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Research article

On the existence and uniqueness of Q^0-weak solution to linear conditional mean-field fractional SDE

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Abstract: With the help of the reference probability measure concept in filtering theory, we proved the existence and uniqueness of the Q^0-weak solution to a linear conditional mean-field fractional SDE with Hurst parameter $H \in (\frac{1}{2}, 1)$, which will be a foundation of investigating equilibrium of insider trading driven by fractional Brownian motion.

Keywords: conditional mean-field fractional SDE; Q^0-weak solution; filtering theory
Mathematics Subject Classification: Primary 60H10, 91G80; Secondary 60G35, 93E11

1. Introduction

In the study of a model of insider trading, Aase, Bjuland, and Øksendal [1] encountered a new type of stochastic differential equation (SDE) satisfying:

$$dy_t = (v - E[v|\mathcal{F}_t^y])\beta_t dt + \sigma_t dB_t, y_0 = 0 \tag{1.1}$$

where v is normally distributed, B is a standard Brownian motion (in short, Bm) independent of v, both β and σ are real functions of time t, and $E[v|\mathcal{F}_t^y]$ is the conditional expectation of v under the information $\mathcal{F}_t^y = \sigma(y_s, 0 \le s \le t)$. By Kalman-Bucy filtering theory [2, 3], the authors [1] proved the existence of strong solution y to the above dynamic (1.1). Later, Ma, Sun, and Zhou [4] took advantage of the reference probability measure concept in filtering theory [5] and obtained the existence and uniqueness of Q_0-weak solution to a more general signal-observation system, called linear conditional mean-field SDE. The perspective of the existence and uniqueness theory of Q_0-weak solution construction is important, because it has significant application potential on stochastic inverse problems. Of course, many researchers were already engaged in this field and had applied these theoretical works to practical problems, such as in financial insider information and physical system parameters. In 2019, Wang et al. [6] developed a conceptual framework of the novel gesture recognition techniques using electromagnetic waves, and some mathematical justifications were presented and extensive numerical examples were provided to validate the effectiveness and efficiency of the methods. There is much relevant literature studying this subject; see [7, 8]. However, the uniqueness of strong solution to the dynamic (1.1) is still an open question. According to their theory, they [1, 4] developed a corresponding insider trading model and obtained the associated solutions. Later, Yang et al. [9] and Xiao et al. [10] extended these insider trading models to cases where the noise signals are driven by fractional Bm, further refining the special structure of the insider trading models.

To understand the impact of noise traders' memories on insider trading, Biagini, Hu, Meyer-Brandis, and Øksendal [11] considered a fractional SDE as follows:

$$dy_t = (v - E[v|\mathcal{F}_t^y])\beta_t dt + \sigma_t dB_t^H, y_0 = 0 \tag{1.2}$$

where a fractional Bm B^H with Hust parameter $H \in (\frac{1}{2}, 1)$ (in short H-fBm) , describing noise traders' memories, takes place of the Bm B in the dynamic (1.1). With the help of filtering theory on fBm [12], they proved the existence of strong solution y to the dynamic (1.2). They also gave some sufficient conditions of uniqueness of its solution with restriction to the form as $y_t = h_1(t) + \int_0^t h_2(t, s)dB_t^H$ for two unknown functions h_1 and h_2. However, the sufficient conditions, as in Proposition 3.1 in [11] are too hard to check. Recently, there have been many experts studying this topic; see [13, 14].

Then, a natural question yields: can the reference probability measure concept in filtering theory [5] be applied to study the well-posedness of the Q^0-weak solution to the dynamic (1.2) driven by fBm? In this paper, our main object is to give a positive answer to this question, which will be a foundation for us to study insider trading driven by fBm in finance.

2. LCMFF SDE with solution concepts

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space satisfying the usual hypotheses [15]. Consider a linear conditional mean-field fractional SDE (in short, LCMFF SDE) as follows, covering the Eq (1.2):

$$dy_t = (\alpha_t v + \beta_t E[v|\mathcal{F}_t^y])dt + \sigma_t dB_t^H, y_0 = 0 \tag{2.1}$$

where v is normally distributed $N(0, 1)$, B^H is an H-fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$ and independent of v, both α and β are continuous and deterministic functions on a finite interval $[0, T]$, σ is a positive, continuous, and deterministic function on $[0, T]$, and $E[v|\mathcal{F}_t^y]$ is the conditional expectation of v under the information $\mathcal{F}_t^y$. In this paper, we understand the integral of deterministic functions with respect to H-fBm in the sense of that described in [16] and others are usual Lebesgue's or Lebesgue-Stieltjes' integral.

Suggested by Ma, Sun, and Zhou [4], we introduce a reference probability space as in the filtering theory [5] below.

Assumption 2.1. There exists a new probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which the above process y_t is a continuous, Gaussian centered process with covariance

$$E^{\mathbb{Q}^0}[y_s y_t] = H(2H-1)\int_0^t \int_0^s \sigma_s \sigma_t |s-t|^{2H-2} ds dt, \quad 0 \le s \le t \le T.$$

The probability measure $\mathbb{Q}^0$ will be referred to as the reference measure, on which y is an H-fBM when $\sigma_t \equiv 1$.

As the remark in [4, Assumption 2.2], our Assumption 2.1 here amounts to saying that the process y is always considered as an observation process and that a *prior* distribution is endowed to the process y, which is not unusual in statistical modeling. To simplify notations in what follows, we shall assume that the volatility function of noise order is $\sigma \equiv 1$, and without confusion, $t \in [0, T]$ will be omitted in many places in the following.

Definition 2.2. A seven-tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a weak solution to the LCMFF SDE (2.1) if

(i) $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space satisfying the usual hypotheses;

(ii) v is normally distributed with $N(0, 1)$ under $\mathbb{P}$;

(iii) B^H is an $\mathcal{F}_t$-adapted H-fBm independent of v under $\mathbb{P}$, and y is an $\mathcal{F}_t$-adapted continuous process;

(iv) (v, y, B^H) satisfies (2.1), $\mathbb{P}$-a.s.

Definition 2.3. A weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a $\mathbb{Q}^0$-weak solution to the LCMFF SDE (2.1) if

(i) there exists a weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that the law of $(v^0, y^0, B^{H,0})$ under $\mathbb{P}^0$ is the same as that of (v, y, B^H) under $\mathbb{P}$; and

(ii) $\mathbb{P}^0 \sim \mathbb{Q}^0$.

In what follows for any $\mathbb{Q}^0$-weak solution, we shall consider only its copy on the reference measurable space $(\Omega, \mathcal{F}^0)$, and we shall still denote the solution by (v, y, B^H). Now, $\mathbb{Q}^0$-*pathwise uniqueness* of the solution to (2.1) is introduced below.

Definition 2.4. The LCMFF SDE (2.3) is said to have $\mathbb{Q}^0$-*pathwise uniqueness* if for any two $\mathbb{Q}^0$-weak solutions $(\Omega, \mathcal{F}^0, \mathbb{F}^0, \mathbb{P}^i; v^i, y^i, B^{H,i})$, $i = 1, 2$, such that

(i) $v^1 = v^2$ with normally distributed $N(0, 1)$; and

(ii) $\mathbb{Q}^0\{y_t^1 = y_t^2, \forall t \in [0, T]\} = 1$,

then it holds that $\mathbb{Q}^0\{B_t^{H,1} = B_t^{H,2}, \forall t \in [0, T]\} = 1$, and $\mathbb{P}^1 = \mathbb{P}^2$.

3. Two lemmas

To establish our theorem in this paper, we need two lemmas: One is about the existence and uniqueness of solution to a related SDE; the other is about the consistence of two filters.

Lemma 3.1. Let α be a deterministic function in $C^2[0, T]$ and β be a deterministic, measurable, and bounded function. Suppose that under a reference probability pace $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, there is a normally distributed random v with $N(0, 1)$, an H-fBm process y independent of v. Then, there is a unique strong solution of process p satisfying the following SDE:

$$p_t = \int_0^t \rho_s \gamma_s (dy_s^* - f_s(p) dl_s) \tag{3.1}$$

where the functions or processes above are defined as follows:

$$\rho_t = \frac{d}{dl_t} \int_0^t k_H(t, s) \alpha_s ds, \quad \gamma_t = \frac{1}{1 + \int_0^t \rho_s^2 dl_s},$$

$$y_t^* = \int_0^t k_H(t, s) dy_s, \quad f_t(p) = \rho_t p_t + \frac{d}{dl_t} \int_0^t k_H(t, s) p_s \beta_s ds$$

with the well-known functions (used frequently in the following)

$$\Gamma(t) = \int_0^{+\infty} x^{t-1}e^{-x}dx, \;\; dl_t = \frac{(2-2H)t^{1-2H}}{2H\Gamma(\frac{3}{2}-H)\Gamma(\frac{1}{2}+H)}dt$$

and

$$k_H(t,s) = \frac{s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}}{2H\Gamma(\frac{3}{2}-H)\Gamma(\frac{1}{2}+H)}, \;\; 0 < s < t \le T.$$

Proof. Note that $\alpha \in C^2[0,T]$, then ρ and γ are also in $C^2[0,T]$ (see a more general result in [11, Lemma 5.1]); and then by applying integration by parts and Fubini's theorem,

$$\int_0^t \rho_s\gamma_s(\frac{d}{dl_s}\int_0^s k_H(s,u)p_u\beta_u du)dl_s = \int_0^t p_s[\rho_t\gamma_t k_H(t,s)\beta_s - \beta_s\int_s^t k_H(u,s)d(\rho_u\gamma_u)]ds.$$

So the process p in system (3.1) can be written as in an integral form:

$$p_t = \int_0^t \rho_s\gamma_s dy_s^* - \int_0^t p_s[\rho_s^2\gamma_s + \rho_t\gamma_t k_H(t,s)\beta_s - \beta_s\int_s^t k_H(u,s)d(\rho_u\gamma_u)]ds.$$

Now, if denote for $0 \le r \le t$,

$$p_r^t = \int_0^r \rho_s\gamma_s dy_s^* - \int_0^r p_s[\rho_s^2\gamma_s + \rho_t\gamma_t k_H(t,s)\beta_s - \beta_s\int_s^t k_H(u,s)d(\rho_u\gamma_u)]ds$$

or in differential form:

$$dp_r^t = \rho_r\gamma_r dy_r^* - p_r^t[\rho_r^2\gamma_r + \rho_t\gamma_t k_H(t,r)\beta_r - \beta_r\int_r^t k_H(u,r)d(\rho_u\gamma_u)]dr,$$

then the above linear Ornstein-Uhlenbeck type SDE has a unique strong solution of p^t on [0,t], $\mathbb{P}$ a.s., since for an H-fBM y, $y_r^* = \int_0^r K_H(r,u)dy_u$ is a continuous Gaussian martingale [16].

Since $p_t = p_t^t$, the SDE (3.1) has a unique strong solution of p.

Now consider the following fractional SDE:

$$dy_t = (\alpha_t v + h_t(y))dt + dB_t^H, y_0 = 0 \tag{3.2}$$

where α_t is a continuous, positive, and deterministic function on [0,T], and $h : [0,T] \times \mathbb{C}([0,T]) \mapsto \mathbb{R}$ is progressively measurable.

Lemma 3.2. Suppose that the fractional SDE (3.2) has a weak solution $(\Omega, \mathcal{F}, \mathbb{P}; v, y, B^H)$ such that $v \sim N(0,1)$, B^H is an H-fBm, and both are independent. Define

$$Q(t) = \frac{d}{dl_t}\int_0^t k_H(t,s)(\alpha_s v + h_s(y))ds, \quad \hat{Q}(t) = \frac{d}{dl_t}\int_0^t k_H(t,s)\alpha_s v ds \tag{3.3}$$

$$\Lambda(T) = e^{-\int_0^T Q(s)dB_s^{H*} - \frac{1}{2}\int_0^T Q^2(s)dl_s}, \quad B_t^{H*} = \int_0^t k_H(t,s)dB_s^H.$$

If the simple paths of Q and $\hat{Q}$ belong $\mathbb{P}$ a.s. to $L^2([0,T], dl)$ and $E\Lambda(T) = 1$, then

$$E[v|\mathcal{F}_t^y] = E[v|\mathcal{F}_t^{\hat{y}}], \qquad \mathbb{P} - a.s. \tag{3.4}$$

where

$$d\hat{y}_t = \alpha_t v dt + dB_t^H, \hat{y}_0 = 0.$$

Proof. Clearly, it suffices to prove $E[v|\mathcal{F}_T^y] = E[v|\mathcal{F}_T^{\hat{y}}]$, as the cases for $t < T$ are analogous.

Define

$$y_t^* = \int_0^t k_H(t,s)dy_s, \qquad \hat{y}_t^* = \int_0^t k_H(t,s)d\hat{y}_s.$$

Then, by [12, Theorem 1], the following two assertions hold:

(i) Both y^* and $\hat{y}^*$ are $(\mathcal{F}_t)$-semimartingales with the decompositions, respectively,

$$y_t^* = \int_0^t Q(s)dl_s + B_t^{H*}, \quad \hat{y}_t^* = \int_0^t \hat{Q}(s)dl_s + B_t^{H*} \tag{3.5}$$

where B^{H*} is a Gaussian martingale whose variance function $\langle B^{H*}\rangle$ satisfies $d\langle B^{H*}\rangle_t = dl_t$.

(ii) $\mathcal{F}_t^y = \mathcal{F}_t^{y^*}$ and $\mathcal{F}_t^{\hat{y}} = \mathcal{F}_t^{\hat{y}^*}$.

Therefore, we need only to prove

$$E[v|\mathcal{F}_t^{y^*}] = E[v|\mathcal{F}_t^{\hat{y}^*}], \qquad \mathbb{P} - a.s.$$

Since $E\Lambda(T) = 1$, then applying the usual Gisanov theorem (or see the proof of [17, Theorem 3]), we obtain that the distribution of the process y^* under probability $\mathbb{P}^*$ with $d\mathbb{P}^* = \Lambda(T)d\mathbb{P}$ is the same as that of the process B^{H*} under $\mathbb{P}$. Furthermore, v is independent of y^* under $\mathbb{P}^*$ since v is independent of B^H under $\mathbb{P}$ (e.g., see [18, Page 39]). Additionally by the expressions of y^* and $\hat{y}^*$ in (3.5), we have

$$\hat{y}_t^* = y_t^* - \int_0^t K_H(s,u)h_t(y)dt.$$

So according to the equality: $\mathcal{F}_t^y = \mathcal{F}_t^{y^*}$, we get that both $h_t(y)$ and $\hat{y}_t^*$ are $\mathcal{F}_T^{y^*}$-measurable and then are independent of v under the probability $\mathbb{P}^*$.

By the Kallianpur-Striebel formula, we have

$$E[v|\mathcal{F}_T^{y^*}] = E^{\mathbb{P}}[v|\mathcal{F}_T^{y^*}] = \frac{E^{\mathbb{P}^*}[v\Lambda^{-1}(T)|\mathcal{F}_T^{y^*}]}{E^{\mathbb{P}^*}[\Lambda^{-1}(T)|\mathcal{F}_T^{y^*}]}. \tag{3.6}$$

Now denote

$$\tilde{\alpha}_t = \frac{d}{dl_t}\int_0^t K_H(t,s)\alpha_s ds, \ \tilde{h}_t(y) = \frac{d}{dl_t}\int_0^t K_H(t,s)h_s(y)ds.$$

Then, $\tilde{\alpha}_t$ is a measurable deterministic function, and $\tilde{h}_t(y)$ is $\mathcal{F}_T^{y^*}$-measurable and independent of v under the probability $\mathbb{P}^*$. Moreover, the process Q in (3.3) and the process y^* in (3.5) can be written, respectively, as

$$Q(t) = \tilde{\alpha}_t v + \tilde{h}_t(y), \ \ y_t^* = \hat{y}_t^* + \int_0^t \tilde{h}_s(y)dl_s.$$

So, the random variable $\Lambda^{-1}(T) = e^{\int_0^T Q(s)dy_s^* - \frac{1}{2}\int_0^T Q^2(s)dl_s}$ can be written as

$$\Lambda^{-1}(T) = \lambda_T(v,\hat{y}^*)\lambda_T^*(y,y^*)$$

where

$$\lambda_T(v, \hat{y}^*) = e^{v \int_0^T \tilde{\alpha}_t d\hat{y}_t^* - \frac{1}{2} v^2 \int_0^T \tilde{\alpha}_t^2 dl_t}, \ \lambda_T^*(y, y^*) = e^{\int_0^T \tilde{h}_t(y) dy_t^* - \frac{1}{2} \int_0^T \tilde{h}_t^2(y) dl_t}.$$

Therefore, the formula (3.6) becomes

$$E[v|\mathcal{F}_T^{y^*}] = E^{\mathbb{P}}[v|\mathcal{F}_T^{y^*}] = \frac{E^{\mathbb{P}^*}[v\lambda_T(v, \hat{y}^*)|\mathcal{F}_T^{y^*}]}{E^{\mathbb{P}^*}[\lambda_T(v, \hat{y}^*)|\mathcal{F}_T^{y^*}]}. \tag{3.7}$$

Since v is independent of $\hat{y}_t^*$, a monotone class argument shows that $E^{\mathbb{P}^*}[\lambda_T(v, \hat{y}^*)|\mathcal{F}_T^{y^*}]$ is $\mathcal{F}_T^{\hat{y}^*}$-measurable. Similarly, $E^{\mathbb{P}^*}[v\lambda_T(v, \hat{y}^*)|\mathcal{F}_T^{y^*}]$ is also $\mathcal{F}_T^{\hat{y}^*}$-measurable. Consequently, $E[v|\mathcal{F}_T^{y^*}]$ is $\mathcal{F}_T^{\hat{y}^*}$-measurable, thanks to (3.7).

Finally, noting $\mathcal{F}_T^{\hat{y}^*} \subseteq \mathcal{F}_T^{y^*}$, we have

$$E[v|\mathcal{F}_T^{y}] = E^{\mathbb{P}}[v|\mathcal{F}_T^{y^*}] = E^{\mathbb{P}}\{E^{\mathbb{P}}[v|\mathcal{F}_T^{y^*}]|\mathcal{F}_T^{\hat{y}^*}\} = E^{\mathbb{P}}[v|\mathcal{F}_T^{\hat{y}^*}] = E[v|\mathcal{F}_T^{\hat{y}}], \tag{3.8}$$

and the proof is complete.

4. Main theorem

To give the main theorem in this paper, we need to emphasize α is a deterministic function in $C^2[0, T]$, and β is a measurable , deterministic, and bounded function. The main theorem is stated below.

Theorem 4.1. Suppose that the assumptions in Lemma 3.1 hold and that

$$E^{\mathbb{Q}^0} e^{\int_0^T Q^*(t) dy_t^* - \frac{1}{2} \int_0^T (Q^*(t))^2 dl_t} = 1$$

where $Q^*(t) = \frac{d}{dl_t} \int_0^t k_H(t, s)(\alpha_s v + \beta_s p_s) dl_s$ belongs to $L^2([0, T], dl)$ $\mathbb{Q}^0$ a.s.

Then, the following LCMFF SDE

$$dy_t = (\alpha_t v + \beta_t E[v|\mathcal{F}_t^y])dt + dB_t^H, y_0 = 0 \tag{4.1}$$

possesses a weak solution, denoted by $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}; v, y, B^H)$; and under $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, the filter $E[v|\mathcal{F}_t^y]$ satisfies the above SDE (3.1).

Moreover, the weak solution can be chosen as a $\mathbb{Q}^0$-weak solution, and the $\mathbb{Q}^0$-pathwise uniqueness holds.

We remark that Theorem 4.1 does not imply that LCMFF SDE (4.1) has a strong solution, as not every weak solution is a $\mathbb{Q}^0$-weak solution.

4.1. *Proof of existence*

Our main idea to prove the existence of the weak solution to (4.1) is by introducing a conjectured system of the filtered state process p_t playing the role of $E[v|\mathcal{F}_t^y]$. The proof is broken into two steps: (1) first, we establish the existence and uniqueness of a strong solution to the conjectured system on a reference probability space; (2) second, based the former result and a Girsanov type formula of [17, Theorem 3], we prove the existence of the weak solution to (4.1):

(1) Let $(\Omega^0, \mathcal{F}^0, \mathbb{Q}^0)$ be a reference probability space on which an observation process y satisfies Assumption 2.1 which is an H-fBm now since $\sigma \equiv 1$ [19]. We consider the following system of SDEs with respect to a pair of processes (B^H, p) on the reference probability space $(\Omega^0, \mathcal{F}^0, \mathbb{Q}^0)$:

$$\begin{cases} dB_t^H = -(\alpha_t v + \beta_t p_t)dt + dy_t, \ B_0^H = 0; \\ dp_t = \rho_t \gamma_t (dy_t^* - f_t(p)dl_t), \ p_0 = 0 \end{cases} \tag{4.2}$$

where all of the functions $\alpha, \beta, \rho, \gamma$, and l_s and the processes f and y^* above are defined the same as those in Theorem 4.1, respectively, and $v \sim N(0, 1)$ is independent of y.

Clearly, by Lemma 3.1, the above system (4.2) of SDEs has a unique strong solution of (B^H, p).

(2) We can now prove the existence of the weak solution to (4.1).

Since by assumption, $E^{\mathbb{Q}^0}\Lambda^*(T) = 1$ where $\Lambda^*(T) = e^{-\int_0^T Q^*(t)dy_t^* - \frac{1}{2}\int_0^T (Q^*(t))^2 dl_t}$, then by the Girsanov type formula of Theorem 3 in [17], under the probability measure $\mathbb{P}$ satisfying $d\mathbb{P} = \Lambda^*(T)d\mathbb{Q}^0$, the probability distribution of the process B^H under $\mathbb{P}$ is the same as that of the process y under $\mathbb{Q}^0$. Then, B^H is an H-fBm under $\mathbb{P}$, and also, is independent of v under $\mathbb{P}$ since y is independent of v under $\mathbb{Q}^0$. So we can see that $(\Omega, \mathcal{F}, \mathbb{P}, v, y, B^H)$ is a weak solution to (4.1) if it holds that

$$p_t = E[v|\mathcal{F}_t^y], t \in [0, T], \mathbb{P} - a.s. \tag{4.3}$$

To prove the above Eq (4.3), we proceed as follows. We consider on the space $(\Omega^0, \mathcal{F}^0, \mathbb{P})$ the following linear filtering problem with v a signal and $\hat{y}$ an observation process:

$$d\hat{y}_t = \alpha_t v dt + dB_t^H, \hat{y}_0 = 0. \tag{4.4}$$

Denote for $t \in [0, T]$, $\hat{p}_t = E[v|\mathcal{F}_t^{\hat{y}}]$, and $\hat{\gamma}_t = E(v - \hat{p}_t)^2$. Then, by the filtering result in [12, Section 5.1] (or see [11]),

$$d\hat{p}_t = \rho_t \hat{\gamma}_t d(\hat{y}_t^* - \rho_t \hat{p}_t dl_t), \qquad \hat{p}_0 = 0, \qquad t \in [0, T] \tag{4.5}$$

where $\hat{y}_t^* = \int_0^t K_H(t, s)d\hat{y}_s$, $\rho_t = \frac{d}{dl_t}\int_0^t K_H(t, s)\alpha_s ds$, and $\hat{\gamma}_t = \gamma_t = \frac{1}{1+\int_0^t \rho_s^2 dl_s}$.

Then, by the expression of p in (4.2) and that of $\hat{p}$ in (4.5), we can get $\Delta p_t = p_t - \hat{p}_t$ satisfying

$$d\Delta p_t = \rho_t^2 \gamma_t \Delta P_t dl_t, \ \ \Delta p_0 = 0.$$

So for any $t \in [0, T]$, $\Delta p_t \equiv 0$ or $p_t \equiv \hat{p}_t$, $\mathbb{P}$-a.s.

By Lemma 3.2, we have

$$E[v|\mathcal{F}_t^y] = E[v|\mathcal{F}_t^{\hat{y}}] = \hat{p}_t, t \in [0, T].$$

Therefore, $p_t = E[v|\mathcal{F}_t^y]$ holds, which proves the existence.

It is worth noting that the weak solution that we have constructed is actually a $\mathbb{Q}^0$-weak solution.

4.2. *Proof of uniqueness*

We now turn to prove the uniqueness of the $\mathbb{Q}^0$-weak solution to (4.1).

Let $(\Omega^0, \mathcal{F}^0, \mathbb{F}^0, \mathbb{P}; v, y, B^H)$ be a $\mathbb{Q}^0$-weak solution to (4.1). Then, without loss of generality, we can assume that $\mathbb{F}^0 = \mathbb{F}^v \vee \mathbb{F}^{B^H}$. Denote $\tilde{P}_t = E[v|\mathcal{F}_t^y]$. We are next to show that $\tilde{P}_t$ satisfies an SDE of the form as that in (4.2) under $\mathbb{Q}^0$, from which the $\mathbb{Q}^0$-pathwise uniqueness can be derived.

To this end, we recall that, as a $\mathbb{Q}^0$-weak solution, one has $\mathbb{P} \sim \mathbb{Q}^0$. Define a $\mathbb{P}$-martingale $Z_t \equiv E^{\mathbb{P}}[\frac{d\mathbb{Q}^0}{d\mathbb{P}}|\mathcal{F}_t]$, $t \geq 0$.

Since

$$dy_t = (\alpha_t v + \beta_t \tilde{P}_t)dt + B_t^H dt, y_0 = 0, \quad t \in [0, T], \tag{4.6}$$

and B^H is an H-fBm under $\mathbb{P}$, then by Theorem 1 in [12],

$$\bar{y}_t = \int_0^t K_H(t,s)dy_s = \int_0^t \bar{Q}(t)dl_t + \bar{B}_t$$

is an $\mathcal{F}_t^0$-semimartingale under P and $\mathcal{F}_t^y = \mathcal{F}_t^{\bar{y}}$, where

$$\bar{Q}(t) = \frac{d}{dl_t}\int_0^t K_H(t,s)(\alpha_s v + \beta_t \tilde{p}_s)ds, \quad \bar{B}_t^H = \int_0^t K_H(t,s)dB_s^H$$

and $\bar{B}$ is a Gaussian martingale with variance function $\langle \bar{B} \rangle$ satisfying $d\langle \bar{B} \rangle_t = dl_t$ for $t \in [0, T]$.

Then, by the Girsanov-Myer theorem or Theorem III-20 in [15], $\bar{y}$ is an $\mathcal{F}_t^0$-semimartingale under $\mathbb{Q}^0$ due to the equivalence of $\mathbb{P}$ and $\mathbb{Q}_0$ and has a unique decomposition:

$$\bar{y}_t = N_t + A_t$$

where N is a $\mathbb{Q}^0$-local martingale of the form

$$N_t = \bar{B}_t - \int_0^t \frac{1}{Z_s}d[Z, \bar{B}]_s, \tag{4.7}$$

and A is a finite variation process. Since, by assumption, y is an H-fBm under $\mathbb{Q}^0$ and $\bar{y}$ is a Gausian continuous martingale under $\mathbb{Q}^0$ [19], we have $A \equiv 0$. In other words, it must hold that

$$\bar{y}_t = \bar{B}_t - \int_0^t \frac{1}{Z_s}d[Z, \bar{B}]_s, t \in [0, T]. \tag{4.8}$$

Consider now a $(\mathbb{F}, \mathbb{P})$-martingale $dM_t = Z_t^{-1}dZ_t$. By applying the Martingale representation theorem we see that there exists a process $\theta \in L^2_{\mathbb{F}}([0, T])$ such that $dM_t = \theta_t d\bar{B}_t$, $t \in [0, T]$. Thus, (4.8) amounts to saying that

$$\bar{y}_t = \bar{B}_t - [M, \bar{B}]_t = \bar{B}^2 - \int_0^t \theta_s dl_s, \quad t \in [0, T].$$

Comparing this to (4.6), we have $\theta_t = \bar{Q}(t)$. That is, Z is the solution to the SDE:

$$dZ_t = Z_t dM_t = -Z_t \bar{Q}(t)d\bar{B}_t, \quad t \in [0, T], \quad Z_0 = 1, \tag{4.9}$$

and, hence, it can be written as the Doléans-Dade stochastic exponential:

$$Z_t = e^{-\int_0^t \bar{Q}(s)d\bar{B}_s - \frac{1}{2}\int_0^t (Q(s))^2 dl_s}. \tag{4.10}$$

Let us now consider again the following filtering problem with respect to v on probability space $(\Omega^0, \mathcal{F}^0, \mathbb{P})$.

$$d\hat{y}_t = \alpha_t v dt + dB_t^H, \quad \hat{y}_0 = 0. \tag{4.11}$$

As before, we know that $\hat{p}_t = E[v|\mathcal{F}_t^{\hat{y}}]$ satisfies the SDE:

$$d\hat{p}_t = \rho_t\gamma_t(d\hat{y}_t^* - \rho_t\hat{p}_t dl_t), \hat{p}_0 = 0, \quad t \in [0, T] \tag{4.12}$$

where $\hat{y}_t^* = \int_0^t K_H(t,s)d\hat{y}_s$, $\rho_t = \frac{d}{dl_t}\int_0^t K_H(t,s)\alpha_s ds$, and $\gamma_t = \frac{1}{1+\int_0^t \rho_s^2 dl_s}$.

Since Z is a $\mathbb{P}$-martingale, we can apply Lemma 3.2 again to conclude that

$$\tilde{p}_t = E[v|\mathcal{F}_t^y] = E[v|\mathcal{F}_t^{\hat{y}}] = \hat{p}_t.$$

Moreover, since

$$\hat{y}_s^* = \int_0^t K_H(t,s)d\hat{y}_s = \int_0^t K_H(t,s)dy_s - \int_0^t K_H(t,s)\beta_s\tilde{p}_s ds$$

which can be written as

$$\hat{y}^* = y_t^* - \int_0^t \frac{d}{dl_s}\int_0^s K_H(s,u)\beta_u\tilde{p}_u dl_s,$$

then bringing it into SDE (4.12) and replacing $\hat{p}$ in (4.12) by $\tilde{p}$, we get the dynamic of $\tilde{p}$:

$$d\tilde{p}_t = \rho_t\gamma_t(dy_s^* - (\rho_t\tilde{p}_t + \frac{d}{dl_t}\int_0^t K_H(t,s)\beta_s\tilde{p}_s ds)dl_t), \quad \tilde{p}_0 = 0, \quad t \in [0, T]. \tag{4.13}$$

That is, $\tilde{p}_t$ satisfies the same SDE as p_t does in (4.2) on the reference space $(\Omega^0, \mathcal{F}^0, \mathbb{Q}^0)$.

To finish the argument, let $(\Omega, \mathcal{F}, \mathbb{P}^i, \mathbb{F}, v^i, y^i, B^{H,i})$, $i = 1, 2$ be any two $\mathbb{Q}^0$-weak solutions, and define $\tilde{p}_t^i \triangleq E^{\mathbb{P}^i}[v^i|\mathcal{F}_t^{y^i}]$, $t \geq 0$, $i = 1, 2$. Then, the arguments above show that $(v^i, B^{H,i}, \tilde{p}^i)$, $i = 1, 2$, are two solutions to the linear system of SDEs (4.2), under $\mathbb{Q}^0$. Thus, if $y^1 \equiv y^2$ under $\mathbb{Q}^0$, then we must have $(v^1, B^{H,1}, \tilde{p}^1) \triangleq (v^2, B^{H,2}, \tilde{P}^2)$, under $\mathbb{Q}^0$, which in turn shows, in light of (4.10), that $\mathbb{P}^1 = \mathbb{P}^2$. This proves the $\mathbb{Q}^0$-pathwise uniqueness of solution to (4.1).

5. Conclusions

In this paper, we establish the existence of both a strong solution and a weak solution on a reference probability space. Notably, it is worth noting that the weak solution that we have constructed is actually a $\mathbb{Q}^0$-weak solution, and the $\mathbb{Q}^0$-pathwise uniqueness holds. However, the uniqueness of the strong solution has not been proven, and this will be a central focus of future work. Additionally, we aim to explore the application of these theoretical findings to real-world economic models.

In addition, we would like to remark that that this work is mainly from the theoretical point of view.

Author contributions

In this paper, all authors have equal contributions.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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