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Research article

Generalized representations, deformations and extensions of 3-Lie superalgebras

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Abstract: In this paper, we introduced generalized representations of 3-Lie superalgebras, which are associated to generalized semidirect product 3-Lie superalgebras. We also developed the corresponding cohomology theory and generalized one-parameter formal deformations. Furthermore, we proved that the infinitesimals and the extensibility of finite-order deformations of generalized one-parameter formal deformations are controlled by the new first and second cohomology groups, respectively. At last, we described split and non-split Abelian extensions by generalized semidirect products and Maurer-Cartan elements, respectively.

Keywords: 3-Lie superalgebra; generalized representation; cohomology; deformation; Abelian extension; Maurer-Cartan element
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1. Introduction

Lie groups, Lie algebras, and related algebras play an important role in geometry and mathematical physics, such as [2, 7, 8]. It is known that many algebraic systems can be described as corresponding canonical structures, such as Lie algebras (see [5]), Lie superalgebras (see [1]), and n-Lie algebras (see [15]). In [9, 14], generalized representations and the corresponding cohomology for 3-Lie algebras and 3-Hom-Lie algebras, respectively, were introduced by method of their canonical structures. The notion of n-Lie superalgebra appeared first in [4] and a more general description was given in [3]. In [13], a 3-Lie superalgebra of parity 0 is called a first-class 3-Lie superalgebra. Unless otherwise indicated, all 3-Lie superalgebras referred to in this paper are the first-class ones. In this paper, we describe a 3-Lie superalgebra structure as a canonical structure, and characterize generalized representations as well as the new cohomology by the canonical structure.

Similar to other algebraic systems (see [6, 10, 11]), we can use deformations and extensions to construct new 3-Lie superalgebras. In [6, 13], the authors gave one-parameter formal deformations, Abelian extensions, and T^*-extensions of 3-Lie superalgebras with respect to the usual cohomology theory. Motivated by the previous work, we give some results for 3-Lie superalgebras with respect to a new cohomology theory. In this paper, we define generalized one-parameter formal deformations of 3-Lie superalgebras. The usual deformations of 3-Lie superalgebras defined in [6] are special cases of generalized deformations. The infinitesimals of generalized deformations are controlled by the new first cohomology groups and the extensibility of finite-order deformations depends on the new second cohomology groups. Moreover, we characterize split Abelian extensions by the generalized semidirect product 3-Lie superalgebras, and the characterization cannot be achieved by the usual semidirect product 3-Lie superalgebras.

The paper is organized as follows. In Section 2, we recall the representation and cohomology of a 3-Lie superalgebra. In Section 3, we construct a $\mathbb{Z}\times\mathbb{Z}_2$-graded Lie algebra on the chain complex of a 3-Lie superalgebra, and then describe a 3-Lie superalgebra as a canonical structure. Section 4 is dedicated to generalized representations and the corresponding cohomology theory of 3-Lie superalgebras. In Section 5, we study the generalized one-parameter formal deformations of 3-Lie superalgebras and characterize them by the new cohomology groups. In Section 6, we prove that split Abelian extensions and non-split Abelian extensions shall be described by generalized semidirect products and Maurer-Cartan elements, respectively.

2. Preliminaries

In this section, we recall representation and cohomology of a 3-Lie superalgebra.

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and we work over $\mathbb{K}$ throughout the paper. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. An element $x \in \mathfrak{g}_k$ is called homogenous with degree k and we denote $|x| = k$. When the notation of $|x|$ occurs, x is supposed to be homogenous. Moreover, x is called even (odd) if $|x| = \bar{0}$ ($|x| = \bar{1}$).

Definition 2.1. ([3]) *A* 3**-Lie superalgebra** *is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ equipped with a trilinear map $[-,-,-] : \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ such that*

$$\begin{aligned} |[x_1, x_2, x_3]| &= |x_1| + |x_2| + |x_3|, && (2.1)\\ [x_1, x_2, x_3] &= -(-1)^{|x_1||x_2|}[x_2, x_1, x_3] = -(-1)^{|x_2||x_3|}[x_1, x_3, x_2], && (2.2)\\ [x_1, x_2, [x_3, x_4, x_5]] &= [[x_1, x_2, x_3], x_4, x_5] + (-1)^{(|x_1|+|x_2|)|x_3|}[x_3, [x_1, x_2, x_4], x_5] \\ &\quad + (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}[x_3, x_4, [x_1, x_2, x_5]], && (2.3) \end{aligned}$$

for $x_i \in \mathfrak{g}$, $i \in \{1, 2, 3, 4, 5\}$.

Let $\mathfrak{g}$ be a $\mathbb{Z}_2$-graded vector space. Then $\wedge^2\mathfrak{g}$ is the set of elements $x_1 \wedge x_2$ such that $x_1 \wedge x_2 = -(-1)^{|x_1||x_2|}x_2 \wedge x_1$.

Definition 2.2. ([13]) *A* **representation** *of a* 3*-Lie superalgebra $\mathfrak{g}$ on $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a linear map $\rho : \wedge^2\mathfrak{g} \to \mathfrak{gl}(V)$, such that the following equalities hold:*

$$\begin{aligned} \rho(x_1, x_2)V_\kappa &\subseteq V_{|x_1|+|x_2|+\kappa}, \kappa \in \mathbb{Z}_2;\\ \rho(x_1, x_2)\rho(x_3, x_4) &= \rho([x_1, x_2, x_3], x_4) + (-1)^{(|x_1|+|x_2|)|x_3|}\rho(x_3, [x_1, x_2, x_4]) \\ &\quad + (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}\rho(x_3, x_4)\rho(x_1, x_2), && (2.4) \end{aligned}$$

$$\begin{aligned}\rho(x_1,[x_2,x_3,x_4]) &= (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}\rho(x_3,x_4)\rho(x_1,x_2)\\ &\quad -(-1)^{|x_1|(|x_2|+|x_4|)+|x_3||x_4|}\rho(x_2,x_4)\rho(x_1,x_3)\\ &\quad +(-1)^{|x_1|(|x_2|+|x_3|)}\rho(x_2,x_3)\rho(x_1,x_4),\end{aligned}\tag{2.5}$$

for $x_1,x_2,x_3,x_4\in\mathfrak{g}$.

Define a linear map $\mathrm{ad}:\wedge^2\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ by

$$\mathrm{ad}(x_1,x_2)x_3=[x_1,x_2,x_3],\ \forall\ x_1,x_2,x_3\in\mathfrak{g}.$$

Then $(\mathfrak{g};\mathrm{ad})$ is a representation of $\mathfrak{g}$, called an **adjoint representation**.

Proposition 2.3. ([13]) *Let* $(\mathfrak{g},\pi)$ *be a* 3*-Lie superalgebra and let* $V=V_{\bar 0}\oplus V_{\bar 1}$ *be a* $\mathbb{Z}_2$*-graded vector space. Suppose that* $\rho:\wedge^2\mathfrak{g}\to\mathfrak{gl}(V)$ *is an even linear map. Then* $(V;\rho)$ *is a representation of* $\mathfrak{g}$ *if and only if there is a* 3*-Lie superalgebra* $(\mathfrak{g}\oplus V,\pi+\bar\rho)$*, where* $\bar\rho:(\mathfrak{g}\oplus V)\times(\mathfrak{g}\oplus V)\times(\mathfrak{g}\oplus V)\to\mathfrak{g}\oplus V$ *is induced by* ρ *via*

$$\begin{aligned}\bar\rho(x_1+u_1,x_2+u_2,x_3+u_3) &= \rho(x_1,x_2)u_3+(-1)^{(|x_1|+|u_2|)|x_3|}\rho(x_3,x_1)u_2\\ &\quad +(-1)^{(|x_2|+|x_3|)|u_1|}\rho(x_2,x_3)u_1.\end{aligned}\tag{2.6}$$

The above 3-Lie superalgebra on $\mathfrak{g}\oplus V$ is called a semi-direct product 3-Lie superalgebra and denoted by $\mathfrak{g}\ltimes_\rho V$.

Let $\mathfrak{g}$ be a 3-Lie superalgebra. The element $\mathfrak{X}=x^1\wedge x^2\in\wedge^2\mathfrak{g}$ is called a **fundamental object** of $\mathfrak{g}$. For $z\in\mathfrak{g}$, $[\mathfrak{X},z]=[x^1,x^2,z]$. There is a bilinear operation $[-,-]_F$:

$$[\mathfrak{X},\mathfrak{Y}]_F=[x^1,x^2,y^1]\wedge y^2+y^1\wedge[x^1,x^2,y^2],\quad\forall\ \mathfrak{X}=x^1\wedge x^2,\mathfrak{Y}=y^1\wedge y^2\in\wedge^2\mathfrak{g}.$$

It follows that $(\wedge^2\mathfrak{g},[-,-]_F)$ is a Leibniz superalgebra.

Let $(V;\rho)$ be a representation of $\mathfrak{g}$. The space of p-cochains is given by

$$C^p(\mathfrak{g},V)=\mathrm{Hom}(\otimes^p(\wedge^2\mathfrak{g})\wedge\mathfrak{g},V).$$

Then $C^p(\mathfrak{g},V)$ is endowed with a natural $\mathbb{Z}_2$-grading as follows: a p-cochain $f\in C^p(\mathfrak{g},V)$ is called homogenous of degree κ if $|f(\mathfrak{X}_1,\cdots,\mathfrak{X}_p,z)|=|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_p|+|z|+\kappa$, for $\mathfrak{X}_1,\cdots,\mathfrak{X}_p\in\wedge^2\mathfrak{g},\ z\in\mathfrak{g}$.

The coboundary operator $\partial:C^p(\mathfrak{g},V)\to C^{p+1}(\mathfrak{g},V)$ is given by

$$\begin{aligned}&(\partial f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_p,z)\\ =&\sum_{1\leqslant j<k\leqslant p}(-1)^j(-1)^{|\mathfrak{X}_j|(|\mathfrak{X}_{j+1}|+\cdots+|\mathfrak{X}_{k-1}|)}f(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}_j},\cdots,\mathfrak{X}_{k-1},[\mathfrak{X}_j,\mathfrak{X}_k]_F,\mathfrak{X}_{k+1},\cdots,\mathfrak{X}_p,z)\\ &+\sum_{j=1}^{p}(-1)^j(-1)^{|\mathfrak{X}_j|(|\mathfrak{X}_{j+1}|+\cdots+|\mathfrak{X}_p|)}f(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}_j},\cdots,\mathfrak{X}_p,[\mathfrak{X}_j,z])\\ &+\sum_{j=1}^{p}(-1)^{j+1}(-1)^{|\mathfrak{X}_j|(|f|+|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_{j-1}|)}\rho(\mathfrak{X}_j)f(\mathfrak{X}_1,\cdots,\widehat{\mathfrak{X}_j},\cdots,\mathfrak{X}_p,z)\\ &+(-1)^{p+1}(-1)^{(|x_p^2|+|z|)(|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_{p-1}|+|x_p^1|)}\rho(x_p^2,z)f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{p-1},x_p^1)\\ &+(-1)^{p+1}(-1)^{(|x_p^1|+|z|)(|\mathfrak{X}_1|+\cdots+|\mathfrak{X}_{p-1}|)+|\mathfrak{X}_p||z|}\rho(z,x_p^1)f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{p-1},x_p^2),\end{aligned}$$

for $\mathfrak{X}_i = x_i^1 \wedge x_i^2 \in \wedge^2\mathfrak{g}$ and $z \in \mathfrak{g}$, $|\mathfrak{X}_i| = |x_i^1| + |x_i^2|$.

Denote the set of p-cocycles by $Z^p(\mathfrak{g}, V)$ and the set of p-coboundaries by $B^p(\mathfrak{g}, V)$, where

$$\begin{aligned} Z^p(\mathfrak{g}, V) &= \{f \in C^p(\mathfrak{g}, V) | \partial f = 0\}, \\ B^p(\mathfrak{g}, V) &= \partial C^{p-1}(\mathfrak{g}, V). \end{aligned}$$

The p-th cohomology group is given by

$$H^p(\mathfrak{g}, V) = Z^p(\mathfrak{g}, V)/B^p(\mathfrak{g}, V).$$

It is natural that $H^p(\mathfrak{g}, V)$ is $\mathbb{Z}_2$-graded.

3. Canonical structures

Lie algebras were described as canonical structures for the Nijenhuis-Richardson bracket in [4]. This is a smart way to characterize the structures and coboundary operators of Lie algebras. Moreover, n-Lie algebras and Lie superalgebras were described as canonical structures for some brackets in [1, 15]. We follow those ideas to describe 3-Lie superalgebras as canonical structures.

Let $L = \oplus_{(p,\kappa)\in\mathbb{Z}\times\mathbb{Z}_2} L_{(p,\kappa)}$ be a $\mathbb{Z}\times\mathbb{Z}_2$-graded vector space. An element $x \in L_{(p,\kappa)}$ is called **homogenous** of bidegree (p, κ), denoted by $||x|| = (p, \kappa)$. Define a linear map $\varphi : \mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z}_2$ by $\varphi(p, \kappa) = (p + \kappa) \bmod 2$. In particular, if $||x|| = (p_1, \kappa_1)$, $||y|| = (p_2, \kappa_2)$,

$$\varphi(||x||||y||) = \varphi(p_1 p_2, \kappa_1\kappa_2) = (p_1 p_2 + \kappa_1\kappa_2) \bmod 2.$$

Definition 3.1. ([1]) *A* $\mathbb{Z}\times\mathbb{Z}_2$**-graded Lie algebra** *is a* $\mathbb{Z}\times\mathbb{Z}_2$*-graded vector space* $L = \oplus_{(p,\kappa)\in\mathbb{Z}\times\mathbb{Z}_2} L_{(p,\kappa)}$ *equipped with a bracket* $[-,-]$ *such that the following identities hold:*

$$\begin{aligned} ||[x, y]|| &= ||x|| + ||y||, \\ [x, y] &= -(-1)^{||x||||y||}[y, x], \\ [x, [y, z]] &= [[x, y], z] + (-1)^{||x||||y||}[y, [x, z]], \end{aligned}$$

where $(-1)^{||x||||y||}$ *stands for* $(-1)^{\varphi(||x||||y||)}$.

Let $\mathfrak{g}$ be a $\mathbb{Z}_2$-graded vector space. Set a $\mathbb{Z} \times \mathbb{Z}_2$-graded vector space

$$\mathcal{L}^*(\mathfrak{g}, \mathfrak{g}) = \oplus_{(p,\kappa)\in\mathbb{Z}\times\mathbb{Z}_2} \mathcal{L}_{(p,\kappa)}(\mathfrak{g}, \mathfrak{g}),$$

where $\mathcal{L}_{(p,\kappa)}(\mathfrak{g}, \mathfrak{g}) = 0$ for $p < 0$ and $\mathcal{L}_{(p,\kappa)} = C^p(\mathfrak{g}, \mathfrak{g})_\kappa$ for $p \geq 0$.

Let $f \in \mathcal{L}_{(p,\kappa_1)}(\mathfrak{g}, \mathfrak{g})$ and $g \in \mathcal{L}_{(q,\kappa_2)}(\mathfrak{g}, \mathfrak{g})$, $p, q \geq 0$, $\kappa_1, \kappa_2 \in \mathbb{Z}_2$. Note that $||f|| = (p, \kappa)$ denotes the bidegree of f in $\mathbb{Z} \times \mathbb{Z}_2$-graded vector space $\mathcal{L}^*(\mathfrak{g}, \mathfrak{g})$, and $|f| = \kappa$ denotes the degree of f in $\mathbb{Z}_2$-graded vector space $C^p(\mathfrak{g}, \mathfrak{g})$. Suppose $J = \{j_1, \cdots, j_q\}_{j_1<\cdots<j_q} \subseteq N \triangleq \{1, 2, \cdots, p + q\}$ and $I = N/J = \{i_1, \cdots, i_p\}_{i_1<\cdots<i_p}$. We define two maps $[-,-]^{3\mathrm{ls}}$, $\circ_{3\mathrm{ls}}$ by

$$[f, g]^{3\mathrm{ls}} = j_g(f) - (-1)^{||f||||g||} j_f(g) = f \circ_{3\mathrm{ls}} g - (-1)^{||f||||g||} g \circ_{3\mathrm{ls}} f, \tag{3.1}$$

and

$$
\begin{aligned}
& f \circ_{3\mathrm{ls}} g(\mathfrak{X}_1, \cdots, \mathfrak{X}_{p+q}, x) \\
= & \sum_{J, j_q < i_{k+1} \leq p+q} (-1)^{(J,I)} (-1)^{\mathcal{K}(J,I;\mathfrak{X})_k} (-1)^{(|\mathfrak{X}_{i_1}| + \cdots + |\mathfrak{X}_{i_k}|)|g|} f(\mathfrak{X}_{i_1}, \cdots, \mathfrak{X}_{i_k}, \\
& \quad g(\mathfrak{X}_{j_1}, \cdots, \mathfrak{X}_{j_q},) \cdot \mathfrak{X}_{i_{k+1}}, \mathfrak{X}_{i_{k+2}}, \cdots, \mathfrak{X}_{i_p}, x) \\
& + \sum_{J} (-1)^{(J,I)} (-1)^{\mathcal{K}(J,I;\mathfrak{X})_p} f(\mathfrak{X}_{i_1}, \cdots, \mathfrak{X}_{i_p}, g(\mathfrak{X}_{j_1}, \cdots, \mathfrak{X}_{j_q}, x)), \qquad (3.2)
\end{aligned}
$$

for all homogenous elements $\mathfrak{X}_i = x_i^1 \wedge x_i^2 \in \wedge^2 \mathfrak{g}$, $x \in \mathfrak{g}$, where

$$
\begin{aligned}
& g(\mathfrak{X}_{j_1}, \cdots, \mathfrak{X}_{j_q},) \cdot \mathfrak{X}_{i_{k+1}} \\
= & g(\mathfrak{X}_{j_1}, \cdots, \mathfrak{X}_{j_q}, x^1_{i_{k+1}}) \wedge x^2_{i_{k+1}} + (-1)^{(|g| + |\mathfrak{X}_{j_1}| + \cdots + |\mathfrak{X}_{j_q}|)|x^1_{i_{k+1}}|} x^1_{i_{k+1}} \wedge g(\mathfrak{X}_{j_1}, \cdots, \mathfrak{X}_{j_q}, x^2_{i_{k+1}}), \\
& \mathcal{K}(J,I;\mathfrak{X})_k = \sum_{1 \leq l \leq p} \sum_{1 \leq s \leq k, j_l < i_s} |\mathfrak{X}_{j_l}||\mathfrak{X}_{i_s}|, \ 0 \leq k \leq p,
\end{aligned}
$$

where k is chosen in the unique way $i_k < j_q < i_{k+1}$ and $(-1)^{(J,I)}$ is the sign of the permutation $(J, I) = (j_1, \cdots, j_q, i_1, \cdots, i_p)$ of N.

Lemma 3.2. *The following equality holds:*

$$
j_{[f,g]^{3\mathrm{ls}}} = j_g j_f - (-1)^{\|f\|\|g\|} j_f j_g,
$$

for all $f \in \mathcal{L}_{(p,\kappa_1)}(\mathfrak{g}, \mathfrak{g})$, $g \in \mathcal{L}_{(q,\kappa_2)}(\mathfrak{g}, \mathfrak{g})$, $p, q \geq 0$, $\kappa_1, \kappa_2 \in \mathbb{Z}_2$.

Proof. We consider $f \in \mathcal{L}_{(p,\kappa_1)}(\mathfrak{g}, \mathfrak{g})$, $g \in \mathcal{L}_{(q,\kappa_2)}(\mathfrak{g}, \mathfrak{g})$, $\xi \in \mathcal{L}_{(r,\kappa_3)}(\mathfrak{g}, \mathfrak{g})$, $p, q, r \geq 0$, $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{Z}_2$. Take homogenous elements $\mathfrak{X}_1, \cdots, \mathfrak{X}_{p+q+r} \in \wedge^2 \mathfrak{g}$ and $x \in \mathfrak{g}$. We set

$$
\begin{aligned}
& J = \{j_1, j_2, \cdots, j_p\}_{j_1 < j_2 < \cdots < j_p} \subset N \triangleq \{1, 2, \cdots, p+q+r\}, \\
& I = \{i_1, i_2, \cdots, i_{q+r}\}_{i_1 < i_2 < \cdots < i_{q+r}} = N/J, \\
& L = \{l_1, l_2, \cdots, l_q\}_{l_1 < l_2 < \cdots < l_q} \subset I \triangleq \{i_1, i_2, \cdots, i_{q+r}\}, \\
& H = \{h_1, h_2, \cdots, h_r\}_{h_1 < h_2 < \cdots < h_r} = I/L.
\end{aligned}
$$

We denote

$$
\begin{aligned}
\mathcal{K}(J,H;\mathfrak{X})_m &= \sum_{1 \leq \tau \leq p} \sum_{1 \leq \varsigma \leq m, j_\tau < h_\varsigma} |\mathfrak{X}_{j_\tau}||\mathfrak{X}_{h_\varsigma}|, \ 0 \leq m \leq r, \\
\mathcal{K}(L,H;\mathfrak{X})_n &= \sum_{1 \leq \epsilon \leq q} \sum_{1 \leq \varsigma \leq n, l_\epsilon < h_\varsigma} |\mathfrak{X}_{l_\epsilon}||\mathfrak{X}_{h_\varsigma}|, \ 0 \leq n \leq r, \\
\mathcal{K}(L,J;\mathfrak{X})_t &= \sum_{1 \leq \epsilon \leq q} \sum_{1 \leq \tau \leq t, l_\epsilon < j_\tau} |\mathfrak{X}_{j_\tau}||\mathfrak{X}_{l_\epsilon}|, \ 0 \leq t \leq p, \\
\mathcal{K}(J,L;\mathfrak{X})_s &= \sum_{1 \leq \tau \leq p} \sum_{1 \leq \epsilon \leq s, j_\tau < l_\epsilon} |\mathfrak{X}_{j_\tau}||\mathfrak{X}_{l_\epsilon}|, \ 0 \leq s \leq q.
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
& j_f(j_g\xi)(\mathfrak{X}_1,\mathfrak{X}_2,\cdots,\mathfrak{X}_{p+q+r},x)=(j_g(\xi))\circ_{3\mathrm{ls}} f(\mathfrak{X}_1,\mathfrak{X}_2,\cdots,\mathfrak{X}_{p+q+r},x)\\
= & \sum_{J,j_p<i_{k+1}\le p+q+r}(-1)^{\{J,I;f\}_k}(j_g(\xi))(\mathfrak{X}_{i_1},\cdots,\mathfrak{X}_{i_k},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\cdot\mathfrak{X}_{i_{k+1}},\mathfrak{X}_{i_{k+2}},\cdots\\
& \mathfrak{X}_{i_{q+r}},x)\\
& +\sum_J(-1)^{\{J,I;f\}_{q+r}}(j_g(\xi))(\mathfrak{X}_{i_1},\cdots,\mathfrak{X}_{i_{q+r}},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},x))\\
= & \sum_{J,j_p<i_{k+1}\le p+q+r}\ \sum_{\substack{L,l_q<h_{m+1}\le i_{q+r},\\ i_{k+1}=h_n,n\le m}}\Omega_1\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_{n-1}},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\cdot\mathfrak{X}_{i_{k+1}},\cdots,\mathfrak{X}_{h_m},\\
& g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},)\cdot\mathfrak{X}_{h_{m+1}},\mathfrak{X}_{h_{m+2}},\cdots,\mathfrak{X}_{h_r},x)\\
& +\sum_{J,j_p<i_{k+1}\le p+q+r}\ \sum_{\substack{L,l_q<h_{m+1}\le i_{q+r},\\ i_{k+1}=h_{m+1}}}\Omega_2\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_m},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},)\cdot(f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\\
& \cdot\mathfrak{X}_{i_{k+1}}),\mathfrak{X}_{h_{m+2}},\cdots,\mathfrak{X}_{h_r},x)\\
& +\sum_{J,j_p<i_{k+1}\le p+q+r}\ \sum_{\substack{L,l_q<h_{m+1}\le i_{q+r},\\ i_{k+1}=h_n,n>m+1}}\Omega_3\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_m},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},)\cdot\mathfrak{X}_{h_{m+1}},\mathfrak{X}_{h_{m+2}},\\
& \cdots,\mathfrak{X}_{h_{n-1}},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\cdot\mathfrak{X}_{i_{k+1}},\mathfrak{X}_{h_{n+1}},\cdots,\mathfrak{X}_{h_r},x)\\
& +\sum_{J,j_p<i_{k+1}\le p+q+r}\ \sum_{\substack{L,l_q<h_{m+1}\le i_{q+r},\\ i_{k+1}=l_s,s\le p}}\Omega_4\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_m},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_{s-1}},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\\
& \cdot\mathfrak{X}_{i_{k+1}},\mathfrak{X}_{l_{s+1}},\cdots,\mathfrak{X}_{l_q},)\cdot\mathfrak{X}_{h_{m+1}},\mathfrak{X}_{h_{m+2}},\cdots,\mathfrak{X}_{h_r},x)\\
& +\sum_{J,j_p<i_{k+1}\le p+q+r}\ \sum_{L,i_{k+1}=h_n}\Omega_5\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_{n-1}},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\cdot\mathfrak{X}_{i_{k+1}},\mathfrak{X}_{h_{n+1}},\cdots,\\
& \mathfrak{X}_{h_r},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},x))\\
& +\sum_{J,j_p<i_{k+1}\le p+q+r}\ \sum_{L,i_{k+1}=l_s}\Omega_6\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_r},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_{s-1}},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\cdot\mathfrak{X}_{i_{k+1}},\\
& \mathfrak{X}_{l_{s+1}},\cdots,\mathfrak{X}_{l_q},x))\\
& +\sum_J\ \sum_{L,l_q<h_{m+1}\le i_{q+r}}\Omega_7\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_m},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},)\cdot\mathfrak{X}_{h_{m+1}},\mathfrak{X}_{h_{m+2}},\cdots,\mathfrak{X}_{h_r},\\
& f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},x))\\
& +\sum_J\sum_L\Omega_8\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_r},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},x))),
\end{aligned}
$$

where

$$
\begin{aligned}
(-1)^{\{J,I;f\}_k} &= (-1)^{(J,I)}(-1)^{\mathcal{K}(J,I;\mathfrak{X})_k}(-1)^{(|\mathfrak{X}_{i_1}|+\cdots+|\mathfrak{X}_{i_k}|)|f|},0\le k\le q+r,\\
\mathcal{K}(J,I;\mathfrak{X})_k &= \sum_{1\le\tau\le p}\ \sum_{1\le\varsigma\le k,j_\tau<i_\varsigma}|\mathfrak{X}_{j_\tau}||\mathfrak{X}_{i_\varsigma}|,\ 0\le k\le q+r,\\
\Omega_1 &= (-1)^{(J,I)}(-1)^{(L,H)}(-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|+|\mathfrak{X}_{j_1}|+\cdots+|\mathfrak{X}_{j_p}|+|f|)}\\
&\times\ (-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_{n-1}}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_{n-1}+\mathcal{K}(L,H;\mathfrak{X})_m+\mathcal{K}(L,J;\mathfrak{X})_p},\\
\Omega_2 &= (-1)^{(J,I)}(-1)^{(L,H)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_q}|)}\\
&\times\ (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_m+\mathcal{K}(L,H;\mathfrak{X})_m+\mathcal{K}(J,L;\mathfrak{X})_q},\\
\Omega_3 &= (-1)^{(J,I)}(-1)^{(L,H)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_{n-1}}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_q}|)}\\
&\times\ (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_{n-1}+\mathcal{K}(L,H;\mathfrak{X})_m+\mathcal{K}(J,L;\mathfrak{X})_q},\\
\Omega_4 &= (-1)^{(J,I)}(-1)^{(L,H)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_{s-1}}|)}\\
&\times\ (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_m+\mathcal{K}(L,H;\mathfrak{X})_m+\mathcal{K}(J,L;\mathfrak{X})_{s-1}},\\
\Omega_5 &= (-1)^{(J,I)}(-1)^{(L,H)}(-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|+|\mathfrak{X}_{j_1}|+\cdots+|\mathfrak{X}_{j_p}|+|f|)}\\
&\times\ (-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_n}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_{n-1}+\mathcal{K}(L,H;\mathfrak{X})_r+\mathcal{K}(L,J;\mathfrak{X})_p},\\
\Omega_6 &= (-1)^{(J,I)}(-1)^{(L,H)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_{s-1}}|)}\\
&\times\ (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_r+\mathcal{K}(L,H;\mathfrak{X})_r+\mathcal{K}(J,L;\mathfrak{X})_{s-1}},
\end{aligned}
$$

$$\begin{aligned}\Omega_7 &= (-1)^{(J,I)}(-1)^{(L,H)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_q}|)}\\ &\times (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_r+\mathcal{K}(L,H;\mathfrak{X})_m+\mathcal{K}(J,L;\mathfrak{X})_q},\\ \Omega_8 &= (-1)^{(J,I)}(-1)^{(L,H)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_q}|)}\\ &\times (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_r+\mathcal{K}(L,H;\mathfrak{X})_r+\mathcal{K}(J,L;\mathfrak{X})_q}.\end{aligned}$$

The action of $j_g j_f$ is similar.

Set

$$\begin{aligned}&O=\{o_1,o_2,\cdots,o_{p+q}\}_{o_1<o_2<\cdots<o_{p+q}}\subseteq N=\{1,2,\cdots,p+q+r\},\\ &H=\{h_1,h_2,\cdots,h_r\}_{h_1<h_2<\cdots<h_r}=N/O,\\ &J=\{j_1,j_2,\cdots,j_p\}_{j_1<j_2<\cdots<j_p}\subseteq O,\\ &L=\{l_1,l_2,\cdots,l_q\}_{l_1<l_2<\cdots<l_q}=O/J.\end{aligned}$$

We have:

$$\begin{aligned}& j_{[f,g]^{3\mathrm{ls}}}(\xi)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{p+q+r},x)\\ =& \sum_{O,o_{p+q}<h_{m+1}\le p+q+r}(-1)^{\{O,H;[f,g]^{3\mathrm{ls}}\}_m}\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_m},[f,g]^{3\mathrm{ls}}(\mathfrak{X}_{o_1},\cdots,\mathfrak{X}_{o_{p+q}},)\\ &\cdot\mathfrak{X}_{h_{m+1}},\mathfrak{X}_{h_{m+2}},\cdots,\mathfrak{X}_{h_r},x)\\ &+\sum_O(-1)^{\{O,H;[f,g]^{3\mathrm{ls}}\}_r}\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_r},[f,g]^{3\mathrm{ls}}(\mathfrak{X}_{o_1},\cdots,\mathfrak{X}_{o_{p+q}},x))\\ =& \sum_{O,o_{p+q}<h_{m+1}\le p+q+r}\ \sum_{L,l_q<j_{t+1}\le o_{p+q}}\Gamma_1\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_m},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_t},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},)\\ &\cdot\mathfrak{X}_{j_{t+1}},\mathfrak{X}_{j_{t+2}},\cdots,\mathfrak{X}_{j_p},)\cdot\mathfrak{X}_{h_{m+1}},\mathfrak{X}_{h_{m+2}},\cdots,\mathfrak{X}_{h_r},x)\\ &-\sum_{O,o_{p+q}<h_{m+1}\le p+q+r}\ \sum_{J,j_p<l_{s+1}\le o_{p+q}}(-1)^{pq+\kappa_1\kappa_2}\Gamma_2\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_m},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_s},\\ &f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\cdot\mathfrak{X}_{l_{s+1}},\mathfrak{X}_{l_{s+2}},\cdots,\mathfrak{X}_{l_q},)\cdot\mathfrak{X}_{h_{m+1}},\cdots,\mathfrak{X}_{h_r},x)\\ &+\sum_{O,o_{p+q}<h_{m+1}\le p+q+r}\ \sum_L\Gamma_3\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_m},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\cdot(g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},)\\ &\cdot\mathfrak{X}_{h_{m+1}}),\mathfrak{X}_{h_{m+2}},\cdots,\mathfrak{X}_{h_r},x)\\ &-\sum_{O,o_{p+q}<h_{m+1}\le p+q+r}\ \sum_J(-1)^{pq+\kappa_1\kappa_2}\Gamma_4\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_m},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},)\cdot\\ &(f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\cdot\mathfrak{X}_{h_{m+1}}),\mathfrak{X}_{h_{m+2}},\cdots,\mathfrak{X}_{h_r},x)\\ &+\sum_O\ \sum_{L,l_q<j_{t+1}\le o_{p+q}}\Gamma_5\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_r},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_t},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},)\cdot\mathfrak{X}_{j_{t+1}},\mathfrak{X}_{j_{t+2}},\\ &\cdots,\mathfrak{X}_{j_p},x))\\ &-\sum_O\ \sum_{J,j_p<l_{s+1}\le o_{p+q}}(-1)^{pq+\kappa_1\kappa_2}\Gamma_6\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_r},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_s},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},)\\ &\cdot\mathfrak{X}_{l_{s+1}},\mathfrak{X}_{l_{s+2}},\cdots,\mathfrak{X}_{l_q},x))\\ &+\sum_O\sum_L\Gamma_7\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_r},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},x))\\ &-\sum_O\sum_J(-1)^{pq+\kappa_1\kappa_2}\Gamma_8\xi(\mathfrak{X}_{h_1},\cdots,\mathfrak{X}_{h_r},g(\mathfrak{X}_{l_1},\cdots,\mathfrak{X}_{l_q},f(\mathfrak{X}_{j_1},\cdots,\mathfrak{X}_{j_p},x)),\end{aligned}$$

where

$$\begin{aligned}(-1)^{\{O,H;[f,g]^{3\mathrm{ls}}\}_m} &= (-1)^{(O,H)}(-1)^{\mathcal{K}(O,H;\mathfrak{X})_m}(-1)^{(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|)(|f|+|g|)},0\le m\le r,\\ \mathcal{K}(O,H;\mathfrak{X})_m &= \sum_{1\le\tau\le p+q}\ \sum_{1\le\varsigma\le m,o_\tau<h_\varsigma}|\mathfrak{X}_{o_\tau}||\mathfrak{X}_{h_\varsigma}|,\ 0\le m\le r,\\ \Gamma_1 &= (-1)^{(O,H)}(-1)^{(L,J)}(-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|+|\mathfrak{X}_{j_1}|+\cdots+|\mathfrak{X}_{j_t}|)}\end{aligned}$$

$$
\begin{array}{rcl}
& \times & (-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_m+\mathcal{K}(L,H;\mathfrak{X})_m+\mathcal{K}(L,J;\mathfrak{X})_t}, \\
\Gamma_2 & = & (-1)^{(O,H)}(-1)^{(J,L)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_s}|)} \\
& \times & (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_m+\mathcal{K}(L,H;\mathfrak{X})_m+\mathcal{K}(J,L;\mathfrak{X})_s}, \\
\Gamma_3 & = & (-1)^{(O,H)}(-1)^{(L,J)}(-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|+|\mathfrak{X}_{j_1}|+\cdots+|\mathfrak{X}_{j_p}|)} \\
& \times & (-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_m+\mathcal{K}(L,H;\mathfrak{X})_m+\mathcal{K}(L,J;\mathfrak{X})_p}, \\
\Gamma_4 & = & (-1)^{(O,H)}(-1)^{(J,L)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_q}|)} \\
& \times & (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_m}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_m+\mathcal{K}(L,H;\mathfrak{X})_m+\mathcal{K}(J,L;\mathfrak{X})_q}, \\
\Gamma_5 & = & (-1)^{(O,H)}(-1)^{(L,J)}(-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|+|\mathfrak{X}_{j_1}|+\cdots+|\mathfrak{X}_{j_t}|)} \\
& \times & (-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_r+\mathcal{K}(L,H;\mathfrak{X})_r+\mathcal{K}(L,J;\mathfrak{X})_t}, \\
\Gamma_6 & = & (-1)^{(O,H)}(-1)^{(J,L)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_s}|)} \\
& \times & (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_r+\mathcal{K}(L,H;\mathfrak{X})_r+\mathcal{K}(J,L;\mathfrak{X})_s}, \\
\Gamma_7 & = & (-1)^{(O,H)}(-1)^{(L,J)}(-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|+|\mathfrak{X}_{j_1}|+\cdots+|\mathfrak{X}_{j_p}|)} \\
& \times & (-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_r+\mathcal{K}(L,H;\mathfrak{X})_r+\mathcal{K}(L,J;\mathfrak{X})_p}, \\
\Gamma_8 & = & (-1)^{(O,H)}(-1)^{(J,L)}(-1)^{|f|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|+|\mathfrak{X}_{l_1}|+\cdots+|\mathfrak{X}_{l_q}|)} \\
& \times & (-1)^{|g|(|\mathfrak{X}_{h_1}|+\cdots+|\mathfrak{X}_{h_r}|)}(-1)^{\mathcal{K}(J,H;\mathfrak{X})_r+\mathcal{K}(L,H;\mathfrak{X})_r+\mathcal{K}(J,L;\mathfrak{X})_q}.
\end{array}
$$

We can check that

$$j_{[f,g]^{3\mathrm{ls}}} = j_g j_f - (-1)^{pq+\kappa_1\kappa_2} j_f j_g = j_g j_f - (-1)^{\|f\|\|g\|} j_f j_g.$$

The proof is finished. □

Theorem 3.3. *The vector space* $\mathcal{L}^*(\mathfrak{g},\mathfrak{g})$ *equipped with the bracket* $[-,-]^{3\mathrm{ls}}$ *is a* $\mathbb{Z}\times\mathbb{Z}_2$*-graded Lie algebra.*

Proof. Take $f \in \mathcal{L}_{(p,\kappa_1)}(\mathfrak{g},\mathfrak{g})$, $g \in \mathcal{L}_{(q,\kappa_2)}(\mathfrak{g},\mathfrak{g})$, and $h \in \mathcal{L}_{(r,\kappa_3)}(\mathfrak{g},\mathfrak{g})$. It is easy to see that $[f,g]^{3\mathrm{ls}} \in \mathcal{L}_{(p+q,\kappa_1+\kappa_2)}(\mathfrak{g},\mathfrak{g})$ and $[f,g]^{3\mathrm{ls}} = -(-1)^{\|f\|\|g\|}[g,f]^{3\mathrm{ls}}$. If $p,q,r \geq 0$, by Lemma 3.2, we have

$$
\begin{array}{rl}
& [f,[g,h]^{3\mathrm{ls}}]^{3\mathrm{ls}} - [[f,g]^{3\mathrm{ls}},h]^{3\mathrm{ls}} - (-1)^{\|f\|\|g\|}[g,[f,h]^{3\mathrm{ls}}]^{3\mathrm{ls}} \\
= & j_{[g,h]^{3\mathrm{ls}}}(f) - (-1)^{p(q+r)+\kappa_1(\kappa_2+\kappa_3)} j_f([g,h]^{3\mathrm{ls}}) \\
& - j_h([f,g]^{3\mathrm{ls}}) + (-1)^{r(p+q)+\kappa_3(\kappa_1+\kappa_2)} j_{[f,g]^{3\mathrm{ls}}}(h) \\
& - (-1)^{pq+\kappa_1\kappa_2} j_{[f,h]^{3\mathrm{ls}}}(g) + (-1)^{qr+\kappa_2\kappa_3} j_g([f,h]^{3\mathrm{ls}}) \\
= & j_h j_g(f) - (-1)^{qr+\kappa_2\kappa_3} j_g j_h(f) - (-1)^{p(q+r)+\kappa_1(\kappa_2+\kappa_3)} j_f([g,h]^{3\mathrm{ls}}) \\
& - j_h([f,g]^{3\mathrm{ls}}) + (-1)^{r(p+q)+\kappa_3(\kappa_1+\kappa_2)} j_g j_f(h) - (-1)^{r(p+q)+\kappa_3(\kappa_1+\kappa_2)}(-1)^{pq+\kappa_1\kappa_2} j_f j_g(h) \\
& - (-1)^{pq+\kappa_1\kappa_2} j_h j_f(g) + (-1)^{p(q+r)+\kappa_1(\kappa_2+\kappa_3)} j_f j_h(g) + (-1)^{qr+\kappa_2\kappa_3} j_g([f,h]^{3\mathrm{ls}}) \\
= & j_h j_g(f) - j_h([f,g]^{3\mathrm{ls}}) - (-1)^{pq+\kappa_1\kappa_2} j_h j_f(g) \\
& + (-1)^{qr+\kappa_2\kappa_3}\big(- j_g j_h(f) + (-1)^{pr+\kappa_1\kappa_3} j_g j_f(h) + j_g([f,h]^{3\mathrm{ls}})\big) \\
& + (-1)^{p(q+r)+\kappa_1(\kappa_2+\kappa_3)}\big(- j_f([g,h]^{3\mathrm{ls}}) - (-1)^{qr+\kappa_2\kappa_3} j_f j_g(h) + j_f j_h(g)\big) \\
= & 0.
\end{array}
$$

Hence $(\mathcal{L}^*(\mathfrak{g},\mathfrak{g}),[-,-]^{3\mathrm{ls}})$ is a $\mathbb{Z}\times\mathbb{Z}_2$-graded Lie algebra. □

Remark 3.4. The subspace $\mathcal{L}^*(\mathfrak{g},\mathfrak{g})_{\bar{0}} = \oplus_{p\in\mathbb{Z}}\mathcal{L}_{(p,\bar{0})}(\mathfrak{g},\mathfrak{g})$ of $\mathcal{L}^*(\mathfrak{g},\mathfrak{g})$ is a $\mathbb{Z}$-graded Lie algebra.

Proposition 3.5. *Let* $\pi : \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ *be an even skew-supersymmetric trilinear map on a* $\mathbb{Z}_2$*-graded vector space* $\mathfrak{g}$*. Then* π *defines a* 3*-Lie superalgebra structure on* $\mathfrak{g}$ *if and only if* $[\pi,\pi]^{3\mathrm{ls}} = 0$.

Moreover, we have

$$[\pi, f]^{3ls} = (-1)^p \partial_{ad} f, \ \forall\, f \in C^p(\mathfrak{g}, \mathfrak{g}), \ p \geq 0.$$

Proof. The pair $(\mathfrak{g}, \pi)$ is a 3-Lie superalgebra if and only if the following equation holds:

$$\begin{aligned} \pi(x_1, x_2, \pi(x_3, x_4, x_5)) \;=\;& \pi(\pi(x_1, x_2, x_3), x_4, x_5) \\ &+ (-1)^{(|x_1|+|x_2|)|x_3|}\pi(x_3, \pi(x_1, x_2, x_4), x_5) \\ &+ (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}\pi(x_3, x_4, \pi(x_1, x_2, x_5)), \end{aligned}$$

that is, $[\pi, \pi]^{3ls} = 0$. The rest also holds by a direct calculation. □

Proposition 3.5 gives a more concise description of the structures and coboundary operators of 3-Lie superalgebras by the bracket $[-, -]^{3ls}$. In this case, we say that 3-Lie superalgebras are **canonical structures** for the bracket $[-, -]^{3ls}$.

By Proposition 2.3 and Proposition 3.5, we immediately obtain:

Proposition 3.6. *Let* $(\mathfrak{g}, \pi)$ *be a* 3*-Lie superalgebra. Suppose that* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *is a* $\mathbb{Z}_2$*-graded vector space and* $\rho : \wedge^2\mathfrak{g} \to \mathfrak{gl}(V)$ *is an even linear map. Then* $(V; \rho)$ *is a representation of* $\mathfrak{g}$ *if and only if*

$$[\pi + \bar{\rho}, \pi + \bar{\rho}]^{3ls} = 0,$$

where $\bar{\rho}$ *is defined by Eq* (2.6)*.*

4. Generalized representations and the corresponding cohomologies of 3-Lie superalgebras

In the section, we introduce generalized representations and the corresponding cohomologies of 3-Lie superalgebras.

Definition 4.1. *Let* $\mathfrak{g}$ *be a* 3*-Lie superalgebra and* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *be a* $\mathbb{Z}_2$*-graded vector space. A* **generalized representation** *of* $\mathfrak{g}$ *on* V *consists of two even linear maps* $\rho : \wedge^2\mathfrak{g} \to \mathfrak{gl}(V)$ *and* $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$*, such that*

$$[\pi + \bar{\rho} + \bar{\nu}, \pi + \bar{\rho} + \bar{\nu}]^{3ls} = 0,$$

where

$$\bar{\rho}, \bar{\nu} : (\mathfrak{g} \oplus V) \times (\mathfrak{g} \oplus V) \times (\mathfrak{g} \oplus V) \to \mathfrak{g} \oplus V$$

are induced by ρ, ν*:*

$$\begin{aligned} \bar{\rho}(x_1 + u_1, x_2 + u_2, x_3 + u_3) =\;& \rho(x_1, x_2)u_3 + (-1)^{(|x_1|+|u_2|)|x_3|}\rho(x_3, x_1)u_2 \\ &+ (-1)^{(|x_2|+|x_3|)|u_1|}\rho(x_2, x_3)u_1, \end{aligned}$$

$$\begin{aligned} \bar{\nu}(x_1 + u_1, x_2 + u_2, x_3 + u_3) =\;& \nu(x_1)(u_2, u_3) + (-1)^{(|x_2|+|u_3|)|u_1|}\nu(x_2)(u_3, u_1) \\ &+ (-1)^{(|u_1|+|u_2|)|x_3|}\nu(x_3)(u_1, u_2), \end{aligned}$$

for all homogenous elements $x_i + u_i \in \mathfrak{g} \oplus V$*,* $|x_i + u_i| = |x_i| = |u_i|$*,* $i = 1, 2, 3$*.*

We will denote the generalized representation by a triple $(V; \rho, \nu)$.

Remark 4.2. *If $\nu = 0$, then the generalized representation $(V;\rho,\nu)$ reduces to a usual representation $(V;\rho)$ of $\mathfrak{g}$. If $\mathfrak{g}_{\bar{1}} = 0$, then $\mathfrak{g}$ is a 3-Lie algebra and $(V;\rho,\nu)$ is a generalized representation of 3-Lie algebra $\mathfrak{g}$. See more details about generalized representations of 3-Lie algebras in* [9, 14].

We denote $[-,-,-]_{(\rho,\nu)} = \pi + \bar{\rho} + \bar{\nu}$.

Theorem 4.3. *Let $(\mathfrak{g},\pi)$ be a 3-Lie superalgebra and $(V;\rho,\nu)$ be a generalized representation of g. Then $(\mathfrak{g}\oplus V,[-,-,-]_{(\rho,\nu)})$ is a 3-Lie superalgebra and we call it a* **generalized semidirect product** *of $\mathfrak{g}$ and V.*

Proof. It is a direct conclusion of Proposition 3.5 and Definition 4.1. □

In the following, we give an equivalent characterization of generalized representations.

Proposition 4.4. *Let $\mathfrak{g}$ be a 3-Lie superalgebra and $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. Let $\rho : \wedge^2\mathfrak{g} \to gl(V)$ and $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$ be two even linear maps. Then $(V;\rho,\nu)$ is a generalized representation of $\mathfrak{g}$ if and only if the following equalities hold:*

$$
\begin{aligned}
\rho(x_1,x_2)\rho(x_3,x_4) &= \rho([x_1,x_2,x_3],x_4) + (-1)^{(|x_1|+|x_2|)|x_3|}\rho(x_3,[x_1,x_2,x_4]) \\
&\quad + (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}\rho(x_3,x_4)\rho(x_1,x_2), \qquad (4.1)\\
\rho(x_1,[x_2,x_3,x_4]) &= (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}\rho(x_3,x_4)\rho(x_1,x_2) \\
&\quad - (-1)^{|x_1|(|x_2|+|x_4|)+|x_3||x_4|}\rho(x_2,x_4)\rho(x_1,x_3) \\
&\quad + (-1)^{|x_1|(|x_2|+|x_3|)}\rho(x_2,x_3)\rho(x_1,x_4), \qquad (4.2)\\
\rho(x_1,x_2)(\nu(x_3)(u_1,u_2)) &= \nu([x_1,x_2,x_3])(u_1,u_2) + (-1)^{(|x_1|+|x_2|)|x_3|}\nu(x_3)(\rho(x_1,x_2)(u_1),u_2) \\
&\quad + (-1)^{(|x_1|+|x_2|)(|x_3|+|u_1|)}\nu(x_3)(u_1,\rho(x_1,x_2)(u_2)), \qquad (4.3)\\
\nu(x_1)(u_1,\rho(x_2,x_3)(u_2)) &= (-1)^{(|x_1|+|u_1|+|x_2|)|x_3|+|u_1||x_2|}\nu(x_3)(\rho(x_1,x_2)(u_1),u_2) \\
&\quad + (-1)^{(|x_1|+|u_1|)(|x_2|+|x_3|)}\nu(x_2)(\rho(x_3,x_1)(u_1),u_2) \\
&\quad + (-1)^{(|x_1|+|u_1|)(|x_2|+|x_3|)}\rho(x_2,x_3)(\nu(x_1)(u_1,u_2)), \qquad (4.4)\\
\nu(x_1)(u_1,\nu(x_2)(u_2,u_3)) &= (-1)^{(|x_1|+|u_1|)|x_2|}\nu(x_2)(\nu(x_1)(u_1,u_2),u_3) \\
&\quad + (-1)^{(|x_1|+|u_1|)(|x_2|+|u_2|)}\nu(x_2)(u_2,\nu(x_1)(u_1,u_3)), \qquad (4.5)\\
\nu(x_1)(\nu(x_2)(u_1,u_2),u_3) &= (-1)^{|x_1||x_2|}\nu(x_2)(\nu(x_1)(u_1,u_2),u_3), \qquad (4.6)
\end{aligned}
$$

for all homogenous elements $x_i \in \mathfrak{g}$, $u_j \in V$, $i,j = 1,2,3$.

Proof. The triple $(V;\rho,\nu)$ is a generalized representation if and only if $[\pi+\bar{\rho}+\bar{\nu},\pi+\bar{\rho}+\bar{\nu}]^{3\mathrm{ls}} = 0$. By straight computations,

$$[\pi+\bar{\rho}+\bar{\nu},\pi+\bar{\rho}+\bar{\nu}]^{3\mathrm{ls}}(x_1,x_2,x_3,x_4,v) = 0$$

is equivalent to Eq (4.1) and

$$[\pi+\bar{\rho}+\bar{\nu},\pi+\bar{\rho}+\bar{\nu}]^{3\mathrm{ls}}(x_1,v,x_2,x_3,x_4,v) = 0$$

is equivalent to Eq (4.2). Other identities can be proved similarly. □

Example 4.5. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a 3-dimensional 3-Lie superalgebra over a field $\mathbb{K}$ equipped with a nonzero bracket $[-,-,-]$. Suppose that $\{e_1, e_2\}$ and $\{e_3\}$ are the basis of $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$, respectively, satisfying $[e_1, e_2, e_3] = e_3$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space with a basis $\{v_1, v_2\}$, where $|v_1| = \bar{0}$ and $|v_2| = \bar{1}$.

Define an even linear map $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$ by

$$\nu(e_1)(v_1, v_2) = l_1 v_2,\ \nu(e_2)(v_1, v_2) = l_2 v_2,\ \nu(e_3)(v_1, v_2) = 0,\ l_1, l_2 \in \mathbb{K}.$$

By direct calculations, $(V; 0, \nu)$ is a generalized representation of $\mathfrak{g}$.

Definition 4.6. *Let $(V_1; \rho_1, \nu_1)$ and $(V_2; \rho_2, \nu_2)$ be two generalized representations of a 3-Lie superalgebra $\mathfrak{g}$. Then they are called equivalent if there is an even linear map $\phi : V_1 \longrightarrow V_2$, satisfying*

$$\phi(\rho_1(x_1, x_2)u) = \rho_2(x_1, x_2)\phi(u), \quad \phi(\nu_1(x)(u_1, u_2)) = \nu_2(x)(\phi(u_1), \phi(u_2)),$$

for all homogenous elements $x, x_1, x_2 \in \mathfrak{g}$, $u, u_1, u_2 \in V_1$.

In the following, we give a new cohomology of a 3-Lie superalgebra, corresponding to a generalized representation.

Let $(V; \rho, \nu)$ be a generalized representation of a 3-Lie superalgebra $(\mathfrak{g}, \pi)$. The space of p-cochains $C^p_>(\mathfrak{g} \oplus V, V)$ is defined as a subset of

$$C^p(\mathfrak{g} \oplus V, V) = \mathrm{Hom}(\otimes^p(\wedge^2(\mathfrak{g} \oplus V)) \wedge (\mathfrak{g} \oplus V), V),$$

such that

$$C^p(\mathfrak{g} \oplus V, V) = C^p_>(\mathfrak{g} \oplus V, V) \oplus C^p(V, V).$$

Then $C^p_>(\mathfrak{g} \oplus V, V)$ is endowed with a natural $\mathbb{Z}_2$-grading.

Define a map $\mathrm{d} : C^p_>(\mathfrak{g} \oplus V, V) \to C^{p+1}_>(\mathfrak{g} \oplus V, V)$ by

$$\mathrm{d}(f) = (-1)^p[\pi + \bar{\rho} + \bar{\nu}, f]^{3\mathrm{ls}},\ f \in C^p_>(\mathfrak{g} \oplus V, V),\ p \geq 0. \tag{4.7}$$

Theorem 4.7. *Let $(V; \rho, \nu)$ be a generalized representation of a 3-Lie superalgebra $\mathfrak{g}$. Then $\mathrm{d} \circ \mathrm{d} = 0$ and $(C^p_>(\mathfrak{g} \oplus V, V), \mathrm{d})$ is a cochain complex of $\mathfrak{g}$.*

Proof. Consider that

$$\begin{aligned} \mathrm{d} \circ \mathrm{d}(f) &= (-1)^{p(p+1)}[\pi + \bar{\rho} + \bar{\nu}, [\pi + \bar{\rho} + \bar{\nu}, f]^{3\mathrm{ls}}]^{3\mathrm{ls}} \\ &= \frac{1}{2}(-1)^{p(p+1)}[[\pi + \bar{\rho} + \bar{\nu}, \pi + \bar{\rho} + \bar{\nu}]^{3\mathrm{ls}}, f]^{3\mathrm{ls}} \\ &= 0, \end{aligned}$$

for all $f \in C^p_>(\mathfrak{g} \oplus V, V)$. □

It is easy to see that $\mathrm{d}(f) = \partial(f)$, for all $f \in C^p(\mathfrak{g}, V)$. A p-cochain $f \in C^p_>(\mathfrak{g} \oplus V, V)$ is called a p-cocycle if $\mathrm{d}(f) = 0$. We denote by $\mathcal{Z}^p(\mathfrak{g}, V)$ the set of p-cocycles. Set $\mathcal{B}^p(\mathfrak{g}, V) = \mathrm{d}C^{p-1}_>(\mathfrak{g} \oplus V, V)$ and call it the set of p-coboundaries. It is easy to see that $\mathcal{B}^p(\mathfrak{g}, V) \subseteq \mathcal{Z}^p(\mathfrak{g}, V)$. The p-th cohomology group is defined by

$$\mathcal{H}^p(\mathfrak{g}, V) = \mathcal{Z}^p(\mathfrak{g}, V)/\mathcal{B}^p(\mathfrak{g}, V).$$

It is obvious that $\mathcal{Z}^p(\mathfrak{g}, V)$, $\mathcal{B}^p(\mathfrak{g}, V)$, and $\mathcal{H}^p(\mathfrak{g}, V)$ are $\mathbb{Z}_2$-graded.

Proposition 4.8. *There is a forgetful map from $\mathcal{H}^p(\mathfrak{g}, V)$ to $H^p(\mathfrak{g}, V)$.*

Proof. It is easy to see that $C^p(\mathfrak{g}, V) \subseteq C^p_>(\mathfrak{g} \oplus V, V)$ and $\mathrm{d}(f) = \partial(f)$, for all $f \in C^p(\mathfrak{g}, V)$. Hence a forgetful map from $\mathcal{H}^p(\mathfrak{g}; V)$ to $H^p(\mathfrak{g}; V)$ can be induced by the projection from $C^p_>(\mathfrak{g} \oplus V, V)$ to $C^p(\mathfrak{g}, V)$. □

Remark 4.9. *It is clear that $(\mathcal{L}^*_>(\mathfrak{g} \oplus V, V)_{\bar{0}} = \oplus_{p\geq 0} C^p_>(\mathfrak{g} \oplus V, V)_{\bar{0}}, [-,-]^{3\mathrm{ls}}, \mathrm{d})$ is a differential graded Lie algebra.*

Proposition 4.10. *A homogenous element $f \in \mathrm{Hom}(\mathfrak{g}, V)$ is a 0-cocycle if and only if the following equalities hold:*

$$\begin{aligned} 0 &= (-1)^{|f||x_1|}\nu(x_1)(f(x_2), u) - (-1)^{(|f|+|x_1|)|x_2|}\nu(x_2)(f(x_1), u) \\ 0 &= f([x_1, x_2, x_3]) - (-1)^{|f|(|x_1|+|x_2|)}\rho(x_1, x_2)f(x_3) - (-1)^{(|f|+|x_1|)(|x_2|+|x_3|)}\rho(x_2, x_3)f(x_1) \\ &\quad - (-1)^{|f|(|x_1|+|x_3|)+|x_2||x_3|}\rho(x_1, x_3)f(x_2), \end{aligned}$$

for all homogenous elements $x_1, x_2, x_3 \in \mathfrak{g}$, $u \in V$.

Proof. For $f : \mathfrak{g} \to V$, it is a 1-cocycle if and only if

$$\begin{aligned} 0 &= (\mathrm{d}f)(x_1, x_2, u) \\ &= (-1)^{|f||x_1|}\nu(x_1)(f(x_2), u) - (-1)^{(|f|+|x_1|)|x_2|}\nu(x_2)(f(x_1), u), \\ 0 &= (\mathrm{d}f)(x_1, x_2, x_3) \\ &= f([x_1, x_2, x_3]) - (-1)^{|f|(|x_1|+|x_2|)}\rho(x_1, x_2)f(x_3) - (-1)^{(|f|+|x_1|)(|x_2|+|x_3|)}\rho(x_2, x_3)f(x_1) \\ &\quad - (-1)^{|f|(|x_1|+|x_3|)+|x_2||x_3|}\rho(x_1, x_3)f(x_2). \end{aligned}$$

The proof is finished. □

Proposition 4.11. *Let $f_1 + f_2 + f_3 \in C^1_>(\mathfrak{g} \oplus V, V)_\kappa$ be a homogenous 1-cochain, where $f_1 \in \mathrm{Hom}(\wedge^2 V \wedge \mathfrak{g}, V)_\kappa$, $f_2 \in \mathrm{Hom}(\wedge^2\mathfrak{g} \wedge V, V)_\kappa$, and $f_3 \in \mathrm{Hom}(\wedge^3\mathfrak{g}, V)_\kappa$, $\kappa \in \mathbb{Z}_2$. Then $f_1 + f_2 + f_3$ is a 1-cocycle if and only if the following equalities hold:*

$$\begin{aligned} 0 &= -(-1)^{\kappa(|x_1|+|x_2|)}\rho(x_1, x_2)f_3(x_3, x_4, x_5) - f_3(x_1, x_2, [x_3, x_4, x_5]) \\ &\quad + (-1)^{(\kappa+|x_1|+|x_2|+|x_3|)(|x_4|+|x_5|)}\rho(x_4, x_5)f_3(x_1, x_2, x_3) + f_3([x_1, x_2, x_3], x_4, x_5) \\ &\quad - (-1)^{(\kappa+|x_1|+|x_2|)(|x_3|+|x_5|)+|x_4||x_5|}\rho(x_3, x_5)f_3(x_1, x_2, x_4) \\ &\quad + (-1)^{(|x_1|+|x_2|)|x_3|}f_3(x_3, [x_1, x_2, x_4], x_5) \\ &\quad + (-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}f_3(x_3, x_4, [x_1, x_2, x_5]) \\ &\quad + (-1)^{(\kappa+|x_1|+|x_2|)(|x_3|+|x_4|)}\rho(x_3, x_4)f_3(x_1, x_2, x_5), \end{aligned} \tag{4.8}$$

$$\begin{aligned} 0 &= (-1)^{(\kappa+|x_1|+|x_2|+|x_3|)|x_4|}\nu(x_4)(f_3(x_1, x_2, x_3), u) \\ &\quad - (-1)^{(\kappa+|x_1|+|x_2|)|x_3|}\nu(x_3)(f_3(x_1, x_2, x_4), u) \\ &\quad + (-1)^{\kappa(|x_1|+|x_2|)}\rho(x_1, x_2)f_2(x_3, x_4, u) \\ &\quad - (-1)^{(\kappa+|x_1|+|x_2|)(|x_3|+|x_4|)}\rho(x_3, x_4)f_2(x_1, x_2, u) \\ &\quad - f_2([x_1, x_2, x_3], x_4, u) + (-1)^{|x_3||x_4|}f_2([x_1, x_2, x_4], x_3, u), \end{aligned} \tag{4.9}$$

$$0 = (-1)^{(\kappa+|x_2|+|x_3|+|x_4|)|u|+\kappa|x_1|}\nu(x_1)(u, f_3(x_2, x_3, x_4))$$

$$
\begin{aligned}
&+(-1)^{(\kappa+|x_1|+|x_2|)(|x_3|+|x_4|)}\rho(x_3,x_4)f_2(x_1,x_2,u)\\
&-(-1)^{(\kappa+|x_1|)(|x_2|+|x_4|)+|x_3||x_4|}\rho(x_2,x_4)f_2(x_1,x_3,u)\\
&+(-1)^{(\kappa+|x_1|)(|x_2|+|x_3|)}\rho(x_2,x_3)f_2(x_1,x_4,u)\\
&+(-1)^{(|x_1|+|x_2|)(|x_3|+|x_4|)}f_2(x_3,x_4,\rho(x_1,x_2)u)\\
&-(-1)^{|x_1|(|x_2|+|x_4|)+|x_3||x_4|}f_2(x_2,x_4,\rho(x_1,x_3)u)-f_2(x_1,[x_2,x_3,x_4],u)\\
&+(-1)^{|x_1|(|x_2|+|x_3|)}f_2(x_2,x_3,\rho(x_1,x_4)u), \qquad (4.10)\\
0 \;=\; &-(-1)^{(\kappa+|x_1|+|x_2|)|x_3|}\nu(x_3)(f_2(x_1,x_2,u_1),u_2)\\
&+(-1)^{(\kappa+|x_1|+|x_2|)|x_3|+|u_1||u_2|}\nu(x_3)(f_2(x_1,x_2,u_2),u_1)\\
&+f_2(x_1,x_2,\nu(x_3)(u_1,u_2))+(-1)^{\kappa(|x_1|+|x_2|)+(|u_1|+|u_2|)|x_3|}\rho(x_1,x_2)f_1(u_1,u_2,x_3)\\
&-(-1)^{(|x_1|+|x_2|)|u_1|+(|u_1|+|u_2|)|x_3|}f_1(u_1,\rho(x_1,x_2)u_2,x_3)\\
&-(-1)^{(|u_1|+|u_2|)|x_3|}f_1(\rho(x_1,x_2)u_1,u_2,x_3)\\
&-(-1)^{(|x_1|+|x_2|+|x_3|)(|u_1|+|u_2|)}f_1(u_1,u_2,[x_1,x_2,x_3]), \qquad (4.11)\\
0 \;=\; &(-1)^{(\kappa+|x_1|+|x_2|+|u_1|)|x_3|+|x_2||u_1|}\nu(x_3)(f_2(x_1,x_2,u_1),u_2)\\
&+(-1)^{\kappa|x_2|+(|x_1|+|u_1|)(|x_2|+|x_3|)}\nu(x_2)(f_2(x_3,x_1,u_1),u_2)\\
&-(-1)^{\kappa(|x_1|+|u_1|)}\nu(x_1)(u_1,f_2(x_2,x_3,u_2))\\
&+(-1)^{(|x_1|+|u_1|)(|x_2|+|x_3|)}f_2(x_2,x_3,\nu(x_1)(u_1,u_2))\\
&+(-1)^{(\kappa+|x_1|+|u_1|)(|x_2|+|x_3|)+|x_1|(|u_1|+|u_2|)}\rho(x_2,x_3)f_1(u_1,u_2,x_1)\\
&+(-1)^{|x_2||u_1|+|x_3||u_2|}f_1(\rho(x_1,x_2)u_1,u_2,x_3)\\
&-(-1)^{|x_1|(|u_1|+|x_2|+|u_2|+|x_3|)}f_1(u_1,\rho(x_2,x_3)u_2,x_1)\\
&-(-1)^{|x_3|(|u_1|+|x_2|)+|x_2||u_2|}f_1(\rho(x_1,x_3)u_1,u_2,x_2), \qquad (4.12)\\
0 \;=\; &(-1)^{|x_2||x_3|}f_2(x_1,x_3,\nu(x_2)(u_1,u_2))-(-1)^{|x_1|(|x_2|+|x_3|)}f_2(x_2,x_3,\nu(x_1)(u_1,u_2))\\
&-f_2(x_1,x_2,\nu(x_3)(u_1,u_2))-(-1)^{\kappa(|x_1|+|x_2|)+(|u_1|+|u_2|)|x_3|}\rho(x_1,x_2)f_1(u_1,u_2,x_3)\\
&+(-1)^{\kappa(|x_1|+|x_3|)+|x_2|(|u_1|+|u_2|+|x_3|)}\rho(x_1,x_3)f_1(u_1,u_2,x_2)\\
&-(-1)^{(\kappa+|x_1|)(|x_2|+|x_3|)+(|u_1|+|u_2|)|x_1|}\rho(x_2,x_3)f_1(u_1,u_2,x_1)\\
&+(-1)^{(|u_1|+|u_2|)(|x_1|+|x_2|+|x_3|)}f_1(u_1,u_2,[x_1,x_2,x_3]), \qquad (4.13)\\
0 \;=\; &-(-1)^{(\kappa+|x_1|)+|u_3||x_1|}\nu(x_2)(f_1(u_1,u_2,x_1),u_3)\\
&+(-1)^{\kappa|x_2|+|x_1||u_2|+|u_2||u_3|}\nu(x_2)(f_1(u_1,u_3,x_1),u_2)\\
&+(-1)^{\kappa(|x_1|+|u_1|)+(|x_1|+|x_2|)(|u_1|+|u_2|+|u_3|)}\nu(x_1)(u_1,f_1(u_2,u_3,x_2))\\
&-(-1)^{(|x_1|+|x_2|)(|u_1|+|u_2|+|u_3|)}f_1(\nu(x_1)(u_1,u_2),u_3,x_2))\\
&+(-1)^{|x_1||x_2|+|x_2||u_1|}f_1(u_1,\nu(x_2)(u_2,u_3),x_1))\\
&+(-1)^{|u_2||u_3|+(|x_1|+|x_2|)(|u_1|+|u_2|+|u_3|)}f_1(\nu(x_1)(u_1,u_3),u_2,x_2)), \qquad (4.14)\\
0 \;=\; &(-1)^{(\kappa+|u_1|+|u_2|+|x_1|)|x_2|}\nu(x_2)(f_1(u_1,u_2,x_1),u_3)\\
&+(-1)^{(\kappa+|u_1|+|u_2|)|x_1|}\nu(x_1)(f_1(u_1,u_2,x_2),u_3)\\
&-(-1)^{(|u_1|+|u_2|)|x_2|+|x_1|(|x_2|+|u_3|)}f_1(\nu(x_2)(u_1,u_2),u_3,x_1)\\
&+(-1)^{|x_1|(|u_1|+|u_2|)+|x_2||u_3|}f_1(\nu(x_1)(u_1,u_2),u_3,x_2), \qquad (4.15)
\end{aligned}
$$

for all homogenous elements $x_1,x_2,x_3,x_4,x_5\in\mathfrak{g}$, $u,u_1,u_2\in V$.

Proof. A homogenous 1-cochain $f_1 + f_2 + f_3$ is a 1-cocycle if and only if $\mathrm{d}(f_1 + f_2 + f_3) = 0$. By straight computations,

$$\begin{aligned}
\mathrm{d}(f_1 + f_2 + f_3)(x_1, x_2, x_3, x_4, x_5) &= 0,\\
\mathrm{d}(f_1 + f_2 + f_3)(x_1, x_2, x_3, x_4, u) &= 0,\\
\mathrm{d}(f_1 + f_2 + f_3)(x_1, u, x_2, x_3, x_4) &= 0,\\
\mathrm{d}(f_1 + f_2 + f_3)(x_1, x_2, u_1, u_2, x_3) &= 0,\\
\mathrm{d}(f_1 + f_2 + f_3)(x_1, u_1, x_2, u_2, x_3) &= 0,\\
\mathrm{d}(f_1 + f_2 + f_3)(u_1, u_2, x_1, x_2, x_3) &= 0,\\
\mathrm{d}(f_1 + f_2 + f_3)(x_1, u_1, u_2, u_3, x_2) &= 0,\\
\mathrm{d}(f_1 + f_2 + f_3)(u_1, u_2, x_1, x_2, u_3) &= 0,
\end{aligned}$$

imply Eqs (4.8)–(4.15) hold. □

5. Generalized one-parameter formal deformations of 3-Lie superalgebras

In this section, we study generalized one-parameter formal deformations of 3-Lie superalgebras with respect to the new cohomology. We will see that the infinitesimals of two generalized deformations depend on the first cohomology groups, and the extendability of finite-order generalized deformations are controlled by the second cohomology groups.

Let $\mathbb{K}[[t]]$ be the power series ring in a variable t over $\mathbb{K}$ and $\mathfrak{g}[[t]]$ be the set of formal power series on a vector space $\mathfrak{g}$. It is natural to extend a $\mathbb{K}$-trilinear map $\varphi : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ to a $\mathbb{K}[[t]]$-trilinear map $\varphi : \mathfrak{g}[[t]] \times \mathfrak{g}[[t]] \times \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$ via

$$\varphi(\sum_{j_1\geq 0} x_1^{j_1} t^{j_1}, \sum_{j_2\geq 0} x_2^{j_2} t^{j_2}, \sum_{j_3\geq 0} x_3^{j_3} t^{j_3}) = \sum_{s\geq 0} \sum_{j_1+j_2+j_3=s} t^s \varphi(x_1^{j_1}, x_2^{j_2}, x_3^{j_3}),$$

for all $x_i^{j_i} \in \mathfrak{g}$, $i = 1, 2, 3$.

Definition 5.1. *Let $(\mathfrak{g}, \pi)$ be a 3-Lie superalgebra over a field $\mathbb{K}$ and $(V; \rho, \nu)$ be a generalized representation of $\mathfrak{g}$. A generalized one-parameter formal deformation of the pair $(\mathfrak{g}, V)$ is an even $\mathbb{K}[[t]]$-trilinear map*

$$[-,-,-]_t : (\mathfrak{g} \oplus V)[[t]] \times (\mathfrak{g} \oplus V)[[t]] \times (\mathfrak{g} \oplus V)[[t]] \to (\mathfrak{g} \oplus V)[[t]]$$

with the form $[-,-,-]_t = \sum_{i\geq 0} t^i [-,-,-]_i$, where $[-,-,-]_i \in \mathrm{Hom}(\wedge^3(\mathfrak{g} \oplus V), V)_{\bar{0}}$ can be extended to a $\mathbb{K}[[t]]$-trilinear map and $[-,-,-]_0 = [-,-,-]_{(\rho,\nu)}$, such that the following equality holds:

$$\begin{aligned}
&[\mathfrak{z}_1, \mathfrak{z}_2, [\mathfrak{z}_3, \mathfrak{z}_4, \mathfrak{z}_5]_t]_t \\
= \; &[[\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_t, \mathfrak{z}_4, \mathfrak{z}_5]_t + (-1)^{(|\mathfrak{z}_1|+|\mathfrak{z}_2|)|\mathfrak{z}_3|} [\mathfrak{z}_3, [\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_4]_t, \mathfrak{z}_5]_t \\
&+ (-1)^{(|\mathfrak{z}_1|+|\mathfrak{z}_2|)(|\mathfrak{z}_3|+|\mathfrak{z}_4)} [\mathfrak{z}_3, \mathfrak{z}_4, [\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_5]_t]_t,
\end{aligned} \tag{5.1}$$

for all homogenous elements $\mathfrak{z}_i \in \mathfrak{g} \oplus V$, $i = 1, 2, 3, 4, 5$.

Recall that a formal deformation of a 3-Lie superalgebra $(\mathfrak{g}, [-,-,-])$ is a power series $[-,-,-]_t^U = \sum_{i=0}^{+\infty}[-,-,-]_i^U t^i$ with $[-,-,-]_i^U \in \mathrm{Hom}(\wedge^3\mathfrak{g}, \mathfrak{g})_{\bar{0}}$ and $[-,-,-]_0 = [-,-,-]$, such that $(\mathfrak{g}, [-,-,-]_t^U)$ is a 3-Lie superalgebra (see [16]). Then we obtain:

Proposition 5.2. *If $V = \mathfrak{g}$, there is a forgetful map from a generalized one-parameter formal deformation $((\mathfrak{g} \oplus \mathfrak{g})[[t]], [-,-,-]_t)$ to a usual one $(\mathfrak{g}[[t]], [-,-,-]_t^U)$.*

Proof. It is obviously that $\mathfrak{g}[[t]] \subseteq (\mathfrak{g} \oplus \mathfrak{g})[[t]]$. Put

$$[x_1, x_2, x_3]_t^U = [(x_1, 0), (x_2, 0), (x_3, 0)]_t. \quad x_i \in \mathfrak{g},\ i = 1, 2, 3.$$

It follows that $[-,-,-]_t^U$ defines a 3-Lie superalgebra structure on $\mathfrak{g}[[t]]$ and it is a one-parameter formal deformation of $\mathfrak{g}$. □

Reviewing the operator " $\circ_{3\mathrm{ls}}$ " given by Eq (3.2), then Eq (5.1) can be written as

$$\sum_{i=0}^{k}[-,-,-]_i \circ_{3\mathrm{ls}} [-,-,-]_{k-i} = 0,\ k = 0, 1, 2, \cdots \tag{5.2}$$

and we call it a deformation equation.

Theorem 5.3. *Let $[-,-,-]_t = \sum_{i\geq 0} t^i[-,-,-]_i$ be a generalized one-parameter formal deformation of $(\mathfrak{g}, V)$. Then the first term $[-,-,-]_1$ is a 1-cocycle, that is, $[-,-,-]_1 \in \mathcal{Z}^1(\mathfrak{g}, V)$.*

Proof. Take $[-,-,-]_1 \in \mathrm{Hom}(\wedge^3\mathfrak{g}, \mathfrak{g})_{\bar{0}}$. For $k = 1$, Eq (5.2) can be written as

$$[-,-,-]_{(\rho,\nu)} \circ_{3\mathrm{ls}} [-,-,-]_1 + [-,-,-]_1 \circ_{3\mathrm{ls}} [-,-,-]_{(\rho,\nu)} = 0,$$

that is, $\mathrm{d}([-,-,-]_1) = 0$. Hence $[-,-,-]_1$ is a 1-cocycle. □

Definition 5.4. *The 1-cocycle $[-,-,-]_1 \in \mathcal{Z}^1(\mathfrak{g}, V)_{\bar{0}}$ is called the* **infinitesimal** *of the generalized one-parameter formal deformation $[-,-,-]_t = \sum_{i\geq 0} t^i[-,-,-]_i$ of $(\mathfrak{g}, V)$.*

Definition 5.5. *Let $\mathfrak{g}$ be a 3-Lie superalgebra and $(V; \rho, \nu)$ be a generalized representation of $\mathfrak{g}$. Suppose that $[-,-,-]_t = \sum_{i\geq 0} t^i[-,-,-]_i$ and $[-,-,-]_t' = \sum_{i\geq 0} t^i[-,-,-]_i'$ are two generalized one-parameter formal deformations of $(\mathfrak{g}, V)$ with*

$$[-,-,-]_0 = [-,-,-]_0' = [-,-,-]_{(\rho,\nu)}.$$

Then we call the two generalized deformations **equivalent** *if there exists a formal isomorphism $\phi_t = \sum\limits_{i\geq 0} \phi_i t^i : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$, where each ϕ_i is an even linear map, such that*

$$\phi_0 = \mathrm{Id}_{\mathfrak{g}\oplus V}, \quad \phi_t([\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_t) = [\phi_t(\mathfrak{z}_1), \phi_t(\mathfrak{z}_2), \phi_t(\mathfrak{z}_3)]_t' \tag{5.3}$$

for all homogenous elements $\mathfrak{z}_i \in \mathfrak{g} \oplus V$, $i = 1, 2, 3$.

A generalized one-parameter formal deformation of $(\mathfrak{g}, V)$ is called trivial if it is equivalent to $[-,-,-]_{(\rho,\nu)}$.

Theorem 5.6. *The infinitesimals of two generalized one-parameter formal deformations correspond to the same cohomology class of $\mathcal{H}^1(\mathfrak{g}, V)$.*

Proof. Suppose that $\phi_t = \mathrm{Id}_{\mathfrak{g}\oplus V} + \sum\limits_{i\geq 1} \phi_i t^i$ is the formal isomorphism such that

$$\phi_t([\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_t) = [\phi_t(\mathfrak{z}_1), \phi_t(\mathfrak{z}_2), \phi_t(\mathfrak{z}_3)]_t^{'}, \tag{5.4}$$

for all homogenous elements $\mathfrak{z}_i \in \mathfrak{g} \oplus V$, $i = 1, 2, 3$. Consider the coefficients of t in the two sides of Eq (5.4). Then we obtain that

$$\begin{aligned}&[\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_1 - [\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_1^{'}\\ =\ &[\phi_1(\mathfrak{z}_1), \mathfrak{z}_2, \mathfrak{z}_3]_{(\rho,\nu)} + [\mathfrak{z}_1, \phi_1(\mathfrak{z}_2), \mathfrak{z}_3]_{(\rho,\nu)} + [\mathfrak{z}_1, \mathfrak{z}_2, \phi_1(\mathfrak{z}_3)]_{(\rho,\nu)} - \phi_1([\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_{(\rho,\nu)}).\end{aligned}$$

That is,

$$[\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_1 - [\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_1^{'} = \mathrm{d}\phi_1(\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3).$$

Hence there is $\psi = \mathrm{d}\phi_1 \in \mathcal{B}^1(\mathfrak{g}, \mathfrak{g})$ such that $\psi = [-,-,-]_1 - [-,-,-]_1^{'}$. □

Theorem 5.7. *If* $\mathcal{H}^1(\mathfrak{g}, V) = 0$*, all generalized one-parameter formal deformations of* $(\mathfrak{g}, V)$ *are trivial.*

Proof. Suppose that $[-,-,-]_t = [-,-,-]_{(\rho,\nu)} + \sum_{i\geq r} t^i[-,-,-]_i (r \geq 1)$ is a generalized one-parameter formal deformation of $(\mathfrak{g}, V)$. It follows from Eq (5.2) that

$$[-,-,-]_{(\rho,\nu)} \circ_{3\mathrm{ls}} [-,-,-]_r + [-,-,-]_r \circ_{3\mathrm{ls}} [-,-,-]_{(\rho,\nu)} = 0.$$

That is, $\mathrm{d}([-,-,-]_r) = 0$.

If $\mathcal{H}^1(\mathfrak{g}, V) = 0$, there exist a 0-cochain ψ such that $[-,-,-]_r = \mathrm{d}(\psi)$. Define $\phi_t = \mathrm{Id}_{\mathfrak{g}\oplus V} - t^r\psi$. Then

$$\phi_t \circ (\mathrm{Id}_{\mathfrak{g}\oplus V} + \sum_{k\geq 1} t^{kr}\psi^k) = (\mathrm{Id}_{\mathfrak{g}\oplus V} + \sum_{k\geq 1} t^{kr}\psi^k) \circ \phi_t = \mathrm{Id}_{\mathfrak{g}\oplus V}$$

and $\phi_t^{-1} = \mathrm{Id}_{\mathfrak{g}\oplus V} + \sum_{k\geq 1} t^{kr}\psi^k$. Set

$$[\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_t^{'} = \phi_t^{-1}([\phi_t(\mathfrak{z}_1), \phi_t(\mathfrak{z}_2), \phi_t(\mathfrak{z}_3)]_t).$$

It follows that $[-,-,-]_t^{'} = \sum_{i\geq 0} t^i[-,-,-]_i^{'}$ is a generalized deformation equivalent to $[-,-,-]_t$.

Note that $\phi_t([\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_t^{'}) = [\phi_t(\mathfrak{z}_1), \phi_t(\mathfrak{z}_2), \phi_t(\mathfrak{z}_3)]_t$, which can be written as

$$(\mathrm{Id}_{\mathfrak{g}\oplus V} - t^r\psi) \sum_{i\geq 0} t^i[\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_i^{'} = \sum_{i\geq 0} t^i[\mathfrak{z}_1 - t^r\psi(\mathfrak{z}_1), \mathfrak{z}_2 - t^r\psi(\mathfrak{z}_2), \mathfrak{z}_3 - t^r\psi(\mathfrak{z}_3)]_i, \tag{5.5}$$

where $[-,-,-]_k = 0$, $1 \leq k \leq r-1$. Comparing the constant terms for two sides of Eq (5.5), we get

$$[\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_0^{'} = [\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_{(\rho,\nu)}.$$

Next we consider the coefficients of t^k $(1 \leq k \leq r-1)$ in Eq (5.5). They yield that

$$[-,-,-]_k^{'} = 0,\ 1 \leq k \leq r-1.$$

For the coefficients of t^r, we obtain from Eq (5.5) that

$$\begin{aligned}&[\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_r^{'} - \psi([\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_{(\rho,\nu)})\\ =\ &[\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]_r - [\psi(\mathfrak{z}_1), \mathfrak{z}_2, \mathfrak{z}_3]_{(\rho,\nu)} - [\mathfrak{z}_1, \psi(\mathfrak{z}_2), \mathfrak{z}_3]_{(\rho,\nu)} - [\mathfrak{z}_1, \mathfrak{z}_2, \psi(\mathfrak{z}_3)]_{(\rho,\nu)}.\end{aligned}$$

By the fact that $[-,-,-]_r = \mathrm{d}(\psi)$, we have $[-,-,-]_r^{'} = 0$. Repeating the procedure, we remove an increasing number of terms of the generalized deformation and obtain that $[-,-,-]_t$ is trivial. □

A generalized one-parameter formal deformation $[-,-,-]_t$ is said to be of order n if $[-,-,-]_t = \sum_{i=0}^{n} t^i[-,-,-]_i$. We say that a generalized deformation of order n is extensible, if it is able to be extended to a generalized deformation of order $n+1$.

Suppose that $\sum_{i=0}^{n} t^i[-,-,-]_i$ is a generalized deformation of order n, satisfying the deformation Eq (5.2). Then it can be extended to a generalized deformation of order $n+1$ if and only if there is a trilinear map $[-,-,-]_{n+1}$ such that

$$\sum_{i=0}^{n+1}[-,-,-]_i \circ_{3ls} [-,-,-]_{n+1-i} = 0. \tag{5.6}$$

Equation (5.6) is equivalent to

$$\mathrm{d}[-,-,-]_{n+1} = \sum_{i=1}^{n}[-,-,-]_i \circ_{3ls} [-,-,-]_{n+1-i}.$$

Set

$$\mathrm{Ob} = \sum_{i=1}^{n}[-,-,-]_i \circ_{3ls} [-,-,-]_{n+1-i}.$$

So Ob is the obstruction to find $[-,-,-]_{n+1}$.

Proposition 5.8. *The* 2*-cochain* Ob *is a* 2*-cocycle, i.e.,* d(Ob) = 0.

Proof. We denote by $[-,-,-]_i = \psi_i$. For $1 \le k \le n$, Eq (5.2) implies that

$$2\mathrm{d}(\psi_k) = \sum_{i=1}^{k-1}[\psi_i, \psi_{k-i}]^{3ls}.$$

So

$$\begin{aligned}
4\mathrm{d}(\mathrm{Ob}) &= 4\sum_{i=1}^{n}[\psi_0, \psi_i \circ_{3ls} \psi_{n+1-i}]^{3ls} \\
&= 2\sum_{i=1}^{n}[\psi_0, [\psi_i, \psi_{n+1-i}]^{3ls}]^{3ls} \\
&= 2\sum_{i=1}^{n}[[\psi_0, \psi_i]^{3ls}, \psi_{n+1-i}]^{3ls} - 2\sum_{i=1}^{n}[\psi_i, [\psi_0, \psi_{n+1-i}]^{3ls}]^{3ls} \\
&= -\sum_{i=1}^{n}\sum_{j=1}^{i-1}[[\psi_j, \psi_{i-j}]^{3ls}, \psi_{n+1-i}]^{3ls} + \sum_{i=1}^{n}\sum_{j=1}^{n-i}[\psi_i, [\psi_j, \psi_{n+1-i-j}]^{3ls}]^{3ls} \\
&= -\sum_{i'=1}^{n}\sum_{j=1}^{n-i'}[[\psi_j, \psi_{n+1-i'-j}]^{3ls}, \psi_{i'}]^{3ls} + \sum_{i=1}^{n}\sum_{j=1}^{n-i}[\psi_i, [\psi_j, \psi_{n+1-i-j}]^{3ls}]^{3ls} \\
&= 2\sum_{i=1}^{n}\sum_{j=1}^{n-i}[\psi_i, [\psi_j, \psi_{n+1-i-j}]^{3ls}]^{3ls} \\
&= \frac{2}{3}\sum_{i=1}^{n}\sum_{j=1}^{n-i}([\psi_i, [\psi_j, \psi_{n+1-i-j}]^{3ls}]^{3ls} + [\psi_j, [\psi_{n+1-i-j}, \psi_i]^{3ls}]^{3ls} \\
&\quad + [\psi_{n+1-i-j}, [\psi_i, \psi_j]^{3ls}]^{3ls}) \\
&= 0.
\end{aligned}$$

Hence Ob is a 2-cocycle. □

Theorem 5.9. *If $\mathcal{H}^2(\mathfrak{g}, V) = 0$, every generalized one-parameter formal deformation of finite order is extensible.*

Proof. Suppose that $\sum_{i=0}^{n} t^i[-,-,-]_i$ is a generalized deformation of order n. If Ob is trivial, there is a 1-cochain ψ such that $\mathrm{Ob} = \mathrm{d}(\psi)$. This yields that

$$\mathrm{d}(\psi) = \sum_{i=1}^{n}[-,-,-]_i \circ [-,-,-]_{n+1-i}.$$

Hence $[-,-,-]_t + t^{n+1}\psi$ is a generalized deformation of order $n+1$ and $[-,-,-]_t$ is extensible.

Conversely, if $[-,-,-]_t$ is extensible, then there is ψ such that $\sum_{i=0}^{n} t^i[-,-,-]_i + t^{n+1}\psi$ is a generalized deformation of order $n+1$. We immediately obtain

$$\mathrm{d}(\psi) = \sum_{i=1}^{n}[-,-,-]_i \circ [-,-,-]_{n+1-i}.$$

It follows that the obstruction class $\mathrm{Ob} = \mathrm{d}(\psi)$ is trivial. □

Corollary 5.10. *If $\mathcal{H}^2(\mathfrak{g}, V) = 0$, every 1-cocycle is the infinitesimal of some generalized one-parameter formal deformation of $(\mathfrak{g}, V)$.*

6. Abelian extensions of 3-Lie superalgebras

In this section, we associate split Abelian extensions to generalized semidirect product 3-Lie superalgebras, and describe non-split Abelian extensions by Maurer-Cartan elements.

Let $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}})$, $(V, [-,-,-]_V)$, and $(\widehat{\mathfrak{g}}, [-,-,-]_{\widehat{\mathfrak{g}}})$ be 3-Lie superalgebras. We say that $\widehat{\mathfrak{g}}$ is an **extension** of $\mathfrak{g}$ by V, if there exists a short exact sequence

$$0 \longrightarrow V \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \longrightarrow 0,$$

where $i : V \to \widehat{\mathfrak{g}}$ and $p : \widehat{\mathfrak{g}} \to \mathfrak{g}$ are two morphisms of 3-Lie superalgebras. If V is an Abelian ideal of $\widehat{\mathfrak{g}}$, i.e., $[V, V, \widehat{\mathfrak{g}}]_{\widehat{\mathfrak{g}}} = 0$, then we say that the extension is an Abelian extension. If there is a homogenous even linear map $\sigma : \mathfrak{g} \to \widehat{\mathfrak{g}}$ such that $p \circ \sigma = \mathrm{Id}_{\mathfrak{g}}$, then the map σ is called a splitting of $\hat{\mathfrak{g}}$. Moreover, if σ is also a homomorphism between 3-Lie superalgebras, we call the corresponding extension a split Abelian extension.

Proposition 6.1. *Every split Abelian extension of a 3-Lie superalgebra is isomorphic to a generalized semidirect product 3-Lie superalgebra.*

Proof. Suppose that $\widehat{\mathfrak{g}}$ is a split Abelian extension of $\mathfrak{g}$ by V and σ is the corresponding splitting. Define two linear maps $\rho : \wedge^2\mathfrak{g} \to V$ and $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$ by

$$\rho(x_1, x_2)(u) = [\sigma(x_1), \sigma(x_2), u]_{\widehat{\mathfrak{g}}}, \tag{6.1}$$

$$\nu(x)(u_1, u_2) = [\sigma(x), u_1, u_2]_{\widehat{\mathfrak{g}}}, \tag{6.2}$$

for all $x, x_1, x_2 \in \mathfrak{g}$, $u, u_1, u_2 \in V$. Then there is 3-Lie superalgebra structure on $\mathfrak{g} \oplus V$ defined by $\widehat{\mathfrak{g}}$

$$\begin{aligned}
& [x_1+u_1, x_2+u_2, x_3+u_3]_{(\rho,\nu)} \\
= \ & [x_1,x_2,x_3]_{\mathfrak{g}} + \rho(x_1,x_2)u_3 + (-1)^{(|x_2|+|x_3|)|u_1|}\rho(x_2,x_3)u_1 + (-1)^{(|x_1|+|u_2|)|x_3|}\rho(x_3,x_1)u_2 \\
& + \nu(x_1)(u_2,u_3) + (-1)^{(|x_2|+|u_3|)|u_1|}\nu(x_2)(u_3,u_1) + (-1)^{(|u_1|+|u_2|)|x_3|}\nu(x_3)(u_1,u_2).
\end{aligned}$$

So $(V;\rho,\nu)$ is a generalized representation of $\mathfrak{g}$ and $(\mathfrak{g}\oplus V, [-,-,-]_{(\rho,\nu)})$ is the corresponding generalized semidirect product 3-Lie superalgebra. □

For a non-split Abelian extension of $\mathfrak{g}$, suppose that ρ and ν are given by Eqs (6.1) and (6.2), respectively. Define a linear map $\vartheta : \wedge^3\mathfrak{g} \to V$ by

$$\vartheta(x_1,x_2,x_3) = [\sigma(x_1),\sigma(x_2),\sigma(x_3)]_{\widehat{\mathfrak{g}}} - \sigma[x_1,x_2,x_3]_{\mathfrak{g}}.$$

Then we transfer the 3-Lie superalgebra structure on $\widehat{\mathfrak{g}}$ to $\mathfrak{g}\oplus V$ by ρ, ν, and ϑ:

$$\begin{aligned}
& [x_1+u_1, x_2+u_2, x_3+u_3]_{(\rho,\nu,\vartheta)} \\
= \ & [x_1,x_2,x_3]_{\mathfrak{g}} + \rho(x_1,x_2)u_3 + (-1)^{(|x_2|+|x_3|)|u_1|}\rho(x_2,x_3)u_1 + (-1)^{(|x_1|+|u_2|)|x_3|}\rho(x_3,x_1)u_2 \\
& + \nu(x_1)(u_2,u_3) + (-1)^{(|x_2|+|u_3|)|u_1|}\nu(x_2)(u_3,u_1) + (-1)^{(|u_1|+|u_2|)|x_3|}\nu(x_3)(u_1,u_2) \\
& + \vartheta(x_1,x_2,x_3),
\end{aligned}$$

for all homogenous elements $x_i + u_i \in \mathfrak{g}\oplus V$, $i = 1,2,3$.

Theorem 6.2. *Under the above notations, the pair* $(\mathfrak{g}\oplus V, [-,-,-]_{(\rho,\nu,\vartheta)})$ *is a* 3*-Lie superalgebra if and only if Eqs* (4.3)–(4.6) *and the following equalities hold:*

$$\begin{aligned}
0 \ = \ & -\rho(x_1,x_2)\vartheta(x_3,x_4,x_5) + (-1)^{(|x_4|+|x_5|)(|x_1|+|x_2|+|x_3|)}\rho(x_4,x_5)\vartheta(x_1,x_2,x_3) \\
& - (-1)^{(|x_3|+|x_5|)(|x_1|+|x_2|)+|x_4||x_5|}\rho(x_3,x_5)\vartheta(x_1,x_2,x_4) \\
& - (-1)^{(|x_3|+|x_4|)(|x_1|+|x_2|)}\rho(x_3,x_4)\vartheta(x_1,x_2,x_5) - \vartheta(x_1,x_2,[x_3,x_4,x_5]) \\
& + \vartheta([x_1,x_2,x_3],x_4,x_5) + (-1)^{|x_3|(|x_1|+|x_2|)}\vartheta(x_3,[x_1,x_2,x_4],x_5) \\
& + (-1)^{(|x_3|+|x_4|)(|x_1|+|x_2|)}\vartheta(x_3,x_4,[x_1,x_2,x_5]), \qquad (6.3)\\
0 \ = \ & (-1)^{(|x_2|+|x_3|+|x_4|)|u|}\nu(x_1)(u,\vartheta(x_2,x_3,x_4)) - \rho(x_1,[x_2,x_3,x_4])u \\
& + (-1)^{(|x_3|+|x_4|)(|x_1|+|x_2|)}\rho(x_3,x_4)\rho(x_1,x_2)u \\
& + (-1)^{(|x_2|+|x_3|)|x_1|}\rho(x_2,x_3)\rho(x_1,x_4)u \\
& - (-1)^{(|x_2|+|x_4|)|x_1|+|x_3||x_4|}\rho(x_2,x_4)\rho(x_1,x_3)u, \qquad (6.4)\\
0 \ = \ & \rho(x_1,x_2)\rho(x_3,x_4)u - (-1)^{(|x_3|+|x_4|)(|x_1|+|x_2|)}\rho(x_3,x_4)\rho(x_1,x_2)u \\
& - \rho([x_1,x_2,x_3],x_4)u + (-1)^{|x_3||x_4|}\rho([x_1,x_2,x_4],x_3)u \\
& + (-1)^{(|x_1|+|x_2|+|x_3|)|x_4|}\nu(x_4)(\vartheta(x_1,x_2,x_3),u) \\
& - (-1)^{(|x_1|+|x_2|)|x_3|}\nu(x_3)(\vartheta(x_1,x_2,x_4),u), \qquad (6.5)
\end{aligned}$$

for all homogenous elements $x_i \in \mathfrak{g}$ *and* $u \in V$, $i = 1,2,3,4,5$.

Proof. The $\mathbb{Z}_2$-graded vector space $\mathfrak{g} \oplus V$ equipped with $[-,-,-]_{(\rho,\nu,\vartheta)}$ is a 3-Lie superalgebra if and only if the following equations hold:

$$
\begin{aligned}
&[[-,-,-]_{(\rho,\nu,\vartheta)},[-,-,-]_{(\rho,\nu,\vartheta)}]^{3\text{ls}}(x_1,x_2,x_3,x_4,x_5)=0,\\
&[[-,-,-]_{(\rho,\nu,\vartheta)},[-,-,-]_{(\rho,\nu,\vartheta)}]^{3\text{ls}}(x_1,u,x_2,x_3,x_4)=0,\\
&[[-,-,-]_{(\rho,\nu,\vartheta)},[-,-,-]_{(\rho,\nu,\vartheta)}]^{3\text{ls}}(x_1,x_2,u,x_3,x_4)=0,\\
&[[-,-,-]_{(\rho,\nu,\vartheta)},[-,-,-]_{(\rho,\nu,\vartheta)}]^{3\text{ls}}(x_1,x_2,u_1,u_2,x_3)=0,\\
&[[-,-,-]_{(\rho,\nu,\vartheta)},[-,-,-]_{(\rho,\nu,\vartheta)}]^{3\text{ls}}(u_1,x_1,u_2,x_2,x_3)=0,\\
&[[-,-,-]_{(\rho,\nu,\vartheta)},[-,-,-]_{(\rho,\nu,\vartheta)}]^{3\text{ls}}(u_1,x_1,u_2,u_3,x_2)=0,\\
&[[-,-,-]_{(\rho,\nu,\vartheta)},[-,-,-]_{(\rho,\nu,\vartheta)}]^{3\text{ls}}(u_1,u_2,u_3,x_1,x_2)=0,
\end{aligned}
$$

which implies Eqs (4.3)–(4.6) and Eqs (6.3)–(6.5) hold. □

We will describe non-split Abelian extensions by Maurer-Cartan elements. Let $(L = \oplus_{k\geq 0}L_k, [-,-], d)$ be a differential graded Lie algebra. We say that $P \in L_1$ is a Maurer-Cartan element ([12]) of L if the following Maurer-Cartan equation holds:

$$dP + \frac{1}{2}[P,P] = 0.$$

Let $(\mathfrak{g}, [-,-,-]_{\mathfrak{g}})$ be a 3-Lie superalgebra and V be a $\mathbb{Z}_2$-graded vector space. Set a bracket

$$[x_1+u_1, x_2+u_2, x_3+u_3]' = [x_1,x_2,x_3]_{\mathfrak{g}}$$

on $\mathfrak{g} \oplus V$ and then $(\mathfrak{g} \oplus V, [-,-,-]')$ is a 3-Lie superalgebra. It follows that $(\mathcal{L}'(\mathfrak{g}\oplus V, \mathfrak{g}\oplus V)_{\bar{0}} = \oplus_{p\geq 0}C^p(\mathfrak{g}\oplus V, V)_{\bar{0}}, [-,-]^{3\text{ls}}, \partial)$ is a differential graded Lie algebra, where ∂ is the usual coboundary operator of 3-Lie superalgebra $\mathfrak{g}\oplus V$. Moreover, $(\mathcal{L}'_{>}(\mathfrak{g}\oplus V, V)_{\bar{0}} = \oplus_{p\geq 0}C^p_{>}(\mathfrak{g}\oplus V, V)_{\bar{0}}, [\cdot,\cdot]^{3\text{ls}}, \partial)$ is a differential graded Lie subalgebra.

Proposition 6.3. *The pair* $(\hat{\mathfrak{g}}, [-,-,-]_{\hat{\mathfrak{g}}})$ *is a non-split Abelian extension of* $\mathfrak{g}$ *by* V *if and only if* $\bar{\rho}+\bar{\nu}+\vartheta$ *is a Maurer-Cartan element of a differential graded Lie algebra* $(\mathcal{L}'_{>}(\mathfrak{g}\oplus V, V)_{\bar{0}}, [-,-]^{3\text{ls}}, \partial)$.

Proof. The pair $(\hat{\mathfrak{g}}, [-,-,-]_{\hat{\mathfrak{g}}})$ is a non-split Abelian extension of $\mathfrak{g}$ by V if and only if $(\mathfrak{g}\oplus V, [-,-,-]_{(\rho,\nu,\vartheta)})$ is a 3-Lie superalgebra. By Lemma 3.5, $(\mathfrak{g}\oplus V, [-,-,-]_{(\rho,\nu,\vartheta)})$ is a 3-Lie superalgebra by showing that

$$
\begin{aligned}
0 &= [[-,-,-]_{(\rho,\nu,\vartheta)},[-,-,-]_{(\rho,\nu,\vartheta)}]^{3\text{ls}}\\
&= [\pi+\bar{\rho}+\bar{\nu}+\vartheta, \pi+\bar{\rho}+\bar{\nu}+\vartheta]^{3\text{ls}}\\
&= [\pi,\pi]^{3\text{ls}} + 2[\pi, \bar{\rho}+\bar{\nu}+\vartheta]^{3\text{ls}} + [\bar{\rho}+\bar{\nu}+\vartheta, \bar{\rho}+\bar{\nu}+\vartheta]^{3\text{ls}}\\
&= 2\partial(\bar{\rho}+\bar{\nu}+\vartheta) + [\bar{\rho}+\bar{\nu}+\vartheta, \bar{\rho}+\bar{\nu}+\vartheta]^{3\text{ls}}.
\end{aligned}
$$

This yields that $\bar{\rho}+\bar{\nu}+\vartheta$ is a Maurer-Cartan element. □

7. Conclusions

In this paper, we introduce generalized representations and the corresponding new cohomology theory of 3-Lie superalgebras, as well as use the new cohomology groups to characterize generalized one-parameter formal deformations and Abelian extensions. First, we describe 3-Lie superalgebras as canonical structures for the bracket $[-,-]^{3ls}$. Second, we define generalized representations and the corresponding cohomology theory by $[-,-]^{3ls}$. Third, we develop generalized one-parameter formal deformations of 3-Lie superalgebras, and prove that the infinitesimals and the extensibility of finite-order deformations of generalized one-parameter formal deformations are controlled by the new first and second cohomology groups, respectively. Finally, we describe split and non-split Abelian extensions of 3-Lie superalgebras by generalized semidirect products and Maurer-Cartan elements, respectively.

Author contributions

J. Zhu and R. Ma wrote the main manuscript text and reviewed the text for spelling mistakes. R. Ma procured funding. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing financial or non-financial interests.

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