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Research article

Characterizing m-Jordan n-derivations of triangular rings

He Yuan[1,*], Chuqi Jia[1] and Lili Ma[2]

[1] College of Mathematics and Computer, Jilin Normal University, Siping, Jilin 136000, China
[2] College of Science, Qiqihar University, Qiqihar, Heilongjiang 161006, China

* **Correspondence:** Email: yuanhe1983@126.com; Tel: +8604343291803; Fax: +8604343291803.

Abstract: This paper explores the relationship between m-derivations and Jordan n-derivations, introducing the concept of m-Jordan n-derivations on triangular rings. By employing the maximal left ring of quotients, we first conduct a detailed analysis of 3-Jordan n-derivations. Subsequently, using an inductive approach, we prove that under specific hypotheses, an m-Jordan n-derivation must be an extremal m-derivation. Finally, we apply the results we obtained to upper triangular matrix rings.

Keywords: triangular ring; 3-Jordan n-derivation; m-Jordan n-derivation; extremal m-derivation; maximal left ring of quotients
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1. Introduction

Let $\mathcal{T}$ be a ring and $\delta : \mathcal{T} \to \mathcal{T}$ be an additive mapping. The mapping δ is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{T}$. If δ satisfies $\delta(a^2) = \delta(a)a + a\delta(a)$ for all $a \in \mathcal{T}$, then it is called a Jordan derivation. Obviously, every derivation is a Jordan derivation, but not any Jordan derivation is a derivation. Herstein [1] proved the fundamental result that every Jordan derivation on a prime ring whose characteristic is not 2 is a derivation. Later, Brešar [2] extended this conclusion to semiprime rings of characteristic not 2. On the basis of previous theoretical studies, Zhang and Yu [3] analyzed the structure of Jordan derivations of triangular algebras, demonstrating all Jordan derivations here are derivations.

It is natural to introduce the definition of a Jordan n-derivation. Qi et al. [4] established that under specific conditions, for a unital ring with a nontrivial idempotent, a mapping on that is a multiplicative Jordan n-derivation if and only if it is an additive Jordan derivation. Building on the work of Qi et al. [4], Liang [5] explored the structure of generalized Jordan n-derivations on a unital algebra $\mathcal{T}$ containing a nontrivial idempotent. He established the structural property of these derivations. Specifically, under mild assumptions, each generalized Jordan n-derivation $\vartheta(x)$ admits a decomposition into $\vartheta(x) = \lambda x + J(x)$, where $\lambda \in Z(\mathcal{T})$ and J is a Jordan n-derivation. Subsequently, Benkovič [6] refined this result using new methods, arriving at a stronger conclusion.

In 1956, Utumi's work [7] gave rise to the notion of maximal left ring of quotients, which is commonly designated as Utumi (left) ring of quotients. Since then, a number of scholars have conducted research by employing the maximal left ring of quotients. For example, Eremita [8] investigated functional identities of degree 2 by using the maximal left ring of quotients. Wang [9] extended Eremita's method, generalizing his conclusions to derive solutions for the functional identities of degree 2 on arbitrary triangular rings. In the same vein, by applying the maximal left ring of quotients, Eremita [10] and Wang [11] respectively investigated the structural characteristics of biderivations on triangular rings, based on which Benkovič's results [12] were improved. In another work, Wang [13] used the maximal left (right) rings of quotients to build a triangular algebra from a preexisting one. As an application, he characterized Lie (Jordan) derivations of arbitrary triangular algebras by means of the constructed triangular algebra. Liu [14] gave a detailed analysis of the structure of Jordan biderivations on triangular rings by using the maximal left ring of quotients, showing that every Jordan biderivation on a triangular ring can be expressed as the sum of an inner biderivation and an extremal biderivation. More recently, Liang and Zhao [15] once again used the maximal left ring of quotients to prove that every bi-Lie n-derivation on specific triangular rings can be decomposed into the sum of an inner biderivation, an extremal biderivation, and an additive central mapping, where the additive central mapping vanishes at the $(n-1)$-th commutators in both components.

Inspired by these results, this paper studies m-Jordan n-derivations on triangular rings of characteristics different from 2 and $n-1$. Section 2 collects preliminary definitions and lemmas. Section 3 is devoted to characterizing 3-Jordan n-derivations by using the maximal left ring of quotients. Finally, Section 4 proves that all m-Jordan n-derivations are extremal m-derivations under certain conditions. This conclusion is applied to upper triangular matrix rings, leading to a corollary.

2. Preliminaries

The necessary definitions are presented first. Let $\mathcal{T}$ be a ring. Define a sequence of polynomials q_n on $\mathcal{T}$ as follows:

$$\begin{aligned} q_1(x_1) &= x_1, \\ q_2(x_1, x_2) &= x_1x_2 + x_2x_1, \\ &\cdots\cdots, \\ q_n(x_1, x_2, \ldots, x_n) &= q_{n-1}(x_1, x_2, \ldots, x_{n-1})x_n + x_nq_{n-1}(x_1, x_2, \ldots, x_{n-1}). \end{aligned}$$

We call q_n the Jordan n-product of $x_1, \ldots, x_n$. An additive mapping $\delta : \mathcal{T} \to \mathcal{T}$ is called a Jordan n-derivation if it satisfies

$$\delta(q_n(x_1, x_2, \ldots, x_n)) = \sum_{i=1}^{n} q_n(x_1, x_2, \ldots, \delta(x_i), \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{T}$.

Letting $m \geq 2$ be an integer, a mapping $\delta : \mathcal{T}^m \to \mathcal{T}$ is called an m-derivation if it is a derivation with respect to each component. Furthermore, if a mapping $\varphi : \mathcal{T}^m \to \mathcal{T}$ is a Jordan n-derivation with respect to each component, then φ is called an m-Jordan n-derivation.

For a ring $\mathcal{T}$ with a nontrivial idempotent p_1 and identity element I, $\mathcal{T}$ can be called a triangular ring if it satisfies both $p_1\mathcal{T}p_2$ is a faithful $(p_1\mathcal{T}p_1, p_2\mathcal{T}p_2)$-bimodule, and $p_2\mathcal{T}p_1 = 0$, where $p_2 = I - p_1$. In particular, we have the following identities:

$$q_n(x, p_1, \ldots, p_1) = 2^{n-1}p_1xp_1 + p_1xp_2,$$
$$q_n(x, p_2, \ldots, p_2) = 2^{n-1}p_2xp_2 + p_1xp_2$$

for all $x \in \mathcal{T}$.

Let $\mathcal{T}$ be a triangular ring. Suppose there exists $a \in p_1\mathcal{T}p_2$ satisfying $[a, [\mathcal{T}, \mathcal{T}]] = 0$. Then, by [16, Remark 1], the mapping $\psi_m : \mathcal{T}^m \to \mathcal{T}$ defined by $\psi_m(x_1, x_2, \ldots, x_m) = [x_1, [x_2, \ldots, [x_m, a] \cdots]]$ is a permuting m-derivation on $\mathcal{T}$, referred to as an extremal m-derivation.

Let $Q_{ml}(\mathcal{T})$ be the maximal left ring of quotients of $\mathcal{T}$, and denote the center of $Q_{ml}(\mathcal{T})$ as $C(\mathcal{T})$, also referred to as the extended center of $\mathcal{T}$. For subsets $A, B \subseteq Q_{ml}(\mathcal{T})$, define $C(A, B) = \{a \in A | ab = ba, \forall b \in B\}$. According to [8], we get

$$C(\mathcal{T}) = \{z \in p_1Q_{ml}(\mathcal{T})p_1 \oplus p_2Q_{ml}(\mathcal{T})p_2 | zp_1xp_2 = p_1xp_2z, \forall x \in \mathcal{T}\} = C(Q_{ml}(\mathcal{T}), \mathcal{T}).$$

Furthermore, there exists a unique ring isomorphism $\tau : C(\mathcal{T})p_1 \to C(\mathcal{T})p_2$ such that $\lambda p_1xp_2 = p_1xp_2\tau(\lambda p_1)$ for all $x \in \mathcal{T}$ and $\lambda \in C(\mathcal{T})$. We summarize here several well-established properties, which can be easily derived through the review of a reputable reference [8].

Proposition 2.1. *Let $\mathcal{T}$ be a unital algebra. Then the following holds:*

(i) $\mathcal{T}$ is a subring of $Q_{ml}(\mathcal{T})$ with the same identity element I.

(ii) For any dense left ideal I of $\mathcal{T}$ and any left $\mathcal{T}$-module homomorphism $h : I \to \mathcal{T}$, there exists $q \in Q_{ml}(\mathcal{T})$ such that h is the right multiplication mapping induced by q.

(iii) $p_1\mathcal{T}$ is a dense left ideal of $\mathcal{T}$, and for any $r \in Q_{ml}(\mathcal{T})$, if $p_1\mathcal{T}p_2r = 0$, then $p_2r = 0$. If $rp_1\mathcal{T}p_2 = 0$, then $rp_1 = 0$.

To study 3-Jordan n-derivations, we first present two results on Jordan n-derivations and conclude this section with them.

Lemma 2.1. *Let $\mathcal{T}$ be a triangular ring with an identity element I. Suppose that the characteristic of $\mathcal{T}$ is not 2 or $n - 1$. Then every Jordan n-derivation $\delta : \mathcal{T} \to \mathcal{T}$ satisfies $\delta(I) = 0$.*

Proof. On the one hand, by the definition of δ, we have

$$\delta(q_n(I, \ldots, I)) = q_n(\delta(I), I, \ldots, I) + \ldots + q_n(I, \ldots, I, \delta(I)) = n \cdot 2^{n-1}\delta(I).$$

On the other hand, we get

$$\delta(q_n(I, \ldots, I)) = \delta(2^{n-1}I) = 2^{n-1}\delta(I).$$

Therefore, $2^{n-1}(n-1)\delta(I) = 0$. Since the characteristic of $\mathcal{T}$ is not 2 or $n-1$, it follows that $\delta(I) = 0$. □

Lemma 2.2. *Let $\mathcal{T}$ be a triangular ring with an identity element I. Assume that the characteristic of $\mathcal{T}$ is not 2 or $n-1$. If $\delta : \mathcal{T} \to \mathcal{T}$ is a Jordan n-derivation, then*

(i) $\delta(xy + yx) = \delta(x)y + x\delta(y) + \delta(y)x + y\delta(x)$,

(ii) $\delta(xyx) = \delta(x)yx + x\delta(y)x + xy\delta(x)$, *and*

(iii) $\delta(xyz + zyx) = \delta(x)yz + x\delta(y)z + xy\delta(z) + \delta(z)yx + z\delta(y)x + zy\delta(x)$

for all $x, y, z \in \mathcal{T}$.

Proof. (i) By the definition of δ, we get

$$\begin{aligned}\delta(q_n(xy + yx, I, \ldots, I)) =&\delta(q_{n+1}(x, y, I, \ldots, I))\\ =&q_{n+1}(\delta(x), y, I, \ldots, I) + q_{n+1}(x, \delta(y), I, \ldots, I)\\ &+ \sum_{i=3}^{n+1} q_{n+1}(x, y, I, \ldots, \underbrace{\delta(I)}_{i-th\ component}, \ldots, I).\end{aligned}$$

According to Lemma 2.1, we have

$$\delta(2^{n-1}(xy + yx)) = 2^{n-1}(\delta(x)y + y\delta(x) + x\delta(y) + \delta(y)x).$$

Since the characteristic of $\mathcal{T}$ is not 2, it follows that

$$\delta(xy + yx) = \delta(x)y + x\delta(y) + \delta(y)x + y\delta(x) \tag{2.1}$$

for all $x, y \in \mathcal{T}$. In particular, setting $y = x$ in (2.1), we obtain

$$\delta(x^2) = \delta(x)x + x\delta(x) \tag{2.2}$$

for all $x \in \mathcal{T}$.

(ii) On the one hand, it follows from (2.1) that

$$\begin{aligned}\delta(x(xy + yx) + (xy + yx)x) =&\delta(x)(xy + yx) + x(\delta(x)y + x\delta(y) + \delta(y)x + y\delta(x))\\ &+ (\delta(x)y + x\delta(y) + \delta(y)x + y\delta(x))x + (xy + yx)\delta(x)\\ =&\delta(x)xy + 2\delta(x)yx + x\delta(x)y + x^2\delta(y) + 2x\delta(y)x\\ &+ 2xy\delta(x) + \delta(y)x^2 + y\delta(x)x + yx\delta(x).\end{aligned} \tag{2.3}$$

On the other hand, it follows from (2.1) and (2.2) that

$$\begin{aligned}\delta(x(xy + yx) + (xy + yx)x) =&\delta((x^2y + yx^2) + 2xyx)\\ =&\delta(x^2)y + x^2\delta(y) + \delta(y)x^2 + y\delta(x^2) + 2\delta(xyx)\\ =&\delta(x)xy + x\delta(x)y + x^2\delta(y) + \delta(y)x^2 + yx\delta(x)\\ &+ y\delta(x)x + 2\delta(xyx).\end{aligned} \tag{2.4}$$

Comparing (2.3) and (2.4), we obtain

$$\delta(xyx) = \delta(x)yx + x\delta(y)x + xy\delta(x) \tag{2.5}$$

for all $x, y \in \mathcal{T}$.

(iii) Applying (2.5) yields that

$$\begin{aligned}\delta((x+z)y(x+z)) =& \delta(x+z)y(x+z) + (x+z)\delta(y)(x+z) + (x+z)y\delta(x+z)\\ =& \delta(x)yx + \delta(x)yz + \delta(z)yx + \delta(z)yz + x\delta(y)x + x\delta(y)z\\ &+ z\delta(y)z + z\delta(y)x + xy\delta(x) + xy\delta(z) + zy\delta(x) + zy\delta(z)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\delta(xyx+zyz) =& \delta(xyx) + \delta(zyz)\\ =& \delta(x)yx + x\delta(y)x + xy\delta(x) + \delta(z)yz + z\delta(y)z + zy\delta(z).\end{aligned} \tag{2.7}$$

Combining $\delta((x+z)y(x+z)) = \delta(xyx + xyz + zyx + zyz)$ with (2.6) and (2.7), we arrive at

$$\delta(xyz + zyx) = \delta(x)yz + \delta(z)yx + x\delta(y)z + z\delta(y)x + xy\delta(z) + zy\delta(x)$$

for all $x, y, z \in \mathcal{T}$. □

3. 3-Jordan n-derivations

This section is devoted to the study of 3-Jordan n-derivations on a triangular ring $\mathcal{T}$. Throughout, we assume that $\mathcal{T}$ has an identity element I, a nontrivial idempotent p_1, and a characteristic that is not 2 or $n-1$. We begin with some pertinent lemmas and, on the basis of Lemma 2.1 and Lemma 2.2, the following results can be established.

Lemma 3.1. *Let $\mathcal{T}$ be a triangular ring. If $\delta : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a 3-Jordan n-derivation, then*

$$\delta(I, x, y) = 0, \qquad \delta(x, I, y) = 0, \qquad \delta(x, y, I) = 0$$

for all $x, y \in \mathcal{T}$.

Lemma 3.2. *Let $\mathcal{T}$ be a triangular ring. If $\delta : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a 3-Jordan n-derivation, then*

$(i)\delta(x_1x_2 + x_2x_1, x_3, x_4) = \delta(x_1, x_3, x_4)x_2 + x_1\delta(x_2, x_3, x_4) + \delta(x_2, x_3, x_4)x_1 + x_2\delta(x_1, x_3, x_4)$,

$(ii)\delta(x_3, x_1x_2 + x_2x_1, x_4) = \delta(x_3, x_1, x_4)x_2 + x_1\delta(x_3, x_2, x_4) + \delta(x_3, x_2, x_4)x_1 + x_2\delta(x_3, x_1, x_4)$,

$(iii)\delta(x_3, x_4, x_1x_2 + x_2x_1) = \delta(x_3, x_4, x_1)x_2 + x_1\delta(x_3, x_4, x_2) + \delta(x_3, x_4, x_2)x_1 + x_2\delta(x_3, x_4, x_1)$,

$(iv)\delta(x_1x_2x_1, x_3, x_4) = \delta(x_1, x_3, x_4)x_2x_1 + x_1\delta(x_2, x_3, x_4)x_1 + x_1x_2\delta(x_1, x_3, x_4)$,

$(v)\delta(x_3, x_1x_2x_1, x_4) = \delta(x_3, x_1, x_4)x_2x_1 + x_1\delta(x_3, x_2, x_4)x_1 + x_1x_2\delta(x_3, x_1, x_4)$,

$(vi)\delta(x_3, x_4, x_1x_2x_1) = \delta(x_3, x_4, x_1)x_2x_1 + x_1\delta(x_3, x_4, x_2)x_1 + x_1x_2\delta(x_3, x_4, x_1)$,

$(vii)\delta(x_1x_2x_3 + x_3x_2x_1, x_4, x_5) = \delta(x_1, x_4, x_5)x_2x_3 + x_1\delta(x_2, x_4, x_5)x_3 + x_1x_2\delta(x_3, x_4, x_5) + \delta(x_3, x_4, x_5)x_2x_1 + x_3\delta(x_2, x_4, x_5)x_1 + x_3x_2\delta(x_1, x_4, x_5)$,

$(viii)\delta(x_4, x_1x_2x_3 + x_3x_2x_1, x_5) = \delta(x_4, x_1, x_5)x_2x_3 + x_1\delta(x_4, x_2, x_5)x_3 + x_1x_2\delta(x_4, x_3, x_5) + \delta(x_4, x_3, x_5)x_2x_1 + x_3\delta(x_4, x_2, x_5)x_1 + x_3x_2\delta(x_4, x_1, x_5)$, *and*

$(ix)\delta(x_4, x_5, x_1x_2x_3 + x_3x_2x_1) = \delta(x_4, x_5, x_1)x_2x_3 + x_1\delta(x_4, x_5, x_2)x_3 + x_1x_2\delta(x_4, x_5, x_3) + \delta(x_4, x_5, x_3)x_2x_1 + x_3\delta(x_4, x_5, x_2)x_1 + x_3x_2\delta(x_4, x_5, x_1)$

for all $x_1, x_2, x_3, x_4, x_5 \in \mathcal{T}$.

Lemma 3.3. *Let $\mathcal{T}$ be a triangular ring. If $\delta : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a 3-Jordan n-derivation, then*

$$[\delta(x_1, x_2, x_3), [x_4, x_5]] + [\delta(x_1, x_4, x_3), [x_2, x_5]] = [\delta(x_5, x_4, x_3), [x_1, x_2]] + [\delta(x_5, x_2, x_3), [x_1, x_4]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in \mathcal{T}$.

Proof. According to Lemma 3.2, on the one hand, we have

$$\begin{aligned} \delta(x_1x_5 + x_5x_1, x_2x_4 + x_4x_2, x_3) &= \delta(x_1, x_2x_4 + x_4x_2, x_3)x_5 + x_1\delta(x_5, x_2x_4 + x_4x_2, x_3) \\ &\quad + \delta(x_5, x_2x_4 + x_4x_2, x_3)x_1 + x_5\delta(x_1, x_2x_4 + x_4x_2, x_3) \\ &= (\delta(x_1, x_2, x_3)x_4 + x_2\delta(x_1, x_4, x_3) + \delta(x_1, x_4, x_3)x_2 + x_4\delta(x_1, x_2, x_3))x_5 \\ &\quad + x_1(\delta(x_5, x_2, x_3)x_4 + x_2\delta(x_5, x_4, x_3) + \delta(x_5, x_4, x_3)x_2 + x_4\delta(x_5, x_2, x_3)) \\ &\quad + (\delta(x_5, x_2, x_3)x_4 + x_2\delta(x_5, x_4, x_3) + \delta(x_5, x_4, x_3)x_2 + x_4\delta(x_5, x_2, x_3))x_1 \\ &\quad + x_5(\delta(x_1, x_2, x_3)x_4 + x_2\delta(x_1, x_4, x_3) + \delta(x_1, x_4, x_3)x_2 + x_4\delta(x_1, x_2, x_3)), \end{aligned} \tag{3.1}$$

and on the other hand, we have

$$\begin{aligned} \delta(x_1x_5 + x_5x_1, x_2x_4 + x_4x_2, x_3) &= \delta(x_1x_5 + x_5x_1, x_2, x_3)x_4 + x_2\delta(x_1x_5 + x_5x_1, x_4, x_3) \\ &\quad + \delta(x_1x_5 + x_5x_1, x_4, x_3)x_2 + x_4\delta(x_1x_5 + x_5x_1, x_2, x_3) \\ &= (\delta(x_1, x_2, x_3)x_5 + x_1\delta(x_5, x_2, x_3) + x_5\delta(x_1, x_2, x_3) + \delta(x_5, x_2, x_3)x_1)x_4 \\ &\quad + x_2(\delta(x_1, x_4, x_3)x_5 + x_1\delta(x_5, x_4, x_3) + x_5\delta(x_1, x_4, x_3) + \delta(x_5, x_4, x_3)x_1) \\ &\quad + (\delta(x_1, x_4, x_3)x_5 + x_1\delta(x_5, x_4, x_3) + x_5\delta(x_1, x_4, x_3) + \delta(x_5, x_4, x_3)x_1)x_2 \\ &\quad + x_4(\delta(x_1, x_2, x_3)x_5 + x_1\delta(x_5, x_2, x_3) + x_5\delta(x_1, x_2, x_3) + \delta(x_5, x_2, x_3)x_1). \end{aligned} \tag{3.2}$$

Combining (3.1) with (3.2), we get

$$[\delta(x_1, x_2, x_3), [x_4, x_5]] + [\delta(x_1, x_4, x_3), [x_2, x_5]] = [\delta(x_5, x_4, x_3), [x_1, x_2]] + [\delta(x_5, x_2, x_3), [x_1, x_4]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in \mathcal{T}$. □

Lemma 3.4. *Let $\mathcal{T}$ be a triangular ring. If $\delta : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a 3-Jordan n-derivation, then*

$$\begin{gathered} \delta(p_1, y, z) \in p_1\mathcal{T}p_2, \quad \delta(y, p_1, z) \in p_1\mathcal{T}p_2, \quad \delta(y, z, p_1) \in p_1\mathcal{T}p_2, \\ \delta(p_2, y, z) \in p_1\mathcal{T}p_2, \quad \delta(y, p_2, z) \in p_1\mathcal{T}p_2, \quad \delta(y, z, p_2) \in p_1\mathcal{T}p_2, \\ \delta(p_1xp_2, y, z) \in p_1\mathcal{T}p_2, \quad \delta(x, p_1yp_2, z) \in p_1\mathcal{T}p_2, \quad \delta(x, y, p_1zp_2) \in p_1\mathcal{T}p_2 \end{gathered}$$

for all $x, y, z \in \mathcal{T}$.

Proof. According to the definition of q_n, we know

$$\delta(q_n(p_1, p_1, \ldots, p_1), y, z) = \delta(2^{n-1}p_1, y, z) = 2^{n-1}\delta(p_1, y, z).$$

Since δ is a 3-Jordan n-derivation, it follows that

$$\delta(q_n(p_1, p_1, \ldots, p_1), y, z) = \sum_{i=1}^{n} q_n(p_1, \ldots, p_1, \underbrace{\delta(p_1, y, z)}_{i-th\ component}, p_1, \ldots, p_1)$$

$$=n\cdot 2^{n-1}p_1\delta(p_1,y,z)p_1+np_1\delta(p_1,y,z)p_2.$$

Comparing the above two relations, we get

$$2^{n-1}\delta(p_1,y,z)=n\cdot 2^{n-1}p_1\delta(p_1,y,z)p_1+np_1\delta(p_1,y,z)p_2.$$

Multiplying the relation on the left and right by p_1, we obtain

$$2^{n-1}p_1\delta(p_1,y,z)p_1=n\cdot 2^{n-1}p_1\delta(p_1,y,z)p_1.$$

The characteristic of $\mathcal{T}$ is not 2 or $n-1$, so $p_1\delta(p_1,y,z)p_1=0$ for all $y,z\in\mathcal{T}$.

On the one hand, according to the definition of δ, we have

$$\delta(0,y,z)=\delta(q_n(0,\ldots,0),y,z)=\sum_{i=1}^{n}q_n(0,\ldots,0,\underbrace{\delta(0,y,z)}_{i-th\ component},0,\ldots,0)=0.$$

On the other hand, using $q_n(p_1,p_2,\ldots,p_2)=0$, we derive

$$\begin{aligned}\delta(0,y,z)=&\delta(q_n(p_1,p_2,\ldots,p_2),y,z)\\=&q_n(\delta(p_1,y,z),p_2,\ldots,p_2)+\sum_{i=2}^{n}q_n(p_1,p_2,\ldots,\underbrace{\delta(p_2,y,z)}_{i-th\ component},\ldots,p_2)\\=&2^{n-1}p_2\delta(p_1,y,z)p_2+p_1\delta(p_1,y,z)p_2+p_1\delta(p_2,y,z)p_2.\end{aligned}$$

Comparing the above two relations, we arrive at

$$2^{n-1}p_2\delta(p_1,y,z)p_2+p_1\delta(p_1,y,z)p_2+p_1\delta(p_2,y,z)p_2=0. \tag{3.3}$$

Multiplying both sides of (3.3) by p_2, we get

$$2^{n-1}p_2\delta(p_1,y,z)p_2=0.$$

Since the characteristic of $\mathcal{T}$ is not 2, we have $p_2\delta(p_1,y,z)p_2=0$. Thus,

$$\delta(p_1,y,z)\in p_1\mathcal{T}p_2 \tag{3.4}$$

for all $y,z\in\mathcal{T}$. Similarly, it can be proved that

$$\begin{gathered}\delta(y,p_1,z)\in p_1\mathcal{T}p_2,\quad \delta(y,z,p_1)\in p_1\mathcal{T}p_2,\\ \delta(p_2,y,z)\in p_1\mathcal{T}p_2,\quad \delta(y,p_2,z)\in p_1\mathcal{T}p_2,\quad \delta(y,z,p_2)\in p_1\mathcal{T}p_2\end{gathered} \tag{3.5}$$

for all $y,z\in\mathcal{T}$.

According to (3.5) and $p_1xp_2=q_n(p_1xp_2,p_2,\ldots,p_2)$, we have

$$\begin{aligned}\delta(p_1xp_2,y,z)=&\delta(q_n(p_1xp_2,p_2,\ldots,p_2),y,z)\\=&q_n(\delta(p_1xp_2,y,z),p_2,\ldots,p_2)+\sum_{i=2}^{n}q_n(p_1xp_2,p_2,\ldots,\underbrace{\delta(p_2,y,z)}_{i-th\ component},\ldots,p_2)\\=&2^{n-1}p_2\delta(p_1xp_2,y,z)p_2+p_1\delta(p_1xp_2,y,z)p_2.\end{aligned} \tag{3.6}$$

Multiplying (3.6) on the left and on the right by p_1, we get $p_1\delta(p_1xp_2, y, z)p_1 = 0$. Similarly, using (3.5) and $p_1xp_2 = q_n(p_1xp_2, p_1, \ldots, p_1)$, we obtain

$$\begin{aligned}\delta(p_1xp_2, y, z) =&\delta(q_n(p_1xp_2, p_1, \ldots, p_1), y, z)\\ =&q_n(\delta(p_1xp_2, y, z), p_1, \ldots, p_1) + \sum_{i=2}^{n} q_n(p_1xp_2, p_1, \ldots, \underbrace{\delta(p_1, y, z)}_{i-th\ component}, \ldots, p_1)\\ =&2^{n-1}p_1\delta(p_1xp_2, y, z)p_1 + p_1\delta(p_1xp_2, y, z)p_2\end{aligned} \tag{3.7}$$

for all $x, y, z \in \mathcal{T}$. It follows from (3.7) that $p_2\delta(p_1xp_2, y, z)p_2 = 0$. Thus,

$$\delta(p_1xp_2, y, z) \in p_1\mathcal{T}p_2.$$

Similarly, we have

$$\delta(x, p_1yp_2, z) \in p_1\mathcal{T}p_2, \qquad \delta(x, y, p_1zp_2) \in p_1\mathcal{T}p_2$$

for all $x, y, z \in \mathcal{T}$. □

Theorem 3.1. *Let $\mathcal{T}$ be a triangular ring, and let $\delta : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be a 3-Jordan n-derivation. Suppose the following conditions hold:*
(1) $C(p_2Q_{ml}(\mathcal{T})p_2, p_2\mathcal{T}p_2) = C(\mathcal{T})p_2$.
(2) Neither $p_1\mathcal{T}C(\mathcal{T})p_1$ nor $p_2\mathcal{T}C(\mathcal{T})p_2$ contains nonzero central ideals.
Then δ is an extremal 3-derivation satisfying $\delta(x, y, z) = [x, [y, [z, \delta(p_1, p_1, p_1)]]]$ for all $x, y, z \in \mathcal{T}$.

Without loss of generality, we assume that $p_1\mathcal{T}C(\mathcal{T})p_1$ has no nonzero central ideals. In Lemma 3.3, let $x_1 = x_2 = x_3 = p_1$, then

$$[\delta(p_1, p_1, p_1), [x_4, x_5]] + [\delta(p_1, x_4, p_1), [p_1, x_5]] = [\delta(x_5, x_4, p_1), [p_1, p_1]] + [\delta(x_5, p_1, p_1), [p_1, x_4]] \tag{3.8}$$

for all $x_4, x_5 \in \mathcal{T}$.

According to Lemma 3.4, we know that $\delta(p_1, x_4, p_1) \in p_1\mathcal{T}p_2$ and $\delta(x_5, p_1, p_1) \in p_1\mathcal{T}p_2$ for all $x_4, x_5 \in \mathcal{T}$. Thus, we can get $[\delta(p_1, p_1, p_1), [x_4, x_5]] = 0$ from (3.8). When $\delta(p_1, p_1, p_1) \neq 0$, we have $0 \neq \delta(p_1, p_1, p_1) = p_1\delta(p_1, p_1, p_1)p_2 \in p_1\mathcal{T}p_2$. Then there exists an extremal 3-derivation ψ such that

$$\psi(x_1, x_2, x_3) = [x_1, [x_2, [x_3, \delta(p_1, p_1, p_1)]]]$$

for all $x_1, x_2, x_3 \in \mathcal{T}$. Obviously,

$$\psi(p_1, p_1, p_1) = [p_1, [p_1, [p_1, \delta(p_1, p_1, p_1)]]] = \delta(p_1, p_1, p_1).$$

Let $\phi = \delta - \psi$, then ϕ is a 3-Jordan n-derivation satisfying $\phi(p_1, p_1, p_1) = 0$. In the following, we prove $\phi = 0$ using the following lemmas. At this point, if $\delta(p_1, p_1, p_1) \neq 0$, then $\delta = \psi$, which is an extremal 3-derivation.

Lemma 3.5. *For any $x, y, z \in \mathcal{T}$, we have*
(i) $\phi(p_1xp_1, p_2yp_2, p_1zp_1) = \phi(p_1xp_1, p_2yp_2, p_2zp_2) = 0$,
(ii) $\phi(p_2xp_2, p_1yp_1, p_1zp_1) = \phi(p_2xp_2, p_1yp_1, p_2zp_2) = 0$, and
(iii) $\phi(p_1xp_1, p_1yp_1, p_2zp_2) = \phi(p_2xp_2, p_2yp_2, p_1zp_1) = 0$.

Proof. According to Lemma 3.2, we get

$$\begin{aligned}\phi(p_1xp_1, p_2yp_2, p_1zp_1) =&\phi(p_1, p_2yp_2, p_1zp_1)p_1xp_1 + p_1\phi(p_1xp_1, p_2yp_2, p_1zp_1)p_1 \\ &+ p_1xp_1\phi(p_1, p_2yp_2, p_1zp_1)\end{aligned} \tag{3.9}$$

for all $x, y, z \in \mathcal{T}$. Multiplying (3.9) on the left and on the right by p_2, we obtain

$$p_2\phi(p_1xp_1, p_2yp_2, p_1zp_1)p_2 = 0. \tag{3.10}$$

In a similar way, we have

$$\begin{aligned}\phi(p_1xp_1, p_2yp_2, p_1zp_1) =&\phi(p_1xp_1, p_2, p_1zp_1)p_2yp_2 + p_2\phi(p_1xp_1, p_2yp_2, p_1zp_1)p_2 \\ &+ p_2yp_2\phi(p_1xp_1, p_2, p_1zp_1)\end{aligned} \tag{3.11}$$

for all $x, y, z \in \mathcal{T}$. Multiplying (3.11) on the left and on the right by p_1, we arrive at

$$p_1\phi(p_1xp_1, p_2yp_2, p_1zp_1)p_1 = 0 \tag{3.12}$$

for all $x, y, z \in \mathcal{T}$. From (3.9), (3.10), and (3.12), and using the fact that $\phi(p_1, p_2, p_1) = \phi(p_1, I, p_1) - \phi(p_1, p_1, p_1) = 0$, it follows that

$$\begin{aligned}\phi(p_1xp_1, p_2yp_2, p_1zp_1) =&p_1\phi(p_1xp_1, p_2yp_2, p_1zp_1)p_2 \\ =&p_1xp_1\phi(p_1, p_2yp_2, p_1zp_1)p_2 \\ =&p_1xp_1(\phi(p_1, p_2yp_2, p_1)p_1zp_1 + p_1\phi(p_1, p_2yp_2, p_1zp_1)p_1 \\ &+ p_1zp_1\phi(p_1, p_2yp_2, p_1))p_2 \\ =&p_1xp_1p_1zp_1\phi(p_1, p_2yp_2, p_1)p_2 \\ =&p_1xp_1p_1zp_1(\phi(p_1, p_2, p_1)p_2yp_2 + p_2\phi(p_1, p_2yp_2, p_1)p_2 \\ &+ p_2yp_2\phi(p_1, p_2, p_1))p_2 \\ =&0\end{aligned}$$

for all $x, y, z \in \mathcal{T}$. Similarly, we get

$$\begin{aligned}\phi(p_1xp_1, p_2yp_2, p_2zp_2) &= \phi(p_2xp_2, p_1yp_1, p_1zp_1) = \phi(p_2xp_2, p_1yp_1, p_2zp_2) \\ &= \phi(p_1xp_1, p_1yp_1, p_2zp_2) = \phi(p_2xp_2, p_2yp_2, p_1zp_1) = 0\end{aligned}$$

for all $x, y, z \in \mathcal{T}$. □

Lemma 3.6. *For any $x, y, z \in \mathcal{T}$, we have*
(i) $\phi(p_1xp_1, p_2yp_2, p_1zp_2) = \phi(p_2xp_2, p_1yp_1, p_1zp_2) = 0$,
(ii) $\phi(p_1xp_1, p_1yp_2, p_2zp_2) = \phi(p_2xp_2, p_1yp_2, p_1zp_1) = 0$,
(iii) $\phi(p_1xp_2, p_1yp_1, p_2zp_2) = \phi(p_1xp_2, p_2yp_2, p_1zp_1) = 0$,
(iv) $\phi(p_1xp_1, p_1yp_1, p_1zp_2) = \phi(p_2xp_2, p_2yp_2, p_1zp_2) = 0$,
(v) $\phi(p_1xp_1, p_1yp_2, p_1zp_1) = \phi(p_2xp_2, p_1yp_2, p_2zp_2) = 0$,
(vi) $\phi(p_1xp_2, p_1yp_1, p_1zp_1) = \phi(p_1xp_2, p_2yp_2, p_2zp_2) = 0$, *and*
(vii) $\phi(p_1xp_1, p_1yp_1, p_1zp_1) = \phi(p_2xp_2, p_2yp_2, p_2zp_2) = 0$.

Proof. Define a mapping $\eta : p_1\mathcal{T} \to p_1\mathcal{T}p_2$ by $\eta(x) = \phi(p_1, p_1xp_2, p_2)$. Lemma 3.4 ensures that η is well-defined and also implies that

$$\begin{aligned}\eta(rx) =&\phi(p_1, p_1rxp_2, p_2)\\ =&\phi(p_1, q_n(p_1rp_1, p_1xp_2, p_2, \ldots, p_2), p_2)\\ =&q_n(\phi(p_1, p_1rp_1, p_2), p_1xp_2, p_2, \ldots, p_2) + q_n(p_1rp_1, \phi(p_1, p_1xp_2, p_2), p_2, \ldots, p_2)\\ &+\sum_{i=3}^{n} q_n(p_1rp_1, p_1xp_2, p_2, \ldots, \underbrace{\phi(p_1, p_2, p_2)}_{i-th\ component}, \ldots, p_2)\\ =&q_n(p_1rp_1, \phi(p_1, p_1xp_2, p_2), p_2, \ldots, p_2)\\ =&p_1rp_1\phi(p_1, p_1xp_2, p_2)p_2\\ =&p_1r\phi(p_1, p_1xp_2, p_2)\\ =&r\eta(x)\end{aligned}$$

for all $r \in \mathcal{T}$ and $x \in p_1\mathcal{T}$. Thus, η is a left $\mathcal{T}$-module homomorphism. According to Property 2.1, there exists $q \in Q_{ml}(\mathcal{T})$ such that $\eta(x) = xq$ for all $x \in p_1\mathcal{T}$. In particular, $\eta(p_1) = p_1q = 0$. It follows that $\eta(x) = xp_2qp_2$ for all $x \in p_1\mathcal{T}$. That is,

$$\phi(p_1, p_1xp_2, p_2) = \eta(p_1x) = p_1xp_2qp_2$$

for all $x \in \mathcal{T}$. According to Lemma 3.4, we get

$$\begin{aligned}\eta(xr) =&\phi(p_1, p_1xrp_2, p_2)\\ =&\phi(p_1, q_n(p_1xp_2, p_2rp_2, p_2, \ldots, p_2), p_2)\\ =&q_n(\phi(p_1, p_1xp_2, p_2), p_2rp_2, p_2, \ldots, p_2) + q_n(p_1xp_2, \phi(p_1, p_2rp_2, p_2), p_2, \ldots, p_2)\\ &+\sum_{i=3}^{n} q_n(p_1xp_2, p_2rp_2, p_2, \ldots, \underbrace{\phi(p_1, p_2, p_2)}_{i-th\ component}, \ldots, p_2)\\ =&q_n(\phi(p_1, p_1xp_2, p_2), p_2rp_2, p_2, \ldots, p_2)\\ =&\phi(p_1, p_1xp_2, p_2)p_2rp_2\\ =&\eta(x)r\end{aligned}$$

for all $r \in p_2\mathcal{T}p_2$ and $x \in p_1\mathcal{T}$. That is, $xp_2rp_2qp_2 = xp_2qp_2rp_2$ for all $x \in p_1\mathcal{T}$. Then $p_1\mathcal{T}(p_2rp_2qp_2 - p_2qp_2rp_2) = 0$ for all $r \in \mathcal{T}$. According to Property 2.1, we get $p_2rp_2p_2qp_2 = p_2qp_2p_2rp_2$ for all $r \in \mathcal{T}$. It follows that $p_2qp_2 \in C(p_2Q_{ml}(\mathcal{T})p_2, p_2\mathcal{T}p_2)$. By the assumption, we obtain $p_2qp_2 \in C(\mathcal{T})p_2$. Let $\lambda = \tau^{-1}(p_2qp_2)$, then $\lambda p_1xp_2 = p_1xp_2p_2qp_2$ for all $x \in p_1\mathcal{T}$. Therefore,

$$\phi(p_1, p_1xp_2, p_2) = \lambda p_1xp_2$$

for all $x \in \mathcal{T}$. According to Lemma 3.3, we have

$$\begin{aligned}&[\phi(p_1, p_1xp_2, p_2), [p_1yp_1, p_1zp_1]] + [\phi(p_1, p_1yp_1, p_2), [p_1xp_2, p_1zp_1]]\\ &= [\phi(p_1zp_1, p_1yp_1, p_2), [p_1, p_1xp_2]] + [\phi(p_1zp_1, p_1xp_2, p_2), [p_1, p_1yp_1]]\end{aligned}$$

for all $x, y, z \in \mathcal{T}$. The above relation can be organized as

$$[p_1yp_1, p_1zp_1]\phi(p_1, p_1xp_2, p_2) = 0.$$

Thus, $\lambda[p_1yp_1, p_1zp_1]p_1xp_2 = 0$. According to Property 2.1, we obtain $\lambda[p_1yp_1, p_1zp_1] = 0$ for all $y, z \in \mathcal{T}$. It follows that $[\lambda p_1\mathcal{T}C(\mathcal{T})p_1, p_1\mathcal{T}C(\mathcal{T})p_1] = 0$. That is, $\lambda p_1\mathcal{T}C(\mathcal{T})p_1$ is a central ideal of $p_1\mathcal{T}C(\mathcal{T})p_1$. By assumption, $p_1\mathcal{T}C(\mathcal{T})p_1$ has no nonzero central ideals. This implies that $\lambda p_1\mathcal{T}C(\mathcal{T})p_1 = 0$. Therefore,

$$\phi(p_1, p_1xp_2, p_2) = 0$$

for all $x \in \mathcal{T}$. Similarly, we conclude

$$\phi(p_2, p_1xp_2, p_1) = \phi(p_1xp_2, p_1, p_2) = \phi(p_1xp_2, p_2, p_1) = \phi(p_1, p_2, p_1xp_2) = \phi(p_2, p_1, p_1xp_2) = 0 \tag{3.13}$$

for all $x \in \mathcal{T}$.

Applying Lemma 3.2, Lemma 3.4, and (3.13) yields

$$\begin{aligned}
\phi(p_1xp_1, p_2yp_2, p_1zp_2) =& \phi(p_1, p_2yp_2, p_1zp_2)p_1xp_1 + p_1\phi(p_1xp_1, p_2yp_2, p_1zp_2)p_1 \\
&+ p_1xp_1\phi(p_1, p_2yp_2, p_1zp_2) \\
=& p_1xp_1\phi(p_1, p_2yp_2, p_1zp_2) \\
=& p_1xp_1(\phi(p_1, p_2, p_1zp_2)p_2yp_2 + p_2\phi(p_1, p_2yp_2, p_1zp_2)p_2 \\
&+ p_2yp_2\phi(p_1, p_2, p_1zp_2)) \\
=& 0, \\
\phi(p_2xp_2, p_1yp_1, p_1zp_2) =& \phi(p_2xp_2, p_1, p_1zp_2)p_1yp_1 + p_1\phi(p_2xp_2, p_1yp_1, p_1zp_2)p_1 \\
&+ p_1yp_1\phi(p_2xp_2, p_1, p_1zp_2) \\
=& p_1yp_1\phi(p_2xp_2, p_1, p_1zp_2) \\
=& p_1yp_1(\phi(p_2, p_1, p_1zp_2)p_2xp_2 + p_2\phi(p_2xp_2, p_1, p_1zp_2)p_2 \\
&+ p_2xp_2\phi(p_2, p_1, p_1zp_2)) \\
=& 0
\end{aligned}$$

for all $x, y, z \in \mathcal{T}$. In a similar manner, it can be proved that

$$\begin{aligned}
\phi(p_1xp_1, p_1yp_2, p_2zp_2) =& \phi(p_2xp_2, p_1yp_2, p_1zp_1) = 0, \\
\phi(p_1xp_2, p_1yp_1, p_2zp_2) =& \phi(p_1xp_2, p_2yp_2, p_1zp_1) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{T}$.

Since $\phi(p_1, p_1, p_1zp_2) = \phi(p_1, I, p_1zp_2) - \phi(p_1, p_2, p_1zp_2) = 0$, Lemma 3.2 and Lemma 3.4 yield that

$$\begin{aligned}
\phi(p_1xp_1, p_1yp_1, p_1zp_2) =& \phi(p_1, p_1yp_1, p_1zp_2)p_1xp_1 + p_1(p_1xp_1, p_1yp_1, p_1zp_2)p_1 \\
&+ p_1xp_1\phi(p_1, p_1yp_1, p_1zp_2) \\
=& p_1xp_1\phi(p_1, p_1yp_1, p_1zp_2) \\
=& p_1xp_1(\phi(p_1, p_1, p_1zp_2)p_1yp_1 + p_1\phi(p_1, p_1yp_1, p_1zp_2)p_1 \\
&+ p_1yp_1\phi(p_1, p_1, p_1zp_2)) \\
=& 0
\end{aligned} \tag{3.14}$$

for all $x, y, z \in \mathcal{T}$. Likewise, we have

$$\begin{aligned}\phi(p_2xp_2, p_2yp_2, p_1zp_2) =&0,\\ \phi(p_1xp_1, p_1yp_2, p_1zp_1) =&\phi(p_2xp_2, p_1yp_2, p_2zp_2) = 0,\\ \phi(p_1xp_2, p_1yp_1, p_1zp_1) =&\phi(p_1xp_2, p_2yp_2, p_2zp_2) = 0\end{aligned}$$

for all $x, y, z \in \mathcal{T}$. Since

$$\phi(p_1xp_1, p_1yp_1, p_1zp_1) = \phi(p_1xp_1, p_1yp_1, I) - \phi(p_1xp_1, p_1yp_1, p_1zp_2) - \phi(p_1xp_1, p_1yp_1, p_2zp_2),$$

it follows from Lemma 3.5 and (3.14) that $\phi(p_1xp_1, p_1yp_1, p_1zp_1) = 0$. Using the same method, we get $\phi(p_2xp_2, p_2yp_2, p_2zp_2)= 0$ for all $x, y, z \in \mathcal{T}$. □

Lemma 3.7. *For any $x, y, z \in \mathcal{T}$, we have*
(i) $\phi(p_1xp_2, p_1yp_2, p_1zp_1) = \phi(p_1xp_2, p_1yp_2, p_2zp_2) = 0$,
(ii) $\phi(p_1xp_1, p_1yp_2, p_1zp_2) = \phi(p_2xp_2, p_1yp_2, p_1zp_2) = 0$,
(iii) $\phi(p_1xp_2, p_1yp_1, p_1zp_2) = \phi(p_1xp_2, p_2yp_2, p_1zp_2) = 0$, and
(iv) $\phi(p_1xp_2, p_1yp_2, p_1zp_2) = 0$.

Proof. Fix $y \in \mathcal{T}$, and define the mapping η_y by $\eta_y(x) = \phi(p_1xp_2, p_1yp_2, p_2)$ for all $x \in p_1\mathcal{T}$. By Lemma 3.4, η_y is a mapping from $p_1\mathcal{T}$ to $p_1\mathcal{T}p_2$. It follows that

$$\begin{aligned}\eta_y(rx) =&\phi(p_1rxp_2, p_1yp_2, p_2)\\ =&\phi(q_n(p_1rp_1, p_1xp_2, p_2, \ldots, p_2), p_1yp_2, p_2)\\ =&q_n(\phi(p_1rp_1, p_1yp_2, p_2), p_1xp_2, p_2, \ldots, p_2) + q_n(p_1rp_1, \phi(p_1xp_2, p_1yp_2, p_2), p_2, \ldots, p_2)\\ &+ \sum_{i=3}^{n} q_n(p_1rp_1, p_1xp_2, p_2, \ldots, \underbrace{\phi(p_2, p_1yp_2, p_2)}_{i-th\ component}, \ldots, p_2)\\ =&p_1rp_1\phi(p_1xp_2, p_1yp_2, p_2)p_2\\ =&r\phi(x)\end{aligned}$$

for all $x \in p_1\mathcal{T}$ and $r \in \mathcal{T}$. Therefore, η_y is a left $\mathcal{T}$-module homomorphism. This implies that there exists q_y such that $\eta_y(x) = xq_y$ for all $x \in p_1\mathcal{T}$. According to $\eta_y(p_1) = p_1q_y = 0$, we get $q_y = p_2q_yp_2$. Consequently,

$$\phi(p_1xp_2, p_1yp_2, p_2) = p_1xp_2q_yp_2$$

for all $x, y \in \mathcal{T}$. Using Lemma 3.4, we obtain

$$\begin{aligned}\eta_y(xr) =&\phi(p_1xrp_2, p_1yp_2, p_2)\\ =&\phi(q_n(p_1xp_2, p_2rp_2, p_2, \ldots, p_2), p_1yp_2, p_2)\\ =&q_n(\phi(p_1xp_2, p_1yp_2, p_2), p_2rp_2, p_2, \ldots, p_2) + q_n(p_1xp_2, \phi(p_2rp_2, p_1yp_2, p_2), p_2, \ldots, p_2)\\ &+ \sum_{i=3}^{n} q_n(p_1xp_2, p_2rp_2, p_2, \ldots, \underbrace{\phi(p_2, p_1yp_2, p_2)}_{i-th\ component}, \ldots, p_2)\\ =&p_1\phi(p_1xp_2, p_1yp_2, p_2)p_2rp_2\end{aligned}$$

$$=\eta_y(x)r$$

for all $x \in p_1\mathcal{T}$ and $r \in p_2\mathcal{T}p_2$. This implies that $xp_2rp_2q_yp_2 = xp_2q_yp_2rp_2$ for all $x \in p_1\mathcal{T}$. That is, $p_1\mathcal{T}(p_2rp_2q_yp_2 - p_2q_yp_2rp_2) = 0$ for all $r \in \mathcal{T}$. This means $p_2rp_2q_yp_2 = p_2q_yp_2rp_2$ for all $r \in \mathcal{T}$. It follows that $p_2q_yp_2 \in C(\mathcal{T})p_2$. Hence,

$$\phi(p_1xp_2, p_1yp_2, p_2) = \tau^{-1}(p_2q_yp_2)p_1xp_2$$

for all $x, y \in \mathcal{T}$.

On the one hand, we have

$$\begin{aligned}
&\phi(q_n(p_1xp_1, p_1x'p_2, p_2, \ldots, p_2), q_n(p_1yp_1, p_1y'p_2, p_2, \ldots, p_2), p_2)\\
&\quad = q_n(\phi(q_n(p_1xp_1, p_1x'p_2, p_2, \ldots, p_2), p_1yp_1, p_2), p_1y'p_2, p_2, \ldots, p_2)\\
&\qquad + q_n(p_1yp_1, \phi(q_n(p_1xp_1, p_1x'p_2, p_2, \ldots, p_2), p_1y'p_2, p_2), p_2, \ldots, p_2)\\
&\qquad + \sum_{i=3}^{n} q_n(p_1yp_1, p_1y'p_2, p_2, \ldots, \underbrace{\phi(q_n(p_1xp_1, p_1x'p_2, p_2, \ldots, p_2), p_2, p_2)}_{i-th\ component}, \ldots, p_2)\\
&\quad = p_1yp_1\phi(q_n(p_1xp_1, p_1x'p_2, p_2, \ldots, p_2), p_1y'p_2, p_2)p_2\\
&\quad = p_1yp_1\phi(p_1xp_1x'p_2, p_1y'p_2, p_2)p_2\\
&\quad = p_1yp_1\phi(p_1xp_1 \cdot p_1x'p_2 + p_1x'p_2 \cdot p_1xp_1, p_1y'p_2, p_2)p_2\\
&\quad = p_1yp_1xp_1\phi(p_1x'p_2, p_1y'p_2, p_2)p_2.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
&\phi(q_n(p_1xp_1, p_1x'p_2, p_2, \ldots, p_2), q_n(p_1yp_1, p_1y'p_2, p_2, \ldots, p_2), p_2)\\
&\quad = q_n(\phi(p_1xp_1, q_n(p_1yp_1, p_1y'p_2, p_2, \ldots, p_2), p_2), p_1x'p_2, p_2, \ldots, p_2)\\
&\qquad + q_n(p_1xp_1, \phi(p_1x'p_2, q_n(p_1yp_1, p_1y'p_2, p_2, \ldots, p_2), p_2), p_2, \ldots, p_2)\\
&\qquad + \sum_{i=3}^{n} q_n(p_1xp_1, p_1x'p_2, p_2, \ldots, \underbrace{\phi(p_2, q_n(p_1yp_1, p_1y'p_2, p_2, \ldots, p_2), p_2)}_{i-th\ component}, \ldots, p_2)\\
&\quad = p_1xp_1\phi(p_1x'p_2, q_n(p_1yp_1, p_1y'p_2, p_2, \ldots, p_2), p_2)p_2\\
&\quad = p_1xp_1\phi(p_1x'p_2, p_1yp_1y'p_2, p_2)p_2\\
&\quad = p_1xp_1yp_1\phi(p_1x'p_2, p_1y'p_2, p_2)p_2
\end{aligned}$$

for all $x, y, x', y' \in \mathcal{T}$. Comparing the above two relations, we arrive at

$$0 = [p_1xp_1, p_1yp_1]\phi(p_1x'p_2, p_1y'p_2, p_2) = \tau^{-1}(p_2q_{y'}p_2)[p_1xp_1, p_1yp_1]p_1x'p_2$$

for all $x, y, x', y' \in \mathcal{T}$. In view of Property 2.1, we obtain $\tau^{-1}(p_2q_{y'}p_2)[p_1\mathcal{T}p_1, p_1\mathcal{T}p_1] = 0$. It follows that

$$[\tau^{-1}(p_2q_{y'}p_2)p_1\mathcal{T}C(\mathcal{T})p_1, p_1\mathcal{T}C(\mathcal{T})p_1] = 0.$$

Since $p_1\mathcal{T}C(\mathcal{T})p_1$ does not contain nonzero central ideals, we have $\tau^{-1}(p_2q_{y'}p_2) = 0$. Therefore,

$$\phi(p_1xp_2, p_1yp_2, p_2) = 0$$

for all $x, y \in \mathcal{T}$. This implies that

$$\phi(p_1xp_2, p_1yp_2, p_1) = \phi(p_1xp_2, p_1yp_2, I) - \phi(p_1xp_2, p_1yp_2, p_2) = 0$$

for all $x, y \in \mathcal{T}$. Using Lemma 3.2 and Lemma 3.4, we arrive at

$$\begin{aligned}\phi(p_1xp_2, p_1yp_2, p_1zp_1) =&\phi(p_1xp_2, p_1yp_2, p_1)p_1zp_1 + p_1\phi(p_1xp_2, p_1yp_2, p_1zp_1)p_1\\ &+ p_1zp_1\phi(p_1xp_2, p_1yp_2, p_1) = 0,\\ \phi(p_1xp_2, p_1yp_2, p_2zp_2) =&\phi(p_1xp_2, p_1yp_2, p_2)p_2zp_2 + p_2\phi(p_1xp_2, p_1yp_2, p_2zp_2)p_2\\ &+ p_2zp_2\phi(p_1xp_2, p_1yp_2, p_2) = 0\end{aligned}$$

for all $x, y, z \in \mathcal{T}$. Analogously, we have

$$\begin{aligned}\phi(p_1xp_1, p_1yp_2, p_1zp_2) = \phi(p_2xp_2, p_1yp_2, p_1zp_2) = 0,\\ \phi(p_1xp_2, p_1yp_1, p_1zp_2) = \phi(p_1xp_2, p_2yp_2, p_1zp_2) = 0\end{aligned}$$

for all $x, y, z \in \mathcal{T}$. Therefore, it can be proved that

$$\begin{aligned}\phi(p_1xp_2, p_1yp_2, p_1zp_2) =&\phi(p_1xp_2, p_1yp_2, I) - \phi(p_1xp_2, p_1yp_2, p_2zp_2)\\ &- \phi(p_1xp_2, p_1yp_2, p_1zp_1) = 0\end{aligned}$$

for all $x, y, z \in \mathcal{T}$. □

Proof of Theorem 3.1. For any $x, y, z \in \mathcal{T}$, we can expand $\phi(x, y, z)$ as follows:

$$\begin{aligned}\phi(x, y, z) =&\phi(p_1xp_1 + p_1xp_2 + p_2xp_2, p_1yp_1 + p_1yp_2 + p_2yp_2, p_1zp_1 + p_1zp_2 + p_2zp_2)\\ =&\phi(p_1xp_1, p_1yp_1, p_1zp_1) + \phi(p_1xp_1, p_1yp_1, p_1zp_2) + \phi(p_1xp_1, p_1yp_1, p_2zp_2)\\ &+ \phi(p_1xp_1, p_1yp_2, p_1zp_1) + \phi(p_1xp_1, p_1yp_2, p_1zp_2) + \phi(p_1xp_1, p_1yp_2, p_2zp_2)\\ &+ \phi(p_1xp_1, p_2yp_2, p_1zp_1) + \phi(p_1xp_1, p_2yp_2, p_1zp_2) + \phi(p_1xp_1, p_2yp_2, p_2zp_2)\\ &+ \phi(p_1xp_2, p_1yp_1, p_1zp_1) + \phi(p_1xp_2, p_1yp_1, p_1zp_2) + \phi(p_1xp_2, p_1yp_1, p_2zp_2)\\ &+ \phi(p_1xp_2, p_1yp_2, p_1zp_1) + \phi(p_1xp_2, p_1yp_2, p_1zp_2) + \phi(p_1xp_2, p_1yp_2, p_2zp_2)\\ &+ \phi(p_1xp_2, p_2yp_2, p_1zp_1) + \phi(p_1xp_2, p_2yp_2, p_1zp_2) + \phi(p_1xp_2, p_2yp_2, p_2zp_2)\\ &+ \phi(p_2xp_2, p_1yp_1, p_1zp_1) + \phi(p_2xp_2, p_1yp_1, p_1zp_2) + \phi(p_2xp_2, p_1yp_1, p_2zp_2)\\ &+ \phi(p_2xp_2, p_1yp_2, p_1zp_1) + \phi(p_2xp_2, p_1yp_2, p_1zp_2) + \phi(p_2xp_2, p_1yp_2, p_2zp_2)\\ &+ \phi(p_2xp_2, p_2yp_2, p_1zp_1) + \phi(p_2xp_2, p_2yp_2, p_1zp_2) + \phi(p_2xp_2, p_2yp_2, p_2zp_2).\end{aligned}$$

According to Lemmas 3.5–3.7, we have $\phi(x, y, z) = 0$ for all $x, y, z \in \mathcal{T}$. Since $\phi = \delta - \psi$, it follows that $\delta = \psi$. This implies that δ is an extremal 3-derivation whenever $\delta(p_1, p_1, p_1) \neq 0$. Moreover, the same lemmas also yield $\delta = 0$ when $\delta(p_1, p_1, p_1) = 0$. Therefore, in all cases, we have $\delta(x, y, z) = [x, [y, [z, \delta(p_1, p_1, p_1)]]]$ for all $x, y, z \in \mathcal{T}$.

4. m-Jordan n-derivations

Theorem 4.1. *Let $\mathcal{T}$ be a triangular ring whose characteristic is not 2 or $n-1$, and let $\zeta : \mathcal{T}^m \to \mathcal{T}$ be an m-Jordan n-derivation, where $m \geq 3$. Suppose that $C(p_2 Q_{ml}(\mathcal{T})p_2, p_2\mathcal{T}p_2) = C(\mathcal{T})p_2$, and that neither $p_1\mathcal{T}C(\mathcal{T})p_1$ nor $p_2\mathcal{T}C(\mathcal{T})p_2$ contains a nonzero central ideal. Then ζ is an extremal m-derivation satisfying $\zeta(a_1, a_2, \ldots, a_m) = [a_1, [a_2, \ldots, [a_m, \zeta(p_1, p_1, \ldots, p_1)] \cdots]]$ for all $a_1, a_2, \ldots, a_m \in \mathcal{T}$.*

Proof. We will prove the theorem by using mathematical induction. According to Theorem 3.1, this conclusion is tenable in the case of $m = 3$. Now assume $m \geq 4$, fix $a_4, \ldots, a_m \in \mathcal{T}$, and let

$$\zeta'(a_1, a_2, a_3) = \zeta(a_1, a_2, a_3, a_4, \ldots, a_m).$$

Then $\zeta'(a_1, a_2, a_3)$ is an extremal 3-derivation. Hence,

$$\zeta'(a_1, a_2, a_3) = [a_1, [a_2, [a_3, \zeta'(p_1, p_1, p_1)]]].$$

Since $\zeta(p_1, p_1, p_1, a_4, \ldots, a_m) = \zeta'(p_1, p_1, p_1)$, we have

$$\zeta(a_1, a_2, \ldots, a_m) = [a_1, [a_2, [a_3, \zeta(p_1, p_1, p_1, a_4, \ldots, a_m)]]]$$

for all $a_1, a_2, \ldots, a_m \in \mathcal{T}$. Obviously, $\zeta(p_1, a_2, a_3, \ldots, a_m)$ is an $(m-1)$-Jordan n-derivation on triangular ring $\mathcal{T}$. Let

$$\zeta(p_1, a_2, a_3, \ldots, a_m) = [a_2, [a_3, \ldots, [a_m, \beta'] \cdots]]$$

for all $a_2, \ldots, a_m \in \mathcal{T}$, where $\beta' = \zeta(p_1, p_1, \ldots, p_1) \in p_1\mathcal{T}p_2$ and $[\beta', [\mathcal{T}, \mathcal{T}]] = 0$. Set $a_2 = a_3 = p_1$, then

$$\zeta(p_1, p_1, p_1, a_4, \ldots, a_m) = [a_4, [a_5, \ldots, [a_m, \beta'] \cdots]]$$

for all $a_4, \ldots, a_m \in \mathcal{T}$. It follows that

$$\zeta(a_1, a_2, \ldots, a_m) = [a_1, [a_2, [a_3, \zeta(p_1, p_1, p_1, a_4, \ldots, a_m)]]] = [a_1, [a_2, \ldots, [a_m, \beta'] \cdots]]$$

for all $a_1, a_2, \ldots, a_m \in \mathcal{T}$. Thus, ζ is an extremal m-derivation. □

Since upper triangular matrix rings are a classic example of triangular rings, we apply Theorem 4.1 to obtain the following corollary.

Corollary 4.1. *Let $T_k(\mathcal{R})$ be an upper triangular matrix ring with $k \geq 3$, and let $\varphi : (T_k(\mathcal{R}))^m \to T_k(\mathcal{R})$ be an m-Jordan n-derivation. Then φ is an extremal m-derivation.*

5. Conclusions

In this paper, we introduced the concept of m-Jordan n-derivations by integrating the notions of m-derivations and Jordan n-derivations. Then we investigated these mappings on triangular rings by utilizing the properties of the maximal left ring of quotients. Our analysis began with a detailed examination of 3-Jordan n-derivations on triangular rings. Subsequently, through an inductive argument, we proved that under specific hypotheses, an m-Jordan n-derivation on a triangular ring is necessarily an extremal m-derivation. Finally, we applied the results we had obtained to the specific context of upper triangular matrix rings.

Author contributions

He Yuan: Conceptualization, funding acquisition, writing-review and editing; Chuqi Jia: Formal analysis, writing-original draft, writing-review and editing; Lili Ma: Formal analysis, writing-original draft. All authors have read and agreed to the published version of the manuscript.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

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