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*Research article*

# Generalized Weierstrass-Enneper representation for minimal surfaces in $\mathbb{R}^4$

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**Abstract:** In this study, we present a generalized Weierstrass-Enneper representation for minimal surfaces in four-dimensional Euclidean space. We derive both parametric (explicit) and algebraic (implicit) representations of several example minimal surfaces, examine their differential-geometric properties, and visualize them through orthogonal projections from $\mathbb{R}^4$ into $\mathbb{R}^3$.

**Keywords:** $\mathbb{R}^4$; minimal surface; generalized Weierstrass-Enneper representation; Gauss map; holomorphic curves
**Mathematics Subject Classification:** 53A10, 53C42

## 1. Introduction

In the 19th century, Weierstrass and Enneper independently developed a fundamental representation for minimal surfaces in $\mathbb{R}^3$, using holomorphic functions to encode geometric data. Now known as the Weierstrass-Enneper representation, it provides an elegant bridge between complex analysis and differential geometry by yielding explicit parametrizations through isotropic null curves in $\mathbb{C}^3$ [1, 2].

There is a close analogy with four-dimensional space, since quaternions can be identified with $\mathbb{R}^4$, and the quaternionic projective line $\mathbb{H}P^1$ provides a model for the conformal 4-sphere. For further details, see Burstall et al. [3]. Friedrich [4] highlighted the role of spinor fields in describing surface immersions via the Dirac equation, establishing natural connections to conformal geometry in four-dimensional space.

Hoffman and Osserman [5] extended the Weierstrass-Enneper representation to minimal surfaces in $\mathbb{R}^4$. De Oliveira [6] constructed examples of nonorientable minimal surfaces in $\mathbb{R}^4$. The $n$-dimensional generalization was subsequently formulated by Jorge and Meeks [7].

In this work, we focus on the case $n = 4$ and construct a representation for conformal minimal immersions. Although the surface remains two-dimensional, its embedding space is richer, possessing two linearly independent normal directions. Our construction uses four holomorphic functions that satisfy a natural isotropy condition in $\mathbb{C}^4$, and we provide explicit inverse formulas for recovering the Weierstrass data from the immersion coordinates.

This representation preserves the geometric essence of the classical case, while revealing new algebraic and differential features that arise in higher dimensions. It further connects to generalized algebraic identities, such as higher-dimensional Pythagorean-type relations, thereby providing a bridge between minimal surface theory and matrix-based curvature frameworks [8].

Section 2 recalls the holomorphic null-curve framework in $\mathbb{R}^4$ and explains in detail why isotropy implies conformality and minimality. Section 3 presents a generalized Weierstrass-Enneper formula in $\mathbb{R}^4$ based on $(f, g, h)$-data, including a fully expanded proof and an explicit immersion regularity criterion. This section constructs orthonormal normal frames and provides geometric intuition for curvature computations. Section 4 develops explicit families $\mathcal{S}_t$ ($t = 1, 2, 3$) and their algebraic projections. Appendix A documents the elimination procedures used to obtain the implicit equations, while the strengthened Gauss-map subsection clarifies the two holomorphic components $g_\pm$ and their geometric significance. Appendix B provides the computation of the orthogonal normal vectors. For completeness, Appendix C contains the construction of the corresponding Monge-type differential relationships in $\mathbb{R}^4$.

## 2. Weierstrass-Enneper representation in $\mathbb{R}^4$

Let $\Sigma \subset \mathbb{C}$ be a simply connected domain. Suppose $f_k : \Sigma \to \mathbb{C}$ for $k = 1, 2, 3, 4$ are holomorphic functions satisfying the isotropy condition

$$f_1^2 + f_2^2 + f_3^2 + f_4^2 = 0.$$

Then the map $X : \Sigma \to \mathbb{R}^4$ defined by

$$X(z) = \mathrm{Re} \int_{z_0}^{z} (f_1(\zeta), f_2(\zeta), f_3(\zeta), f_4(\zeta))\, d\zeta$$

yields a conformal minimal immersion, provided the regularity condition

$$|f_1|^2 + |f_2|^2 + |f_3|^2 + |f_4|^2 \neq 0$$

holds throughout $\Sigma$. This construction provides a natural generalization of the classical Weierstrass-Enneper representation in $\mathbb{R}^3$ to the four-dimensional setting $\mathbb{R}^4$.

### 2.1. *Why the isotropy condition ensures conformality and minimality*

Let $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ be the holomorphic $\mathbb{C}^4$-valued 1-form

$$\partial X \;=\; \Phi\, dz, \qquad \Phi(z) = \big(f_1(z), f_2(z), f_3(z), f_4(z)\big).$$

Then $X(z) = \mathrm{Re}\int_{z_0}^{z} \Phi(\zeta)\, d\zeta$. Write $z = u + \mathrm{i}v$. Because $\Phi$ is holomorphic,

$$X_u \;=\; \mathrm{Re}\,\Phi, \qquad X_v \;=\; -\,\mathrm{Im}\,\Phi, \qquad X_u - \mathrm{i}X_v \;=\; \Phi.$$

Using the complex bilinear extension of the Euclidean inner product on $\mathbb{R}^4$ to $\mathbb{C}^4$, one finds

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle.$$

If $\sum_{k=1}^{4} \phi_k^2 = 0$, then $F = 0$ and $E = G = \frac{1}{2}\sum_{k=1}^{4} |\phi_k|^2$. Hence the immersion is conformal. For conformal immersions, $\mathbf{H} = 0$ iff $\partial_{\bar{z}}\Phi = 0$. Since $\Phi$ is holomorphic, $\mathbf{H} \equiv 0$, so the immersion is minimal.

## 3. Alternative Weierstrass-Enneper representation in $\mathbb{R}^4$

We now present a generalization that mirrors the classical Weierstrass-Enneper formulas in $\mathbb{R}^3$.

**Theorem 3.1** (Generalized Weierstrass-Enneper representation in $\mathbb{R}^4$)**.** *Let $f$, $g$, and $h$ be holomorphic functions on a domain $D \subset \mathbb{C}$. Then the mapping*

$$X(\omega, \bar{\omega}) = \mathrm{Re} \int \left( \frac{1}{2} f \left( 1 - g^2 - h^2 \right), \frac{\mathrm{i}}{2} f \left( 1 + g^2 + h^2 \right), fg, fh \right) d\omega \tag{3.1}$$

*defines a conformal minimal immersion $X : D \to \mathbb{R}^4$, where $\omega = u + \mathrm{i}v \in \mathbb{C}$.*

*Proof of Theorem 3.1 (nullity, conformality, minimality, regularity).* Set

$$\Phi(\omega) = (\phi_1, \phi_2, \phi_3, \phi_4)\, d\omega, \quad \phi_1 = \frac{1}{2} f(1 - g^2 - h^2), \;\; \phi_2 = \frac{\mathrm{i}}{2} f(1 + g^2 + h^2), \;\; \phi_3 = fg, \;\; \phi_4 = fh,$$

with $f, g, h$ holomorphic on $D \subset \mathbb{C}$.
**Nullity.** A direct computation gives

$$\phi_1^2 + \phi_2^2 \;=\; -f^2(g^2 + h^2), \qquad \phi_3^2 + \phi_4^2 \;=\; f^2(g^2 + h^2),$$

hence $\sum_{k=1}^4 \phi_k^2 = 0$. Thus, $\Phi$ is a holomorphic null curve in $\mathbb{C}^4$.
**Conformality.** With $X(\omega) = \mathrm{Re} \int^{\omega} \Phi$, we have $X_u - \mathrm{i} X_v = \Phi$. The nullity implies $F = 0$ and $E = G = \frac{1}{2} \sum_{k=1}^4 |\phi_k|^2$.
**Minimality.** Since $\Phi$ is holomorphic, $\partial_{\bar{\omega}} \Phi = 0$, so the mean curvature vector of a conformal immersion vanishes identically. Therefore, $X$ is minimal.
**Regularity (immersion).** We compute

$$E = G = \frac{1}{2} \sum_{k=1}^{4} |\phi_k|^2 = \frac{1}{2} |f|^2 \Big( \frac{1}{4} |\, 1 - g^2 - h^2 \,|^2 + \frac{1}{4} |\, 1 + g^2 + h^2 \,|^2 + |g|^2 + |h|^2 \Big).$$

The bracket is strictly positive for all $g, h \in \mathbb{C}$; hence, $E > 0$ wherever $f \neq 0$. More generally, $E > 0$ at a point whenever $(\phi_1, \phi_2, \phi_3, \phi_4) \neq 0$. Choosing $f$ so that its zeros dominate possible poles of $g$ and $h$ gives meromorphic Weierstrass data with at most isolated branch points. □

Note the following simple regularity criterion: If $f$ is holomorphic and never vanishes on $D$, and $g, h$ are holomorphic on $D$, then the immersion defined by (3.1) is regular on $D$ (i.e., has $E = G > 0$ everywhere). In particular, $X$ is a conformal minimal immersion without branch points.

Indeed, this criterion is true: from the expression for $E$ above, $E > 0$ wherever $f \neq 0$, since the bracketed term is strictly positive for all $g, h \in \mathbb{C}$.

### *3.1. The two Gauss maps and their meaning*

Let $X : \Sigma \to \mathbb{R}^4$ be a conformal minimal immersion with $\partial X = \Phi = (\phi_1, \phi_2, \phi_3, \phi_4)\, dz$ holomorphic and $\sum_{k=1}^4 \phi_k^2 = 0$. The projective class $[\Phi] \in \mathbb{C}P^3$ lies on the nonsingular quadric

$$Q^2 = \{[w] \in \mathbb{C}P^3 : \; w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0\}.$$

The inverse Segre map identifies $Q^2 \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$, yielding two holomorphic maps

$$g_+(z) = [\phi_1 + \mathrm{i}\phi_2 : \ \phi_3 + \mathrm{i}\phi_4], \qquad g_-(z) = [\phi_1 - \mathrm{i}\phi_2 : \ \phi_3 - \mathrm{i}\phi_4].$$

Geometrically, $g_+$ and $g_-$ encode the oriented tangent plane as a *decomposable* 2-form, split into its self-dual and anti-self-dual components. Via stereographic projection one obtains $S^2$-valued maps $\tilde{g}_\pm$, so the oriented Gauss map $G : \Sigma \to G_{2,4} \cong S^2 \times S^2$ factorizes as $G = (\tilde{g}_+, \tilde{g}_-)$. For a conformal immersion in $\mathbb{R}^4$, minimality is equivalent to *both* $g_+$ and $g_-$ being (meromorphic) holomorphic.

Consider $(f, g, h) = (2\omega^\ell, \omega^\mathfrak{m}, \omega^\mathfrak{n})$. For $\ell = 0, \ \mathfrak{m} = 1, \ \mathfrak{n} = 1$, we have

$$\Phi = \Big(\frac{1}{2}(1 - 2\omega^2), \ \frac{\mathrm{i}}{2}(1 + 2\omega^2), \ \omega, \ \omega\Big)\, d\omega.$$

Hence

$$g_+(z) = \Big[\frac{1}{2}(1 - 2\omega^2) + \frac{\mathrm{i}}{2}(1 + 2\omega^2) \ : \ \omega + \mathrm{i}\omega\Big].$$

Both $g_+$ and $g_-$ are rational (hence meromorphic) functions of $\omega$, so the corresponding $S^2$-valued Gauss maps are holomorphic away from branch points.

## 4. Geometric beauty in $\mathbb{R}^4$: minimal surface constructions

In this section, we consider the minimal surface family $\mathcal{S}_t$ defined by the choice $f(\omega) = 2\omega^\ell$, $g(\omega) = \omega^\mathfrak{m}$, and $h(\omega) = \omega^\mathfrak{n}$ in (3.1) for natural numbers $\ell, \mathfrak{m}, \mathfrak{n}$.

**Definition 4.1.** *Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A minimal curve is a holomorphic map $\Phi : \mathcal{U} \to \mathbb{C}^4$ such that $\langle \Phi'(\omega), \Phi'(\omega)\rangle = 0$ for all $\omega \in \mathcal{U}$, where $\Phi'(\omega) := \dfrac{\mathrm{d}\Phi}{\mathrm{d}\omega}$ denotes the complex derivative of $\Phi$. If, in addition, $\langle \Phi'(\omega), \overline{\Phi'(\omega)}\rangle \neq 0$ for every $\omega \in \mathcal{U}$, then $\Phi$ is called a regular minimal curve.*

Firstly, we present the complex minimal curves family in $\mathbb{C}^4$ as follows:

$$X(\omega) = \begin{pmatrix} \dfrac{\omega^{\ell+1}}{\ell+1} - \dfrac{\omega^{\ell+2\mathfrak{m}+1}}{\ell+2\mathfrak{m}+1} - \dfrac{\omega^{\ell+2\mathfrak{n}+1}}{\ell+2\mathfrak{n}+1} \\ \mathrm{i}\left(\dfrac{\omega^{\ell+1}}{\ell+1} + \dfrac{\omega^{\ell+2\mathfrak{m}+1}}{\ell+2\mathfrak{m}+1} + \dfrac{\omega^{\ell+2\mathfrak{n}+1}}{\ell+2\mathfrak{n}+1}\right) \\ \dfrac{2\,\omega^{\ell+\mathfrak{m}+1}}{\ell+\mathfrak{m}+1} \\ \dfrac{2\,\omega^{\ell+\mathfrak{n}+1}}{\ell+\mathfrak{n}+1} \end{pmatrix}. \tag{4.1}$$

Its real part corresponds to the minimal surface family, while its imaginary part corresponds to the conjugate minimal surface family in $\mathbb{R}^4$.

### 4.1. Mean curvature vector and Gaussian curvature for surfaces in $\mathbb{R}^4$

Let $X\colon M^2 \to \mathbb{R}^4$ be an immersed surface with local coordinates $(u^1, u^2)$ and tangent vectors

$$X_i = \frac{\partial X}{\partial u^i}, \qquad g_{ij} = \langle X_i, X_j\rangle, \qquad g^{ik} g_{kj} = \delta^i{}_j.$$

We work with a local orthonormal normal frame $\{\mathbf{n}_1, \mathbf{n}_2\}$ and write $X_i = \partial_i X$. Without invoking Christoffel symbols, the second derivatives decompose into tangent and normal parts via orthogonal projections

$$X_{ij} = (X_{ij})^\top + (X_{ij})^\perp,$$

where

$$(X_{ij})^\top = \sum_{k,l=1}^{2} \langle X_{ij}, X_k \rangle \, g^{kl} \, X_l,$$

$$(X_{ij})^\perp = \sum_{\alpha=1}^{2} h_{ij}^{(\alpha)} \, \mathbf{n}_\alpha, \qquad h_{ij}^{(\alpha)} = \langle X_{ij}, \mathbf{n}_\alpha \rangle.$$

Hence,

$$X_{ij} = \sum_{k,l=1}^{2} \langle X_{ij}, X_k \rangle \, g^{kl} \, X_l + \sum_{\alpha=1}^{2} h_{ij}^{(\alpha)} \, \mathbf{n}_\alpha.$$

The mean curvature vector is

$$\mathbf{H} = \frac{1}{2} g^{ij} (h_{ij}^{(1)} \, \mathbf{n}_1 + h_{ij}^{(2)} \, \mathbf{n}_2),$$

so minimality is equivalent to $\mathbf{H} \equiv \mathbf{0}$. The Gaussian curvature $K$ is

$$K = \frac{\det h^{(1)} + \det h^{(2)}}{\det g}, \qquad \det h^{(\alpha)} = h_{11}^{(\alpha)} h_{22}^{(\alpha)} - (h_{12}^{(\alpha)})^2.$$

**Example 4.1.** *Let* $(f, g, h) = (2, \omega, \omega)$ $(\ell = 0, \mathfrak{m} = 1, \mathfrak{n} = 1)$. *Then, the real parametric minimal surface* $\mathcal{S}_1 \subset \mathbb{R}^4$ *(See Figure 1 for the projection of* $\mathcal{S}_1$ *into xyw-space and onto coordinate planes) is given by*

$$X(u, v) = (\, x(u, v),\ y(u, v),\ z(u, v),\ w(u, v)\,) = \Big(\, u - \frac{2}{3}u^3 + 2uv^2,\ -v - 2u^2 v + \frac{2}{3}v^3,\ u^2 - v^2,\ u^2 - v^2 \,\Big).$$

*The tangent vectors are*

$$X_u = (1 - 2u^2 + 2v^2,\ -4uv,\ 2u,\ 2u), \quad X_v = (4uv,\ -1 - 2u^2 + 2v^2,\ -2v,\ -2v).$$

*Thus*

$$E = G = (2(u^2 + v^2) + 1)^2, \qquad F = 0,$$

*so the parametrization is orthogonal and conformal. The orthogonal normal vectors of the surface are determined by*

$$\mathbf{n}_1(u, v) = \frac{1}{(2(u^2 + v^2) + 1)\sqrt{2(u^2 + v^2)}} \Big(4uv,\ 4v^2,\ -u - v - 2u^3 + 2v^3,\ u - v + 2u^3 + 4uv^2 + 2v^3\Big),$$

$$\mathbf{n}_2(u, v) = \frac{\sqrt{2(u^2 + v^2)}}{2(u^2 + v^2) + 1} \Big(2u^2 - 2v^2 + 1,\ 4uv,\ 2v,\ -2v\Big),$$

*with* $\langle \mathbf{n}_1, \mathbf{n}_2 \rangle = 0$ *and* $\|\mathbf{n}_1\| = \|\mathbf{n}_2\| = 1$. *A direct computation gives*

$$K(u, v) = -\frac{4}{(2(u^2 + v^2) + 1)^4}, \qquad \mathbf{H} \equiv 0.$$

*An implicit equation for the projection to* $(x, y, w)$ *($z = w$) is the degree-9 polynomial* $Q_1(x, y, w) = 0$.

*See Appendix A for the implicit equation* $Q_1$*, and also Figure 2 for the projections of it into three-dimensional space.*

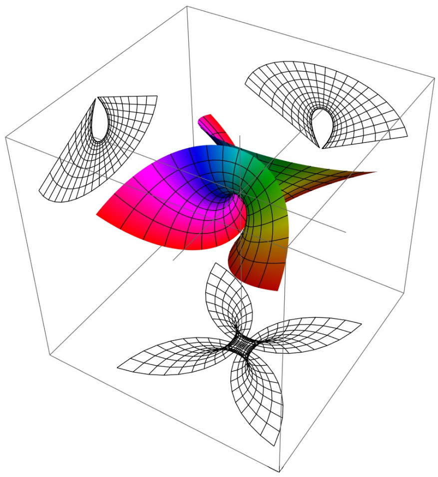

**Figure 1.** Projections of the parametric minimal surface $\mathcal{S}_1$ (classical Enneper minimal surface) into *xyw*-space and onto the coordinate planes.

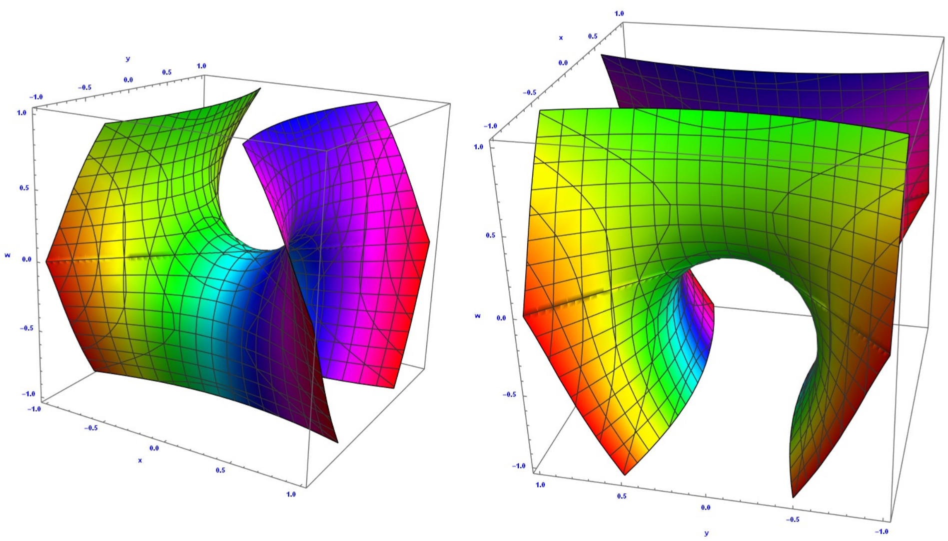

**Figure 2.** Algebraic surface $Q_1(x, y, w) = 0$ (Enneper's implicit minimal surface).

**Example 4.2.** *Let* $(f, g, h) = (2, \omega, \omega^2)$ $(\ell = 0, \mathfrak{m} = 1, \mathfrak{n} = 2)$*. Therefore, the real parametric minimal surface* $\mathcal{S}_2 \subset \mathbb{R}^4$ *(See Figures 3–6 for the projections of* $\mathcal{S}_2$ *into three-dimensional spaces and onto coordinate planes) is determined by*

$$X(u,v)=\begin{pmatrix}x(u,v)\\y(u,v)\\z(u,v)\\w(u,v)\end{pmatrix}=\begin{pmatrix}-\frac{1}{5}u^5+2u^3v^2-uv^4-\frac{1}{3}u^3+uv^2+u\\-u^4v-\frac{1}{5}v^5+2u^2v^3-u^2v+\frac{1}{3}v^3-v\\u^2-v^2\\\frac{2}{3}u^3-2uv^2\end{pmatrix}.$$

*An implicit equation for the projection to* $(x, y, z, w)$ *is the degree-*8 *polynomial* $Q_2(x, y, w) = 0$.

*See Appendix B for the implicit equation* $Q_2$*, and also Figures 7–10 for the projections of it into three-dimensional spaces.*

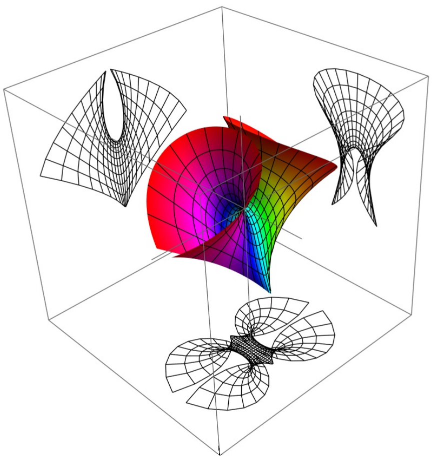

**Figure 3.** Projections of the parametric surface $\mathcal{S}_2$ into *xyz*-space and onto the coordinate planes.

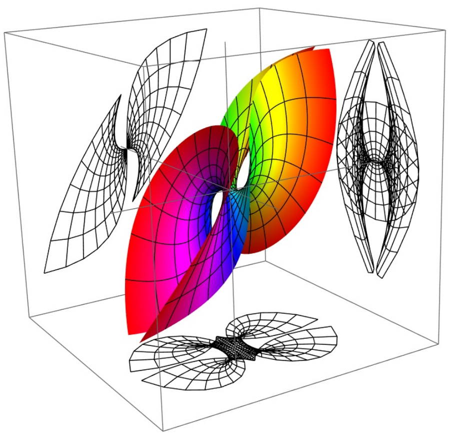

**Figure 4.** Projections of the parametric minimal surface $\mathcal{S}_2$ into *xyw*-space and onto the coordinate planes.

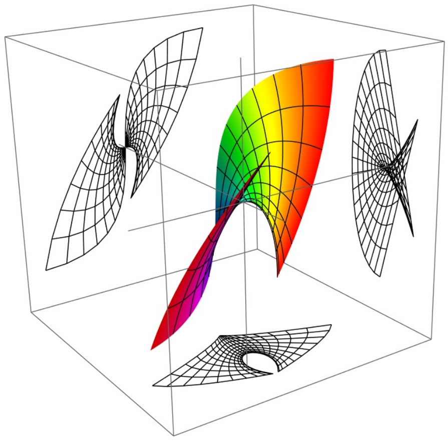

**Figure 5.** Projections of the parametric minimal surface $\mathcal{S}_2$ into $xzw$-space and onto the coordinate planes.

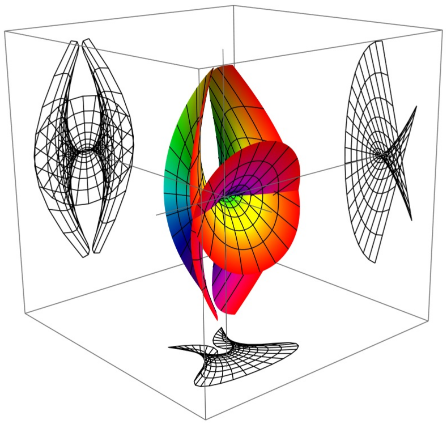

**Figure 6.** Projections of the parametric minimal surface $\mathcal{S}_2$ into $yzw$-space and onto the coordinate planes.

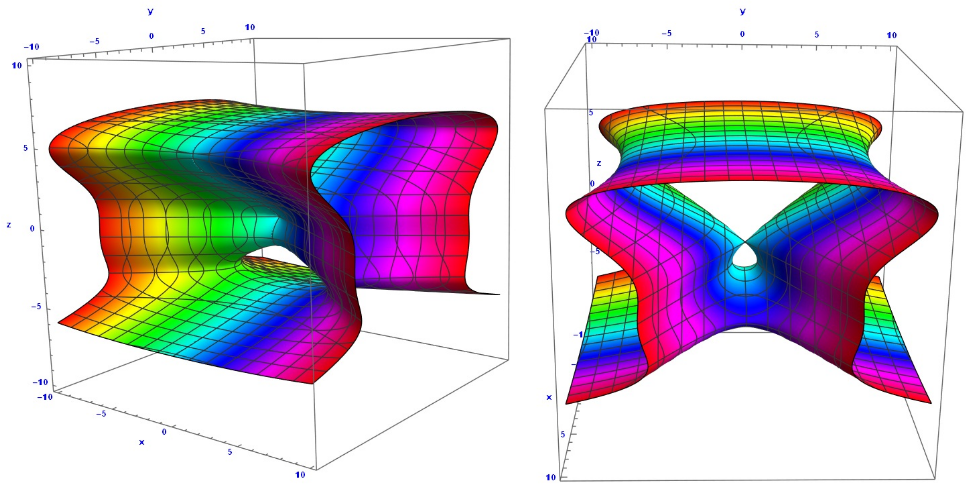


**Figure 7.** Two views of the projection of the algebraic minimal surface $Q_2(x, y, z, w) = 0$ into the $xyz$-space.

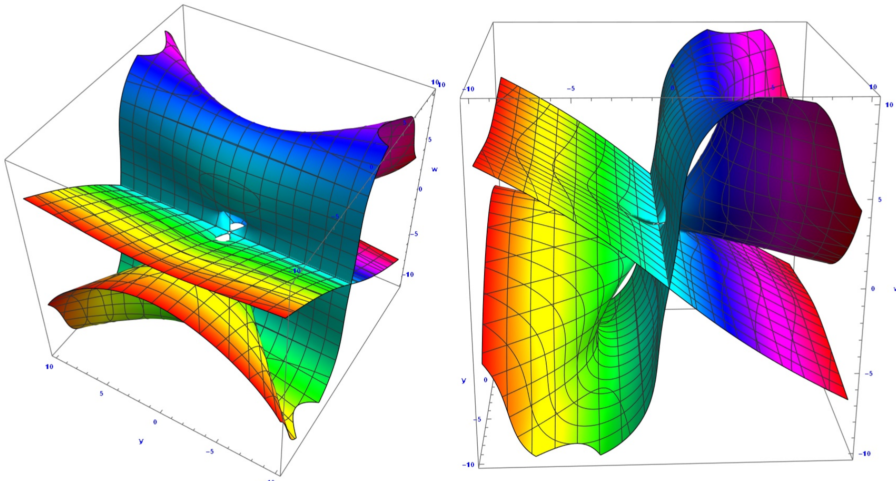

**Figure 8.** Two views of the projection of the algebraic minimal surface $Q_2(x, y, z, w) = 0$ into the *xyw*-space.

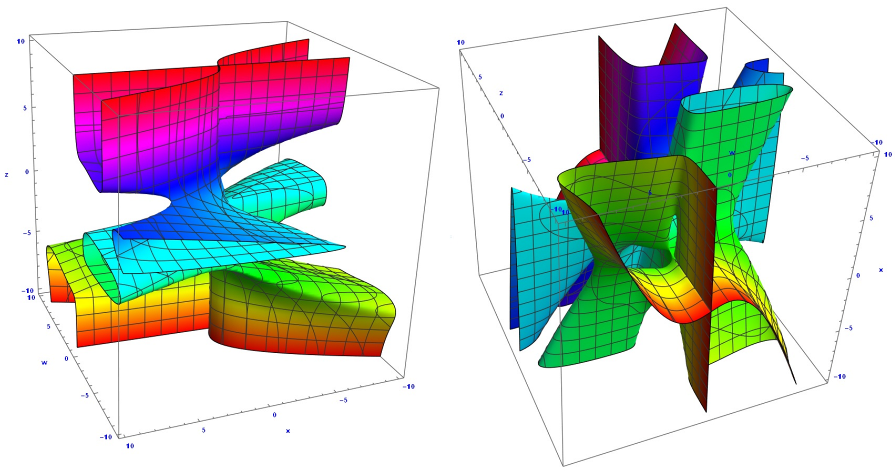

**Figure 9.** Two views of the projection of the algebraic minimal surface $Q_2(x, y, z, w) = 0$ into the *xzw*-space.

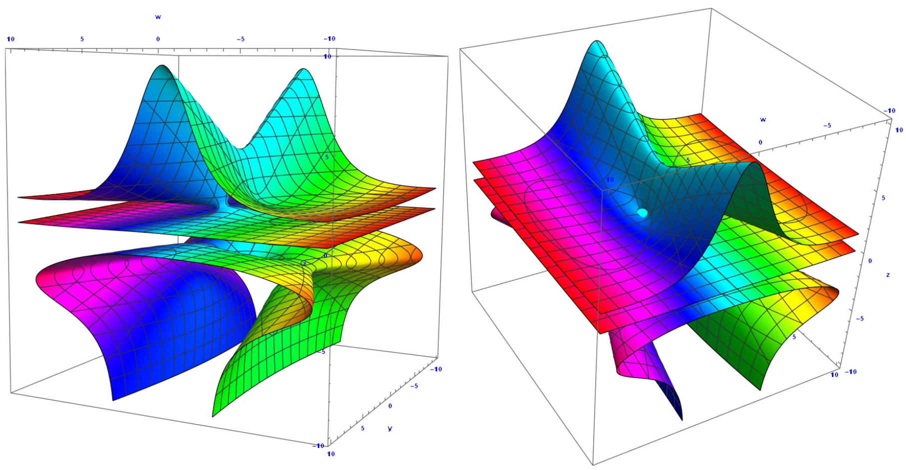

**Figure 10.** Two views of the projection of the algebraic minimal surface $Q_2(x, y, z, w) = 0$ into the *yzw*-space.

**Example 4.3.** *Let* $(f, g, h) = (2\omega, \omega, \omega^2)$ $(\ell = 1, \mathfrak{m} = 1, \mathfrak{n} = 2)$. *Then, the real parametric minimal surface* $\mathcal{S}_3 \subset \mathbb{R}^4$ *(See Figures 11–14 for the projections of* $\mathcal{S}_3$ *into three-dimensional spaces and onto coordinate planes) is described by*

$$X(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \\ w(u,v) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}u^2 - \frac{1}{2}v^2 - \frac{1}{4}u^4 + \frac{3}{2}u^2v^2 - \frac{1}{4}v^4 - \frac{1}{6}u^6 + \frac{5}{2}u^4v^2 - \frac{5}{2}u^2v^4 + \frac{1}{6}v^6 \\ -uv - u^3v + uv^3 - u^5v + \frac{10}{3}u^3v^3 - uv^5 \\ \frac{2}{3}u^3 - 2uv^2 \\ \frac{1}{2}u^4 - 3u^2v^2 + \frac{1}{2}v^4 \end{pmatrix}.$$

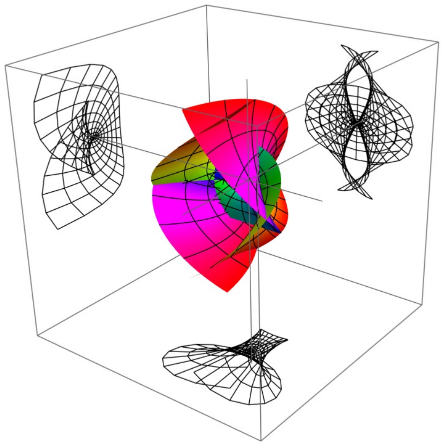

**Figure 11.** Projections of the parametric minimal surface $\mathcal{S}_3$ into $xyz$-space and onto the coordinate planes.

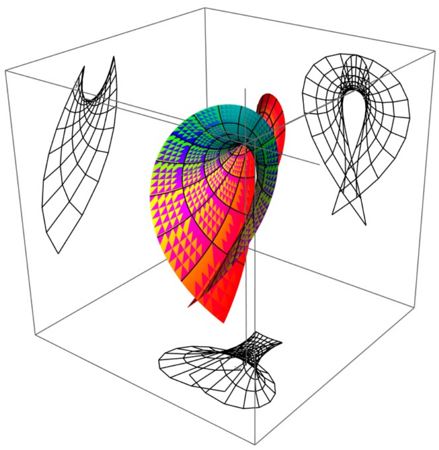

**Figure 12.** Projections of the parametric minimal surface $\mathcal{S}_3$ into $xyw$-space and onto the coordinate planes.

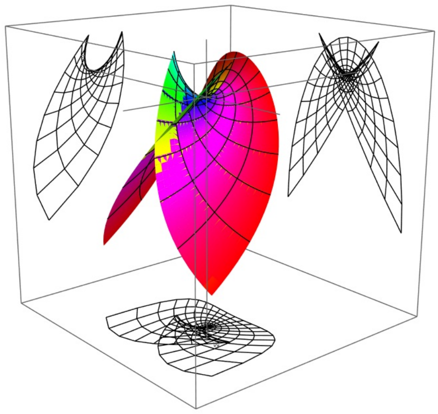

**Figure 13.** Projections of the parametric minimal surface $\mathcal{S}_3$ into *xzw*-space and onto the coordinate planes.

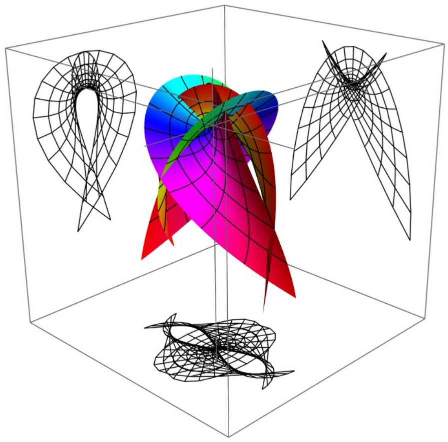

**Figure 14.** Projections of the parametric minimal surface $\mathcal{S}_3$ into *yzw*-space and onto the coordinate planes.

### *4.2. Geometric intuition for the normal fields.*

In $\mathbb{R}^4$, the tangent plane is 2-dimensional, so its orthogonal complement is also two-dimensional. The Gram-Schmidt procedure above produces an *orthonormal* normal frame $\{\mathbf{n}_1, \mathbf{n}_2\}$. One verifies explicitly

$$\langle \mathbf{n}_1, \mathbf{n}_1 \rangle = \langle \mathbf{n}_2, \mathbf{n}_2 \rangle = 1, \qquad \langle \mathbf{n}_1, \mathbf{n}_2 \rangle = 0,$$

and that both $\mathbf{n}_1, \mathbf{n}_2$ are orthogonal to $X_u$ and $X_v$. Thus, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}_1, \mathbf{n}_2\}$ forms a complete orthonormal frame for curvature computations.

## 5. Conclusions and open questions

The Weierstrass-Enneper representation admits a natural generalization to minimal surfaces in $\mathbb{R}^4$. Thus, such surfaces can be expressed locally in terms of pairs of meromorphic and holomorphic functions or, equivalently, through twistor-theoretic and harmonic map data. Despite recent advances, many fundamental questions remain unresolved in local and global settings alike.

One of the main challenges is the global classification of minimal surfaces in $\mathbb{R}^4$. While the local representation is explicit, it is not at all clear which meromorphic data produce complete and embedded minimal surfaces. In particular, the classification of minimal tori arising from the generalized Weierstrass representation, analogous to the spectral curve methods successfully applied in $\mathbb{R}^3$, remains open. This problem is closely related to understanding the structure of the moduli space of minimal surfaces of a fixed topological type.

Another area of active investigation involves the nature of the Gauss map. In $\mathbb{R}^4$, the oriented Gauss map takes values in the Grassmannian $G_{2,4} \cong S^2 \times S^2$, which yields two holomorphic components. A complete classification of minimal surfaces with algebraic Gauss maps, especially in the finite total curvature case, is still lacking. Furthermore, the global interplay between the two components of the Gauss map, and the precise constraints needed for completeness and regularity are not well understood. A central question remains whether Osserman-type results relating the total curvature to the conformal type, which are known for $\mathbb{R}^3$ would truly extend to minimal surfaces in $\mathbb{R}^4$.

Overall, in terms of topological and geometric properties, including surfaces with embedded ends, the study of stable minimal surfaces in $\mathbb{R}^4$ is far from being fully developed, and we expect to see many decades of new results of growing significance and meaning.

## Author contributions

Magdalena Toda: Conceptualization, investigation, software, writing-original draft, writing-review and editing, supervision, funding acquisition; Erhan Güler: Visualization, software, investigation, writing-review and editing, formal analysis, supervision. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article, except for Grammarly to correct linguistic and typographical errors.

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## Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix A. Computational methods for implicit equations

**Goal.** Given a parametric immersion $(u, v) \mapsto (x(u, v), y(u, v), z(u, v), w(u, v))$, obtain low-degree implicit equations for its 3-dimensional orthogonal projections.

**Method 1: resultants.** For a 3-coordinate projection, form $F_1(u, v) = x(u, v) - X$, $F_2(u, v) = y(u, v) - Y$ (and, if needed, $F_3 = z(u, v) - Z$ or $w(u, v) - W$). Eliminate $(u, v)$ by successive univariate resultants:

$$R_1(T) = \mathrm{Res}_v(F_1(u, v), F_2(u, v)), \qquad Q(X, Y, W) = \mathrm{Res}_u(R_1(u),\ w(u, v) - W),$$

evaluated symbolically. The final polynomial $Q = 0$ defines the projected algebraic surface. We optimized the elimination order to minimize total degree.

**Method 2: Gröbner bases.** Alternatively, work over a field $\mathbb{K}$ with polynomial ring $\mathbb{K}[X, Y, Z, W, u, v]$ and the ideal

$$I = \langle x(u, v) - X,\ y(u, v) - Y,\ z(u, v) - Z,\ w(u, v) - W \rangle$$

in a lexicographic order eliminating $(u, v)$. The reduced Gröbner basis of $I$ contains the desired eliminant polynomial(s) in $(X, Y, Z, W)$ alone. In our examples $\mathcal{S}_1$ and $\mathcal{S}_2$, this reproduces the degree-9 and degree-8 equations reported in the text with reduced intermediate swell when using sparsity-aware monomial orders.

**Why implicit forms matter.** Implicit equations expose the global algebraic geometry of the projected image (degree, singular locus, and intersection behavior), enabling comparison with classical $\mathbb{R}^3$ minimal surfaces and facilitating accurate rendering and slicing.

Therefore, the implicit form of the parametric minimal surface $\mathcal{S}_1$ in Example 4.1 is determined by

$$\begin{aligned} Q_1(x,y,w) = & -4096\,w^9 + 6912\,w^6x^2 - 6912\,w^6y^2 + 18432\,w^7 + 31104\,w^5x^2 + 31104\,w^5y^2 \\ & + 4860\,w^3x^4 + 25272\,w^3x^2y^2 + 4860\,w^3y^4 + 25920\,w^4x^2 - 25920\,w^4y^2 + 8748\,w^2x^4 \\ & - 8748\,w^2y^4 + 729\,x^6 - 2187\,x^4y^2 + 2187\,x^2y^4 - 729\,y^6 - 20736\,w^5 - 7776\,w^3x^2 \\ & - 7776\,w^3y^2 - 729\,wx^4 + 1458\,wx^2y^2 - 729\,wy^4. \end{aligned}$$

In addition, the implicit form of the parametric minimal surface $\mathcal{S}_2$ in Example 4.2 is given by

$$\begin{aligned} Q_2(x,y,z,w) = & \; 116640\,w^4z^4 - 13122\,w^6z + 563760\,w^4z^3 + 641520\,w^3xz^3 + 47520\,w^2z^5 \\ & + 123840\,wxz^5 - 5760\,x^2z^5 + 2187\,w^6 - 17496\,wx + 369360\,w^4z^2 + 205740\,w^3xz^2 \\ & + 1044900\,w^2z^2 - 283500\,w^2y^2z^2 - 273360\,w^2z^4 + 350880\,wxz^4 + 240000\,x^2z^4 \\ & - 428760\,w^2z + 635580\,w^3xz + 1937250\,wx^2z - 303750\,w^2y^2z - 641200\,w^2z^3 \\ & + 526500\,wx^3z - 607500\,wxy^2z - 480400\,wxz^3 + 185600\,x^2z^3 - 75150\,w^4 \\ & - 404100\,w^3x + 338400\,wx^2 + 45000\,w^2y^2 - 398400\,w^2z^2 + 477000\,w^3 \\ & - 225000\,wxy^2 - 322800\,wxz^2 + 81000\,x^4 - 225000\,x^2y^2 + 300000\,x^2z^2 \\ & - 432000\,w^2z - 1044000\,wxz - 9000\,x^2z + 351000\,w^2 - 108000\,wx. \end{aligned}$$

## Appendix B. Computation of the normals

We set $\mu_1 = (-X_{u2}, X_{u1}, -X_{u4}, X_{u3})$, which is orthogonal to $X_u = (X_{u1}, X_{u2}, X_{u3}, X_{u4})$, and $\mu_2 = (-X_{v2}, X_{v1}, -X_{v4}, X_{v3})$, which is orthogonal to $X_v = (X_{v1}, X_{v2}, X_{v3}, X_{v4})$. Then,

$$p = \langle X_u, X_u \rangle = \langle X_v, X_v \rangle = \langle \mu_1, \mu_1 \rangle = \langle \mu_2, \mu_2 \rangle,$$

and

$$q = \langle X_u, \mu_2 \rangle = -\langle X_v, \mu_1 \rangle.$$

To construct an orthonormal basis for the normal space, we apply Gram-Schmidt. Setting

$$\mathbf{e}_1 = \frac{X_u}{\sqrt{p}} \quad \text{and} \quad \mathbf{e}_2 = \frac{X_v}{\sqrt{p}},$$

we have the normal vectors

$$\mathbf{n}_1 = \sqrt{\frac{p}{p^2-q^2}}\left(\frac{q}{p}X_v + \mu_1\right) \qquad \text{and} \qquad \mathbf{n}_2 = \sqrt{\frac{p}{p^2-q^2}}\left(-\frac{q}{p}X_u + \mu_2\right).$$

## Appendix C. Monge-type equations in $\mathbb{R}^4$

We derive in detail the Monge-type minimal surface equations in $\mathbb{R}^4$. Consider the Monge graph

$$X(u,v) = (u, v, p(u,v), q(u,v)) \subset \mathbb{R}^4,$$

where $p, q : \mathbb{R}^2 \to \mathbb{R}$ are smooth functions.

The tangent vectors are

$$X_u = (1, 0, p_u, q_u), \qquad X_v = (0, 1, p_v, q_v).$$

Hence

$$E = 1 + p_u^2 + q_u^2, \qquad F = p_u p_v + q_u q_v, \qquad G = 1 + p_v^2 + q_v^2.$$

The area element is

$$dA = \sqrt{EG - F^2}\, du\, dv.$$

The area functional becomes

$$A[p, q] = \iint \mathcal{L}(p, q, p_u, p_v, q_u, q_v)\, du\, dv,$$

where

$$\mathcal{L} = \sqrt{(1 + p_u^2 + q_u^2)(1 + p_v^2 + q_v^2) - (p_u p_v + q_u q_v)^2}.$$

Let

$$\Delta = EG - F^2, \qquad \mathcal{L} = \sqrt{\Delta}.$$

Since $\mathcal{L}$ depends only on the first derivatives of $p$ and $q$, the Euler-Lagrange equations reduce to

$$\partial_u\Big(\frac{\partial \mathcal{L}}{\partial p_u}\Big) + \partial_v\Big(\frac{\partial \mathcal{L}}{\partial p_v}\Big) = 0, \qquad \partial_u\Big(\frac{\partial \mathcal{L}}{\partial q_u}\Big) + \partial_v\Big(\frac{\partial \mathcal{L}}{\partial q_v}\Big) = 0.$$

Computing the partial derivatives, the divergence form of the system is

$$\partial_u\Big(\frac{G p_u - F p_v}{\sqrt{\Delta}}\Big) + \partial_v\Big(\frac{E p_v - F p_u}{\sqrt{\Delta}}\Big) = 0, \qquad \partial_u\Big(\frac{G q_u - F q_v}{\sqrt{\Delta}}\Big) + \partial_v\Big(\frac{E q_v - F q_u}{\sqrt{\Delta}}\Big) = 0.$$

Differentiating the above expressions and simplifying, using the identities $E, F, G$, one finds that all mixed terms cancel, leaving only second-order derivatives. The resulting system is

$$\frac{G p_{uu} - 2F p_{uv} + E p_{vv}}{\sqrt{\Delta}} = 0, \qquad \frac{G q_{uu} - 2F q_{uv} + E q_{vv}}{\sqrt{\Delta}} = 0.$$

Since $\Delta > 0$ for a regular graph, this reduces to the compact equations

$$G p_{uu} - 2F p_{uv} + E p_{vv} = 0, \qquad G q_{uu} - 2F q_{uv} + E q_{vv} = 0.$$

Substituting the expressions for $E, F, G$ yields the Monge-type PDE system

$$(1 + p_v^2 + q_v^2) p_{uu} - 2(p_u p_v + q_u q_v) p_{uv} + (1 + p_u^2 + q_u^2) p_{vv} = 0,$$
$$(1 + p_v^2 + q_v^2) q_{uu} - 2(p_u p_v + q_u q_v) q_{uv} + (1 + p_u^2 + q_u^2) q_{vv} = 0.$$

These two equations represent the Monge-type minimal surface system in $\mathbb{R}^4$. They generalize the classical minimal surface equation in $\mathbb{R}^3$, which involves a single function $z = z(x, y)$, to the case of two dependent functions $(p, q)$ defining a graph in four dimensions. Thus the variational approach via the area functional leads naturally to the coupled nonlinear PDE system above.