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Research article

A compendium of expressions for dependence measures of bivariate copulas

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Abstract: Copulas are important because they allow for modeling and analyzing the dependence structure between random variables, providing insights into complex relationships beyond linear correlations. In this paper, we produced a compendium of expressions for four of the most popular dependence measures of bivariate copulas. Over twenty-five families of bivariate copulas were considered.

Keywords: Blomqvist's β^*; Hoeffding's Φ^2; Kendall's τ; Spearman's ρ
Mathematics Subject Classification: 62E99

1. Introduction

In statistics, a copula is a mathematical concept used to describe the dependence structure between random variables. It allows researchers to model and understand the relationships between variables, particularly their joint distribution, without making assumptions about the marginal distributions. A copula is a function that links univariate marginal distribution functions to form a multivariate distribution function. Formally, a copula C is a multivariate cumulative distribution function with uniform marginals on the interval $[0, 1]$.

Sklar's theorem [33] is fundamental in the theory of copulas. It states that for any bivariate joint distribution function H with marginals F_1, F_2, there exists a copula C such that

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$

If F_1, F_2 are continuous, the copula C is unique. Conversely, given a copula C and marginal distribution functions F_1, F_2, the function H defined above is a joint distribution function with F_1, F_2 as its marginals. Let $c(x, y) = \frac{\partial^2}{\partial x \partial y} C(x, y)$ denote the copula density.

Application areas of copulas include finance, insurance, hydrology, engineering and economics. Some recent papers on applications of copulas include [5, 26, 27, 30, 32]. Comprehensive accounts of the theory of copulas can be found in [25, 14, 21].

The aim of this paper is to derive expressions for dependence measures for bivariate copulas. Given a copula $C : [0, 1] \times [0, 1] \to [0, 1]$, the four most popular dependence measures are Blomqvist's β^* [4], Kendall's τ [22], Spearman's ρ and Hoeffding's Φ^2 [20], see also [1]. They are defined by

$$\beta^* = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1, \tag{1.1}$$

$$\tau = 4 \int_0^1 \int_0^1 C(x, y) dC(x, y) - 1, \tag{1.2}$$

$$\rho = 12 \int_0^1 \int_0^1 xy dC(x, y) - 3 \tag{1.3}$$

and

$$\Phi^2 = 90 \int_0^1 \int_0^1 \left[C(x, y) - xy\right]^2 dxdy, \tag{1.4}$$

respectively.

The ranges of these dependence measures are $-1 \leq \beta^* \leq 3$, $-1 \leq \tau \leq 1$, $-1 \leq \rho \leq 1$ and $0 \leq \Phi^2 \leq 1$. Furthermore,

$$\Psi(\tau) \leq \rho \leq \Psi(-\tau),$$

where

$$\Psi(x) = \begin{cases} -1, & \text{if } x = -1, \\ \Psi_n(x), & \text{if } \frac{2-n}{n} \leq x \leq \frac{3-n}{n-1} \text{ for some } n \geq 2, \end{cases}$$

where

$$\Psi_n(x) = -1 - \frac{4}{n^2} + \frac{3}{n} + \frac{3x}{n} - \frac{(n-2)(n-2+nx)^{\frac{3}{2}}}{\sqrt{2(n-1)}n^2}.$$

This result is due to [28].

Blomqvist's β^*, Kendall's τ, Spearman's ρ and Hoeffding's Φ^2 are statistical measures that assess the relationship between two variables, but they capture different aspects of this relationship. Blomqvist's β^* is a measure of concordance that focuses on the median of the joint distribution, making it robust to outliers and suitable for non-parametric settings. Kendall's τ measures the strength of monotonic relationships by comparing the number of concordant and discordant pairs of observations, offering an intuitive interpretation and resilience to nonlinear associations. Spearman's ρ is a rank-based correlation coefficient that assesses the monotonic relationship between variables, sensitive to all rank orders, making it more influenced by extreme values than Kendall's τ. Hoeffding's Φ^2 is a measure of dependence that detects nonlinear relationships by evaluating the joint distribution's deviation from independence, making it powerful for identifying complex associations.

While all these measures share the goal of quantifying dependence, they differ in sensitivity and scope. Kendall's τ and Spearman's ρ are closely related and often yield similar results, but Kendall's τ provides a more natural probabilistic interpretation. Blomqvist's β^* is simpler and focuses on central relationships, while Hoeffding's Φ^2 stands out by detecting broader nonlinear dependencies at the cost of being less interpretable in common applications. Choosing among these measures depends on the type of relationship, robustness needed, and interpretive clarity desired.

Having exact expressions for (1.1) to (1.4) provides significant advantages in statistical analysis and dependence modeling. Exact expressions enable precise computation without relying on approximations, which is especially beneficial for small sample sizes or when assessing subtle dependencies. They facilitate analytical comparisons among measures, enhancing the understanding of their behavior and interrelationships under different data scenarios. Moreover, exact formulas allow for efficient implementation in software, reducing computational overhead and improving the accuracy of results in real-world applications, such as hypothesis testing and copula modeling. These benefits make them invaluable tools for rigorous and reproducible statistical research.

Closed form expressions for (1.1) to (1.4) are not known for many bivariate copulas. In this paper, we state expressions enabling closed form expressions for (1.1) to (1.4), see Section 2. The expressions in Section 2 excluding Sections 2.1, 2.2, 2.5, 2.26 and 2.27 are new and original. Their derivations can be obtained from the corresponding author. Nearly thirty classes of bivariate copulas are considered. Conclusions and the use of the expressions in Section 2 are discussed in Section 3.

2. Exact expressions for dependence measures

In this section, we state without derivations general expressions for (1.1) to (1.4) for survival copulas, Bernstein copulas, power type copulas, Archimedean copulas, [3]'s copulas, [16]'s copulas, Farlie-Gumbel-Morgenstern (FGM) type copulas, Chesneau [6]'s copulas, Chesneau [7, 8]'s copulas, linear combination of copulas, power combination of copulas, [13]'s copulas, [15]'s copulas, [23]'s perturbed copulas, [29]'s copulas, Gaussian copula and Student's t copula. Most of these expressions are new and original.

2.1. Survival copulas

Survival copulas are a type of copula function used to model the dependence structure between random variables by focusing on their joint survival (upper tail) probabilities rather than their lower tail behavior. They are particularly useful in risk management, reliability engineering, and actuarial science, where understanding the likelihood of extreme co-occurring events (for example, simultaneous failures or high losses) is crucial. For a survival copula given by

$$C(x, y) = x + y - 1 + \mathcal{B}(1 - x, 1 - y)$$

for $0 < x < 1$ and $0 < y < 1$, where $\mathcal{B}$ is a valid copula, we have

$$c(x, y) = b(1 - x, 1 - y),$$

$$\beta^* = 4\mathcal{B}\left(\frac{1}{2}, \frac{1}{2}\right) - 1,$$

$$\tau = 4\int_0^1 \int_0^1 \left[x + y - 1 + \mathcal{B}(1-x, 1-y)\right] b(1-x, 1-y)dxdy - 1,$$

$$\rho = 12\int_0^1 \int_0^1 xyb(1-x, 1-y)dxdy - 3$$

and

$$\Phi^2 = 90\int_0^1 \int_0^1 \left[\mathcal{B}(1-x, 1-y) - (1-x)(1-y)\right]^2 dxdy,$$

where $b(x, y) = \frac{\partial^2}{\partial x \partial y}\mathcal{B}(x, y)$.

2.2. *Bernstein copulas*

A Bernstein copula is a type of copula function that leverages Bernstein polynomials to approximate the underlying copula structure, offering a non-parametric way to model dependency between random variables. These copulas are particularly useful for capturing complex, nonlinear dependencies in bivariate data. Bernstein copulas are flexible and can approximate any copula as the degree of the Bernstein polynomial increases, making them powerful tools for modeling relationships without strong assumptions about the form of dependence. They are often employed in statistical and financial applications where understanding joint distributions is crucial.

For Bernstein copulas specified given by

$$C(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha\left(\frac{i}{m}, \frac{j}{n}\right)\binom{m}{i} x^i (1-x)^{m-i}\binom{n}{j} y^j (1-y)^{n-j}$$

for $0 < x < 1$ and $0 < y < 1$, where $\alpha\left(\frac{i}{m}, \frac{j}{n}\right)$ is a real valued constant indexed by (i, j) such that $0 \leq i \leq m$, $0 \leq j \leq n$, we have

$$\begin{aligned} c(x, y) = &\sum_{i=0}^{m} \sum_{j=0}^{n} \alpha\left(\frac{i}{m}, \frac{j}{n}\right)\binom{m}{i}\left[ix^i(1-x)^{m-i} - (m-i)x^i(1-x)^{m-i-1}\right] \\ &\cdot \binom{n}{j}\left[jy^j(1-y)^{n-j} - (n-j)y^j(1-y)^{n-j-1}\right], \end{aligned}$$

$$\beta^* = 2^{2-m-n} \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha\left(\frac{i}{m}, \frac{j}{n}\right)\binom{m}{i}\binom{n}{j} - 1,$$

$$\tau = 4\sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{m} \sum_{\ell=0}^{n} \alpha\left(\frac{i}{m}, \frac{j}{n}\right)\alpha\left(\frac{k}{m}, \frac{\ell}{n}\right)\binom{m}{i}\binom{m}{k}\binom{n}{j}\binom{n}{\ell}$$

$$\cdot\,[iB(i+k+1,2m-i-k+1)-(m-i)B(i+k+1,2m-i-k)]$$
$$\cdot\,\left[jB(j+\ell+1,2n-j-\ell+1)-(n-j)B(j+\ell+1,2n-j-\ell)\right],$$

$$\rho = 12\sum_{i=0}^{m}\sum_{j=0}^{n}\alpha\left(\frac{i}{m},\frac{j}{n}\right)\binom{m}{i}[iB(i+2,m-i+1)-(m-i)B(i+2,m-i)]$$
$$\cdot\binom{n}{j}\left[jB(j+2,n-j+1)-(n-j)B(j+2,n-j)\right]-3$$

and

$$\Phi^2 = 90\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\alpha\left(\frac{i}{m},\frac{j}{n}\right)\alpha\left(\frac{k}{m},\frac{\ell}{n}\right)\binom{m}{i}\binom{m}{k}\binom{n}{j}\binom{n}{\ell}$$
$$\cdot\,B(i+k+1,2m-i-k+1)B(j+\ell+1,2n-j-\ell+1)$$
$$-180\sum_{i=0}^{m}\sum_{j=0}^{n}\alpha\left(\frac{i}{m},\frac{j}{n}\right)\binom{m}{i}B(i+2,m-i+1)\binom{n}{j}B(j+2,n-j+1)+10,$$

where $B(a,b)$ denotes the beta function.

2.3. Power type copulas [9, 10]

For the copulas given by

$$C(x,y) = x^{P(x,y)}y^{Q(x,y)}$$

for $0 < x < 1$ and $0 < y < 1$, where $P(x,y)$ and $Q(x,y)$ are suitable functions such that $C(0,0) = 0$, $C(0,1) = 0$, $C(1,0) = 0$ and $C(1,1) = 1$, we have

$$c(x,y) = x^{P(x,y)-1}y^{Q(x,y)}P(x,y)\frac{\partial P(x,y)}{\partial y}\log x + x^{P(x,y)}y^{Q(x,y)}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}(\log x)^2$$
$$+\,x^{P(x,y)-1}y^{Q(x,y)}\frac{\partial P(x,y)}{\partial y} + x^{P(x,y)}y^{Q(x,y)}\frac{\partial^2 P(x,y)}{\partial x\partial y}\log x$$
$$+\,x^{P(x,y)-1}y^{Q(x,y)-1}P(x,y)Q(x,y) + x^{P(x,y)-1}y^{Q(x,y)}P(x,y)\frac{\partial Q(x,y)}{\partial y}\log y$$
$$+\,x^{P(x,y)}y^{Q(x,y)-1}\frac{\partial P(x,y)}{\partial x}Q(x,y)\log x + x^{P(x,y)}y^{Q(x,y)}\frac{\partial P(x,y)}{\partial x}\frac{\partial Q(x,y)}{\partial y}\log x\log y$$
$$+\,x^{P(x,y)}y^{Q(x,y)}\frac{\partial P(x,y)}{\partial y}\frac{\partial Q(x,y)}{\partial x}\log x\log y$$
$$+\,x^{P(x,y)}y^{Q(x,y)-1}\frac{\partial Q(x,y)}{\partial x}Q(x,y)\log y + x^{P(x,y)}y^{Q(x,y)}\frac{\partial Q(x,y)}{\partial x}\frac{\partial Q(x,y)}{\partial y}(\log y)^2$$
$$+\,x^{P(x,y)}y^{Q(x,y)-1}\frac{\partial Q(x,y)}{\partial x} + x^{P(x,y)}y^{Q(x,y)}\frac{\partial^2 Q(x,y)}{\partial x\partial y}\log y,$$

$$\beta^* = 2^{2-P\left(\frac{1}{2},\frac{1}{2}\right)-Q\left(\frac{1}{2},\frac{1}{2}\right)}-1,$$

$$
\begin{aligned}
\tau = 4 &\int_0^1 \int_0^1 x^{2P(x,y)-1} y^{2Q(x,y)} P(x,y) \frac{\partial P(x,y)}{\partial y} \log x dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)} y^{2Q(x,y)} \frac{\partial P(x,y)}{\partial x} \frac{\partial P(x,y)}{\partial y} (\log x)^2 dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)-1} y^{2Q(x,y)} \frac{\partial P(x,y)}{\partial y} dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)} y^{2Q(x,y)} \frac{\partial^2 P(x,y)}{\partial x \partial y} \log x dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)-1} y^{2Q(x,y)-1} P(x,y) Q(x,y) dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)-1} y^{2Q(x,y)} P(x,y) \frac{\partial Q(x,y)}{\partial y} \log y dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)} y^{2Q(x,y)-1} \frac{\partial P(x,y)}{\partial x} Q(x,y) \log x dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)} y^{2Q(x,y)} \frac{\partial P(x,y)}{\partial x} \frac{\partial Q(x,y)}{\partial y} \log x \log y dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)} y^{2Q(x,y)} \frac{\partial P(x,y)}{\partial y} \frac{\partial Q(x,y)}{\partial x} \log x \log y dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)} y^{2Q(x,y)-1} \frac{\partial Q(x,y)}{\partial x} Q(x,y) \log y dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)} y^{2Q(x,y)} \frac{\partial Q(x,y)}{\partial x} \frac{\partial Q(x,y)}{\partial y} (\log y)^2 dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)} y^{2Q(x,y)-1} \frac{\partial Q(x,y)}{\partial x} dx dy \\
&+ 4 \int_0^1 \int_0^1 x^{2P(x,y)} y^{2Q(x,y)} \frac{\partial^2 Q(x,y)}{\partial x \partial y} \log y dx dy - 1,
\end{aligned}
$$

$$
\begin{aligned}
\rho = 12 &\int_0^1 \int_0^1 x^{P(x,y)} y^{Q(x,y)+1} P(x,y) \frac{\partial P(x,y)}{\partial y} \log x dx dy \\
&+ 12 \int_0^1 \int_0^1 x^{P(x,y)+1} y^{Q(x,y)+1} \frac{\partial P(x,y)}{\partial x} \frac{\partial P(x,y)}{\partial y} (\log x)^2 dx dy \\
&+ 12 \int_0^1 \int_0^1 x^{P(x,y)} y^{Q(x,y)+1} \frac{\partial P(x,y)}{\partial y} dx dy \\
&+ 12 \int_0^1 \int_0^1 x^{P(x,y)+1} y^{Q(x,y)+1} \frac{\partial^2 P(x,y)}{\partial x \partial y} \log x dx dy \\
&+ 12 \int_0^1 \int_0^1 x^{P(x,y)} y^{Q(x,y)} P(x,y) Q(x,y) dx dy \\
&+ 12 \int_0^1 \int_0^1 x^{P(x,y)} y^{Q(x,y)+1} P(x,y) \frac{\partial Q(x,y)}{\partial y} \log y dx dy
\end{aligned}
$$

$$
\begin{aligned}
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{Q(x,y)}\frac{\partial P(x,y)}{\partial x}Q(x,y)\log x dxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{Q(x,y)+1}\frac{\partial P(x,y)}{\partial x}\frac{\partial Q(x,y)}{\partial y}\log x\log y dxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{Q(x,y)+1}\frac{\partial P(x,y)}{\partial y}\frac{\partial Q(x,y)}{\partial x}\log x\log y dxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{Q(x,y)}\frac{\partial Q(x,y)}{\partial x}Q(x,y)\log y dxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{Q(x,y)+1}\frac{\partial Q(x,y)}{\partial x}\frac{\partial Q(x,y)}{\partial y}(\log y)^2 dxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{Q(x,y)}\frac{\partial Q(x,y)}{\partial x}dxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{Q(x,y)+1}\frac{\partial^2 Q(x,y)}{\partial x\partial y}\log y dxdy-3
\end{aligned}
$$

and

$$\Phi^2 = 90\int_0^1\int_0^1 x^{2P(x,y)}y^{2Q(x,y)}dxdy - 180\int_0^1\int_0^1 x^{P(x,y)+1}y^{Q(x,y)+1}dxdy + 10.$$

2.4. Power type copulas [9, 10]

We consider the particular case in Section 2.3 for $P(x, y) = Q(x, y)$. For the copulas given by

$$C(x, y) = x^{P(x,y)}y^{P(x,y)} \tag{2.1}$$

for $0 < x < 1$, $0 < y < 1$ and $P(x, y)$ in a suitable function, we have

$$
\begin{aligned}
c(x,y) = {}& x^{P(x,y)-1}y^{P(x,y)}P(x,y)\frac{\partial P(x,y)}{\partial y}\log x + x^{P(x,y)}y^{P(x,y)}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}(\log x)^2\\
&+ x^{P(x,y)-1}y^{P(x,y)}\frac{\partial P(x,y)}{\partial y} + x^{P(x,y)}y^{P(x,y)}\frac{\partial^2 P(x,y)}{\partial x\partial y}\log x\\
&+ x^{P(x,y)-1}y^{P(x,y)-1}P^2(x,y) + x^{P(x,y)-1}y^{P(x,y)}P(x,y)\frac{\partial P(x,y)}{\partial y}\log y\\
&+ x^{P(x,y)}y^{P(x,y)-1}\frac{\partial P(x,y)}{\partial x}P(x,y)\log x + 2x^{P(x,y)}y^{P(x,y)}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}\log x\log y\\
&+ x^{P(x,y)}y^{P(x,y)-1}\frac{\partial P(x,y)}{\partial x}P(x,y)\log y + x^{P(x,y)}y^{P(x,y)}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}(\log y)^2\\
&+ x^{P(x,y)}y^{P(x,y)-1}\frac{\partial P(x,y)}{\partial x} + x^{P(x,y)}y^{P(x,y)}\frac{\partial^2 P(x,y)}{\partial x\partial y}\log y,
\end{aligned}
$$

$$\beta^* = 2^{2-2P\left(\frac{1}{2},\frac{1}{2}\right)} - 1,$$

$$\tau = 4\int_0^1\int_0^1 x^{2P(x,y)-1}y^{2P(x,y)}P(x,y)\frac{\partial P(x,y)}{\partial y}\log x dxdy$$

$$
\begin{aligned}
&+4\int_0^1\int_0^1 x^{2P(x,y)}y^{2P(x,y)}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}(\log x)^2dxdy\\
&+4\int_0^1\int_0^1 x^{2P(x,y)-1}y^{2P(x,y)}\frac{\partial P(x,y)}{\partial y}dxdy\\
&+4\int_0^1\int_0^1 x^{2P(x,y)}y^{2P(x,y)}\frac{\partial^2 P(x,y)}{\partial x\partial y}\log xdxdy\\
&+4\int_0^1\int_0^1 x^{2P(x,y)-1}y^{2P(x,y)-1}P^2(x,y)dxdy\\
&+4\int_0^1\int_0^1 x^{2P(x,y)-1}y^{2P(x,y)}P(x,y)\frac{\partial P(x,y)}{\partial y}\log ydxdy\\
&+4\int_0^1\int_0^1 x^{2P(x,y)}y^{2P(x,y)-1}\frac{\partial P(x,y)}{\partial x}P(x,y)\log xdxdy\\
&+8\int_0^1\int_0^1 x^{2P(x,y)}y^{2P(x,y)}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}\log x\log ydxdy\\
&+4\int_0^1\int_0^1 x^{2P(x,y)}y^{2P(x,y)-1}\frac{\partial P(x,y)}{\partial x}P(x,y)\log ydxdy\\
&+4\int_0^1\int_0^1 x^{2P(x,y)}y^{2P(x,y)}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}(\log y)^2dxdy\\
&+4\int_0^1\int_0^1 x^{2P(x,y)}y^{2P(x,y)-1}\frac{\partial P(x,y)}{\partial x}dxdy\\
&+4\int_0^1\int_0^1 x^{2P(x,y)}y^{2P(x,y)}\frac{\partial^2 P(x,y)}{\partial x\partial y}\log ydxdy-1,
\end{aligned}
$$

$$
\begin{aligned}
\rho=&12\int_0^1\int_0^1 x^{P(x,y)}y^{P(x,y)+1}P(x,y)\frac{\partial P(x,y)}{\partial y}\log xdxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{P(x,y)+1}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}(\log x)^2dxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)}y^{P(x,y)+1}\frac{\partial P(x,y)}{\partial y}dxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{P(x,y)+1}\frac{\partial^2 P(x,y)}{\partial x\partial y}\log xdxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)}y^{P(x,y)}P^2(x,y)dxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)}y^{P(x,y)+1}P(x,y)\frac{\partial P(x,y)}{\partial y}\log ydxdy\\
&+12\int_0^1\int_0^1 x^{P(x,y)+1}y^{P(x,y)}\frac{\partial P(x,y)}{\partial x}P(x,y)\log xdxdy\\
&+24\int_0^1\int_0^1 x^{P(x,y)+1}y^{P(x,y)+1}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}\log x\log ydxdy
\end{aligned}
$$

$$+ 12\int_0^1\int_0^1 x^{P(x,y)+1}y^{P(x,y)}\frac{\partial P(x,y)}{\partial x}P(x,y)\log y dxdy$$
$$+ 12\int_0^1\int_0^1 x^{P(x,y)+1}y^{P(x,y)+1}\frac{\partial P(x,y)}{\partial x}\frac{\partial P(x,y)}{\partial y}(\log y)^2 dxdy$$
$$+ 12\int_0^1\int_0^1 x^{P(x,y)+1}y^{P(x,y)}\frac{\partial P(x,y)}{\partial x}dxdy$$
$$+ 12\int_0^1\int_0^1 x^{P(x,y)+1}y^{P(x,y)+1}\frac{\partial^2 P(x,y)}{\partial x\partial y}\log y dxdy - 3$$

and

$$\Phi^2 = 90\int_0^1\int_0^1 x^{2P(x,y)}y^{2P(x,y)}dxdy - 180\int_0^1\int_0^1 x^{P(x,y)+1}y^{P(x,y)+1}dxdy + 10.$$

An example of (2.1) with $P(x,y) = 1 + a(1-x)(1-y)$, $0 \leq a \leq 1$ is given by Proposition 1 in Chesneau [11].

2.5. Archimedean copulas

Archimedean copulas are a family of copulas used in statistics to model the dependence structure between random variables, defined through a generator function that is both decreasing and convex. They are popular for their simplicity and flexibility, allowing for a wide range of dependency patterns by varying the generator function. A copula C is called Archimedean if it admits the representation

$$C(x,y;\theta) = g^{-1}\left(g(x;\theta) + g(y;\theta);\theta\right),$$

where $g : [0,1]\times\Theta \to [0,\infty)$ is a continuous, strictly decreasing and convex function such that $g(1;\theta) = 0$, θ is a parameter within some parameter space Θ, and g is the so-called generator function and g^{-1} is its pseudo-inverse defined by

$$g^{-1}(t;\theta) = \begin{cases} g^{-1}(t;\theta), & \text{if } 0 \leq t \leq g(0;\theta), \\ \\ 0, & \text{if } g(0;\theta) \leq t \leq \infty. \end{cases}$$

For Archimedean copulas specified by

$$C(x,y) = g^{-1}\left(g(x) + g(y)\right)$$

for $0 < x < 1$ and $0 < y < 1$, we have

$$c(x,y) = -\frac{g^{'}(x)g^{'}(y)g^{''}\left(g^{-1}\left(g(x)+g(y)\right)\right)}{\left[g^{'}\left(g^{-1}\left(g(x)+g(y)\right)\right)\right]^3},$$

$$\beta^* = 4g^{-1}\left(2g\left(\frac{1}{2}\right)\right) - 1,$$

$$\tau = 1 + 4\int_0^1 \frac{g(t)}{g'(t)} dt,$$

$$\rho = -12\int_0^1\int_0^1 \frac{xyg'(x)g'(y)g''\left(g^{-1}\left(g(x)+g(y)\right)\right)}{\left[g'\left(g^{-1}\left(g(x)+g(y)\right)\right)\right]^3} dxdy - 3$$

and

$$\Phi^2 = 90\int_0^1\int_0^1 \left[g^{-1}\left(g(x)+g(y)\right)\right]^2 dxdy - 180\int_0^1\int_0^1 xyg^{-1}\left(g(x)+g(y)\right) dxdy + 10.$$

We now give three examples. For Clayton copula [12], $g(t) = \frac{t^{-\theta}-1}{\theta}$, $g^{-1}(t) = (1+\theta t)^{-\frac{1}{\theta}}$, $g'(t) = -t^{-\theta-1}$ and $g''(t) = (1+\theta)t^{-\theta-2}$ for $\theta \in [-1,\infty)\backslash\{0\}$. So,

$$C(x,y) = \begin{cases} \left(x^{-\theta}+y^{-\theta}-1\right)^{-\frac{1}{\theta}}, & \text{if } 0<x<1, 0<y<1, x^{-\theta}+y^{-\theta}>1, \\ 0, & \text{otherwise,} \end{cases}$$

$$c(x,y) = \begin{cases} (\theta+1)x^{-\theta-1}y^{-\theta-1}\left(x^{-\theta}+y^{-\theta}-1\right)^{-\frac{1}{\theta}-2}, & \text{if } 0<x<1, 0<y<1, x^{-\theta}+y^{-\theta}>1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta^* = 4\left(2^{\theta+1}-1\right)^{-\frac{1}{\theta}} - 1,$$

$$\tau = \frac{\theta}{\theta+2},$$

$$\rho = 12(\theta+1)\int_0^1\int_0^1 I\left\{x^{-\theta}+y^{-\theta}>1\right\} x^{-\theta}y^{-\theta}\left(x^{-\theta}+y^{-\theta}-1\right)^{-\frac{1}{\theta}-2} dxdy - 3$$

and

$$\begin{aligned} \Phi^2 = {} & 90\int_0^1\int_0^1 I\left\{x^{-\theta}+y^{-\theta}>1\right\}\left(x^{-\theta}+y^{-\theta}-1\right)^{-\frac{2}{\theta}} dxdy \\ & - 180\int_0^1\int_0^1 I\left\{x^{-\theta}+y^{-\theta}>1\right\} xy\left(x^{-\theta}+y^{-\theta}-1\right)^{-\frac{1}{\theta}} dxdy + 10. \end{aligned}$$

For Ali, Mikhail and Haq's copula [2], $g(t) = \log\left[\frac{1-\theta(1-t)}{t}\right]$, $g^{-1}(t) = \frac{1-\theta}{\exp(t)-\theta}$, $g'(t) = -\frac{1-\theta}{t[1-\theta(1-t)]}$ and $g''(t) = \frac{(1-\theta)(1-\theta+2\theta t)}{t^2[1-\theta(1-t)]^2}$ for $-1 \le \theta \le 1$. So,

$$C(x,y) = \frac{xy}{1-\theta(1-x)(1-y)},$$

$$c(x,y)=\frac{1}{1-\theta(1-x)(1-y)}-\frac{\theta(1-x)y}{[1-\theta(1-x)(1-y)]^2}-\frac{\theta x(1-2y)}{[1-\theta(1-x)(1-y)]^2}+\frac{2\theta^2xy(1-x)(1-y)}{[1-\theta(1-x)(1-y)]^3},$$

$$\beta^*=\frac{4(1-\theta)}{(2-\theta)^2-\theta}-1,$$

$$\tau=\frac{2(1-\theta)^3\log(1-\theta)}{3\theta^2}-\frac{4\left[1-(1-\theta)^3\right]}{9\theta^2}+\frac{(1-\theta)\left[1-(1-\theta)^2\right]}{4\theta^2}-\frac{5\theta}{9},$$

$$\begin{aligned}\rho=&\int_0^1\int_0^1\frac{12xy}{1-\theta(1-x)(1-y)}dxdy-\int_0^1\int_0^1\frac{12\theta x(1-x)y^2}{[1-\theta(1-x)(1-y)]^2}dxdy\\&-\int_0^1\int_0^1\frac{12\theta x^2y(1-2y)}{[1-\theta(1-x)(1-y)]^2}dxdy\\&+\int_0^1\int_0^1\frac{24\theta^2x^2y^2(1-x)(1-y)}{[1-\theta(1-x)(1-y)]^3}dxdy-3\end{aligned}$$

and

$$\begin{aligned}\Phi^2=&\,90\int_0^1\int_0^1\frac{(1-\theta)^2x^2y^2}{\{[1-\theta(1-x)]\,[1-\theta(1-y)]-\theta xy\}^2}dxdy\\&-180\int_0^1\int_0^1\frac{(1-\theta)x^2y^2}{[1-\theta(1-x)]\,[1-\theta(1-y)]-\theta xy}dxdy+10.\end{aligned}$$

For Gumbel–Hougaard's copula [25], $g(t)=(-\log t)^\theta$, $g^{-1}(t)=\exp\left(-t^{\frac{1}{\theta}}\right)$, $g'(t)=-\frac{\theta}{t}(-\log t)^{\theta-1}$ and $g''(t)=\frac{\theta}{t^2}(-\log t)^{\theta-1}+\frac{\theta(\theta-1)}{t^2}(-\log t)^{\theta-2}$ for $\theta\geq 1$. So,

$$C(x,y)=\exp\left\{-\left[(-\log x)^\theta+(-\log y)^\theta\right]^{\frac{1}{\theta}}\right\},$$

$$\begin{aligned}c(x,y)=&\,C(x,y)\frac{1}{xy}(-\log x)^{\theta-1}(-\log y)^{\theta-1}\left[(-\log x)^\theta+(-\log y)^\theta\right]^{\frac{2}{\theta}-2}\\&-(1-\theta)C(x,y)\frac{1}{xy}(-\log x)^{\theta-1}(-\log y)^{\theta-1}\left[(-\log x)^\theta+(-\log y)^\theta\right]^{\frac{1}{\theta}-2},\end{aligned}$$

$$\beta^*=2^{2-2^{\frac{1}{\theta}}}-1,$$

$$\tau=1-\frac{1}{\theta},$$

$$\rho=12\int_0^1\int_0^1 C(x,y)\frac{1}{xy}(-\log x)^{\theta-1}(-\log y)^{\theta-1}\left[(-\log x)^\theta+(-\log y)^\theta\right]^{\frac{2}{\theta}-2}dxdy$$

$$- 12(1-\theta)\int_0^1\int_0^1 C(x,y)\frac{1}{xy}(-\log x)^{\theta-1}(-\log y)^{\theta-1}\left[(-\log x)^{\theta}+(-\log y)^{\theta}\right]^{\frac{1}{\theta}-2}dxdy-3$$

and

$$\Phi^2 = 90\int_0^1\int_0^1 \exp\left\{-2\left[(-\log x)^{\theta}+(-\log y)^{\theta}\right]^{\frac{1}{\theta}}\right\}dxdy$$
$$- 180\int_0^1\int_0^1 xy\exp\left\{-\left[(-\log x)^{\theta}+(-\log y)^{\theta}\right]^{\frac{1}{\theta}}\right\}dxdy+10.$$

2.6. *Alzaid and Alhadlaq [3]'s copulas*

For the copulas given by

$$C(x,y) = F^{-1}\left(F(x)F(y)\right)$$

for $0 < x < 1$ and $0 < y < 1$, where F is a strictly increasing log-concave cumulative distribution function, we have

$$c(x,y) = \frac{f(x)f(y)\left[f\left(F(x)F(y)\right)-F(x)F(y)f'\left(F(x)F(y)\right)\right]}{\left[f\left(F(x)F(y)\right)\right]^2},$$

$$\beta^* = 4F^{-1}\left(\left[F\left(\frac{1}{2}\right)\right]^2\right)-1,$$

$$\tau = 4\int_0^1\int_0^1 \frac{F^{-1}\left(F(x)+F(y)\right)f(x)f(y)\left[f\left(F(x)F(y)\right)-F(x)F(y)f'\left(F(x)F(y)\right)\right]}{\left[f\left(F(x)F(y)\right)\right]^2}dxdy-1,$$

$$\rho = 12\int_0^1\int_0^1 \frac{xyf(x)f(y)\left[f\left(F(x)F(y)\right)-F(x)F(y)f'\left(F(x)F(y)\right)\right]}{\left[f\left(F(x)F(y)\right)\right]^2}dxdy-3$$

and

$$\Phi^2 = 90\int_0^1\int_0^1\left[F^{-1}\left(F(x)F(y)\right)\right]^2dxdy-180\int_0^1\int_0^1 xyF^{-1}\left(F(x)F(y)\right)dxdy+10.$$

2.7. *El Ktaibi et al. [16]'s copulas*

For the copulas given by

$$C(x,y) = \frac{xy}{1-\theta\phi(x,y)} = xy\sum_{i=0}^{\infty}\theta^i\left[\phi(x,y)\right]^i$$

for $0 < x < 1$, $0 < y < 1$, and suitable values of θ and $\phi(x,y)$ in a suitable function, we have

$$\begin{aligned}
c(x,y) &= \frac{1}{1-\theta\phi(x,y)} + \frac{\theta y}{[1-\theta\phi(x,y)]^2}\frac{\partial\phi(x,y)}{\partial y} \\
&\quad + \frac{\theta x}{[1-\theta\phi(x,y)]^2}\frac{\partial\phi(x,y)}{\partial x} + \frac{\theta xy}{[1-\theta\phi(x,y)]^2}\frac{\partial^2\phi(x,y)}{\partial x\partial y} \\
&\quad + \frac{2\theta^2 xy}{[1-\theta\phi(x,y)]^3}\frac{\partial\phi(x,y)}{\partial x}\frac{\partial\phi(x,y)}{\partial y} \\
&= \sum_{i=0}^{\infty}\theta^i\,[\phi(x,y)]^i + \theta y\sum_{i=0}^{\infty}(i+1)\theta^i\,[\phi(x,y)]^i\,\frac{\partial\phi(x,y)}{\partial y} \\
&\quad + \theta x\sum_{i=0}^{\infty}(i+1)\theta^i\,[\phi(x,y)]^i\,\frac{\partial\phi(x,y)}{\partial x} \\
&\quad + \theta xy\sum_{i=0}^{\infty}(i+1)\theta^i\,[\phi(x,y)]^i\,\frac{\partial^2\phi(x,y)}{\partial x\partial y} \\
&\quad + 2\theta^2 xy\sum_{i=0}^{\infty}(i+1)(i+2)\theta^i\,[\phi(x,y)]^i\,\frac{\partial\phi(x,y)}{\partial x}\frac{\partial\phi(x,y)}{\partial y},
\end{aligned}$$

$$\beta^* = \frac{\theta\phi\left(\frac{1}{2},\frac{1}{2}\right)}{1-\theta\phi\left(\frac{1}{2},\frac{1}{2}\right)},$$

$$\begin{aligned}
\tau &= 4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\theta^{i+j}\int_0^1\int_0^1 xy\,[\phi(x,y)]^{i+j}\,dxdy \\
&\quad + 4\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)\theta^{i+j}\int_0^1\int_0^1 xy^2\,[\phi(x,y)]^{i+j}\,\frac{\partial\phi(x,y)}{\partial y}dxdy \\
&\quad + 4\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)\theta^{i+j}\int_0^1\int_0^1 x^2y\,[\phi(x,y)]^{i+j}\,\frac{\partial\phi(x,y)}{\partial x}dxdy \\
&\quad + 4\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)\theta^{i+j}\int_0^1\int_0^1 x^2y^2\,[\phi(x,y)]^{i+j}\,\frac{\partial^2\phi(x,y)}{\partial x\partial y}dxdy \\
&\quad + 8\theta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)(j+2)\theta^{i+j}\int_0^1\int_0^1 x^2y^2\,[\phi(x,y)]^{i+j}\,\frac{\partial\phi(x,y)}{\partial x}\frac{\partial\phi(x,y)}{\partial y}dxdy - 1,
\end{aligned}$$

$$\begin{aligned}
\rho &= 12\sum_{i=0}^{\infty}\theta^i\int_0^1\int_0^1 xy\,[\phi(x,y)]^i\,dxdy + 12\theta\sum_{i=0}^{\infty}(i+1)\theta^i\int_0^1\int_0^1 xy^2\,[\phi(x,y)]^i\,\frac{\partial\phi(x,y)}{\partial y}dxdy \\
&\quad + 12\theta\sum_{i=0}^{\infty}(i+1)\theta^i\int_0^1\int_0^1 x^2y\,[\phi(x,y)]^i\,\frac{\partial\phi(x,y)}{\partial x}dxdy
\end{aligned}$$

$$+ 12\theta \sum_{i=0}^{\infty} (i+1)\theta^i \int_0^1 \int_0^1 x^2 y^2 \left[\phi(x,y)\right]^i \frac{\partial^2 \phi(x,y)}{\partial x \partial y} dxdy$$
$$+ 24\theta^2 \sum_{i=0}^{\infty} (i+1)(i+2)\theta^i \int_0^1 \int_0^1 x^2 y^2 \left[\phi(x,y)\right]^i \frac{\partial \phi(x,y)}{\partial x} \frac{\partial \phi(x,y)}{\partial y} dxdy - 3$$

and

$$\Phi^2 = 90 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \theta^{i+j} \int_0^1 \int_0^1 x^2 y^2 \left[\phi(x,y)\right]^{i+j} dxdy - 180 \sum_{i=0}^{\infty} \theta^i \int_0^1 \int_0^1 x^2 y^2 \left[\phi(x,y)\right]^i dxdy + 10.$$

2.8. El Ktaibi et al. [16]'s copulas

We consider the particular case in Section 2.7 for $\phi(x,y) = f(x)g(y)$. For the copulas given by

$$C(x,y) = \frac{xy}{1 - \theta f(x)g(y)} = xy \sum_{i=0}^{\infty} \theta^i \left[f(x)g(y)\right]^i$$

for $0 < x < 1$, $0 < y < 1$, and suitable values of θ and f, g in suitable functions, we have

$$\begin{aligned}
c(x,y) &= \frac{1}{1 - \theta f(x)g(y)} + \frac{\theta f(x) y}{\left[1 - \theta f(x)g(y)\right]^2} \frac{\partial g(y)}{\partial y} \\
&\quad + \frac{\theta x}{\left[1 - \theta f(x)g(y)\right]^2} \frac{\partial f(x)}{\partial x} + \frac{\theta xy}{\left[1 - \theta f(x)g(y)\right]^2} \frac{\partial f(x)}{\partial x} \frac{\partial g(y)}{\partial y} \\
&\quad + \frac{2\theta^2 xy f(x)g(y)}{\left[1 - \theta f(x)g(y)\right]^3} \frac{\partial f(x)}{\partial x} \frac{\partial g(y)}{\partial y} \\
&= \sum_{i=0}^{\infty} \theta^i \left[f(x)g(y)\right]^i + \theta y \sum_{i=0}^{\infty} (i+1)\theta^i \left[f(x)\right]^{i+1} \left[g(y)\right]^i \frac{\partial g(y)}{\partial y} \\
&\quad + \theta x \sum_{i=0}^{\infty} (i+1)\theta^i \left[f(x)\right]^i \left[g(y)\right]^{i+1} \frac{\partial f(x)}{\partial x} \\
&\quad + \theta xy \sum_{i=0}^{\infty} (i+1)\theta^i \left[f(x)\right]^i \left[g(y)\right]^i \frac{\partial f(x)}{\partial x} \frac{\partial g(y)}{\partial y} \\
&\quad + 2\theta^2 xy \sum_{i=0}^{\infty} (i+1)(i+2)\theta^i \left[f(x)g(y)\right]^{i+1} \frac{\partial f(x)}{\partial x} \frac{\partial g(y)}{\partial y},
\end{aligned}$$

$$\beta^* = \frac{\theta f\left(\frac{1}{2}\right) g\left(\frac{1}{2}\right)}{1 - \theta f\left(\frac{1}{2}\right) g\left(\frac{1}{2}\right)},$$

$$\tau = 4 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \theta^{i+j} \int_0^1 \int_0^1 xy \left[f(x)g(y)\right]^{i+j} dxdy$$

$$+ 4\theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (j+1)\theta^{i+j} \int_0^1 \int_0^1 xy^2 \left[f(x)\right]^{i+j+1} \left[g(y)\right]^{i+j} \frac{\partial g(y)}{\partial y} dxdy$$
$$+ 4\theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (j+1)\theta^{i+j} \int_0^1 \int_0^1 x^2 y \left[f(x)\right]^{i+j} \left[g(y)\right]^{i+j+1} \frac{\partial f(x)}{\partial x} dxdy$$
$$+ 4\theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (j+1)\theta^{i+j} \int_0^1 \int_0^1 x^2 y^2 \left[f(x)\right]^{i+j} \left[g(y)\right]^{i+j} \frac{\partial f(x)}{\partial x} \frac{\partial g(y)}{\partial y} dxdy$$
$$+ 8\theta^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (j+1)(j+2)\theta^{i+j} \int_0^1 \int_0^1 x^2 y^2 \left[f(x)\right] \left[g(y)\right]^{i+j+1} \frac{\partial f(x)}{\partial x} \frac{\partial g(y)}{\partial y} dxdy - 1,$$

$$\rho = 12 \sum_{i=0}^{\infty} \theta^i \int_0^1 \int_0^1 xy \left[f(x)g(y)\right]^i dxdy + 12\theta \sum_{i=0}^{\infty} (i+1)\theta^i \int_0^1 \int_0^1 xy^2 \left[f(x)\right]^{i+1} \left[g(y)\right]^i \frac{\partial g(y)}{\partial y} dxdy$$
$$+ 12\theta \sum_{i=0}^{\infty} (i+1)\theta^i \int_0^1 \int_0^1 x^2 y \left[f(x)\right]^i \left[g(y)\right]^{i+1} \frac{\partial f(x)}{\partial x} dxdy$$
$$+ 12\theta \sum_{i=0}^{\infty} (i+1)\theta^i \int_0^1 \int_0^1 x^2 y^2 \left[f(x)\right]^i \left[g(y)\right]^i \frac{\partial f(x)}{\partial x} \frac{\partial g(y)}{\partial y} dxdy$$
$$+ 24\theta^2 \sum_{i=0}^{\infty} (i+1)(i+2)\theta^i \int_0^1 \int_0^1 x^2 y^2 \left[f(x)\right] \left[g(y)\right]^{i+1} \frac{\partial f(x)}{\partial x} \frac{\partial g(y)}{\partial y} dxdy - 3$$

and

$$\Phi^2 = 90 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \theta^{i+j} \int_0^1 \int_0^1 x^2 y^2 \left[f(x)g(y)\right]^{i+j} dxdy - 180 \sum_{i=0}^{\infty} \theta^i \int_0^1 \int_0^1 x^2 y^2 \left[f(x)g(y)\right]^i dxdy + 10.$$

2.9. El Ktaibi et al. [16]'s copulas

We consider the particular case in Section 2.7 for $\phi(x, y) = f(x)f(y)$. For the copulas given by

$$C(x, y) = \frac{xy}{1 - \theta f(x)f(y)} = xy \sum_{i=0}^{\infty} \theta^i \left[f(x)f(y)\right]^i$$

for $0 < x < 1$, $0 < y < 1$, and suitable values of θ and f, g in suitable functions, we have

$$\begin{aligned} c(x, y) &= \frac{1}{1 - \theta f(x)f(y)} + \frac{\theta f(x)y}{\left[1 - \theta f(x)f(y)\right]^2} \frac{\partial f(y)}{\partial y} \\ &\quad + \frac{\theta x}{\left[1 - \theta f(x)f(y)\right]^2} \frac{\partial f(x)}{\partial x} + \frac{\theta xy}{\left[1 - \theta f(x)f(y)\right]^2} \frac{\partial f(x)}{\partial x} \frac{\partial f(y)}{\partial y} \\ &\quad + \frac{2\theta^2 xy f(x)f(y)}{\left[1 - \theta f(x)f(y)\right]^3} \frac{\partial f(x)}{\partial x} \frac{\partial f(y)}{\partial y} \\ &= \sum_{i=0}^{\infty} \theta^i \left[f(x)f(y)\right]^i + \theta y \sum_{i=0}^{\infty} (i+1)\theta^i \left[f(x)\right]^{i+1} \left[f(y)\right]^i \frac{\partial f(y)}{\partial y} \end{aligned}$$

$$+\,\theta x\sum_{i=0}^{\infty}(i+1)\theta^{i}\left[f(x)\right]^{i}\left[f(y)\right]^{i+1}\frac{\partial f(x)}{\partial x}$$
$$+\,\theta xy\sum_{i=0}^{\infty}(i+1)\theta^{i}\left[f(x)\right]^{i}\left[f(y)\right]^{i}\frac{\partial f(x)}{\partial x}\frac{\partial f(y)}{\partial y}$$
$$+\,2\theta^{2}xy\sum_{i=0}^{\infty}(i+1)(i+2)\theta^{i}\left[f(x)f(y)\right]^{i+1}\frac{\partial f(x)}{\partial x}\frac{\partial f(y)}{\partial y},$$

$$\beta^{*}=\frac{\theta\left[f\left(\frac{1}{2}\right)\right]^{2}}{1-\theta\left[f\left(\frac{1}{2}\right)\right]^{2}},$$

$$\begin{aligned}\tau=\;&4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)f(y)\right]^{i+j}dxdy\\&+4\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)\theta^{i+j}\int_{0}^{1}\int_{0}^{1}xy^{2}\left[f(x)\right]^{i+j+1}\left[f(y)\right]^{i+j}\frac{\partial f(y)}{\partial y}dxdy\\&+4\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)\theta^{i+j}\int_{0}^{1}\int_{0}^{1}x^{2}y\left[f(x)\right]^{i+j}\left[f(y)\right]^{i+j+1}\frac{\partial f(x)}{\partial x}dxdy\\&+4\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)\theta^{i+j}\int_{0}^{1}\int_{0}^{1}x^{2}y^{2}\left[f(x)\right]^{i+j}\left[f(y)\right]^{i+j}\frac{\partial f(x)}{\partial x}\frac{\partial f(y)}{\partial y}dxdy\\&+8\theta^{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)(j+2)\theta^{i+j}\int_{0}^{1}\int_{0}^{1}x^{2}y^{2}\left[f(x)f(y)\right]^{i+j+1}\frac{\partial f(x)}{\partial x}\frac{\partial f(y)}{\partial y}dxdy-1,\end{aligned}$$

$$\begin{aligned}\rho=\;&12\sum_{i=0}^{\infty}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)f(y)\right]^{i}dxdy+12\theta\sum_{i=0}^{\infty}(i+1)\theta^{i}\int_{0}^{1}\int_{0}^{1}xy^{2}\left[f(x)\right]^{i+1}\left[f(y)\right]^{i}\frac{\partial f(y)}{\partial y}dxdy\\&+12\theta\sum_{i=0}^{\infty}(i+1)\theta^{i}\int_{0}^{1}\int_{0}^{1}x^{2}y\left[f(x)\right]^{i}\left[f(y)\right]^{i+1}\frac{\partial f(x)}{\partial x}dxdy\\&+12\theta\sum_{i=0}^{\infty}(i+1)\theta^{i}\int_{0}^{1}\int_{0}^{1}x^{2}y^{2}\left[f(x)\right]^{i}\left[f(y)\right]^{i}\frac{\partial f(x)}{\partial x}\frac{\partial f(y)}{\partial y}dxdy\\&+24\theta^{2}\sum_{i=0}^{\infty}(i+1)(i+2)\theta^{i}\int_{0}^{1}\int_{0}^{1}x^{2}y^{2}\left[f(x)\right]\left[f(y)\right]^{i+1}\frac{\partial f(x)}{\partial x}\frac{\partial f(y)}{\partial y}dxdy-3\end{aligned}$$

and

$$\Phi^{2}=90\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}x^{2}y^{2}\left[f(x)f(y)\right]^{i+j}dxdy-180\sum_{i=0}^{\infty}\theta^{i}\int_{0}^{1}\int_{0}^{1}x^{2}y^{2}\left[f(x)f(y)\right]^{i}dxdy+10.$$

2.10. *FGM type copulas [17, 18, 19, 24]*

FGM type copulas are a class of bivariate copulas that provide a simple way to model dependence between two random variables. FGM copulas are limited in capturing strong dependence but are valued for their simplicity, interpretability, and suitability in modeling relationships with low to moderate dependence. They are primarily used in theoretical studies and cases where simplicity outweighs the need for flexibility in capturing tail dependence. For FGM type copulas given by

$$C(x,y) = xy\,[1+\theta a(x,y)]^p = xy\sum_{i=0}^{\infty}\binom{p}{i}\theta^i\,[a(x,y)]^i$$

for $0 < x < 1$, $0 < y < 1$, $p > 0$, $-1 < \theta < 1$ and $a(x,y)$ in a suitable function, we have

$$\begin{aligned}
c(x,y) &= [1+\theta a(x,y)]^p + \theta px\,[1+\theta a(x,y)]^{p-1}\,\frac{\partial a(x,y)}{\partial x} + \theta py\,[1+\theta a(x,y)]^{p-1}\,\frac{\partial a(x,y)}{\partial y} \\
&\quad + \theta^2 p(p-1)xy\,[1+\theta a(x,y)]^{p-2}\,\frac{\partial a(x,y)}{\partial x}\frac{\partial a(x,y)}{\partial y} \\
&\quad + \theta pxy\,[1+\theta a(x,y)]^{p-1}\,\frac{\partial^2 a(x,y)}{\partial x\partial y} \\
&= \sum_{i=0}^{\infty}\binom{p}{i}\theta^i\,[a(x,y)]^i + \theta px\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i\,[a(x,y)]^i\,\frac{\partial a(x,y)}{\partial x} \\
&\quad + \theta py\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i\,[a(x,y)]^i\,\frac{\partial a(x,y)}{\partial y} \\
&\quad + \theta^2 p(p-1)xy\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^i\,[a(x,y)]^i\,\frac{\partial a(x,y)}{\partial x}\frac{\partial a(x,y)}{\partial y} \\
&\quad + \theta pxy\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i\,[a(x,y)]^i\,\frac{\partial^2 a(x,y)}{\partial x\partial y},
\end{aligned}$$

$$\beta^* = \left[1+\theta a\left(\frac{1}{2},\frac{1}{2}\right)\right]^p - 1,$$

$$\begin{aligned}
\tau &= 4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1 xy\,[a(x,y)]^{i+j}\,dxdy \\
&\quad + 4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-1}{j}\theta^{i+j}\int_0^1\int_0^1 x^2y\,[a(x,y)]^{i+j}\,\frac{\partial a(x,y)}{\partial x}dxdy \\
&\quad + 4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-1}{j}\theta^{i+j}\int_0^1\int_0^1 xy^2\,[a(x,y)]^{i+j}\,\frac{\partial a(x,y)}{\partial y}dxdy \\
&\quad + 4\theta^2 p(p-1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-2}{j}\theta^{i+j}\int_0^1\int_0^1 x^2y^2\,[a(x,y)]^{i+j}\,\frac{\partial a(x,y)}{\partial x}\frac{\partial a(x,y)}{\partial y}dxdy
\end{aligned}$$

$$+ 4\theta p \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{p}{i}\binom{p-1}{j}\theta^{i+j} \int_0^1 \int_0^1 x^2 y^2 \left[a(x,y)\right]^{i+j} \frac{\partial^2 a(x,y)}{\partial x \partial y} dxdy - 1,$$

$$\begin{aligned}
\rho = 12 \sum_{i=0}^{\infty} \binom{p}{i}\theta^{i} \int_0^1 \int_0^1 xy \left[a(x,y)\right]^{i} dxdy
&+ 12\theta p \sum_{i=0}^{\infty} \binom{p-1}{i}\theta^{i} \int_0^1 \int_0^1 x^2 y \left[a(x,y)\right]^{i} \frac{\partial a(x,y)}{\partial x} dxdy \\
&+ 12\theta p \sum_{i=0}^{\infty} \binom{p-1}{i}\theta^{i} \int_0^1 \int_0^1 xy^2 \left[a(x,y)\right]^{i} \frac{\partial a(x,y)}{\partial y} dxdy \\
&+ 12\theta^2 p(p-1) \sum_{i=0}^{\infty} \binom{p-2}{i}\theta^{i} \int_0^1 \int_0^1 x^2 y^2 \left[a(x,y)\right]^{i} \frac{\partial a(x,y)}{\partial x}\frac{\partial a(x,y)}{\partial y} dxdy \\
&+ 12\theta p \sum_{i=0}^{\infty} \binom{p-1}{i}\theta^{i} \int_0^1 \int_0^1 x^2 y^2 \left[a(x,y)\right]^{i} \frac{\partial^2 a(x,y)}{\partial x \partial y} dxdy - 3,
\end{aligned}$$

and

$$\begin{aligned}
\Phi^2 = 90 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{p}{i}\binom{p}{j}\theta^{i+j} \int_0^1 \int_0^1 x^2 y^2 \left[a(x,y)\right]^{i+j} dxdy \\
- 180 \sum_{i=0}^{\infty} \binom{p}{i}\theta^{i} \int_0^1 \int_0^1 x^2 y^2 \left[a(x,y)\right]^{i} dxdy + 10.
\end{aligned}$$

If $p > 0$ is an integer then

$$\begin{aligned}
c(x,y) = \sum_{i=0}^{p} \binom{p}{i}\theta^{i} \left[a(x,y)\right]^{i} + \theta p x \sum_{i=0}^{p-1} \binom{p-1}{i}\theta^{i} \left[a(x,y)\right]^{i} \frac{\partial a(x,y)}{\partial x} \\
+ \theta p y \sum_{i=0}^{p-1} \binom{p-1}{i}\theta^{i} \left[a(x,y)\right]^{i} \frac{\partial a(x,y)}{\partial y} \\
+ \theta^2 p(p-1) xy \sum_{i=0}^{p-2} \binom{p-2}{i}\theta^{i} \left[a(x,y)\right]^{i} \frac{\partial a(x,y)}{\partial x}\frac{\partial a(x,y)}{\partial y} \\
+ \theta p xy \sum_{i=0}^{p-1} \binom{p-1}{i}\theta^{i} \left[a(x,y)\right]^{i} \frac{\partial^2 a(x,y)}{\partial x \partial y},
\end{aligned}$$

$$\begin{aligned}
\tau = 4 \sum_{i=0}^{p} \sum_{j=0}^{p} \binom{p}{i}\binom{p}{j}\theta^{i+j} \int_0^1 \int_0^1 xy \left[a(x,y)\right]^{i+j} dxdy \\
+ 4\theta p \sum_{i=0}^{p} \sum_{j=0}^{p-1} \binom{p}{i}\binom{p-1}{j}\theta^{i+j} \int_0^1 \int_0^1 x^2 y \left[a(x,y)\right]^{i+j} \frac{\partial a(x,y)}{\partial x} dxdy
\end{aligned}$$

$$+ 4\theta p \sum_{i=0}^{p} \sum_{j=0}^{p-1} \binom{p}{i}\binom{p-1}{j} \theta^{i+j} \int_0^1 \int_0^1 xy^2 \left[a(x,y)\right]^{i+j} \frac{\partial a(x,y)}{\partial y} dxdy$$
$$+ 4\theta^2 p(p-1) \sum_{i=0}^{p} \sum_{j=0}^{p-2} \binom{p}{i}\binom{p-2}{j} \theta^{i+j} \int_0^1 \int_0^1 x^2y^2 \left[a(x,y)\right]^{i+j} \frac{\partial a(x,y)}{\partial x}\frac{\partial a(x,y)}{\partial y} dxdy$$
$$+ 4\theta p \sum_{i=0}^{p} \sum_{j=0}^{p-1} \binom{p}{i}\binom{p-1}{j} \theta^{i+j} \int_0^1 \int_0^1 x^2y^2 \left[a(x,y)\right]^{i+j} \frac{\partial^2 a(x,y)}{\partial x \partial y} dxdy - 1,$$

$$\begin{aligned}
\rho = 12 \sum_{i=0}^{p} \binom{p}{i} \theta^i \int_0^1 \int_0^1 xy \left[a(x,y)\right]^i dxdy \\
&+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 x^2 y \left[a(x,y)\right]^i \frac{\partial a(x,y)}{\partial x} dxdy \\
&+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 xy^2 \left[a(x,y)\right]^i \frac{\partial a(x,y)}{\partial y} dxdy \\
&+ 12\theta^2 p(p-1) \sum_{i=0}^{p-2} \binom{p-2}{i} \theta^i \int_0^1 \int_0^1 x^2y^2 \left[a(x,y)\right]^i \frac{\partial a(x,y)}{\partial x}\frac{\partial a(x,y)}{\partial y} dxdy \\
&+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 x^2y^2 \left[a(x,y)\right]^i \frac{\partial^2 a(x,y)}{\partial x \partial y} dxdy - 3,
\end{aligned}$$

and

$$\begin{aligned}
\Phi^2 = 90 \sum_{i=0}^{p} \sum_{j=0}^{p} \binom{p}{i}\binom{p}{j} \theta^{i+j} \int_0^1 \int_0^1 x^2y^2 \left[a(x,y)\right]^{i+j} dxdy \\
&- 180 \sum_{i=0}^{p} \binom{p}{i} \theta^i \int_0^1 \int_0^1 x^2y^2 \left[a(x,y)\right]^i dxdy + 10.
\end{aligned}$$

2.11. FGM type copulas [17, 18, 19, 24]

We consider the particular case in Section 2.10 for $a(x,y) = \phi(x)\psi(y)$. For the copulas given by

$$C(x,y) = xy\left[1 + \theta\phi(x)\psi(y)\right]^p$$

for $0 < x < 1$, $0 < y < 1$, $p > 0$, $-1 < \theta < 1$ and $\phi(x)$, $\psi(y)$ in suitable functions, we have

$$\begin{aligned}
c(x,y) = \left[1 + \theta\phi(x)\psi(y)\right]^p + p\theta\phi(x)y\left[1 + \theta\phi(x)\psi(y)\right]^{p-1} \frac{\partial \psi(y)}{\partial y} \\
&+ p\theta x\psi(y)\left[1 + \theta\phi(x)\psi(y)\right]^{p-1} \frac{\partial \phi(x)}{\partial x} + p\theta xy\left[1 + \theta\phi(x)\psi(y)\right]^{p-1} \frac{\partial \phi(x)}{\partial x}\frac{\partial \psi(y)}{\partial y} \\
&+ p(p-1)\theta^2 xy\phi(x)\psi(y)\left[1 + \theta\phi(x)\psi(y)\right]^{p-2} \frac{\partial \phi(x)}{\partial x}\frac{\partial \psi(y)}{\partial y}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty}\binom{p}{i}\theta^i\left[\phi(x)\right]^i\left[\psi(y)\right]^i + p\theta y\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i\left[\phi(x)\right]^{i+1}\left[\psi(y)\right]^i\frac{\partial\psi(y)}{\partial y}\\
&\quad + p\theta x\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i\left[\phi(x)\right]^i\left[\psi(y)\right]^{i+1}\frac{\partial\phi(x)}{\partial x}\\
&\quad + p\theta xy\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i\left[\phi(x)\right]^i\left[\psi(y)\right]^i\frac{\partial\phi(x)}{\partial x}\frac{\partial\psi(y)}{\partial y}\\
&\quad + p(p-1)\theta^2 xy\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^i\left[\phi(x)\right]^{i+1}\left[\psi(y)\right]^{i+1}\frac{\partial\phi(x)}{\partial x}\frac{\partial\psi(y)}{\partial y},
\end{aligned}
$$

$$
\beta^* = \left[1+\theta\phi\left(\frac{1}{2}\right)\psi\left(\frac{1}{2}\right)\right]^p - 1,
$$

$$
\begin{aligned}
\tau &= 4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}I_{1,i+j,\phi}I_{1,i+j,\psi}\\
&\quad + 4p\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-1}{j}\theta^{i+j}I_{1,i+j+1,\phi}J_{2,i+j,\psi}\\
&\quad + 4p\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-1}{j}\theta^{i+j}J_{2,i+j,\phi}I_{1,i+j+1,\psi}\\
&\quad + 4p\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-1}{j}\theta^{i+j}J_{2,i+j,\phi}J_{2,i+j,\psi}\\
&\quad + 4p(p-1)\theta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-2}{j}\theta^{i+j}J_{2,i+j+1,\phi}J_{2,i+j+1,\psi} - 1,
\end{aligned}
$$

$$
\begin{aligned}
\rho &= 12\sum_{i=0}^{\infty}\binom{p}{i}\theta^i I_{1,i,\phi}I_{1,i,\psi} + 12p\theta\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i I_{1,i+1,\phi}J_{2,i,\psi}\\
&\quad + 12p\theta\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i J_{2,i,\phi}I_{1,i+1,\psi}\\
&\quad + 12p\theta\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i J_{2,i,\phi}J_{2,i,\psi} + 12p(p-1)\theta^2\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^i J_{2,i+1,\phi}J_{2,i+1,\psi} - 3
\end{aligned}
$$

and

$$
\Phi^2 = 90\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}I_{2,i+j,\phi}I_{2,i+j,\psi} - 180\sum_{i=0}^{\infty}\binom{p}{i}\theta^i I_{2,i+j,\phi}I_{2,i+j,\psi} + 10,
$$

where

$$
I_{m,n,\varphi} = \int_0^1 x^m\left[\varphi(x)\right]^n dx
$$

and

$$J_{m,n,\varphi} = \int_0^1 x^m \left[\varphi(x)\right]^n \frac{d\varphi(x)}{dx} dx.$$

If $p > 0$ is an integer then

$$\begin{aligned}
c(x,y) &= \sum_{i=0}^{p} \binom{p}{i} \theta^i \left[\phi(x)\right]^i \left[\psi(y)\right]^i + p\theta y \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \left[\phi(x)\right]^{i+1} \left[\psi(y)\right]^i \frac{\partial \psi(y)}{\partial y} \\
&+ p\theta x \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \left[\phi(x)\right]^i \left[\psi(y)\right]^{i+1} \frac{\partial \phi(x)}{\partial x} \\
&+ p\theta xy \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \left[\phi(x)\right]^i \left[\psi(y)\right]^i \frac{\partial \phi(x)}{\partial x} \frac{\partial \psi(y)}{\partial y} \\
&+ p(p-1)\theta^2 xy \sum_{i=0}^{p-2} \binom{p-2}{i} \theta^i \left[\phi(x)\right]^{i+1} \left[\psi(y)\right]^{i+1} \frac{\partial \phi(x)}{\partial x} \frac{\partial \psi(y)}{\partial y},
\end{aligned}$$

$$\begin{aligned}
\tau &= 4 \sum_{i=0}^{p} \sum_{j=0}^{p} \binom{p}{i} \binom{p}{j} \theta^{i+j} I_{1,i+j,\phi} I_{1,i+j,\psi} \\
&+ 4p\theta \sum_{i=0}^{p} \sum_{j=0}^{p-1} \binom{p}{i} \binom{p-1}{j} \theta^{i+j} I_{1,i+j+1,\phi} J_{2,i+j,\psi} \\
&+ 4p\theta \sum_{i=0}^{p} \sum_{j=0}^{p-1} \binom{p}{i} \binom{p-1}{j} \theta^{i+j} J_{2,i+j,\phi} I_{1,i+j+1,\psi} \\
&+ 4p\theta \sum_{i=0}^{p} \sum_{j=0}^{p-1} \binom{p}{i} \binom{p-1}{j} \theta^{i+j} J_{2,i+j,\phi} J_{2,i+j,\psi} \\
&+ 4p(p-1)\theta^2 \sum_{i=0}^{p} \sum_{j=0}^{p-2} \binom{p}{i} \binom{p-2}{j} \theta^{i+j} J_{2,i+j+1,\phi} J_{2,i+j+1,\psi} - 1,
\end{aligned}$$

$$\begin{aligned}
\rho &= 12 \sum_{i=0}^{p} \binom{p}{i} \theta^i I_{1,i,\phi} I_{1,i,\psi} + 12p\theta \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i I_{1,i+1,\phi} J_{2,i,\psi} \\
&+ 12p\theta \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i J_{2,i,\phi} I_{1,i+1,\psi} \\
&+ 12p\theta \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i J_{2,i,\phi} J_{2,i,\psi} + 12p(p-1)\theta^2 \sum_{i=0}^{p-2} \binom{p-2}{i} \theta^i J_{2,i+1,\phi} J_{2,i+1,\psi} - 3
\end{aligned}$$

and

$$\Phi^2 = 90 \sum_{i=0}^{p} \sum_{j=0}^{p} \binom{p}{i} \binom{p}{j} \theta^{i+j} I_{2,i+j,\phi} I_{2,i+j,\psi} - 180 \sum_{i=0}^{p} \binom{p}{i} \theta^i I_{2,i+j,\phi} I_{2,i+j,\psi} + 10,$$

2.12. FGM type copulas [17, 18, 19, 24]

We consider the particular case in Section 2.10 for $a(x,y) = \psi(x)\psi(y)$. For the copulas given by

$$C(x,y) = xy\left[1+\theta\psi(x)\psi(y)\right]^p$$

for $0 < x < 1$, $0 < y < 1$, $p > 0$, $-1 < \theta < 1$ and ψ in a suitable function, we have

$$\begin{aligned}
c(x,y) &= \left[1+\theta\psi(x)\psi(y)\right]^p + p\theta\psi(x)y\left[1+\theta\psi(x)\psi(y)\right]^{p-1}\frac{\partial\psi(y)}{\partial y} \\
&\quad + p\theta x\psi(y)\left[1+\theta\psi(x)\psi(y)\right]^{p-1}\frac{\partial\psi(x)}{\partial x} + p\theta xy\left[1+\theta\psi(x)\psi(y)\right]^{p-1}\frac{\partial\psi(x)}{\partial x}\frac{\partial\psi(y)}{\partial y} \\
&\quad + p(p-1)\theta^2 xy\psi(x)\psi(y)\left[1+\theta\psi(x)\psi(y)\right]^{p-2}\frac{\partial\psi(x)}{\partial x}\frac{\partial\psi(y)}{\partial y} \\
&= \sum_{i=0}^{\infty}\binom{p}{i}\theta^i\left[\psi(x)\right]^i\left[\psi(y)\right]^i + p\theta y\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i\left[\psi(x)\right]^{i+1}\left[\psi(y)\right]^i\frac{\partial\psi(y)}{\partial y} \\
&\quad + p\theta x\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i\left[\psi(x)\right]^i\left[\psi(y)\right]^{i+1}\frac{\partial\psi(x)}{\partial x} \\
&\quad + p\theta xy\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i\left[\psi(x)\right]^i\left[\psi(y)\right]^i\frac{\partial\psi(x)}{\partial x}\frac{\partial\psi(y)}{\partial y} \\
&\quad + p(p-1)\theta^2 xy\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^i\left[\psi(x)\right]^{i+1}\left[\psi(y)\right]^{i+1}\frac{\partial\psi(x)}{\partial x}\frac{\partial\psi(y)}{\partial y},
\end{aligned}$$

$$\beta^* = \left[1+\theta\psi^2\left(\frac{1}{2}\right)\right]^p - 1,$$

$$\begin{aligned}
\tau &= 4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}I^2_{1,i+j,\psi} + 8p\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-1}{j}\theta^{i+j}I_{1,i+j+1,\psi}J_{2,i+j,\psi} \\
&\quad + 4p\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-1}{j}\theta^{i+j}J^2_{2,i+j,\psi} \\
&\quad + 4p(p-1)\theta^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p-2}{j}\theta^{i+j}J^2_{2,i+j+1,\psi} - 1,
\end{aligned}$$

$$\begin{aligned}
\rho &= 12\sum_{i=0}^{\infty}\binom{p}{i}\theta^i I^2_{1,i,\psi} + 24p\theta\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i I_{1,i+1,\psi}J_{2,i,\psi} \\
&\quad + 12p\theta\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^i J^2_{2,i,\psi} + 12p(p-1)\theta^2\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^i J^2_{2,i+1,\psi} - 3
\end{aligned}$$

and

$$\Phi^2 = 90\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}I_{2,i+j,\psi}^2 - 180\sum_{i=0}^{\infty}\binom{p}{i}\theta^i I_{2,i+j,\psi}^2 + 10.$$

If $p > 0$ is an integer then

$$\begin{aligned} c(x,y) = & \sum_{i=0}^{p}\binom{p}{i}\theta^i\left[\psi(x)\right]^i\left[\psi(y)\right]^i + p\theta y\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^i\left[\psi(x)\right]^{i+1}\left[\psi(y)\right]^i\frac{\partial\psi(y)}{\partial y} \\ & + p\theta x\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^i\left[\psi(x)\right]^i\left[\psi(y)\right]^{i+1}\frac{\partial\psi(x)}{\partial x} \\ & + p\theta xy\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^i\left[\psi(x)\right]^i\left[\psi(y)\right]^i\frac{\partial\psi(x)}{\partial x}\frac{\partial\psi(y)}{\partial y} \\ & + p(p-1)\theta^2 xy\sum_{i=0}^{p-2}\binom{p-2}{i}\theta^i\left[\psi(x)\right]^{i+1}\left[\psi(y)\right]^{i+1}\frac{\partial\psi(x)}{\partial x}\frac{\partial\psi(y)}{\partial y}, \end{aligned}$$

$$\begin{aligned} \tau = & 4\sum_{i=0}^{p}\sum_{j=0}^{p}\binom{p}{i}\binom{p}{j}\theta^{i+j}I_{1,i+j,\psi}^2 + 8p\theta\sum_{i=0}^{p}\sum_{j=0}^{p-1}\binom{p}{i}\binom{p-1}{j}\theta^{i+j}I_{1,i+j+1,\psi}J_{2,i+j,\psi} \\ & + 4p\theta\sum_{i=0}^{p}\sum_{j=0}^{p-1}\binom{p}{i}\binom{p-1}{j}\theta^{i+j}J_{2,i+j,\psi}^2 \\ & + 4p(p-1)\theta^2\sum_{i=0}^{p}\sum_{j=0}^{p-2}\binom{p}{i}\binom{p-2}{j}\theta^{i+j}J_{2,i+j+1,\psi}^2 - 1, \end{aligned}$$

$$\begin{aligned} \rho = & 12\sum_{i=0}^{p}\binom{p}{i}\theta^i I_{1,i,\psi}^2 + 24p\theta\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^i I_{1,i+1,\psi}J_{2,i,\psi} \\ & + 12p\theta\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^i J_{2,i,\psi}^2 + 12p(p-1)\theta^2\sum_{i=0}^{p-2}\binom{p-2}{i}\theta^i J_{2,i+1,\psi}^2 - 3 \end{aligned}$$

and

$$\Phi^2 = 90\sum_{i=0}^{p}\sum_{j=0}^{p}\binom{p}{i}\binom{p}{j}\theta^{i+j}I_{2,i+j,\psi}^2 - 180\sum_{i=0}^{p}\binom{p}{i}\theta^i I_{2,i+j,\psi}^2 + 10.$$

2.13. Chesneau [6]'s copula

For the copulas given by

$$C(x,y) = \sin(\pi x)\sin(\pi y)\phi(x+y)$$

for $0 < x < 1$ and $0 < y < 1$, we have

$$\begin{aligned} c(x,y) &= \pi^2 \cos(\pi x)\cos(\pi y)\phi(x+y) + \pi\cos(\pi x)\sin(\pi y)\phi^{'}(x+y) \\ &\quad + \pi\sin(\pi x)\cos(\pi y)\phi^{'}(x+y) + \sin(\pi x)\sin(\pi y)\phi^{''}(x+y), \end{aligned}$$

$$\beta^* = 4\phi(1) - 1,$$

$$\begin{aligned} \tau &= 4\pi^2 \int_0^1 \int_0^1 \sin(\pi x)\sin(\pi y)\cos(\pi x)\cos(\pi y)\left[\phi(x+y)\right]^2 dxdy \\ &\quad + 4\pi \int_0^1 \int_0^1 \sin(\pi x)\cos(\pi x)\sin^2(\pi y)\phi(x+y)\phi^{'}(x+y)dxdy \\ &\quad + 4\pi \int_0^1 \int_0^1 \sin^2(\pi x)\sin(\pi y)\cos(\pi y)\phi(x+y)\phi^{'}(x+y)dxdy \\ &\quad + 4 \int_0^1 \int_0^1 \sin^2(\pi x)\sin^2(\pi y)\phi(x+y)\phi^{''}(x+y)dxdy - 1, \end{aligned}$$

$$\begin{aligned} \rho &= 12\pi^2 \int_0^1 \int_0^1 xy\cos(\pi x)\cos(\pi y)\phi(x+y)dxdy \\ &\quad + 12\pi \int_0^1 \int_0^1 xy\cos(\pi x)\sin(\pi y)\phi^{'}(x+y)dxdy \\ &\quad + 12\pi \int_0^1 \int_0^1 xy\sin(\pi x)\cos(\pi y)\phi^{'}(x+y)dxdy \\ &\quad + 12 \int_0^1 \int_0^1 xy\sin(\pi x)\sin(\pi y)\phi^{''}(x+y)dxdy - 3 \end{aligned}$$

and

$$\Phi^2 = 90 \int_0^1 \int_0^1 \sin^2(\pi x)\sin^2(\pi y)\left[\phi(x,y)\right]^2 dxdy - 180 \int_0^1 \int_0^1 xy\sin(\pi x)\sin(\pi y)\phi(x,y)dxdy + 10.$$

2.14. Chesneau [7, 8]'s copula

For the copulas given by

$$C(x,y) = x^y y^x \phi(x,y)$$

for $0 < x < 1$, $0 < y < 1$ and $\phi(x,y)$ in a suitable function, we have

$$\begin{aligned} c(x,y) &= x^{y-1}y^{1+x}\log x\phi(x,y) + (1+x)x^{y-1}y^x\phi(x,y) + x^{y-1}y^{1+x}\frac{\partial\phi(x,y)}{\partial y} \\ &\quad + x^y y^x \log x \log y\phi(x,y) + x^{y+1}y^{x-1}\log y\phi(x,y) + x^y y^{x-1}\phi(x,y) \\ &\quad + x^y y^x \log y\frac{\partial\phi(x,y)}{\partial y} + x^y y^x \log x\frac{\partial\phi(x,y)}{\partial x} + x^{y+1}y^{x-1}\frac{\partial\phi(x,y)}{\partial x} \end{aligned}$$

$$+ x^y y^x \frac{\partial^2 \phi(x,y)}{\partial x \partial y},$$

$$\beta^* = 2\phi\left(\frac{1}{2}, \frac{1}{2}\right) - 1,$$

$$\begin{aligned}
\tau = 4\int_0^1 \int_0^1 x^{2y-1} y^{1+2x} \log x \left[\phi(x,y)\right]^2 dxdy + 4\int_0^1 \int_0^1 (1+x) x^{2y-1} y^{2x} \left[\phi(x,y)\right]^2 dxdy \\
+ 4\int_0^1 \int_0^1 x^{2y-1} y^{1+2x} \phi(x,y) \frac{\partial \phi(x,y)}{\partial y} dxdy + 4\int_0^1 \int_0^1 x^{2y} y^{2x} \log x \log y \left[\phi(x,y)\right]^2 dxdy \\
+ 4\int_0^1 \int_0^1 x^{2y+1} y^{2x-1} \log y \left[\phi(x,y)\right]^2 dxdy + 4\int_0^1 \int_0^1 x^{2y} y^{2x-1} \left[\phi(x,y)\right]^2 dxdy \\
+ 4\int_0^1 \int_0^1 x^{2y} y^{2x} \log y \phi(x,y) \frac{\partial \phi(x,y)}{\partial y} dxdy + 4\int_0^1 \int_0^1 x^{2y} y^{2x} \log x \phi(x,y) \frac{\partial \phi(x,y)}{\partial x} dxdy \\
+ 4\int_0^1 \int_0^1 x^{2y+1} y^{2x-1} \phi(x,y) \frac{\partial \phi(x,y)}{\partial x} dxdy + 4\int_0^1 \int_0^1 x^{2y} y^{2x} \phi(x,y) \frac{\partial^2 \phi(x,y)}{\partial x \partial y} dxdy - 1,
\end{aligned}$$

$$\begin{aligned}
\rho = 12\int_0^1 \int_0^1 x^y y^{2+x} \log x \phi(x,y) dxdy + 12\int_0^1 \int_0^1 (1+x) x^y y^{x+1} \phi(x,y) dxdy \\
+ 12\int_0^1 \int_0^1 x^y y^{2+x} \frac{\partial \phi(x,y)}{\partial y} dxdy + 12\int_0^1 \int_0^1 x^{y+1} y^{x+1} \log x \log y \phi(x,y) dxdy \\
+ 12\int_0^1 \int_0^1 x^{y+2} y^x \log y \phi(x,y) dxdy + 12\int_0^1 \int_0^1 x^{y+1} y^x \phi(x,y) dxdy \\
+ 12\int_0^1 \int_0^1 x^{y+1} y^{x+1} \log y \frac{\partial \phi(x,y)}{\partial y} dxdy + 12\int_0^1 \int_0^1 x^{y+1} y^{x+1} \log x \frac{\partial \phi(x,y)}{\partial x} dxdy \\
+ 12\int_0^1 \int_0^1 x^{y+2} y^x \frac{\partial \phi(x,y)}{\partial x} dxdy + x^{y+1} y^{x+1} \frac{\partial^2 \phi(x,y)}{\partial x \partial y} dxdy - 3
\end{aligned}$$

and

$$\Phi^2 = 90\int_0^1 \int_0^1 x^{2y} y^{2x} \left[\phi(x,y)\right]^2 dxdy - 180\int_0^1 \int_0^1 x^{y+1} y^{x+1} \phi(x,y) dxdy + 10.$$

2.15. *Linear combination of copulas*

For the copulas given by

$$C(x,y) = \sum_{i=1}^{n} w_i C_i(x,y)$$

for $0 < x < 1$ and $0 < y < 1$, where $C_i(x,y)$, $i = 1, 2, \ldots, n$ are valid copulas and w_i, $i = 1, 2, \ldots, n$ are non-negative weights summing to 1, we have

$$c(x,y) = \sum_{i=1}^{n} w_i c_i(x,y),$$

$$\beta^* = 4\sum_{i=1}^{n} w_i C_i\left(\frac{1}{2}, \frac{1}{2}\right) - 1,$$

$$\tau = 4\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \int_0^1\int_0^1 C_i(x,y)dC_j(x,y) - 1,$$

$$\rho = 12\sum_{i=1}^{n} w_i \int_0^1\int_0^1 xydc_i(x,y) - 3$$

and

$$\Phi^2 = 90\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \int_0^1\int_0^1 C_i(x,y)C_j(x,y)dxdy - 180\sum_{i=1}^{n} w_i \int_0^1\int_0^1 xyC_i(x,y)dxdy + 10.$$

2.16. Power combination of copulas

For the copulas given by

$$C(x,y) = \prod_{i=1}^{n}\left[C_i(x,y)\right]^{w_i}$$

for $0 < x < 1$ and $0 < y < 1$, where $C_i(x,y)$, $i = 1, 2, \ldots, n$ are valid copulas and w_i, $i = 1, 2, \ldots, n$ are non-negative, we have

$$\begin{aligned} c(x,y) = &\sum_{i=1}^{n} w_i\left(w_i - 1\right)\left[C_i(x,y)\right]^{w_i-1}\frac{\partial C_i(x,y)}{\partial x}\frac{\partial C_i(x,y)}{\partial y}\prod_{j=1, j\neq i}^{n}\left[C_j(x,y)\right]^{w_j} \\ &+ \sum_{i=1}^{n} w_i\left[C_i(x,y)\right]^{w_i-1} c_i(x,y)\prod_{j=1, j\neq i}^{n}\left[C_j(x,y)\right]^{w_j} \\ &+ \sum_{i=1}^{n} w_i\left[C_i(x,y)\right]^{w_i-2}\frac{\partial C_i(x,y)}{\partial x}\sum_{j=1, j\neq i}^{n} w_j\left[C_j(x,y)\right]^{w_j-1}\frac{\partial C_j(x,y)}{\partial y}\prod_{k=1, k\neq i,j}^{n}\left[C_k(x,y)\right]^{w_k}, \end{aligned}$$

$$\beta^* = 4\prod_{i=1}^{n}\left[C_i\left(\frac{1}{2}, \frac{1}{2}\right)\right]^{w_i} - 1,$$

$$\begin{aligned} \tau = &4\sum_{i=1}^{n} w_i\left(w_i - 1\right)\prod_{j=1, j\neq i}^{n}\int_0^1\int_0^1\left[C_i(x,y)\right]^{w_i-1}\left[C_j(x,y)\right]^{w_j}\prod_{\ell=1}^{n}\left[C_\ell(x,y)\right]^{w_\ell}\frac{\partial C_i(x,y)}{\partial x}\frac{\partial C_i(x,y)}{\partial y}dxdy \\ &+ 4\sum_{i=1}^{n} w_i\prod_{j=1, j\neq i}^{n}\int_0^1\int_0^1\left[C_i(x,y)\right]^{w_i-1}\left[C_j(x,y)\right]^{w_j}\prod_{\ell=1}^{n}\left[C_\ell(x,y)\right]^{w_\ell} c_i(x,y)dxdy \\ &+ 4\sum_{i=1}^{n} w_i\sum_{j=1, j\neq i}^{n} w_j\prod_{k=1, k\neq i,j}^{n}\int_0^1\int_0^1\left[C_i(x,y)\right]^{w_i-2}\left[C_j(x,y)\right]^{w_j-1}\left[C_k(x,y)\right]^{w_k}\prod_{\ell=1}^{n}\left[C_\ell(x,y)\right]^{w_\ell}\frac{\partial C_i(x,y)}{\partial x}\frac{\partial C_j(x,y)}{\partial y}dxdy - 1, \end{aligned}$$

$$\tau = 12\sum_{i=1}^{n} w_i\,(w_i-1)\prod_{j=1,j\neq i}^{n}\int_0^1\int_0^1 xy\,[C_i(x,y)]^{w_i-1}\left[C_j(x,y)\right]^{w_j}\frac{\partial C_i(x,y)}{\partial x}\frac{\partial C_i(x,y)}{\partial y}dxdy$$
$$+\,12\sum_{i=1}^{n} w_i\prod_{j=1,j\neq i}^{n}\int_0^1\int_0^1 xy\,[C_i(x,y)]^{w_i-1}\left[C_j(x,y)\right]^{w_j} c_i(x,y)dxdy$$
$$+\,12\sum_{i=1}^{n} w_i\sum_{j=1,j\neq i}^{n} w_j\prod_{k=1,k\neq i,j}^{n}\int_0^1\int_0^1 xy\,[C_i(x,y)]^{w_i-2}\left[C_j(x,y)\right]^{w_j-1}[C_k(x,y)]^{w_k}\frac{\partial C_i(x,y)}{\partial x}\frac{\partial C_j(x,y)}{\partial y}dxdy-3$$

and

$$\Phi^2 = 90\int_0^1\int_0^1\prod_{i=1}^{n}\prod_{j=1}^{n}[C_i(x,y)]^{w_i}\left[C_j(x,y)\right]^{w_j}dxdy-180\int_0^1\int_0^1 xy\prod_{i=1}^{n}[C_i(x,y)]^{w_i}dxdy+10.$$

2.17. Durante [13]'s copulas

For the copulas given by

$$C(x,y) = A\left(x^{\alpha},y^{\beta}\right)\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)$$

for $0<x<1$, $0<y<1$, $0<\alpha<1$ and $0<\beta<1$, where A and $\mathcal{B}$ are valid copulas, we have

$$c(x,y) = \alpha\beta x^{\alpha-1}y^{\beta-1}a\left(x^{\alpha},y^{\beta}\right)\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)$$
$$+\,\alpha(1-\beta)x^{\alpha-1}y^{-\beta}\frac{\partial A\left(x^{\alpha},y^{\beta}\right)}{\partial x^{\alpha}}\frac{\partial\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)}{\partial y^{1-\beta}}$$
$$+\,(1-\alpha)\beta x^{-\beta}y^{\beta-1}\frac{\partial A\left(x^{\alpha},y^{\beta}\right)}{\partial y^{\beta}}\frac{\partial\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)}{\partial x^{1-\alpha}}$$
$$+\,(1-\alpha)(1-\beta)x^{-\alpha}y^{-\beta}A\left(x^{\alpha},y^{\beta}\right)b\left(x^{1-\alpha},y^{1-\beta}\right),$$

$$\beta^{*} = 4A\left(2^{-\alpha},2^{-\beta}\right)\mathcal{B}\left(2^{\alpha-1},2^{\beta-1}\right)-1,$$

$$\rho = 4\alpha\beta\int_0^1\int_0^1 x^{\alpha-1}y^{\beta-1}a\left(x^{\alpha},y^{\beta}\right)A\left(x^{\alpha},y^{\beta}\right)\left[\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)\right]^2dxdy$$
$$+\,4\alpha(1-\beta)\int_0^1\int_0^1 x^{\alpha-1}y^{-\beta}\frac{\partial A\left(x^{\alpha},y^{\beta}\right)}{\partial x^{\alpha}}\frac{\partial\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)}{\partial y^{1-\beta}}A\left(x^{\alpha},y^{\beta}\right)\mathcal{B}\left(x^{1-\beta},y^{1-\beta}\right)dxdy$$
$$+\,4(1-\alpha)\beta\int_0^1\int_0^1 x^{-\alpha}y^{\beta-1}\frac{\partial A\left(x^{\alpha},y^{\beta}\right)}{\partial y^{\beta}}\frac{\partial\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)}{\partial x^{1-\alpha}}A\left(x^{\alpha},y^{\beta}\right)\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)dxdy$$
$$+\,4(1-\alpha)(1-\beta)\int_0^1\int_0^1 x^{-\alpha}y^{-\beta}\left[A\left(x^{\alpha},y^{\beta}\right)\right]^2b\left(x^{1-\alpha},y^{1-\beta}\right)\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)dxdy-1,$$

$$\tau = 12\alpha\beta\int_0^1\int_0^1 x^{\alpha}y^{\beta}a\left(x^{\alpha},y^{\beta}\right)\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)dxdy$$

$$+ 12\alpha(1-\beta)\int_0^1\int_0^1 x^{\alpha}y^{1-\beta}\frac{\partial A\left(x^{\alpha},y^{\beta}\right)}{\partial x^{\alpha}}\frac{\partial \mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)}{\partial y^{1-\beta}}dxdy$$
$$+ 12(1-\alpha)\beta\int_0^1\int_0^1 x^{1-\alpha}y^{\beta}\frac{\partial A\left(x^{\alpha},y^{\beta}\right)}{\partial y^{\beta}}\frac{\partial \mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)}{\partial x^{1-\alpha}}dxdy$$
$$+ 12(1-\alpha)(1-\beta)\int_0^1\int_0^1 x^{1-\alpha}y^{1-\beta}A\left(x^{\alpha},y^{\beta}\right)b\left(x^{1-\alpha},y^{1-\beta}\right)dxdy-3$$

and

$$\Phi^2 = 90\int_0^1\int_0^1\left[A\left(x^{\alpha},y^{\beta}\right)\right]^2\left[\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)\right]^2 dxdy$$
$$- 180\int_0^1\int_0^1 xyA\left(x^{\alpha},y^{\beta}\right)\mathcal{B}\left(x^{1-\alpha},y^{1-\beta}\right)dxdy+10.$$

2.18. Durante et al. [15]'s copulas

For the copulas given by

$$C(x,y) = xf\left(\frac{1}{x}f^{-1}(y)\right)$$

for $0<x<1$, $0<y<1$ and $f:[0,\infty]\to[0,1]$ as a surjective, monotonic function, we have

$$c(x,y) = -\frac{1}{x^2}f''\left(\frac{1}{x}f^{-1}(y)\right)\frac{df^{-1}(y)}{dy}f^{-1}(y),$$

$$\beta^* = 2f\left(2f^{-1}\left(\frac{1}{2}\right)\right)-1,$$

$$\tau = -4\int_0^1\int_0^1\frac{1}{x}f\left(\frac{1}{x}f^{-1}(y)\right)f''\left(\frac{1}{x}f^{-1}(y)\right)\frac{df^{-1}(y)}{dy}f^{-1}(y)dxdy-1,$$

$$\rho = -12\int_0^1\int_0^1\frac{y}{x}f''\left(\frac{1}{x}f^{-1}(y)\right)\frac{df^{-1}(y)}{dy}f^{-1}(y)dxdy-3$$

and

$$\Phi^2 = 90\int_0^1\int_0^1 x^2\left[f\left(\frac{1}{x}f^{-1}(y)\right)\right]^2 dxdy-180\int_0^1\int_0^1 x^2yf\left(\frac{1}{x}f^{-1}(y)\right)dxdy+10.$$

2.19. Kumar [23]'s perturbed copulas

For the copulas given by

$$C(x,y) = xy+a\left[x-\mathcal{B}(x,y)\right]\left[y-\mathcal{B}(x,y)\right]$$

for $0 < x < 1$, $0 < y < 1$ and $0 \le a \le 1$, where B is a valid copula, we have

$$\begin{aligned} c(x,y) = 1 + a - a\frac{\partial\mathcal{B}(x,y)}{\partial x} \\ - a\frac{\partial\mathcal{B}(x,y)}{\partial y} + 2a\frac{\partial\mathcal{B}(x,y)}{\partial x}\frac{\partial\mathcal{B}(x,y)}{\partial y} \\ + 2ab(x,y)\mathcal{B}(x,y) - a\,(x+y)\,b(x,y), \end{aligned}$$

$$\beta^* = a\left[1 - 2\mathcal{B}\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2,$$

$$\begin{aligned} \tau = (1+a)^2 - 4a(1+a)\int_0^1\int_0^1 xy\frac{\partial\mathcal{B}(x,y)}{\partial x}dxdy \\ - 4a(1+a)\int_0^1\int_0^1 xy\frac{\partial\mathcal{B}(x,y)}{\partial y}dxdy + 8a(1+a)\int_0^1\int_0^1 xy\frac{\partial\mathcal{B}(x,y)}{\partial x}\frac{\partial\mathcal{B}(x,y)}{\partial y}dxdy \\ + 8a(1+a)\int_0^1\int_0^1 xyb(x,y)\mathcal{B}(x,y)dxdy - 4a(1+a)\int_0^1\int_0^1 xy\,(x+y)\,b(x,y)dxdy \\ + 4a(1+a)\int_0^1\int_0^1 (x+y)\mathcal{B}(x,y)dxdy - 4a^2\int_0^1\int_0^1 (x+y)\mathcal{B}(x,y)\frac{\partial\mathcal{B}(x,y)}{\partial x}dxdy \\ - 4a^2\int_0^1\int_0^1 (x+y)\mathcal{B}(x,y)\frac{\partial\mathcal{B}(x,y)}{\partial y}dxdy + 8a^2\int_0^1\int_0^1 (x+y)\mathcal{B}(x,y)\frac{\partial\mathcal{B}(x,y)}{\partial x}\frac{\partial\mathcal{B}(x,y)}{\partial y}dxdy \\ + 8a^2\int_0^1\int_0^1 (x+y)b(x,y)\mathcal{B}^2(x,y)dxdy - 4a^2\int_0^1\int_0^1 (x+y)^2\,b(x,y)\mathcal{B}(x,y)dxdy \\ 4a(1+a)\int_0^1\int_0^1 \mathcal{B}^2(x,y)dxdy - 4a^2\int_0^1\int_0^1 \mathcal{B}^2(x,y)\frac{\partial\mathcal{B}(x,y)}{\partial x}dxdy \\ - 4\int_0^1\int_0^1 a^2\mathcal{B}^2(x,y)\frac{\partial\mathcal{B}(x,y)}{\partial y}dxdy + 8a^2\int_0^1\int_0^1 \mathcal{B}^2(x,y)\frac{\partial\mathcal{B}(x,y)}{\partial x}\frac{\partial\mathcal{B}(x,y)}{\partial y}dxdy \\ + 8a^2\int_0^1\int_0^1 b(x,y)\mathcal{B}^3(x,y)dxdy - 4a^2\int_0^1\int_0^1 (x+y)\,b(x,y)\mathcal{B}^2(x,y)dxdy - 1, \end{aligned}$$

$$\begin{aligned} \tau = a + 4ac(c-1) - 4a(1+a)\int_0^1\int_0^1 \frac{\partial\mathcal{B}(x,y)}{\partial x}dxdy \\ + 8a(1+a)\int_0^1\int_0^1 xy\frac{\partial\mathcal{B}(x,y)}{\partial x}\frac{\partial\mathcal{B}(x,y)}{\partial y}dxdy + 8a(1+a)c\int_0^1\int_0^1 xyb(x,y)dxdy \\ - 4a(1+a)\int_0^1\int_0^1 xy\,(x+y)\,b(x,y)dxdy + 4a^2\int_0^1\int_0^1 (x+y)\frac{\partial\mathcal{B}(x,y)}{\partial x}dxdy \\ - 8a^2\int_0^1\int_0^1 (x+y)\frac{\partial\mathcal{B}(x,y)}{\partial x}\frac{\partial\mathcal{B}(x,y)}{\partial y}dxdy - 8a^2c\int_0^1\int_0^1 (x+y)b(x,y)dxdy \\ + 4a^2\int_0^1\int_0^1 (x+y)^2\,b(x,y)dxdy - 4a^2c^2\int_0^1\int_0^1 \frac{\partial\mathcal{B}(x,y)}{\partial x}dxdy \end{aligned}$$

$$+ 8a^2c^2 \int_0^1 \int_0^1 \frac{\partial \mathcal{B}(x,y)}{\partial x} \frac{\partial \mathcal{B}(x,y)}{\partial y} dxdy + 8a^2c^3 \int_0^1 \int_0^1 b(x,y) dxdy$$
$$- 4a^2c^2 \int_0^1 \int_0^1 (x+y)\, b(x,y) dxdy,$$

$$\begin{aligned} \rho = {} & 3(1+a) - 12a \int_0^1 \int_0^1 xy \frac{\partial \mathcal{B}(x,y)}{\partial x} dxdy \\ & - 12a \int_0^1 \int_0^1 xy \frac{\partial \mathcal{B}(x,y)}{\partial y} dxdy + 24a \int_0^1 \int_0^1 xy \frac{\partial \mathcal{B}(x,y)}{\partial x} \frac{\partial \mathcal{B}(x,y)}{\partial y} dxdy \\ & + 24a \int_0^1 \int_0^1 xy b(x,y) \mathcal{B}(x,y) dxdy - 12a \int_0^1 \int_0^1 xy\,(x+y)\, b(x,y) dxdy - 3 \end{aligned}$$

and

$$\Phi^2 = 90a^2 \int_0^1 \int_0^1 \left[x - \mathcal{B}(x,y)\right]^2 \left[y - \mathcal{B}(x,y)\right]^2 dxdy.$$

2.20. Sharifonnasabi et al. [29]'s copulas

For copulas given by

$$C(x,y) = K(x,y) \exp\left[\theta \phi(x,y)\right]$$

for $0 < x < 1$, $0 < y < 1$, $-1 < \theta < 1$ and $\phi(x,y)$ in a suitable function, where $K(x,y)$ is a valid copula, we have

$$\begin{aligned} c(x,y) = {} & \theta^2 e^{\theta\phi(x,y)} K(x,y) \frac{\partial \phi(x,y)}{\partial x} \frac{\partial \phi(x,y)}{\partial y} + \theta e^{\theta\phi(x,y)} \frac{\partial \phi(x,y)}{\partial x} \frac{\partial K(x,y)}{\partial y} \\ & + \theta e^{\theta\phi(x,y)} \frac{\partial \phi(x,y)}{\partial y} \frac{\partial K(x,y)}{\partial x} + \theta e^{\theta\phi(x,y)} K(x,y) \frac{\partial^2 \phi(x,y)}{\partial x \partial y} \\ & + e^{\theta\phi(x,y)} \frac{\partial^2 K(x,y)}{\partial x \partial y}, \end{aligned}$$

$$\beta^* = 4\, K\left(\frac{1}{2}, \frac{1}{2}\right) \exp\left[\theta \phi\left(\frac{1}{2}, \frac{1}{2}\right)\right],$$

$$\begin{aligned} \tau = {} & 4\theta^2 \int_0^1 \int_0^1 K(x,y) e^{2\theta\phi(x,y)} K(x,y) \frac{\partial \phi(x,y)}{\partial x} \frac{\partial \phi(x,y)}{\partial y} dxdy \\ & + 4\theta \int_0^1 \int_0^1 K(x,y) e^{2\theta\phi(x,y)} \frac{\partial \phi(x,y)}{\partial x} \frac{\partial K(x,y)}{\partial y} dxdy \\ & + 4\theta \int_0^1 \int_0^1 K(x,y) e^{2\theta\phi(x,y)} \frac{\partial \phi(x,y)}{\partial y} \frac{\partial K(x,y)}{\partial x} dxdy \\ & + 4\theta \int_0^1 \int_0^1 K^2(x,y) e^{2\theta\phi(x,y)} \frac{\partial^2 \phi(x,y)}{\partial x \partial y} \end{aligned}$$

$$+ 4\int_0^1\int_0^1 K(x,y)e^{2\theta\phi(x,y)}\frac{\partial^2 K(x,y)}{\partial x\partial y}dxdy - 1,$$

$$\begin{aligned}\rho = {} & 12\theta^2\int_0^1\int_0^1 xye^{\theta\phi(x,y)}K(x,y)\frac{\partial\phi(x,y)}{\partial x}\frac{\partial\phi(x,y)}{\partial y}dxdy \\ & + 12\theta\int_0^1\int_0^1 xye^{\theta\phi(x,y)}\frac{\partial\phi(x,y)}{\partial x}\frac{\partial K(x,y)}{\partial y}dxdy \\ & + 12\theta\int_0^1\int_0^1 xye^{\theta\phi(x,y)}\frac{\partial\phi(x,y)}{\partial y}\frac{\partial K(x,y)}{\partial x}dxdy \\ & + 12\theta\int_0^1\int_0^1 xye^{\theta\phi(x,y)}K(x,y)\frac{\partial^2\phi(x,y)}{\partial x\partial y}dxdy \\ & + 12\int_0^1\int_0^1 xye^{\theta\phi(x,y)}\frac{\partial^2 K(x,y)}{\partial x\partial y}dxdy - 3\end{aligned}$$

and

$$\Phi^2 = 90\int_0^1\int_0^1 \left[K(x,y)\right]^2 e^{2\theta\phi(x,y)}dxdy - 180\int_0^1\int_0^1 xyK(x,y)e^{\theta\phi(x,y)}dxdy + 10.$$

2.21. Sharifonnasabi et al. [29]'s copulas

Consider a particular case of Section 2.20 for $\phi(x,y) = f(x)g(y)$. For copulas given by

$$C(x,y) = K(x,y)\exp\left[\theta f(x)g(y)\right]$$

for $0 < x < 1$, $0 < y < 1$, $-1 < \theta < 1$ and $f(x)$, $g(y)$ in suitable functions, where $K(x,y)$ is a valid copula, we have

$$\begin{aligned}c(x,y) = {} & \theta^2 e^{\theta f(x)g(y)}K(x,y)f(x)g(y)f^{'}(x)g^{'}(y) + \theta e^{\theta f(x)g(y)}f^{'}(x)g(y)\frac{\partial K(x,y)}{\partial y} \\ & + \theta e^{\theta f(x)g(y)}f(x)g^{'}(y)\frac{\partial K(x,y)}{\partial x} + \theta e^{\theta f(x)g(y)}K(x,y)f^{'}(x)g^{'}(y) \\ & + e^{\theta f(x)g(y)}\frac{\partial^2 K(x,y)}{\partial x\partial y},\end{aligned}$$

$$\beta^* = 4\,K\left(\frac{1}{2},\frac{1}{2}\right)\exp\left[\theta f\left(\frac{1}{2}\right)g\left(\frac{1}{2}\right)\right],$$

$$\begin{aligned}\tau = {} & 4\theta^2\int_0^1\int_0^1 K(x,y)e^{2\theta f(x)g(y)}K(x,y)f(x)g(y)f^{'}(x)g^{'}(y)dxdy \\ & + 4\theta\int_0^1\int_0^1 K(x,y)e^{2\theta f(x)g(y)}f^{'}(x)g(y)\frac{\partial K(x,y)}{\partial y}dxdy \\ & + 4\theta\int_0^1\int_0^1 K(x,y)e^{2\theta f(x)g(y)}f(x)g^{'}(y)\frac{\partial K(x,y)}{\partial x}dxdy\end{aligned}$$

$$+ 4\theta \int_0^1 \int_0^1 K^2(x,y) e^{2\theta f(x) g(y)} f'(x) g'(y) dxdy$$
$$+ 4 \int_0^1 \int_0^1 K(x,y) e^{2\theta f(x) g(y)} \frac{\partial^2 K(x,y)}{\partial x \partial y} dxdy - 1,$$

$$\rho = 12\theta^2 \int_0^1 \int_0^1 xy e^{\theta f(x) g(y)} K(x,y) f(x) g(y) f'(x) g'(y) dxdy$$
$$+ 12\theta \int_0^1 \int_0^1 xy e^{\theta f(x) g(y)} f'(x) g(y) \frac{\partial K(x,y)}{\partial y} dxdy$$
$$+ 12\theta \int_0^1 \int_0^1 xy e^{\theta f(x) g(y)} f(x) g'(y) \frac{\partial K(x,y)}{\partial x} dxdy$$
$$+ 12\theta \int_0^1 \int_0^1 xy e^{\theta f(x) g(y)} K(x,y) f'(x) g'(y) dxdy$$
$$+ 12 \int_0^1 \int_0^1 xy e^{\theta f(x) g(y)} \frac{\partial^2 K(x,y)}{\partial x \partial y} dxdy - 3$$

and

$$\Phi^2 = 90 \int_0^1 \int_0^1 [K(x,y)]^2 e^{2\theta f(x) g(y)} dxdy - 180 \int_0^1 \int_0^1 xy K(x,y) e^{\theta f(x) g(y)} dxdy + 10.$$

2.22. Sharifonnasabi et al. [29]'s copulas

Consider a particular case of Section 2.20 for $\phi(x,y) = f(x) f(y)$. For copulas given by

$$C(x,y) = K(x,y) \exp\left[\theta f(x) f(y)\right]$$

for $0 < x < 1$, $0 < y < 1$, $-1 < \theta < 1$ and f in a suitable function, where $K(x,y)$ is a valid copula, we have

$$c(x,y) = \theta^2 e^{\theta f(x) f(y)} K(x,y) f(x) f(y) f'(x) f'(y) + \theta e^{\theta f(x) f(y)} f'(x) f(y) \frac{\partial K(x,y)}{\partial y}$$
$$+ \theta e^{\theta f(x) f(y)} f(x) f'(y) \frac{\partial K(x,y)}{\partial x} + \theta e^{\theta f(x) f(y)} K(x,y) f'(x) f'(y)$$
$$+ e^{\theta f(x) f(y)} \frac{\partial^2 K(x,y)}{\partial x \partial y},$$

$$\beta^* = 4\, K\left(\frac{1}{2}, \frac{1}{2}\right) \exp\left[\theta f^2\left(\frac{1}{2}\right)\right],$$

$$\tau = 4\theta^2 \int_0^1 \int_0^1 K(x,y) e^{2\theta f(x) f(y)} K(x,y) f(x) f(y) f'(x) f'(y) dxdy$$
$$+ 4\theta \int_0^1 \int_0^1 K(x,y) e^{2\theta f(x) f(y)} f'(x) f(y) \frac{\partial K(x,y)}{\partial y} dxdy$$

$$+ 4\theta \int_0^1 \int_0^1 K(x,y) e^{2\theta f(x)f(y)} f(x) f'(y) \frac{\partial K(x,y)}{\partial x} dxdy$$
$$+ 4\theta \int_0^1 \int_0^1 K^2(x,y) e^{2\theta f(x)f(y)} f'(x) f'(y) dxdy$$
$$+ 4 \int_0^1 \int_0^1 K(x,y) e^{2\theta f(x)f(y)} \frac{\partial^2 K(x,y)}{\partial x \partial y} dxdy - 1,$$

$$\begin{aligned}\rho = {} & 12\theta^2 \int_0^1 \int_0^1 xy e^{\theta f(x)f(y)} K(x,y) f(x) f(y) f'(x) f'(y) dxdy \\ & + 12\theta \int_0^1 \int_0^1 xy e^{\theta f(x)f(y)} f'(x) f(y) \frac{\partial K(x,y)}{\partial y} dxdy \\ & + 12\theta \int_0^1 \int_0^1 xy e^{\theta f(x)f(y)} f(x) f'(y) \frac{\partial K(x,y)}{\partial x} dxdy \\ & + 12\theta \int_0^1 \int_0^1 xy e^{\theta f(x)f(y)} K(x,y) f'(x) f'(y) dxdy \\ & + 12 \int_0^1 \int_0^1 xy e^{\theta f(x)f(y)} \frac{\partial^2 K(x,y)}{\partial x \partial y} dxdy - 3\end{aligned}$$

and

$$\Phi^2 = 90 \int_0^1 \int_0^1 [K(x,y)]^2 e^{2\theta f(x)f(y)} dxdy - 180 \int_0^1 \int_0^1 xy K(x,y) e^{\theta f(x)f(y)} dxdy + 10.$$

2.23. Sharifonnasabi et al. [29]'s copulas

For copulas given by

$$C(x,y) = K(x,y) [1 + \theta \phi(x,y)]^p = K(x,y) \sum_{i=0}^{\infty} \binom{p}{i} \theta^i [\phi(x,y)]^i$$

for $0 < x < 1$, $0 < y < 1$, $-1 < \theta < 1$, $p > 0$ and $\phi(x,y)$ in a suitable function, where $K(x,y)$ is a valid copula, we have

$$\begin{aligned}c(x,y) = {} & [1 + \theta \phi(x,y)]^p \frac{\partial^2 K(x,y)}{\partial x \partial y} \\ & + \theta p \frac{\partial K(x,y)}{\partial y} \frac{\partial \phi(x,y)}{\partial x} [1 + \theta \phi(x,y)]^{p-1} \\ & + \theta p \frac{\partial K(x,y)}{\partial x} \frac{\partial \phi(x,y)}{\partial y} [1 + \theta \phi(x,y)]^{p-1} \\ & + \theta p K(x,y) \frac{\partial^2 \phi(x,y)}{\partial x \partial y} [1 + \theta \phi(x,y)]^{p-1} \\ & + \theta^2 p(p-1) K(x,y) [1 + \theta \phi(x,y)]^{p-2} \frac{\partial \phi(x,y)}{\partial x} \frac{\partial \phi(x,y)}{\partial y}\end{aligned}$$

$$
\begin{aligned}
= \sum_{i=0}^{\infty}\binom{p}{i}\theta^{i}\left[\phi(x,y)\right]^{i}\frac{\partial^{2}K(x,y)}{\partial x\partial y} \\
&+ \theta p \sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\left[\phi(x,y)\right]^{i}\frac{\partial K(x,y)}{\partial y}\frac{\partial \phi(x,y)}{\partial x} \\
&+ \theta p \sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\left[\phi(x,y)\right]^{i}\frac{\partial K(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y} \\
&+ \theta p \sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\left[\phi(x,y)\right]^{i}K(x,y)\frac{\partial^{2}\phi(x,y)}{\partial x\partial y} \\
&+ \theta^{2}p(p-1)\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^{i}\left[\phi(x,y)\right]^{i}K(x,y)\frac{\partial \phi(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y},
\end{aligned}
$$

$$
\beta^{*} = 4K\left(\frac{1}{2},\frac{1}{2}\right)\left[1+\theta\phi\left(\frac{1}{2},\frac{1}{2}\right)\right]^{p} - 1,
$$

$$
\begin{aligned}
\tau = 4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[\phi(x,y)\right]^{i+j}K(x,y)\frac{\partial^{2}K(x,y)}{\partial x\partial y}dxdy \\
&+ 4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[\phi(x,y)\right]^{i+j}K(x,y)\frac{\partial K(x,y)}{\partial y}\frac{\partial \phi(x,y)}{\partial x}dxdy \\
&+ 4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[\phi(x,y)\right]^{i+j}K(x,y)\frac{\partial K(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y}dxdy \\
&+ 4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[\phi(x,y)\right]^{i+j}K^{2}(x,y)\frac{\partial^{2}\phi(x,y)}{\partial x\partial y}dxdy \\
&+ 4\theta^{2}p(p-1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-2}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[\phi(x,y)\right]^{i+j}K^{2}(x,y)\frac{\partial \phi(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y}dxdy - 1,
\end{aligned}
$$

$$
\begin{aligned}
\rho = 12\sum_{i=0}^{\infty}\binom{p}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[\phi(x,y)\right]^{i}\frac{\partial^{2}K(x,y)}{\partial x\partial y}dxdy \\
&+ 12\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[\phi(x,y)\right]^{i}\frac{\partial K(x,y)}{\partial y}\frac{\partial \phi(x,y)}{\partial x}dxdy \\
&+ 12\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[\phi(x,y)\right]^{i}\frac{\partial K(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y}dxdy \\
&+ 12\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[\phi(x,y)\right]^{i}K(x,y)\frac{\partial^{2}\phi(x,y)}{\partial x\partial y}dxdy \\
&+ 12\theta^{2}p(p-1)\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[\phi(x,y)\right]^{i}K(x,y)\frac{\partial \phi(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y}dxdy - 3
\end{aligned}
$$

and

$$\Phi^2 = 90\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1 [K(x,y)]^2 [\phi(x,y)]^{i+j}\, dxdy$$
$$- 180\sum_{i=0}^{\infty}\binom{p}{i}\theta^{i}\int_0^1\int_0^1 xyK(x,y) [\phi(x,y)]^{i}\, dxdy + 10.$$

If $p > 0$ is an integer then

$$C(x,y) = K(x,y)\sum_{i=0}^{p}\binom{p}{i}\theta^{i} [\phi(x,y)]^{i},$$

$$\begin{aligned} c(x,y) = &\sum_{i=0}^{p}\binom{p}{i}\theta^{i} [\phi(x,y)]^{i} \frac{\partial^2 K(x,y)}{\partial x\partial y} \\ &+ \theta p\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^{i} [\phi(x,y)]^{i} \frac{\partial K(x,y)}{\partial y}\frac{\partial \phi(x,y)}{\partial x} \\ &+ \theta p\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^{i} [\phi(x,y)]^{i} \frac{\partial K(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y} \\ &+ \theta p\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^{i} [\phi(x,y)]^{i} K(x,y)\frac{\partial^2 \phi(x,y)}{\partial x\partial y} \\ &+ \theta^2 p(p-1)\sum_{i=0}^{p-2}\binom{p-2}{i}\theta^{i} [\phi(x,y)]^{i} K(x,y)\frac{\partial \phi(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y}, \end{aligned}$$

$$\begin{aligned} \tau = 4&\sum_{i=0}^{p}\sum_{j=0}^{p}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1 [\phi(x,y)]^{i+j} K(x,y)\frac{\partial^2 K(x,y)}{\partial x\partial y}dxdy \\ &+ 4\theta p\sum_{i=0}^{p-1}\sum_{j=0}^{p}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1 [\phi(x,y)]^{i+j} K(x,y)\frac{\partial K(x,y)}{\partial y}\frac{\partial \phi(x,y)}{\partial x}dxdy \\ &+ 4\theta p\sum_{i=0}^{p-1}\sum_{j=0}^{p}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1 [\phi(x,y)]^{i+j} K(x,y)\frac{\partial K(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y}dxdy \\ &+ 4\theta p\sum_{i=0}^{p-1}\sum_{j=0}^{p}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1 [\phi(x,y)]^{i+j} K^2(x,y)\frac{\partial^2 \phi(x,y)}{\partial x\partial y}dxdy \\ &+ 4\theta^2 p(p-1)\sum_{i=0}^{p-2}\sum_{j=0}^{p}\binom{p-2}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1 [\phi(x,y)]^{i+j} K^2(x,y)\frac{\partial \phi(x,y)}{\partial x}\frac{\partial \phi(x,y)}{\partial y}dxdy - 1, \end{aligned}$$

$$\rho = 12\sum_{i=0}^{p}\binom{p}{i}\theta^{i}\int_0^1\int_0^1 xy [\phi(x,y)]^{i} \frac{\partial^2 K(x,y)}{\partial x\partial y}dxdy$$

$$+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 xy\,[\phi(x,y)]^i \,\frac{\partial K(x,y)}{\partial y} \frac{\partial \phi(x,y)}{\partial x} dxdy$$

$$+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 xy\,[\phi(x,y)]^i \,\frac{\partial K(x,y)}{\partial x} \frac{\partial \phi(x,y)}{\partial y} dxdy$$

$$+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 xy\,[\phi(x,y)]^i \,K(x,y) \frac{\partial^2 \phi(x,y)}{\partial x \partial y} dxdy$$

$$+ 12\theta^2 p(p-1) \sum_{i=0}^{p-2} \binom{p-2}{i} \theta^i \int_0^1 \int_0^1 xy\,[\phi(x,y)]^i \,K(x,y) \frac{\partial \phi(x,y)}{\partial x} \frac{\partial \phi(x,y)}{\partial y} dxdy - 3$$

and

$$\Phi^2 = 90 \sum_{i=0}^{p} \sum_{j=0}^{p} \binom{p}{i} \binom{p}{j} \theta^{i+j} \int_0^1 \int_0^1 [K(x,y)]^2 \,[\phi(x,y)]^{i+j} \,dxdy$$

$$- 180 \sum_{i=0}^{p} \binom{p}{i} \theta^i \int_0^1 \int_0^1 xyK(x,y)\,[\phi(x,y)]^i \,dxdy + 10.$$

2.24. Sharifonnasabi et al. [29]'s copulas

Consider a particular case of Section 2.23 for $\phi(x,y) = f(x)g(y)$. For copulas given by

$$C(x,y) = K(x,y)\,[1 + \theta f(x)g(y)]^p = K(x,y) \sum_{i=0}^{\infty} \binom{p}{i} \theta^i \,[f(x)g(y)]^i$$

for $0 < x < 1$, $0 < y < 1$, $-1 < \theta < 1$, $p > 0$ and $f(x)$, $g(y)$ in suitable functions, where $K(x,y)$ is a valid copula, we have

$$\begin{aligned}
c(x,y) &= [1 + \theta f(x)g(y)]^p \,\frac{\partial^2 K(x,y)}{\partial x \partial y} \\
&\quad + \theta p \frac{\partial K(x,y)}{\partial y} f^{'}(x)g(y)\,[1 + \theta f(x)g(y)]^{p-1} \\
&\quad + \theta p \frac{\partial K(x,y)}{\partial x} f(x)g^{'}(y)\,[1 + \theta f(x)g(y)]^{p-1} \\
&\quad + \theta p K(x,y) f^{'}(x)g^{'}(y)\,[1 + \theta f(x)g(y)]^{p-1} \\
&\quad + \theta^2 p(p-1)K(x,y)\,[1 + \theta f(x)g(y)]^{p-2} \,f(x)g(y)f^{'}(x)g^{'}(y) \\
&= \sum_{i=0}^{\infty} \binom{p}{i} \theta^i \,[f(x)g(y)]^i \,\frac{\partial^2 K(x,y)}{\partial x \partial y} \\
&\quad + \theta p \sum_{i=0}^{\infty} \binom{p-1}{i} \theta^i \,[f(x)g(y)]^i \,\frac{\partial K(x,y)}{\partial y} f^{'}(x)g(y) \\
&\quad + \theta p \sum_{i=0}^{\infty} \binom{p-1}{i} \theta^i \,[f(x)g(y)]^i \,\frac{\partial K(x,y)}{\partial x} f(x)g^{'}(y)
\end{aligned}$$

$$+\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\left[f(x)g(y)\right]^{i}K(x,y)f^{'}(x)g^{'}(y)$$
$$+\theta^{2}p(p-1)\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^{i}\left[f(x)g(y)\right]^{i}K(x,y)f(x)g(y)f^{'}(x)g^{'}(y),$$

$$\beta^{*}=4K\left(\frac{1}{2},\frac{1}{2}\right)\left[1+\theta f\left(\frac{1}{2}\right)g\left(\frac{1}{2}\right)\right]^{p}-1,$$

$$\begin{aligned}\tau=4&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K(x,y)\frac{\partial^{2}K(x,y)}{\partial x\partial y}dxdy\\&+4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K(x,y)\frac{\partial K(x,y)}{\partial y}f^{'}(x)g(y)dxdy\\&+4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K(x,y)\frac{\partial K(x,y)}{\partial x}f(x)g^{'}(y)dxdy\\&+4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K^{2}(x,y)f^{'}(x)g^{'}(y)dxdy\\&+4\theta^{2}p(p-1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-2}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K^{2}(x,y)f(x)g(y)f^{'}(x)g^{'}(y)dxdy-1,\end{aligned}$$

$$\begin{aligned}\rho=12&\sum_{i=0}^{\infty}\binom{p}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)g(y)\right]^{i}\frac{\partial^{2}K(x,y)}{\partial x\partial y}dxdy\\&+12\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)g(y)\right]^{i}\frac{\partial K(x,y)}{\partial y}f^{'}(x)g(y)dxdy\\&+12\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)g(y)\right]^{i}\frac{\partial K(x,y)}{\partial x}f(x)g^{'}(y)dxdy\\&+12\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)g(y)\right]^{i}K(x,y)f^{'}(x)g^{'}(y)dxdy\\&+12\theta^{2}p(p-1)\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)g(y)\right]^{i}K(x,y)f(x)g(y)f^{'}(x)g^{'}(y)dxdy-3\end{aligned}$$

and

$$\begin{aligned}\Phi^{2}=90&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[K(x,y)\right]^{2}\left[f(x)g(y)\right]^{i+j}dxdy\\&-180\sum_{i=0}^{\infty}\binom{p}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xyK(x,y)\left[f(x)g(y)\right]^{i}dxdy+10.\end{aligned}$$

If $p > 0$ is an integer then

$$C(x,y) = K(x,y)\sum_{i=0}^{p}\binom{p}{i}\theta^{i}\left[f(x)g(y)\right]^{i},$$

$$\begin{aligned}
c(x,y) = {} & \sum_{i=0}^{p}\binom{p}{i}\theta^{i}\left[f(x)g(y)\right]^{i}\frac{\partial^{2}K(x,y)}{\partial x\partial y} \\
& + \theta p\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^{i}\left[f(x)g(y)\right]^{i}\frac{\partial K(x,y)}{\partial y}f^{'}(x)g(y) \\
& + \theta p\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^{i}\left[f(x)g(y)\right]^{i}\frac{\partial K(x,y)}{\partial x}f(x)g^{'}(y) \\
& + \theta p\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^{i}\left[f(x)g(y)\right]^{i}K(x,y)f^{'}(x)g^{'}(y) \\
& + \theta^{2}p(p-1)\sum_{i=0}^{p-2}\binom{p-2}{i}\theta^{i}\left[f(x)g(y)\right]^{i}K(x,y)f(x)g(y)f^{'}(x)g^{'}(y),
\end{aligned}$$

$$\begin{aligned}
\tau = {} & 4\sum_{i=0}^{p}\sum_{j=0}^{p}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K(x,y)\frac{\partial^{2}K(x,y)}{\partial x\partial y}dxdy \\
& + 4\theta p\sum_{i=0}^{p-1}\sum_{j=0}^{p}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K(x,y)\frac{\partial K(x,y)}{\partial y}f^{'}(x)g(y)dxdy \\
& + 4\theta p\sum_{i=0}^{p-1}\sum_{j=0}^{p}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K(x,y)\frac{\partial K(x,y)}{\partial x}f(x)g^{'}(y)dxdy \\
& + 4\theta p\sum_{i=0}^{p-1}\sum_{j=0}^{p}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K^{2}(x,y)f^{'}(x)g^{'}(y)dxdy \\
& + 4\theta^{2}p(p-1)\sum_{i=0}^{p-2}\sum_{j=0}^{p}\binom{p-2}{i}\binom{p}{j}\theta^{i+j}\int_{0}^{1}\int_{0}^{1}\left[f(x)g(y)\right]^{i+j}K^{2}(x,y)f(x)g(y)f^{'}(x)g^{'}(y)dxdy-1,
\end{aligned}$$

$$\begin{aligned}
\rho = {} & 12\sum_{i=0}^{p}\binom{p}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)g(y)\right]^{i}\frac{\partial^{2}K(x,y)}{\partial x\partial y}dxdy \\
& + 12\theta p\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)g(y)\right]^{i}\frac{\partial K(x,y)}{\partial y}f^{'}(x)g(y)dxdy \\
& + 12\theta p\sum_{i=0}^{p-1}\binom{p-1}{i}\theta^{i}\int_{0}^{1}\int_{0}^{1}xy\left[f(x)g(y)\right]^{i}\frac{\partial K(x,y)}{\partial x}f(x)g^{'}(y)dxdy
\end{aligned}$$

$$+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 xy\,[f(x)g(y)]^i\, K(x,y) f'(x) g'(y) dxdy$$
$$+ 12\theta^2 p(p-1) \sum_{i=0}^{p-2} \binom{p-2}{i} \theta^i \int_0^1 \int_0^1 xy\,[f(x)g(y)]^i\, K(x,y) f(x)g(y) f'(x) g'(y) dxdy - 3$$

and

$$\Phi^2 = 90 \sum_{i=0}^{p} \sum_{j=0}^{p} \binom{p}{i}\binom{p}{j} \theta^{i+j} \int_0^1 \int_0^1 [K(x,y)]^2\,[f(x)g(y)]^{i+j}\, dxdy$$
$$- 180 \sum_{i=0}^{p} \binom{p}{i} \theta^i \int_0^1 \int_0^1 xyK(x,y)\,[f(x)g(y)]^i\, dxdy + 10.$$

2.25. Sharifonnasabi et al. [29]'s copulas

Consider a particular case of Section 2.23 for $\phi(x, y) = f(x)f(y)$. For copulas given by

$$C(x,y) = K(x,y)\,[1 + \theta f(x) f(y)]^p = K(x,y) \sum_{i=0}^{\infty} \binom{p}{i} \theta^i\,[f(x)f(y)]^i$$

for $0 < x < 1$, $0 < y < 1$, $-1 < \theta < 1$, $p > 0$ and f in a suitable function, where $K(x, y)$ is a valid copula, we have

$$c(x,y) = [1 + \theta f(x) f(y)]^p\, \frac{\partial^2 K(x,y)}{\partial x \partial y}$$
$$+ \theta p \frac{\partial K(x,y)}{\partial y} f'(x) f(y)\,[1 + \theta f(x) f(y)]^{p-1}$$
$$+ \theta p \frac{\partial K(x,y)}{\partial x} f(x) f'(y)\,[1 + \theta f(x) f(y)]^{p-1}$$
$$+ \theta p K(x,y) f'(x) f'(y)\,[1 + \theta f(x) f(y)]^{p-1}$$
$$+ \theta^2 p(p-1) K(x,y)\,[1 + \theta f(x) f(y)]^{p-2}\, f(x) f(y) f'(x) f'(y)$$
$$= \sum_{i=0}^{\infty} \binom{p}{i} \theta^i\,[f(x)f(y)]^i\, \frac{\partial^2 K(x,y)}{\partial x \partial y}$$
$$+ \theta p \sum_{i=0}^{\infty} \binom{p-1}{i} \theta^i\,[f(x)f(y)]^i\, \frac{\partial K(x,y)}{\partial y} f'(x) f(y)$$
$$+ \theta p \sum_{i=0}^{\infty} \binom{p-1}{i} \theta^i\,[f(x)f(y)]^i\, \frac{\partial K(x,y)}{\partial x} f(x) f'(y)$$
$$+ \theta p \sum_{i=0}^{\infty} \binom{p-1}{i} \theta^i\,[f(x)f(y)]^i\, K(x,y) f'(x) f'(y)$$
$$+ \theta^2 p(p-1) \sum_{i=0}^{\infty} \binom{p-2}{i} \theta^i\,[f(x)f(y)]^i\, K(x,y) f(x) f(y) f'(x) f'(y),$$

$$\beta^{*}=4K\left(\frac{1}{2},\frac{1}{2}\right)\left[1+\theta f^{2}\left(\frac{1}{2}\right)\right]^{p}-1,$$

$$\begin{aligned}\tau=4\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1\left[f(x)f(y)\right]^{i+j}K(x,y)\frac{\partial^2K(x,y)}{\partial x\partial y}dxdy\\
+4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1\left[f(x)f(y)\right]^{i+j}K(x,y)\frac{\partial K(x,y)}{\partial y}f^{'}(x)f(y)dxdy\\
+4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1\left[f(x)f(y)\right]^{i+j}K(x,y)\frac{\partial K(x,y)}{\partial x}f(x)f^{'}(y)dxdy\\
+4\theta p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-1}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1\left[f(x)f(y)\right]^{i+j}K^2(x,y)f^{'}(x)f^{'}(y)dxdy\\
+4\theta^2p(p-1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p-2}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1\left[f(x)f(y)\right]^{i+j}K^2(x,y)f(x)f(y)f^{'}(x)f^{'}(y)dxdy-1,\end{aligned}$$

$$\begin{aligned}\rho=12\sum_{i=0}^{\infty}\binom{p}{i}\theta^{i}\int_0^1\int_0^1 xy\left[f(x)f(y)\right]^{i}\frac{\partial^2K(x,y)}{\partial x\partial y}dxdy\\
+12\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\int_0^1\int_0^1 xy\left[f(x)f(y)\right]^{i}\frac{\partial K(x,y)}{\partial y}f^{'}(x)f(y)dxdy\\
+12\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\int_0^1\int_0^1 xy\left[f(x)f(y)\right]^{i}\frac{\partial K(x,y)}{\partial x}f(x)f^{'}(y)dxdy\\
+12\theta p\sum_{i=0}^{\infty}\binom{p-1}{i}\theta^{i}\int_0^1\int_0^1 xy\left[f(x)f(y)\right]^{i}K(x,y)f^{'}(x)f^{'}(y)dxdy\\
+12\theta^2p(p-1)\sum_{i=0}^{\infty}\binom{p-2}{i}\theta^{i}\int_0^1\int_0^1 xy\left[f(x)f(y)\right]^{i}K(x,y)f(x)f(y)f^{'}(x)f^{'}(y)dxdy-3\end{aligned}$$

and

$$\begin{aligned}\Phi^2=90\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1\left[K(x,y)\right]^2\left[f(x)f(y)\right]^{i+j}dxdy\\
-180\sum_{i=0}^{\infty}\binom{p}{i}\theta^{i}\int_0^1\int_0^1 xyK(x,y)\left[f(x)f(y)\right]^{i}dxdy+10.\end{aligned}$$

If $p>0$ is an integer then

$$C(x,y)=K(x,y)\sum_{i=0}^{p}\binom{p}{i}\theta^{i}\left[f(x)f(y)\right]^{i},$$

$$
\begin{aligned}
c(x,y) = \sum_{i=0}^{p} \binom{p}{i} \theta^i \left[f(x)f(y)\right]^i \frac{\partial^2 K(x,y)}{\partial x \partial y} \\
&+ \theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \left[f(x)f(y)\right]^i \frac{\partial K(x,y)}{\partial y} f^{'}(x)f(y) \\
&+ \theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \left[f(x)f(y)\right]^i \frac{\partial K(x,y)}{\partial x} f(x)f^{'}(y) \\
&+ \theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \left[f(x)f(y)\right]^i K(x,y)f^{'}(x)f^{'}(y) \\
&+ \theta^2 p(p-1) \sum_{i=0}^{p-2} \binom{p-2}{i} \theta^i \left[f(x)f(y)\right]^i K(x,y)f(x)f(y)f^{'}(x)f^{'}(y),
\end{aligned}
$$

$$
\begin{aligned}
\tau = 4 \sum_{i=0}^{p} \sum_{j=0}^{p} \binom{p}{i}\binom{p}{j} \theta^{i+j} \int_0^1 \int_0^1 \left[f(x)f(y)\right]^{i+j} K(x,y) \frac{\partial^2 K(x,y)}{\partial x \partial y} dxdy \\
&+ 4\theta p \sum_{i=0}^{p-1} \sum_{j=0}^{p} \binom{p-1}{i}\binom{p}{j} \theta^{i+j} \int_0^1 \int_0^1 \left[f(x)f(y)\right]^{i+j} K(x,y) \frac{\partial K(x,y)}{\partial y} f^{'}(x)f(y)dxdy \\
&+ 4\theta p \sum_{i=0}^{p-1} \sum_{j=0}^{p} \binom{p-1}{i}\binom{p}{j} \theta^{i+j} \int_0^1 \int_0^1 \left[f(x)f(y)\right]^{i+j} K(x,y) \frac{\partial K(x,y)}{\partial x} f(x)f^{'}(y)dxdy \\
&+ 4\theta p \sum_{i=0}^{p-1} \sum_{j=0}^{p} \binom{p-1}{i}\binom{p}{j} \theta^{i+j} \int_0^1 \int_0^1 \left[f(x)f(y)\right]^{i+j} K^2(x,y) f^{'}(x)f^{'}(y)dxdy \\
&+ 4\theta^2 p(p-1) \sum_{i=0}^{p-2} \sum_{j=0}^{p} \binom{p-2}{i}\binom{p}{j} \theta^{i+j} \int_0^1 \int_0^1 \left[f(x)f(y)\right]^{i+j} K^2(x,y) f(x)f(y)f^{'}(x)f^{'}(y)dxdy - 1,
\end{aligned}
$$

$$
\begin{aligned}
\rho = 12 \sum_{i=0}^{p} \binom{p}{i} \theta^i \int_0^1 \int_0^1 xy \left[f(x)f(y)\right]^i \frac{\partial^2 K(x,y)}{\partial x \partial y} dxdy \\
&+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 xy \left[f(x)f(y)\right]^i \frac{\partial K(x,y)}{\partial y} f^{'}(x)f(y)dxdy \\
&+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 xy \left[f(x)f(y)\right]^i \frac{\partial K(x,y)}{\partial x} f(x)f^{'}(y)dxdy \\
&+ 12\theta p \sum_{i=0}^{p-1} \binom{p-1}{i} \theta^i \int_0^1 \int_0^1 xy \left[f(x)f(y)\right]^i K(x,y) f^{'}(x)f^{'}(y)dxdy \\
&+ 12\theta^2 p(p-1) \sum_{i=0}^{p-2} \binom{p-2}{i} \theta^i \int_0^1 \int_0^1 xy \left[f(x)f(y)\right]^i K(x,y)f(x)f(y)f^{'}(x)f^{'}(y)dxdy - 3
\end{aligned}
$$

and

$$\Phi^2 = 90\sum_{i=0}^{p}\sum_{j=0}^{p}\binom{p}{i}\binom{p}{j}\theta^{i+j}\int_0^1\int_0^1 [K(x,y)]^2\,[f(x)f(y)]^{i+j}\,dxdy$$
$$-180\sum_{i=0}^{p}\binom{p}{i}\theta^{i}\int_0^1\int_0^1 xyK(x,y)\,[f(x)f(y)]^{i}\,dxdy + 10.$$

2.26. Gaussian copula

A Gaussian copula is a statistical tool used to model and describe the dependence structure between two variables by linking their marginal distributions through a bivariate normal distribution. For the Gaussian copula given by

$$C(x,y) = \Phi_2\left(\Phi^{-1}(x), \Phi^{-1}(y)\right)$$

for $0 < x < 1$ and $0 < y < 1$, where

$$\Phi(x) = \int_{-\infty}^{x}\phi(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{t^2}{2}\right)dt$$

and

$$\Phi_2(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x}\phi_2(s,t)dsdt = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{y}\int_{-\infty}^{x}\exp\left[-\frac{s^2-2\rho st+t^2}{2\left(1-\rho^2\right)}\right]dsdt,$$

we have

$$c(x,y) = \frac{\phi_2\left(\Phi^{-1}(x), \Phi^{-1}(y)\right)}{\phi\left(\Phi^{-1}(x)\right)\phi\left(\Phi^{-1}(y)\right)},$$

$$\beta^* = \frac{2}{\pi}\arcsin\rho,$$

$$\tau = \frac{2}{\pi}\arcsin\rho,$$

$$\rho = \frac{6}{\pi}\arcsin\frac{\rho}{2}$$

and

$$\Phi^2 = 90\int_0^1\int_0^1\left[\Phi_2\left(\Phi^{-1}(x), \Phi^{-1}(y)\right) - xy\right]^2 dxdy.$$

2.27. Student's t copula

The Student's t copula is a bivariate copula derived from the Student's t distribution, used to model dependence structures with tail dependence, allowing for stronger correlations in the tails compared to the Gaussian copula. For the Student's t copula given by

$$C(x,y) = \overline{T}_2\left(T_a^{-1}(x), T_b^{-1}(y)\right)$$

for $0 < x < 1$ and $0 < y < 1$, where

$$T_a(x) = \int_{-\infty}^{x} t_a(t)dt = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\Gamma\left(\frac{a}{2}\right)} \int_{-\infty}^{x} \left(1 + \frac{t^2}{a}\right)^{-\frac{a+1}{2}} dt$$

and

$$\overline{T}_2(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} \bar{t}_2(s,t)dsdt = \frac{\Gamma\left(\frac{a+2}{2}\right)}{a\pi\Gamma\left(\frac{a}{2}\right)\sqrt{1-\rho^2}} \int_{-\infty}^{y} \int_{-\infty}^{x} \left[1 + \frac{s^2 - 2\rho st + t^2}{2a\left(1-\rho^2\right)}\right]^{-\frac{a+2}{2}} dsdt,$$

we have

$$c(x,y) = \frac{\bar{t}_2\left(T_a^{-1}(x), T_a^{-1}(y)\right)}{t_a\left(T_a^{-1}(x)\right) t_a\left(T_a^{-1}(y)\right)},$$

$$\beta^* = 4\overline{T}_2(0,0) - 1,$$

$$\tau = 4\int_0^1 \int_0^1 \overline{T}_2\left(T_a^{-1}(x), T_b^{-1}(y)\right) \frac{\bar{t}_2\left(T_a^{-1}(x), T_a^{-1}(y)\right)}{t_a\left(T_a^{-1}(x)\right) t_a\left(T_a^{-1}(y)\right)} dxdy - 1,$$

$$\rho = 12\int_0^1 \int_0^1 \frac{xy\bar{t}_2\left(T_a^{-1}(x), T_a^{-1}(y)\right)}{t_a\left(T_a^{-1}(x)\right) t_a\left(T_a^{-1}(y)\right)} dxdy - 3$$

and

$$\Phi^2 = 90\int_0^1 \int_0^1 \left[\overline{T}_2\left(T_a^{-1}(x), T_b^{-1}(y)\right) - xy\right]^2 dxdy.$$

3. Conclusions

In this paper, we have provided general expressions for Blomqvist's β^*, Kendall's τ, Spearman's ρ and Hoeffding's Φ^2 for twenty-seven families of bivariate copulas. These general expressions can be used to derive a wide class of closed form expressions. For example, if C is the standard FGM copula then $\psi(x) = 1 - x$ in Section 2.12. By noting that

$$I_{m,n,\psi} = \int_0^1 x^m (1-x)^n dx = B(m+1, n+1)$$

and

$$J_{m,n,\psi} = -\int_0^1 x^m (1-x)^n \, dx = -B(m+1, n+1),$$

we see that Section 2.12 gives explicit expressions for the four dependence measures. If C is the copula due to Shih and Emura [31] then $\psi(x) = (1-x^p)^q$ in Section 2.12. By noting that

$$I_{m,n,\psi} = \int_0^1 x^m (1-x^p)^{nq} \, dx = \frac{1}{p} B\left(\frac{m+1}{p}, nq+1\right)$$

and

$$J_{m,n,\psi} = -pq \int_0^1 x^{m+p-1} (1-x)^{nq+q-1} \, dx = -qB\left(\frac{m}{p}+1, (n+1)q\right),$$

we see that Section 2.12 gives explicit expressions for the four dependence measures.

Future work is to derive expressions for dependence measures for multivariate copulas, complex variate copulas, matrix variate copulas and fuzzy copulas.

Author contributions

Both authors contributed equally to the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

Both authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

Both authors declare no conflicts of interest.

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