$S \simeq \mathbb{F}_2 \times T$$

with T having no $\mathbb{F}_2$-factor, then

$$U(S) - U(S) = \{0\} \times T.$$

If

$$S \simeq \mathbb{F}_2 \times \mathbb{F}_2,$$

then

$$U(S) = \{(1, 1)\}.$$

Proof. If $q_i > 2$, then

$$1, a - 1 \in U(\mathbb{F}_{q_i}) \text{ and } a = 1 + (a - 1)$$

expresses every $a \in \mathbb{F}_{q_i}$ as a difference of units. Thus,

$$U(S) - U(S) = S$$

whenever no $\mathbb{F}_2$-factor appears.

In $\mathbb{F}_2$, we have

$$U(\mathbb{F}_2) = \{1\} \text{ and } 1 - 1 = 0,$$

so no nonzero element is a difference of units. Componentwise calculation shows the remaining statements. □

Theorem 5.5. *Let R be a finite commutative ring and G a finite abelian group. Then:*

(1) If RG has no quotient isomorphic to $\mathbb{Z}_2$, then $\operatorname{diam}(\Upsilon(RG)) = 2$.
(2) If RG is local with residue field of order 2, then $\operatorname{diam}(\Upsilon(RG)) = 2$.
(3) If RG has a quotient $\mathbb{Z}_2$ but no quotient $\mathbb{Z}_2 \times \mathbb{Z}_2$, and is not local with residue field 2, then $\operatorname{diam}(\Upsilon(RG)) = 3$.
(4) If RG has a quotient $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Upsilon(RG)$ is disconnected.

Proof. (1) **No $\mathbb{Z}_2$ quotient.** Write

$$RG/J(RG) \simeq \prod_i \mathbb{F}_{q_i}.$$

No q_i equals 2, so Lemma 5.4 gives

$$RG/J(RG) = U(RG/J) - U(RG/J).$$

Lemma 5.3 implies

$$RG = U(RG) - U(RG).$$

By Lemma 5.2, diam = 2.

(2) **Local with residue field 2.** Let M be the maximal ideal. Then

$$U(RG) = 1 + M.$$

Every $x \in M$ satisfies

$$x + 1 \in U(RG),$$

hence $x \sim 1$ and $1 \sim 0$. Thus, every vertex is within distance ≤ 2 of every other.

(3) **One $\mathbb{Z}_2$ quotient, nonlocal.** Write

$$S = RG/J(RG) \simeq \mathbb{F}_2 \times T$$

with T having no $\mathbb{F}_2$-factor.

Lower bound. Lemma 5.4 gives

$$U(S) - U(S) = \{0\} \times T \neq S.$$

Thus, some vertices are at distance > 2 in S by Lemma 5.2; hence, diam ≥ 3 in $\Upsilon(RG)$ by Lemma 5.1.

Upper bound. We show

$$\operatorname{diam}(\Upsilon(S)) \leq 3.$$

Write

$$C_0 = \{0\} \times T, \qquad C_1 = \{1\} \times T.$$

Vertices in the same part differ by an element of

$$U(S) - U(S) = \{0\} \times T,$$

so they are at distance ≤ 2 by Lemma 5.2. For

$$a = (0, t) \in C_0 \text{ and } b = (1, s) \in C_1,$$

choose

$$z_1 = (1, t) \text{ and } z_2 = (0, s).$$

Then

$$a + z_1 = (1, 0) \in U(S), \quad z_1 + z_2 = (1, t + s) \in U(S), \quad z_2 + b = (1, 0) \in U(S),$$

so $a \sim z_1 \sim z_2 \sim b$ is a path of length 3. Thus, diam = 3 for RG.

(4) **A $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient.** Let

$$S = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Since

$$U(S) = \{(1, 1)\},$$

adjacency is $x \sim y$ iff

$$x + y = (1, 1).$$

Each vertex has exactly one neighbor, so $\Upsilon(S)$ is a perfect match and, hence, disconnected. Lemma 5.1 implies that $\Upsilon(RG)$ is disconnected. □

Proposition 5.6. *Assume* $2 \cdot 1_{RG} \notin U(RG)$ *and let*

$$H = \langle U(RG) \rangle_{\text{add}}$$

be the additive subgroup generated by the units. Then:

(1) If $H = RG$, *then* $\text{diam}(\Upsilon(RG)) \leq 2$.
(2) If $H \subsetneq RG$, *then every connected component has diameter* ≤ 2.
(3) The maps $x \mapsto x + h$ $(h \in H)$ *and* $x \mapsto -x$ *preserve adjacency and act transitively on each connected component.*

Proof. (1) If $H = RG$ then every element of RG is a sum of unit differences. Thus,

$$RG = U(RG) - U(RG)$$

and the two-step criterion (Lemma 5.2) gives diameter ≤ 2.

(2) If $H \subsetneq RG$, each component is an H-coset. Inside a coset $x + H$, every pair of vertices differs by an element of H, and every $h \in H$ is a sum of unit differences. Thus, any two vertices in the same component are at distance ≤ 2.

(3) Translation by $h \in H$ maps $x \sim y$ to $(x + h) \sim (y + h)$ since

$$(x + h) + (y + h) = x + y + 2h,$$

and $2h \notin U(RG)$ affects only adjacency when it annihilates units, which is excluded by hypothesis. Negation preserves adjacency since $x \sim y$ if $x + y \in U(RG)$, equivalently

$$(-x) + (-y) = -(x + y) \in U(RG).$$

These actions preserve components and act transitively on each coset. □

Proposition 5.7. *If $U(RG)$ is a dominating set in $\Upsilon(RG)$, then $\Upsilon(RG)$ is connected and*

$$\operatorname{diam}(\Upsilon(RG)) \leq 4,$$

and the eccentricity of 0 *is at most* 2*.*

Proof. If $U(RG)$ is dominating, every $x \in RG$ has a neighbor of the form $x \sim u_x$ with $u_x \in U(RG)$. Similarly, 0 has a neighbor $u_0 \in U(RG)$.

For any x, we have a path

$$x \sim u_x \sim u_0 \sim 0.$$

Thus, $\operatorname{dist}(x, 0) \leq 2$.

For $x, y \in RG$, use

$$x \sim u_x, \quad y \sim u_y, \quad u_x, u_y \in U(RG).$$

Since u_x and u_y are units, they differ by a difference of units. Hence,

$$\operatorname{dist}(u_x, u_y) \leq 2$$

by Lemma 5.2. Therefore,

$$\operatorname{dist}(x, y) \leq 1 + 2 + 1 = 4.$$

Thus, the graph is connected and has diameter at most 4. □

Proposition 5.8. *Let $\Upsilon[U(RG)]$ denote the unit-induced subgraph. If $U(RG)$ is dominating, then*

$$\operatorname{diam}(\Upsilon(RG)) \;\leq\; \min\Big\{4,\; 2 + \operatorname{diam}(\Upsilon[U(RG)])\Big\}.$$

Proof. For $x, y \in RG$, pick $u_x, u_y \in U(RG)$ with $x \sim u_x$ and $y \sim u_y$. Any path in $\Upsilon[U(RG)]$ from u_x to u_y has length at most $\operatorname{diam}(\Upsilon[U(RG)])$. Thus, every x, y satisfy

$$\operatorname{dist}(x, y) \leq 1 + \operatorname{diam}(\Upsilon[U(RG)]) + 1 = 2 + \operatorname{diam}(\Upsilon[U(RG)]).$$

Combining with Proposition 5.7 yields the stated bound. □

Corollary 5.9. *Let F be a finite field of order $q > 2$ and let G be a finite p-group with*

$$p = \operatorname{char}(F).$$

Then the closed unit graph $\overline{\Upsilon}(FG)$ satisfies

$$\operatorname{diam} \leq 2.$$

Proof. Since G is a p-group and

$$p = \operatorname{char}(F),$$

the group algebra FG is local with residue field F. Because $q > 2$, Lemma 5.4 shows that

$$F = U(F) - U(F).$$

Passing to the radical quotient and applying Lemma 5.3 yields

$$FG = U(FG) - U(FG).$$

By the two-step criterion (Lemma 5.2), every pair of vertices in the closed unit graph is at distance at most 2. □

6. Characterization of cycle graphs

Having studied the local and global structure of $\Upsilon(RG)$ through vertex degrees, edge counts, and complement graphs, we now determine precisely when the unit graph becomes a simple cycle. This turns out to be a very restrictive situation, occurring only for a few small group rings whose unit groups have exactly two elements.

Theorem 6.1. *Let R be a finite ring with identity and G a finite group. Write*

$$A = (RG, +) \text{ and } U = U(RG).$$

The unit graph $\Upsilon(RG)$ is a (connected) cycle graph $C_{|RG|}$ if and only if the following equivalent conditions hold:

(1) $2 \cdot 1_{RG} \notin U(RG)$.
(2) $|U(RG)| = 2$.
(3) The additive group $(RG, +)$ is generated by $U(RG)$, i.e. $\langle U(RG)\rangle_{\text{add}} = A$.

In this case, $\Upsilon(RG)$ is the Cayley graph $\text{Cay}((RG, +), U(RG))$ and is the simple cycle $C_{|RG|}$.

Proof. ($\Rightarrow$) Suppose $\Upsilon(RG)$ is a cycle. Then it is 2-regular and connected.

If

$$2 \cdot 1_{RG} \in U(RG),$$

then by Proposition 3.6 we have

$$\deg(x) = \begin{cases} |U(RG)| - 1, & x \in U(RG), \\ |U(RG)|, & x \notin U(RG), \end{cases}$$

so the degrees are not all equal unless

$$|RG| = 2,$$

a trivial case that we exclude. Hence,

$$2 \cdot 1_{RG} \notin U(RG).$$

By Proposition 3.6, every vertex then has degree $|U(RG)|$. Since $\Upsilon(RG)$ is a cycle, every vertex has degree 2, so

$$|U(RG)| = 2.$$

Write

$$U(RG) = \{u, v\}.$$

The adjacency condition $x \sim y$ if and only if $x + y \in U(RG)$ is equivalent to $y - x \in \{u, v\}$, so $\Upsilon(RG)$ is the Cayley graph $\text{Cay}(A, \{u, v\})$ on

$$A = (RG, +).$$

A Cayley graph on an abelian group with a symmetric generating set of size 2 is a disjoint union of cycles, each corresponding to a coset of the subgroup generated by $\{u, v\}$. Since $\Upsilon(RG)$ is connected, we must have

$$\langle U(RG)\rangle_{\text{add}} = A.$$

($\Leftarrow$) Conversely, assume the three conditions hold. Since

$$2 \cdot 1_{RG} \notin U(RG),$$

Proposition 3.6 gives

$$\deg(x) = |U(RG)| = 2$$

for every vertex x. The adjacency condition $x \sim y$ if and only if $y - x \in U(RG)$ shows that $\Upsilon(RG)$ is the Cayley graph $\mathrm{Cay}(A, U(RG))$. Because

$$\langle U(RG) \rangle_{\mathrm{add}} = A,$$

this Cayley graph is connected and 2-regular. Hence, a single cycle on the vertex set A, that is,

$$\Upsilon(RG) \cong C_{|A|} = C_{|RG|}.$$

This completes the proof. □

Corollary 6.2. *Let R be a finite ring with identity and G a finite group. If*

$$2 \cdot 1_{RG} \notin U(RG) \ \text{ and } \ |U(RG)| = 2,$$

then

$$\Upsilon(RG) = \mathrm{Cay}(A, U(RG))$$

is a disjoint union of cycles, each supported on a coset of the subgroup

$$\langle U(RG) \rangle_{\mathrm{add}} \leq A = (RG, +).$$

Moreover, $\Upsilon(RG)$ is a single cycle if and only if

$$\langle U(RG) \rangle_{\mathrm{add}} = A.$$

Proof. Under the hypotheses, the graph has no loops, and every vertex has degree 2, by Proposition 3.6. As in the proof of Theorem 6.1, $\Upsilon(RG)$ is isomorphic to the Cayley graph $\mathrm{Cay}(A, U)$ with

$$U := U(RG).$$

In $\mathrm{Cay}(A, U)$, any walk starting at $a \in A$ adds elements of U at each step. Hence, this remains in the coset $a + \langle U \rangle_{\mathrm{add}}$. Conversely, any element of $a + \langle U \rangle_{\mathrm{add}}$ is a finite sum of elements of U, and can, therefore, be reached from a by a path. Thus, the connected component of a is exactly the coset $a + \langle U \rangle_{\mathrm{add}}$, and we have a decomposition

$$\mathrm{Cay}(A, U) \ = \bigsqcup_{a \in A / \langle U \rangle_{\mathrm{add}}} \mathrm{Cay}(a + \langle U \rangle_{\mathrm{add}}, \, U).$$

Each component is a finite 2-regular simple graph, hence a cycle. There is only one component, and therefore a single cycle, when

$$\langle U \rangle_{\mathrm{add}} = A.$$

This completes the proof. □

Corollary 6.3. *The unit graph $\Upsilon(RG)$ is a connected cycle if and only if RG is isomorphic to one of the following three rings:*

$$\mathbb{Z}_4, \quad \mathbb{Z}_6, \quad \mathbb{F}_2[t]/(t^2) \ (\cong \mathbb{Z}_2 C_2).$$

In these cases,

$$U(\mathbb{Z}_4) = \{1, 3\}, \quad U(\mathbb{Z}_6) = \{1, 5\}, \quad U(\mathbb{F}_2[t]/(t^2)) = \{1, 1 + t\},$$

and

$$\Upsilon(RG) \cong C_{|RG|}.$$

Proof. By Theorem 6.1, we require

$$2 \cdot 1_{RG} \notin U(RG), \quad |U(RG)| = 2, \quad \text{and} \quad \langle U(RG) \rangle_{\text{add}} = (RG, +).$$

If $(RG, +)$ is cyclic, then

$$RG \cong \mathbb{Z}_n$$

for some $n \geq 1$. Having

$$|U(\mathbb{Z}_n)| = 2$$

means

$$\varphi(n) = 2,$$

so $n \in \{3, 4, 6\}$. In $\mathbb{Z}_3$, however, 2 is a unit, so $2 \cdot 1_{\mathbb{Z}_3} \in U(\mathbb{Z}_3)$ and this case is excluded. Thus, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ arise.

If $(RG, +)$ is not cyclic but

$$|U(RG)| = 2,$$

the smallest possibility is

$$|RG| = 4$$

with additive group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Among such rings, one checks that only $\mathbb{F}_2[t]/(t^2)$ has exactly two units, namely $\{1, 1 + t\}$, and that

$$\langle U(RG) \rangle_{\text{add}} = (RG, +)$$

holds in this case. No larger noncyclic additive groups can be generated by a set of two elements of size 2 and satisfy the conditions of Theorem 6.1, so these three rings are the only possibilities. Direct inspection shows that, in each case, $\Upsilon(RG)$ is a single cycle. □

Proposition 6.4. *Let R be a finite ring with identity and G a finite group. If*

$$2 \cdot 1_{RG} \in U(RG) \ \textit{and} \ |RG| > 2,$$

then $\Upsilon(RG)$ cannot be 2-regular and, therefore, cannot be a cycle.

Proof. When

$$2 \cdot 1_{RG} \in U(RG),$$

Proposition 3.6 shows that

$$\deg(x) = \begin{cases} |U(RG)| - 1, & x \in U(RG), \\ |U(RG)|, & x \notin U(RG), \end{cases}$$

so there are at least two distinct vertex degrees whenever $|RG| > 2$. Hence, the graph cannot be 2-regular, and cannot, in particular be a cycle. □

Corollary 6.5. *If $\Upsilon(RG)$ is a cycle, then it is the cycle $C_{|RG|}$ of length $|RG|$. Moreover, $\Upsilon(RG)$ is bipartite if and only if $|RG|$ is even; in particular, the graphs C_4 and C_6 arising from $\mathbb{Z}_4$, $\mathbb{Z}_6$, and $\mathbb{F}_2[t]/(t^2)$ are bipartite.*

Proof. If $\Upsilon(RG)$ is a cycle, then by Theorem 6.1, the three conditions above hold, and we know

$$\Upsilon(RG) \cong C_{|RG|}.$$

A cycle C_n is bipartite if and only if n is even, since a graph is bipartite exactly when it contains no odd cycle. Applying this with

$$n = |RG|$$

yields the stated criterion. In the explicit examples

$$RG = \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{F}_2[t]/(t^2),$$

we have $|RG| \in \{4, 6\}$, so the corresponding unit graphs are C_4 and C_6, and both are bipartite. □

Theorem 6.6. *Let R be a finite ring with identity and G a finite group. Then the unit graph $\Upsilon(RG)$ is a complete graph $K_{|RG|}$ if and only if every nonzero element of RG is a unit; equivalently, RG is a finite field.*

Proof. If $\Upsilon(RG)$ is complete, then for all distinct $x, y \in RG$, we have $x + y \in U(RG)$. Taking $y = 0$ gives $x \in U(RG)$ for all nonzero x, so every nonzero element is a unit, and RG is a field. Conversely, if RG is a field, then for $x \neq y$ we have $x + y \neq 0$, so $x + y \in U(RG)$ and every pair of distinct vertices is adjacent. Hence,

$$\Upsilon(RG) \cong K_{|RG|}.$$

This completes the proof. □

Proposition 6.7. *Let R be a finite ring with identity and G a finite group. The unit graph $\Upsilon(RG)$ has no edges (i.e., is empty) if and only if*

$$U(RG) = \{0\},$$

which is impossible for a unital ring. Hence, a nontrivial group ring RG never has an empty unit graph.

Proof. If $\Upsilon(RG)$ has no edges, then for all $x \neq y$, $x + y \notin U(RG)$. Taking $x = 0$ and $y \neq 0$ shows $y \notin U(RG)$, so

$$U(RG) = \{0\}.$$

However, every ring with identity has $1_{RG} \in U(RG)$, which is a contradiction. Thus, no such graph exists. □

7. Girth of the unit graph $\Upsilon(RG)$

In this section, we study short cycles in the unit graph $\Upsilon(RG)$ and determine when $\Upsilon(RG)$ is triangle-free. The key point is that the presence of a factor $\mathbb{F}_2$ in the semisimple quotient $RG/J(RG)$ forbids triangles, whereas in the absence of such a factor, one quickly finds 3-cycles, except in the special case of $\mathbb{F}_3$.

Throughout, let R be a finite commutative ring and G a finite abelian group, and set

$$S \;=\; RG/J(RG) \;\cong\; \prod_{i=1}^{t} \mathbb{F}_{q_i}$$

for prime powers q_i. We write

$$J = J(RG)$$

and note that, by Lemma 5.1, the natural projection $\pi\colon RG \twoheadrightarrow S$ induces a graph homomorphism $\Upsilon(RG) \to \Upsilon(S)$ that preserves adjacency, and triangles lift modulo J in finite rings.

We first analyze triangles in the unit graph of a finite product of fields.

Lemma 7.1. *Let*

$$S = \prod_{i=1}^{t} \mathbb{F}_{q_i}$$

and let $\Upsilon(S)$ be the unit graph on S (with adjacency $x \sim y$ iff $x + y \in U(S)$). Then:

(1) If some factor is $\mathbb{F}_2$, then $\Upsilon(S)$ is triangle-free.
(2) If $S \cong \mathbb{F}_3$, then $\Upsilon(S)$ is a path on three vertices and in particular has no cycles.
(3) If S has no $\mathbb{F}_2$ factor and $S \ncong \mathbb{F}_3$, then $\Upsilon(S)$ contains a triangle.

Proof. Write elements of S as

$$a = (a_i)_i,$$

and note that

$$a \sim b \iff a_i + b_i \neq 0 \text{ for all } i,$$

since a tuple is a unit if all coordinates are nonzero.

(1) Presence of an $\mathbb{F}_2$ factor. Suppose some coordinate i corresponds to

$$\mathbb{F}_2 = \{0, 1\}.$$

If a, b, and c are vertices forming a triangle, then, in that coordinate, we must have

$$a_i \neq b_i, \quad b_i \neq c_i, \quad c_i \neq a_i,$$

because $a_i + b_i$, $b_i + c_i$ and $c_i + a_i$ must all be nonzero. This is impossible in a 2-element set, so no triangles can occur in $\Upsilon(S)$.

(2) The case $S \cong \mathbb{F}_3$. In $\mathbb{F}_3$, we have

$$U(\mathbb{F}_3) = \{1, 2\} \text{ and } 1 + 2 = 0 \notin U(\mathbb{F}_3).$$

Thus,

$$0 \sim 1, \quad 0 \sim 2, \quad 1 \nsim 2.$$

So $\Upsilon(\mathbb{F}_3)$ is the path 0–1–2 on three vertices and has no cycles.

(3) No $\mathbb{F}_2$ factor and $S \ncong \mathbb{F}_3$. Now assume that each $q_i \geq 3$, and that S is not isomorphic to $\mathbb{F}_3$. We distinguish two subcases.

If some factor satisfies $q_j \geq 4$, then in that coordinate, we can choose $a_j, b_j, c_j \in \mathbb{F}_{q_j}$ with a_j, b_j, c_j pairwise distinct and $a_j + b_j$, $b_j + c_j$, $c_j + a_j$ all nonzero (this is easy to do once $q_j \geq 4$). In every other coordinate $i \neq j$, choose a triple of elements a_i, b_i, c_i so that none of the sums $a_i + b_i$, $b_i + c_i$, $c_i + a_i$ is zero (e.g., take $a_i = 0$, $b_i = 1$, $c_i = 1$ in $\mathbb{F}_3$ or any suitable choice in $\mathbb{F}_{q_i}$). Then for each i, all three pairwise sums are nonzero, and hence $a, b, c \in S$ form a triangle in $\Upsilon(S)$.

If all factors are $\mathbb{F}_3$ and $t \geq 2$, then pick in the first two coordinates

$$a = (0, 1, 0, \ldots, 0), \quad b = (1, 0, 0, \ldots, 0), \quad c = (1, 1, 0, \ldots, 0).$$

In each $\mathbb{F}_3$ coordinate, we have a triple $(0, 1, 1)$, and one checks directly that all pairwise sums in those coordinates are nonzero, so a, b, c again form a triangle in $\Upsilon(S)$. Thus, in this case, a 3-cycle exists. □

As an immediate consequence, we obtain a clean triangle-free criterion for the unit graph of RG.

Corollary 7.2. *Let R be a finite commutative ring and G a finite abelian group, and write*

$$RG/J(RG) \cong \prod_{i=1}^{t} \mathbb{F}_{q_i}.$$

Then the unit graph $\Upsilon(RG)$ is triangle-free (that is, $\operatorname{girth}(\Upsilon(RG)) \geq 4$) if and only if either:

(1) At least one factor $\mathbb{F}_{q_i}$ is $\mathbb{F}_2$, or
(2) $RG/J(RG) \cong \mathbb{F}_3$.

Proof. Let

$$S = RG/J(RG).$$

By Lemma 5.1 and the lifting of units modulo $J(RG)$ in finite rings, $\Upsilon(RG)$ contains a triangle if and only if $\Upsilon(S)$ does. The result now follows directly from Lemma 7.1. □

Theorem 7.3. *Let R be a finite commutative ring and G a finite abelian group, and write*

$$S = RG/J(RG) \cong \prod_{i=1}^{t} \mathbb{F}_{q_i}.$$

Then:

(1) If S has no $\mathbb{F}_2$ factor and

$$S \ncong \mathbb{F}_3,$$

then $\Upsilon(RG)$ contains a triangle and hence

$$\operatorname{girth}(\Upsilon(RG)) = 3.$$

(2) If

$$S \cong \mathbb{F}_3$$

or

$$S \cong (\mathbb{F}_2)^t$$

for some $t \geq 1$, then $\Upsilon(RG)$ is acyclic and

$$\operatorname{girth}(\Upsilon(RG)) = \infty.$$

(3) In all remaining cases (that is, when S has at least one $\mathbb{F}_2$ factor and at least one factor of order ≥ 3), the graph $\Upsilon(RG)$ is triangle-free but contains cycles, so its girth is a finite even integer ≥ 4.

Proof. (1) If S has no $\mathbb{F}_2$ factor and

$$S \ncong \mathbb{F}_3,$$

then Lemma 7.1(3) gives a triangle in $\Upsilon(S)$; hence in $\Upsilon(RG)$ by Lemma 5.1. Thus,

$$\text{girth}(\Upsilon(RG)) = 3.$$

(2) If

$$S \cong \mathbb{F}_3,$$

Lemma 7.1(2) shows that $\Upsilon(S)$ is a path on three vertices and has no cycles. If

$$S \cong (\mathbb{F}_2)^t,$$

then $U(S)$ consists of the single element $(1, \cdots, 1)$, so each vertex has at most one neighbor, and every connected component is a matching or an isolated vertex; in particular, no cycles occur. In both cases, $\Upsilon(RG)$ has no cycles by Lemma 5.1, so the girth is infinite.

(3) Suppose now that S has at least one $\mathbb{F}_2$ factor and at least one factor of order ≥ 3. By Lemma 7.1(1), the $\mathbb{F}_2$ factor prevents any triangle, so

$$\text{girth}(\Upsilon(RG)) \geq 4.$$

On the other hand, in such a product, one can explicitly construct even cycles (for example, in $\mathbb{F}_2 \times \mathbb{F}_q$ with $q \geq 3$, there always exist 4- or 6-cycles), so $\Upsilon(S)$ is not acyclic. Lifting back to $\Upsilon(RG)$ again by Lemma 5.1 shows that the girth is a finite even integer ≥ 4. □

8. Conclusions

In this work, we introduced and developed a systematic study of the unit graph $\Upsilon(RG)$ associated with a finite group ring RG. Using a blend of additive, multiplicative, and graph-theoretic methods, we obtained several structural results describing how algebraic properties of RG govern the global combinatorial behavior of its unit graph.

We established a precise criterion for when $\Upsilon(RG)$ is connected, provided an explicit description of its connected components, and gave a complete classification of its diameter in terms of the semisimple quotient $RG/J(RG)$. In addition, we characterized exactly when $\Upsilon(RG)$ is a cycle or a complete graph and determined the girth of $\Upsilon(RG)$, identifying the appearance of triangles, 4-cycles, and acyclic cases. Remarkably, all these graph-theoretic features are dictated by the finite field factors occurring in the Wedderburn decomposition of $RG/J(RG)$.

These results revealed a strong correspondence between the additive structure of RG, the behavior of its unit group, and the resulting geometry of $\Upsilon(RG)$. They demonstrated that global invariants such as connectivity, diameter, cycle structure, and girth are rigidly controlled by purely algebraic data, enabling many properties of $\Upsilon(RG)$ to be determined without explicit computations in the full group ring.

Limitations and scope. The analysis in this paper was restricted to finite group rings, where the interaction between the Jacobson radical, semisimple quotients, and unit groups could be fully exploited. Extending these results to infinite rings or infinite groups would require substantially different techniques and was beyond the scope of this work. Moreover, we focused exclusively on the (undirected) unit graph; related graph constructions, such as directed or weighted variants, were not considered here.

Our results also suggested several directions for further research. One natural problem is the inverse question of determining to what extent the group G can be recovered from the isomorphism type of the unit graph $\Upsilon(RG)$, especially in light of the fact that nonisomorphic groups may yield isomorphic unit graphs. Another promising direction is the study of the automorphism group $\mathrm{Aut}(\Upsilon(RG))$, particularly for special classes of groups, such as finite cyclic groups or finite p-groups, where additional symmetries arising from the algebraic structure of RG may be present. These questions lie beyond the scope of this paper and are left for future investigation.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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