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*Research article*

# Denseness of soft spaces

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**Abstract:** In a soft environment, various structures associated with sets and topological spaces have been rigorously analyzed, including soft open (closed) sets, soft separation axioms, soft connectedness, and so on. The practical applications of soft set theory underscore its significant impact on real-world problems, offering innovative solutions that enhance decision-making processes across diverse domains. In this study, we built upon existing research by introducing new concepts within the realm of soft spaces, which served as an extension of classical topology. Our focus was on soft isolated sets and soft dense sets (spaces) in themselves. We investigated the fundamental properties and characterizations of these concepts, bolstered by rigorous proofs of theorems and relevant counterexamples. We explored the interrelations between these concepts and the notion of soft closure. Additionally, we showed that a soft topology was classified as soft dense in itself if, and only if, each of its parametric topologies was similarly classified. We also examined the behavior of soft dense spaces in themselves under various operations, including the formation of soft subspaces, the soft topological sum, and the image and inverse image under specific soft mappings.

**Keywords:** soft set; soft point; soft topology; parametric topology; soft limit point; soft isolated point; soft dense set in itself; soft dense space in itself
**Mathematics Subject Classification:** 54E99, 54F65

## Abbreviations:

$T, U$: sets of parameters; $\tilde{\theta}$: a soft topology; $\tilde{\lambda}, \tilde{\mu}, \tilde{\upsilon}$: soft sets; $\theta_t$: a parametric topology; $S(Q)_T$: the set of all soft sets over $Q$ with respect to $T$; $Cl_{\tilde{\theta}}(\tilde{\lambda})$: the soft closure of $\tilde{\lambda}$ relative to $\tilde{\theta}$; $\tilde{\Phi}$: a null soft set; $Int_{\tilde{\theta}}(\tilde{\lambda})$: the soft interior of $\tilde{\lambda}$ relative to $\tilde{\theta}$; $\tilde{Q}, \tilde{W}$: absolute soft sets; $D_{\tilde{\theta}}(\tilde{\lambda})$: a derived soft set of $\tilde{\mu}$ relative to $\tilde{\theta}$; $\tilde{\lambda}_t^q$: a soft point; $D_{\theta_t}(\mu(t))$: a derived set of $\mu(t)$ in the parametric topology $\theta_t$; $\Omega(Q)_T$: the set of all soft points over $Q$; $D(\tilde{\mu})$: $D(\tilde{\mu}) = \{(t, D_{\theta_t}(\mu(t)))$ for all $t \in T\}$; **STS**: a soft topological space; $is_{\tilde{\theta}}(\tilde{\mu})$: the union of all soft isolated points of $\tilde{\mu}$ relative to $\tilde{\theta}$; **SDTS**: a soft dense topological space in itself; $is_{\theta_t}(\mu(t))$: the set of all isolated points of $\mu(t)$ in the parametric topology $\theta_t$; **SDS**: a soft dense set in itself; $is(\tilde{\mu})$: $is(\tilde{\mu}) = \{(t, is_{\theta_t}(\mu(t)))$ for all $t \in T\}$

## 1. Introduction

Addressing and mitigating uncertainty is pivotal for enhancing the reliability of data analysis results. However, effectively reducing the existing uncertainty often poses considerable challenges. Consequently, many mathematical techniques developed for data analysis have fallen short of meeting these objectives. Prominent examples of such techniques include fuzzy sets [1], rough sets [2], and probability theory. Inadequate representation of uncertainty has posed challenges for decision-making procedures, which struggle to effectively capture it. To overcome limitations in representing uncertainty and improve decision-making procedures and in pursuit of improved results, Molodtsove [3] proposed soft sets as a parameterization tool to address these challenges. Soft sets offer an effective mathematical model for decision-making procedures, particularly in selecting the best alternative, as they facilitate the description of objects supplying parameters. The rapid development of soft set theory has contributed to significant advancements in the field of decision-making problems and medical diagnosis and treatment planning (see [4–7]). The practical applications of soft set theory have demonstrated its influence is evident in real-world applications, where it has provided innovative solutions to complex problems, improved decision-making processes, and enhanced outcomes in various domains.

Numerous mathematical branches have been explored within the context of soft-set frameworks. Soft topology is indeed a branch of mathematics that has been introduced in [8, 9] as an extension of classical topology within the framework of soft set theory. It provides a fresh generality for studying topological properties and operations in uncertain environments. The purpose of this study is to present new concepts in soft topological spaces and establish their generality as extensions of classical topology. Specifically, we introduce and investigate the notions of soft isolated points, soft dense sets (spaces) in themselves. These concepts aim to capture and analyze important properties of soft topological spaces, drawing inspiration from the studies conducted by scholars in classical topology (see [10, 11]). The motivation for studying more concepts of this soft topology arises from its practical applications, the need to handle uncertainty, the development of fuzzy theory, theoretical development, interdisciplinary connections, and educational value. By furthering our understanding of soft topology, we can make valuable contributions to addressing real-world challenges, propelling the development of mathematical theory, and promoting collaboration across disciplines.

In 2011, Shabir and Naz [8] introduced the concept of soft topological space and conducted a comprehensive exploration of various soft set concepts, including different types such as soft open and closed sets, as well as soft closure, soft neighborhood, soft subspaces, and soft separation axioms in relation to ordinary points. In [12], the author identified certain gaps in Shabir's work and proceeded to address and rectify them. In 2012, the authors [13] delved into the fundamental principles of soft topological spaces, focusing on essential concepts such as the soft interior, soft exterior, and soft boundary. Zarlutana et al. [14] conducted an exploration and study of the concept of soft continuous mapping between two collections of soft sets. Furthermore, they introduced the concept of compact soft spaces. In [15], a significant contribution was made by the authors in the form of introducing initial and product soft topology, which is derived from a family of soft mapping and soft projection mappings, respectively. Furthermore, they introduced the concept of soft compactness space. In

2013, Lin [16] examined the concepts of soft connected spaces and soft paracompact spaces. In 2015, Hussain [17] conducted a thorough investigation of soft open, soft closed, and soft homeomorphism mappings in the context of soft topological spaces. In [18], he provided a comprehensive definition of soft connected spaces and explored their properties. In 2018, Bayramov et al. [19] conducted a study on the approach of separability in soft topological spaces . In 2020, the authors [20] introduced the notion of the sum of soft topological spaces by utilizing pairwise disjoint soft topological spaces and proceeded to examine its fundamental properties. The classes of soft sets, including soft dense, soft codense, soft nowhere dense, and soft somewhat dense open, have been extensively studied by multiple authors (see [21–24]). A considerable number of authors have dedicated their research to the study and examination of various methods for constructing soft topology using classical topology. These different approaches have been examined and explored in multiple works (see [25–28]). In recent years, several kinds of soft topological spaces have been studied, including soft submaximal spaces [29], soft expandable spaces [30], soft primal topology [31], soft nodense spaces [32], cluster soft topologies [33], density soft topologies [34, 35], and soft door spaces [36].

Extensive research has been conducted on soft topological spaces and ongoing investigation continues to advance the field. Soft topology, which encompasses the study of soft sets and their corresponding topological structures, has garnered substantial interest among researchers due to its capacity to effectively address uncertainty, imprecision, and vagueness with a high degree of flexibility.

The paper follows a structured outline. The literature review of studying soft topology is presented as a subsection in this section. Following that, the second section focuses on the definitions and results derived from set theory and soft topological spaces which hold relevance to the present study. Section 3 outlines the methodology employed in this study. Section 4 comprises two subsections. The first subsection introduces the new theorems related to soft limit points and the concept of soft isolated points, along with their properties. The second subsection explores the concept of soft dense sets and soft spaces in themselves, examining their distinctive characteristics. Sections 5 and 6 encompass a thorough discussion and conclusion of this work.

## 2. Preliminaries

This section references the fundamental definitions and results required for comprehending this work. Throughout this work, we adopt the notations where $Q$ represents an initial universe set, $\mathcal{P}(Q)$ signifies its power set, and we use the symbol $\emptyset$ to refer to the empty set.

**Definition 1.** *[3] A soft set over an initial universe set $Q$ is defined as:*

$$\tilde{\lambda} = \{(t, \lambda(t)) : t \in T \text{ and } \lambda(t) \subseteq Q\},$$

*where $\lambda$ is a set-valued function that maps a nonempty subset $T$ of a set of parameters $E$ into the power set $\mathcal{P}(Q)$.*

The set $S(Q)_T$ represents the class of all soft sets over $Q$ with respect to the set of parameters $T$. In this paper, we consider soft sets over $Q$ that are defined concerning a fixed set of parameters.

**Definition 2.** *[37] Let $\tilde{\lambda} \in S(Q)_T$.*

*(1) If $\lambda(t) = \emptyset$ for all $t \in T$, then $\tilde{\lambda}$ is said to be a null soft set symbolized by $\tilde{\Phi}$.*

*(2) If $\lambda(t) = Q$ for all $t \in T$, then $\tilde{\lambda}$ is said to be an absolute soft set symbolized by $\tilde{Q}$.*

**Definition 3.** *[38] Let $\tilde{\lambda} \in S(Q)_T$. A soft complement of $\tilde{\lambda}$ is a soft set $\tilde{\lambda}^c \in S(Q)_T$ such that $\lambda^c(t) = \lambda(t)^c$ for all $t \in T$.*

**Definition 4.** *[12] Let $\tilde{\lambda}, \tilde{\mu} \in S(Q)_T$. Then, $\tilde{\lambda}$ is a soft subset of $\tilde{\mu}$, symbolized by $\tilde{\lambda} \tilde{\subseteq} \tilde{\mu}$, if $\lambda(t) \subseteq \mu(t)$ for all $t \in T$. The soft sets $\tilde{\lambda}$ and $\tilde{\mu}$ are soft equal if $\tilde{\lambda} \tilde{\subseteq} \tilde{\mu}$ and $\tilde{\mu} \tilde{\subseteq} \tilde{\lambda}$.*

**Definition 5.** *[38] Let $\tilde{\lambda}, \tilde{\mu} \in S(Q)_T$. The intersection of two soft sets $\tilde{\lambda}$ and $\tilde{\mu}$, symbolized by $\tilde{\lambda} \tilde{\cap} \tilde{\mu}$, is a soft set $\tilde{\upsilon} \in S(Q)_T$, such that $\upsilon(t) = \lambda(t) \cap \mu(t)$ for all $t \in T$.*

**Definition 6.** *[37] Let $\tilde{\lambda}, \tilde{\mu} \in S(Q)_T$. The union of two soft sets $\tilde{\lambda}$ and $\tilde{\mu}$, symbolized by $\tilde{\lambda} \tilde{\cup} \tilde{\mu}$, is a soft set $\tilde{\upsilon} \in S(Q)_T$, such that $\upsilon(t) = \lambda(t) \cup \mu(t)$ for all $t \in T$.*

**Proposition 1.** *[37] Let $\tilde{\lambda}, \tilde{\mu}$, and $\tilde{\upsilon} \in S(Q)_T$.*

*(1) $\tilde{\lambda} \tilde{\cup} \tilde{\Phi} = \tilde{\lambda}$ and $\tilde{\lambda} \tilde{\cap} \tilde{\Phi} = \tilde{\Phi}$.*
*(2) $\tilde{\lambda} \tilde{\cup} \tilde{Q} = \tilde{Q}$ and $\tilde{\lambda} \tilde{\cap} \tilde{Q} = \tilde{\lambda}$.*
*(3) $\tilde{\lambda} \tilde{\cup} \tilde{\lambda} = \tilde{\lambda}$ and $\tilde{\lambda} \tilde{\cap} \tilde{\lambda} = \tilde{\lambda}$.*
*(4) $\tilde{\lambda} \tilde{\cup} (\tilde{\mu} \tilde{\cap} \tilde{\upsilon}) = (\tilde{\lambda} \tilde{\cup} \tilde{\mu}) \tilde{\cap} (\tilde{\lambda} \tilde{\cup} \tilde{\upsilon})$.*
*(5) $\tilde{\lambda} \tilde{\cap} (\tilde{\mu} \tilde{\cup} \tilde{\upsilon}) = (\tilde{\lambda} \tilde{\cap} \tilde{\mu}) \tilde{\cup} (\tilde{\lambda} \tilde{\cap} \tilde{\upsilon})$.*

**Definition 7.** *[39] Let $T$ and $U$ be sets of parameters and $Q$ and $W$ be initial universe sets. Let $H : Q \to W$ and $L : T \to U$ be mappings. A soft mapping*

$$\Upsilon_{H,L} : S(Q)_T \to S(W)_U$$

*is defined by:*

*(1) The image of $\tilde{\lambda} \in S(Q)_T$ is a soft set in $S(W)_U$ such that for all $u \in U$,*

$$\Upsilon_{H,L}(\tilde{\lambda})(u) = \begin{cases} \bigcup\limits_{t \in L^{-1}(u)} H(\lambda(t)) & \text{if } L^{-1}(u) \neq \emptyset, \\ \emptyset & \text{if } L^{-1}(u) = \emptyset. \end{cases}$$

*(2) The inverse image of $\tilde{\mu} \in S(W)_U$, symbolized by $\Upsilon^{-1}_{H,L}(\tilde{\mu})$, is the soft set $\tilde{\lambda}$ such that*

$$\lambda(t) = H^{-1}(\mu(L(t)))$$

*for all $t \in T$.*

**Theorem 1.** *[39] Let $\Upsilon_{H,L} : S(Q)_T \to S(W)_U$ be a soft mapping.*

*(1) $\Upsilon_{H,L}(\tilde{\Phi}) = \tilde{\Phi} \in S(W)_U$ and $\Upsilon_{H,L}(\tilde{Q}) \tilde{\subseteq} \tilde{W}$.*
*(2) $\Upsilon^{-1}_{H,L}(\tilde{\Phi}) = \tilde{\Phi} \in S(Q)_T$ and $\Upsilon^{-1}_{H,L}(\tilde{W}) = \tilde{Q}$.*
*(3) $\Upsilon_{H,L}(\tilde{\lambda}_1 \tilde{\cup} \tilde{\lambda}_2) = \Upsilon_{H,L}(\tilde{\lambda}_1) \tilde{\cup} \Upsilon_{H,L}(\tilde{\lambda}_2)$, $\tilde{\lambda}_1, \tilde{\lambda}_2 \in S(Q)_T$.*
*(4) $\Upsilon_{H,L}(\tilde{\lambda}_1) \tilde{\cap} \Upsilon_{H,L}(\tilde{\lambda}_2) \tilde{\subseteq} \Upsilon_{H,L}(\tilde{\lambda}_1 \tilde{\cap} \tilde{\lambda}_2)$, $\tilde{\lambda}_1, \tilde{\lambda}_2 \in S(Q)_T$.*
*(5) $\Upsilon^{-1}_{H,L}(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) = \Upsilon^{-1}_{H,L}(\tilde{\mu}_1) \tilde{\cup} \Upsilon^{-1}_{H,L}(\tilde{\mu}_2)$, $\tilde{\mu}_1, \tilde{\mu}_2 \in S(W)_U$.*
*(6) $\Upsilon^{-1}_{H,L}(\tilde{\mu}_1) \tilde{\cap} \Upsilon^{-1}_{H,L}(\tilde{\mu}_2) = \Upsilon^{-1}_{H,L}(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2)$, $\tilde{\mu}_1, \tilde{\mu}_2 \in S(W)_U$.*

**Remark 1.** *[14] The soft mapping $\Upsilon_{H,L}$ is classified as injective (resp, surjective, bijective) based on whether the mappings $H$ and $L$ are injective (resp, surjective, bijective) mappings. The properties of $H$ and $L$ determine the corresponding properties on the map $\Upsilon_{H,L}$.*

**Theorem 2.** *[14] Let $\Upsilon_{H,L} : S(Q)_T \to S(W)_U$ be a soft mapping.*

*(1) $\Upsilon_{H,L} \circ \Upsilon^{-1}_{H,L}(\tilde{\mu}) \tilde{\subseteq} \tilde{\mu}$ for any $\tilde{\mu} \in S(W)_U$, and $\Upsilon_{H,L} \circ \Upsilon^{-1}_{H,L}(\tilde{\mu}) = \tilde{\mu}$ If $\Upsilon_{H,L}$ is surjective.*
*(2) $\tilde{\lambda} \tilde{\subseteq} \Upsilon^{-1}_{H,L} \circ \Upsilon_{H,L}(\tilde{\lambda})$ for any $\tilde{\lambda} \in S(Q)_T$, and $\tilde{\lambda} = \Upsilon^{-1}_{H,L} \circ \Upsilon_{H,L}(\tilde{\lambda})$ If $\Upsilon_{H,L}$ is injective.*

**Definition 8.** *[40, 41] Let $\tilde{\lambda} \in S(Q)_T$. Then, $\tilde{\lambda}$ is called a soft point, symbolized by $\tilde{\lambda}^q_t$, if there exist $t \in T$ and $q \in Q$ such that $\lambda(t) = \{q\}$ and $\lambda(r) = \emptyset$ for all $r \in T - \{t\}$. The set of all soft points over $Q$ is symbolized by $\Omega(Q)_T$.*

**Definition 9.** *[40] Let $\tilde{\mu} \in S(Q)_T$ and $\tilde{\lambda}^q_t \in \Omega(Q)_T$. If $q \in \mu(t)$, then we say that $\tilde{\lambda}^q_t$ belongs to the soft set $\tilde{\mu}$ and is symbolized by $\tilde{\lambda}^q_t \tilde{\in} \tilde{\mu}$.*

**Definition 10.** *[8] A subfamily $\tilde{\theta}$ of $S(Q)_T$ is referred to as a soft topology on $Q$ when it fulfills the following conditions:*

*(1) $\tilde{\theta}$ includes both $\tilde{Q}$ and $\tilde{\Phi}$.*
*(2) $\tilde{\theta}$ is closed under arbitrary unions.*
*(3) $\tilde{\theta}$ is closed under finite intersections.*

The pair $(Q, \tilde{\theta})_T$ will be referred to as a soft topological space (abbreviated as **STS**), wherein each element of $\tilde{\theta}$ is referred to as a soft open set, and its soft complement is termed as a soft closed set. When $\tilde{\theta} = \{\tilde{\Phi}, \tilde{Q}\}$, it is referred to as the soft indiscrete space; conversely, when $\tilde{\theta} = S(Q)_T$, it is known as the soft discrete space.

**Definition 11.** *[8, 13] Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\lambda} \in S(Q)_T$.*

*(1) The soft closure of $\tilde{\lambda}$ is the intersection of every soft closed sets which contain $\tilde{\lambda}$ and is symbolized by $Cl_{\tilde{\theta}}(\tilde{\lambda})$.*
*(2) The soft interior of $\tilde{\lambda}$ is the union of every soft open sets which are contained in $\tilde{\lambda}$ and is symbolized by $Int_{\tilde{\theta}}(\tilde{\lambda})$.*

**Proposition 2.** *[8] Let $(Q, \tilde{\theta})_T$ be an **STS**. For all $t \in T$, $\theta_t = \{\upsilon(t) : \tilde{\upsilon} \in \tilde{\theta}\}$ induces a topology on $Q$. This particular topology will be referred to as a parametric topology.*

**Definition 12.** *[8] Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\lambda} \in S(Q)_T$. Then,*

$$Cl(\tilde{\lambda}) = \{(t, Cl_{\theta_t}(\lambda(t)) : \text{for all } t \in T\},$$

*where $Cl_{\theta_t}(\lambda(t))$ is the closure of $\lambda(t)$ in $\theta_t$.*

**Proposition 3.** *[8] Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\lambda} \in S(Q)_T$. Then, $Cl(\tilde{\lambda}) \tilde{\subseteq} Cl_{\tilde{\theta}}(\tilde{\lambda})$, and the equality holds if and only if $Cl(\tilde{\lambda})$ is a soft closed set.*

**Definition 13.** *[21] Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\lambda} \in S(Q)_T$. Then, $\tilde{\lambda}$ is said to be a soft nowhere dense set if $Int_{\tilde{\theta}}(Cl_{\tilde{\theta}}(\tilde{\lambda})) = \tilde{\Phi}$.*

**Definition 14.** *[40] Let $(Q, \tilde{\theta}_1)_T$ and $(Q, \tilde{\theta}_2)_T$ be two soft topological spaces over $Q$. Then, $\tilde{\theta}_1$ is a soft finer than $\tilde{\theta}_2$ if $\tilde{\theta}_2 \subseteq \tilde{\theta}_1$.*

**Definition 15.** *[15] Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{B} \subseteq \tilde{\theta}$. Then, $\tilde{B}$ is said to be a soft base for $\tilde{\theta}$ if every element of $\tilde{\theta}$ is a union of elements of $\tilde{B}$.*

**Definition 16.** *[40] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS*** *and* $\tilde{\mu} \in S(Q)_T$*. Let* $\tilde{\lambda}_t^q \in \Omega(Q)_T$*. Then,* $\tilde{\mu}$ *is said to be a soft neighborhood of* $\tilde{\lambda}_t^q$ *if there exists a soft open set* $\tilde{\upsilon}$ *such that* $\tilde{\lambda}_t^q \tilde{\in} \tilde{\upsilon} \tilde{\subseteq} \tilde{\mu}$*. The set of all soft neighborhoods of* $\tilde{\lambda}_t^q$ *in* $\tilde{\theta}$ *will be symbolized by* $N_{\tilde{\theta}}(\tilde{\lambda}_t^q)$*.*

**Proposition 4.** *[40] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS*** *and* $\tilde{\lambda} \in S(Q)_T$*. Then,* $\tilde{\lambda} \in \tilde{\theta}$ *if, and only if,* $\tilde{\lambda}$ *is a soft neighborhood of all its soft points.*

**Definition 17.** *[9, 40] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS*** *and* $\tilde{\mu} \in S(Q)_T$*. Let* $\tilde{\lambda}_t^q \in \Omega(Q)_T$*. Then,* $\tilde{\lambda}_t^q$ *is said to be a soft limit point of the soft set* $\tilde{\mu}$ *if*

$$(\tilde{\upsilon} - \tilde{\lambda}_t^q) \tilde{\cap} \tilde{\mu} \neq \tilde{\Phi},$$

*for all* $\tilde{\upsilon} \in N_{\tilde{\theta}}(\tilde{\lambda}_t^q)$*, where* $(\tilde{\upsilon} - \tilde{\lambda}_t^q)$ *is a soft set defined by* $(\upsilon - \lambda_t^q)(t) = \upsilon(t) - \lambda_t^q(t)$ *for all* $t \in T$*. The union of all soft limit points of* $\tilde{\mu}$ *is called the derived soft set of* $\tilde{\mu}$ *and is symbolized by* $D_{\tilde{\theta}}(\tilde{\mu})$*.*

**Proposition 5.** *[40] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS*** *and* $\tilde{\lambda} \in S(Q)_T$*. Then,* $\tilde{\lambda}$ *is a soft closed set if, and only if, it contains all its soft limit points.*

**Proposition 6.** *[9, 40] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS****. Let* $\tilde{\lambda}, \tilde{\mu} \in S(Q)_T$*.*

*(1)* $D_{\tilde{\theta}}(\tilde{\Phi}) = \tilde{\Phi}$
*(2) If* $\tilde{\lambda} \tilde{\subseteq} \tilde{\mu}$*, then* $D_{\tilde{\theta}}(\tilde{\lambda}) \tilde{\subseteq} D_{\tilde{\theta}}(\tilde{\mu})$*.*
*(3)* $D_{\tilde{\theta}}(\tilde{\lambda} \tilde{\cup} \tilde{\mu}) = D_{\tilde{\theta}}(\tilde{\lambda}) \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu})$*.*
*(4)* $D_{\tilde{\theta}}(\tilde{\lambda} \tilde{\cap} \tilde{\mu}) \tilde{\subseteq} D_{\tilde{\theta}}(\tilde{\lambda}) \tilde{\cap} D_{\tilde{\theta}}(\tilde{\mu})$*.*

**Theorem 3.** *[9] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS*** *and* $\tilde{\mu} \in S(Q)_T$*. Then,* $\tilde{\mu} \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu})$ *is a soft closed set.*

**Theorem 4.** *[9] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS*** *and* $\tilde{\mu} \in S(Q)_T$*. Then,*

$$Cl_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\mu} \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu}).$$

**Definition 18.** *[8] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS*** *and* $Z$ *be a nonempty subset of* $Q$*. Then,*

$$\tilde{\theta}_Z = \{\tilde{\mu} : \tilde{\mu} = \tilde{\upsilon} \tilde{\cap} \tilde{Z} \text{ for all } \tilde{\upsilon} \in \tilde{\theta}\}$$

*is called a soft relative topology on* $Z$*, where* $\tilde{Z} = \{(t, Z) : \text{for all } t \in T\}$*.*

**Definition 19.** *[42] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS****. If there exist two soft open sets* $\tilde{\upsilon}_1$ *and* $\tilde{\upsilon}_2$ *such that*

$$\tilde{\upsilon}_1 \tilde{\cap} \tilde{\upsilon}_2 = \tilde{\Phi} \text{ and } \tilde{\upsilon}_1 \tilde{\cup} \tilde{\upsilon}_2 = \tilde{Q},$$

*then* $(Q, \tilde{\theta})_T$ *is said to be a soft disconnected space. Otherwise,* $(Q, \tilde{\theta})_T$ *is said to be a soft connected space.*

**Proposition 7.** *[40] Let* $(Q, \tilde{\theta})_T$ *be an* ***STS*** *and* $\theta_t$ *be a parametric topology on* $Q$ *as in Proposition 2. Then,*

$$\theta^* = \{\tilde{\lambda} : \lambda(t) \in \theta_t, \text{for all } t \in T\}$$

*is a soft topology on* $Q$ *and is called an extended soft topology.*

**Definition 20.** *[25] Let* $(Q, \theta)$ *be an ordinary topology on* $Q$*. Then,*

$$\tilde{B} = \{\tilde{\lambda}_{q,C} : \text{for all } q \in Q \text{ and } C \in \theta \text{ such that } q \in C\} \tilde{\cup} \tilde{\Phi}$$

*is a soft base of a soft topology on $Q$. This particular soft topology is said to be a soft topology generated by soft single points on $Q$ with respect to $\theta$ and is symbolized by* $(STGP)_{(Q,\theta)}$.

In the following, we present the main results that pertain to the soft mappings between two collections of soft sets.

**Definition 21.** *[17] Let $(Q, \tilde{\theta}_1)_T$ and $(W, \tilde{\theta}_2)_U$ be two **STS**s. Let*

$$\Upsilon_{H,L} : S(Q)_T \to S(W)_U$$

*be a soft mapping. Then,*

*(1) $\Upsilon_{H,L}$ is said to be a soft open if for all $\tilde{\lambda} \in \tilde{\theta}_1$, $\Upsilon_{H,L}(\tilde{\lambda}) \in \tilde{\theta}_2$.*
*(2) $\Upsilon_{H,L}$ is said to be a soft closed if for all soft closed $\tilde{\lambda}$ in $\tilde{\theta}_1$, $\Upsilon_{H,L}(\tilde{\lambda})$ is a soft closed in $\tilde{\theta}_2$.*

**Theorem 5.** *[14] Let $(Q, \tilde{\theta}_1)_T$ and $(W, \tilde{\theta}_2)_U$ be two **STS**s. A soft mapping*

$$\Upsilon_{H,L} : S(Q)_T \to S(W)_U$$

*is said to be a soft continuous if, and only if, for all $\tilde{\mu} \in \tilde{\theta}_2$, $\Upsilon^{-1}_{H,L}(\tilde{\mu}) \in \tilde{\theta}_1$.*

**Definition 22.** *[17] A soft mapping $\Upsilon_{H,L} : S(Q)_T \to S(W)_U$ is said to be a soft homeomorphism if $\Upsilon_{H,L}$ is a soft continuous, $\Upsilon_{H,L}$ is a bijective mapping, and $\Upsilon^{-1}_{H,L}$ is a soft continuous.*

**Proposition 8.** *[20] Let $\{(Q_i, \tilde{\theta}_i)_T$ be a family of pairwise disjoint **STS**s. Let $Q = \bigcup_{i \in I} Q_i$. Then,*

$$\tilde{\theta} = \{\tilde{\upsilon} : \tilde{\upsilon} \tilde{\cap} \tilde{Q}_i \in \theta_i \text{ for all } i \in I\},$$

*where $\tilde{Q}_i = \{(t, Q_i) : \text{for all } t \in T\}$ is a soft topology on $Q$ with the set of parameter $T$, and is called the sum soft topological spaces, is symbolized by $(\bigoplus_{i \in I} Q_i, \tilde{\theta})_T$.*

## 3. Methodology

By scrutinizing the concepts and properties of soft topological spaces, we can conduct a more comprehensive analysis of the characterization of soft sets, leading to a thorough exploration. A key aspect of this examination involves focusing on the local behavior of soft sets in relation to specific points, which provides valuable insights into their structure and relationships within the soft topological space. This acquired knowledge has the potential to advance the development of soft topological structures, propose new neighborhood structures for soft sets, and enhance our overall comprehension of soft set theory. In this study, we employ conventional and classical theories as our methods and strategies. Our approach encompasses the study and comparison of previous findings in the literature concerning limit points, isolated points, and dense sets (spaces) in themselves in ordinary topology. By drawing upon these previous results, our objective is to develop new theorems within the domain of soft topological spaces. Findings may have implications for applications in other fields, such as decision-making processes in uncertain environments where soft sets are utilized. This work will lay the groundwork for future studies, encouraging further exploration into soft topological spaces and their properties.

## 4. Results

In this section, we will present the key findings, which will be organized into two subsections. The first subsection focuses on introducing the concept of soft isolated points and some results about soft limit points, along with the findings derived from them. In the second subsection, we discuss the concepts of soft dense set in itself and soft dense space in itself, and examine their properties.

### *4.1. Soft limit points and soft isolated points*

This subsection presents the concept of a soft isolated point, providing an in-depth discussion of its characteristics.

Before we begin presenting our results, we draw attention to these important remarks that we will use in the proof of our results.

**Remark 2.** *Let $\tilde{\lambda}, \tilde{\mu} \in S(Q)_T$. Then, $\tilde{\lambda} \tilde{\subseteq} \tilde{\mu}$ if, and only if, all $\tilde{\lambda}_t^q \tilde{\in} \tilde{\lambda}$, $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$.*

**Remark 3.** *Directly from Definition 17, $\tilde{\lambda}_t^q$ is said to be a soft limit point of $\tilde{\mu}$ if, and only if, all soft open neighborhood of $\tilde{\lambda}_t^q$ intersects $\tilde{\mu}$ in some soft subsets other than $\tilde{\lambda}_t^q$.*

**Theorem 6.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in S(Q)_T$. Let*

$$D(\tilde{\mu}) = \{(t, D_{\theta_t}(\mu(t))) : \text{for all } t \in T\},$$

*where $D_{\theta_t}(\mu(t))$ is the derived set of $\mu(t)$ in the parametric topology $\theta_t$. Then, $D(\tilde{\mu}) \tilde{\subseteq} D_{\tilde{\theta}}(\tilde{\mu})$.*

*Proof.* Suppose that $\tilde{\lambda}_t^q \tilde{\in} D(\tilde{\mu})$. Then, $q \in D_{\theta_t}(\mu(t))$. Therefore, $(G - \{q\}) \cap \mu(t) \neq \emptyset$ for all open neighborhood $G$ of $q$ in $\theta_t$. We assume that $\tilde{\lambda}_t^q \tilde{\notin} D_{\tilde{\theta}}(\tilde{\mu})$. Then, by Theorem 3 there exists a soft open neighborhood $\tilde{\upsilon}$ of $\tilde{\lambda}_t^q$ such that

$$(\tilde{\upsilon} - \tilde{\lambda}_t^q) \tilde{\cap} \tilde{\mu} = \tilde{\Phi}.$$

Therefore,

$$(\upsilon(t) - \{q\}) \cap \mu(t) = \emptyset.$$

However, $\upsilon(t)$ is an open neighborhood of $q$ in $\theta_t$. It follows that $q \notin D_{\theta_t}(\mu(t))$. This is a contradiction since $\tilde{\lambda}_t^q \tilde{\in} D(\tilde{\mu})$. Hence, $D(\tilde{\mu}) \tilde{\subseteq} D_{\tilde{\theta}}(\tilde{\mu})$. □

The opposite of the theorem stated above is generally not true, as shown by the following example.

**Example 1.** *Consider $Q = \{q_1, q_2, q_3\}$, $T = \{t_1, t_2\}$, and $\tilde{\theta}$ to be the soft indiscrete space. Let*

$$\tilde{\mu}_1 = \{(t_1, \{q_1, q_2\}), (t_2, \{q_3\}).$$

*Then, $D_{\tilde{\theta}}(\tilde{\mu}_1) = \tilde{Q}$. We have $\theta_{t_1} = \theta_{t_2} = \{Q, \emptyset\}$. Therefore,*

$$D(\tilde{\mu}) = \{(t_1, Q\}), (t_2, \{q_1, q_2\})\}.$$

*Hence, $D(\tilde{\mu}) \tilde{\subset} D_{\tilde{\theta}}(\tilde{\mu})$.*

In the subsequent discussion, we will define the soft isolated points and explore their inherent characteristics.

**Definition 23.** *Let $(Q, \tilde{\theta})_T$ be an **STS**. Let $\tilde{\mu} \in S(Q)_T$ and $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$. Then, $\tilde{\lambda}_t^q$ is said to be a soft isolated point of $\tilde{\mu}$ if there exists a soft open set $\tilde{\upsilon}$ such that $\tilde{\upsilon} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q$. The union of all soft isolated points of $\tilde{\mu}$ is symbolized by $is_{\tilde{\theta}}(\tilde{\mu})$.*

**Remark 4.** *(1) For any soft topological space $(Q, \tilde{\theta})_T$, $\tilde{\lambda}_t^q$ is said to be a soft isolated point of $\tilde{\theta}$ if $\tilde{\lambda}_t^q \in \tilde{\theta}$, and the union of all soft isolated points of $\tilde{\theta}$ is symbolized by $is_{\tilde{\theta}}(\tilde{Q})$.*

*(2) If $is_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\mu}$, then $\tilde{\mu}$ is said to be a soft isolated set.*

**Example 2.** *Let $(Q, \tilde{\theta})$ be the soft discrete topology on $Q$. Then, for all $\tilde{\lambda}_t^q \in \Omega(Q)_T$, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{Q})$.*

**Example 3.** *Let $\mathbb{R}$ be the set of real numbers and $\mathcal{U}$ be the usual topology on $\mathbb{R}$. Let $T = \{a, b\}$ be a set of parameters. Then,*

$$\tilde{\theta} = \{\tilde{\lambda} : \tilde{\lambda} = \{(a, G), (b, G)\} : G \in \mathcal{U}\}$$

*is a soft topology on $\mathbb{R}$. It is clear that $is_{\tilde{\theta}}(\tilde{\mathbb{R}}) = \tilde{\Phi}$. Let $\tilde{\mu} = \{(a, \{2\}), (b, \{3\})\} \in S(\mathbb{R})_T$. Then, there exist two soft open sets of the form*

$$\tilde{\upsilon}_1 = \{(a, (0, 3), (b, (0, 3)\}, \tilde{\upsilon}_2 = \{(a, (2, 4), (b, (2, 4)\},$$

*and $\tilde{\mu} \tilde{\cap} \tilde{\upsilon}_1 = \tilde{\lambda}_a^2$, $\tilde{\mu} \tilde{\cap} \tilde{\upsilon}_2 = \tilde{\lambda}_b^3$. Therefore, $is_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\mu}$.*

**Proposition 9.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in S(Q)_T$. Then,*

*(1) $is_{\tilde{\theta}}(\tilde{\Phi}) = \tilde{\Phi}$,*

*(2) $is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} \tilde{\mu}$.*

*Proof.* The proof follows directly from the definition. □

**Theorem 7.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in S(Q)_T$. Then, $is_{\tilde{\theta}}(is_{\tilde{\theta}}(\tilde{\mu})) = is_{\tilde{\theta}}(\tilde{\mu})$.*

*Proof.* Suppose that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(is_{\tilde{\theta}}(\tilde{\mu}))$. It follows that there exists a soft open set $\tilde{\nu}$ such that

$$\tilde{\nu} \tilde{\cap} is_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\lambda}_t^q.$$

Therefore, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$. Hence,

$$is_{\tilde{\theta}}(is_{\tilde{\theta}}(\tilde{\mu})) \tilde{\subseteq} is_{\tilde{\theta}}(\tilde{\mu}).$$

Conversely, suppose that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$. Therefore, there exists a soft open set $\tilde{\nu}$ such that

$$\tilde{\nu} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q.$$

It follows that $\tilde{\lambda}_t^q \tilde{\in} \tilde{\nu}$. Since $is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} \tilde{\mu}$,

$$\tilde{\lambda}_t^q \tilde{\in} \tilde{\nu} \tilde{\cap} is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} \tilde{\nu} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q.$$

Thus, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(is_{\tilde{\theta}}(\tilde{\mu}))$. Hence,

$$is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} is_{\tilde{\theta}}(is_{\tilde{\theta}}(\tilde{\mu})).$$

Consequently, the theorem's proof has been completed. □

**Theorem 8.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\mu}_1, \tilde{\mu}_2 \in S(Q)_T$. Then,*

*(1) $is_{\tilde{\theta}}(\tilde{\mu}_1) \tilde{\cap} is_{\tilde{\theta}}(\tilde{\mu}_2) \tilde{\subseteq} is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2)$.*

*(2)* $is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) \tilde{\subseteq} is_{\tilde{\theta}}(\tilde{\mu}_1) \tilde{\cup} is_{\tilde{\theta}}(\tilde{\mu}_2)$.

*Proof.* (1) Suppose that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_1) \tilde{\cap} is_{\tilde{\theta}}(\tilde{\mu}_2)$. Therefore, there exist two soft open sets $\tilde{\upsilon}_1$ and $\tilde{\upsilon}_2$ such that

$$\tilde{\upsilon}_1 \tilde{\cap} \tilde{\mu}_1 = \tilde{\lambda}_t^q \text{ and } \tilde{\upsilon}_2 \tilde{\cap} \tilde{\mu}_2 = \tilde{\lambda}_t^q.$$

Then, $\tilde{\lambda}_t^q \tilde{\in} \tilde{\upsilon}_1 \tilde{\cap} \tilde{\upsilon}_2$ and $\tilde{\upsilon}_1 \tilde{\cap} \tilde{\upsilon}_2 \in \tilde{\theta}$. Thus,

$$(\tilde{\upsilon}_1 \tilde{\cap} \tilde{\upsilon}_2) \tilde{\cap} (\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) = \tilde{\lambda}_t^q.$$

Hence, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2)$. This means that

$$is_{\tilde{\theta}}(\tilde{\mu}_1) \tilde{\cap} is_{\tilde{\theta}}(\tilde{\mu}_2) \tilde{\subseteq} is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2).$$

(2) Suppose that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2)$. Therefore, there exists a soft open set $\tilde{\upsilon}$ such that

$$\tilde{\upsilon} \tilde{\cap} (\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) = \tilde{\lambda}_t^q.$$

It follows that

$$\tilde{\upsilon} \tilde{\cap} \tilde{\mu}_1 = \tilde{\lambda}_t^q \text{ or } \tilde{\upsilon} \tilde{\cap} \tilde{\mu}_2 = \tilde{\lambda}_t^q.$$

Therefore, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_1)$ or $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_2)$. Thus, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_1) \tilde{\cup} is_{\tilde{\theta}}(\tilde{\mu}_2)$. Hence,

$$is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) \tilde{\subseteq} is_{\tilde{\theta}}(\tilde{\mu}_1) \tilde{\cup} is_{\tilde{\theta}}(\tilde{\mu}_2).$$

□

The subsequent example illustrates that the opposite of the above theorem generally does not hold.

**Example 4.** *Let* $Q = \{q_1, q_2, q_3, q_4\}$ *and* $T = \{t_1, t_2\}$. *Let* $\tilde{\theta} = \{\tilde{Q}, \tilde{\Phi}, \tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3, \}$ *be a soft topology on* $Q$, *where*

$$\tilde{\nu}_1 = \{(t_1, \{q_1\}), (t_2, \{q_3\})\}, \quad \tilde{\nu}_2 = \{(t_1, \{q_2\}), (t_2, \{q_1\})\},$$

*and*

$$\tilde{\nu}_3 = \{(t_1, \{q_1, q_2\}), (t_2, \{q_1, q_3\})\}.$$

*Let* $\tilde{\mu}_1, \tilde{\mu}_1 \in S(Q)_T$, *where*

$$\tilde{\mu}_1 = \{(t_1, \{q_1, q_2\}), (t_2, \{q_3\})\} \text{ and } \tilde{\mu}_2 = \{(t_1, \{q_2\}), (t_2, \{q_3, q_4\})\}.$$

*Then,*

$$\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2 = \{(t_1, \{q_2\}), (t_2, \{q_3\})\} \text{ and } \tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2 = \{(t_1, \{q_1, q_2\}), (t_2, \{q_3, q_4\})\}.$$

*Therefore,*

$$is_{\tilde{\theta}}(\tilde{\mu}_1) = \{(t_1, \{q_2\}), (t_2, \emptyset)\}, \; is_{\tilde{\theta}}(\tilde{\mu}_2) = \{(t_1, \{q_2\}), (t_2, \{q_3\})\},$$

$$is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) = \{(t_1, \{q_2\}), (t_2, \{q_3\})\} \text{ and } is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) = \{(t_1, \{q_2\}), (t_2, \emptyset)\}.$$

*Hence,*

$$is_{\tilde{\theta}}(\tilde{\lambda}) \tilde{\cap} is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subset} is_{\tilde{\theta}}(\tilde{\lambda} \tilde{\cap} \tilde{\mu}) \text{ and } is_{\tilde{\theta}}(\tilde{\lambda} \tilde{\cup} \tilde{\mu}) \tilde{\subset} is_{\tilde{\theta}}(\tilde{\lambda}) \tilde{\cup} is_{\tilde{\theta}}(\tilde{\mu}).$$

**Theorem 9.** *Let $(Q,\tilde{\theta})_T$ be an **STS**. Let $\tilde{\mu} \in S(Q)_T$ and $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$. Then, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$ if and only if $\tilde{\lambda}_t^q \tilde{\notin} D_{\tilde{\theta}}(\tilde{\mu})$.*

*Proof. Necessity.* Suppose that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$. Then, there exists a soft open set $\tilde{\upsilon}$ such that $\tilde{\upsilon} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q$. Therefore, $\tilde{\lambda}_t^q \tilde{\in} \tilde{\upsilon}$ and $(\tilde{\upsilon} - \tilde{\lambda}_t^q) \tilde{\cap} \tilde{\mu} = \tilde{\Phi}$. By Remark 3, $\tilde{\lambda}_t^q \tilde{\notin} D_{\tilde{\theta}}(\tilde{\mu})$.

*Sufficiency.* Suppose that $\tilde{\lambda}_t^q \tilde{\notin} D_{\tilde{\theta}}(\tilde{\mu})$. Then, there exists a soft open set $\tilde{\upsilon}$ such that $(\tilde{\upsilon} - \tilde{\lambda}_t^q) \tilde{\cap} \tilde{\mu} = \tilde{\Phi}$. Since $\tilde{\lambda}_t^q \tilde{\in} \tilde{\upsilon}$ and $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$, then $\tilde{\upsilon} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q$. Hence, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$. □

**Corollary 1.** *Let $(Q,\tilde{\theta})_T$ be an **STS** and $\tilde{\lambda} \in S(Q)_T$. If $is_{\tilde{\theta}}(\tilde{\lambda}) = \tilde{\Phi}$, then $\tilde{\lambda} \tilde{\subseteq} D_{\tilde{\theta}}(\tilde{\lambda})$.*

**Theorem 10.** *Let $(Q,\tilde{\theta})_T$ be an **STS**. Let $\tilde{\mu} \in S(Q)_T$ and $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$. If $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(Cl_{\tilde{\theta}}(\tilde{\mu}))$, then $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$.*

*Proof.* Suppose that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(Cl_{\tilde{\theta}}(\tilde{\mu}))$. We assume that $\tilde{\lambda}_t^q \tilde{\notin} is_{\tilde{\theta}}(\tilde{\mu})$. By Theorem 9, $\tilde{\lambda}_t^q \tilde{\in} D_{\tilde{\theta}}(\tilde{\mu})$. By Proposition 6, $D_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} D_{\tilde{\theta}}(Cl_{\tilde{\theta}}(\tilde{\mu}))$ since $\tilde{\mu} \tilde{\subseteq} Cl_{\tilde{\theta}}(\tilde{\mu})$. Then, $\tilde{\lambda}_t^q \tilde{\in} D_{\tilde{\theta}}(Cl_{\tilde{\theta}}(\tilde{\mu}))$. By Theorem 9, $\tilde{\lambda}_t^q \tilde{\notin} is_{\tilde{\theta}}(Cl_{\tilde{\theta}}(\tilde{\mu}))$. This contradicts the assumption that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(Cl_{\tilde{\theta}}(\tilde{\mu}))$. Hence, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$. □

**Theorem 11.** *Let $(Q,\tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in \tilde{\theta}$. Then, $is_{\tilde{\theta}}(\tilde{\mu}) \in \tilde{\theta}$.*

*Proof.* Let $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$. Then, there exits a soft open set $\tilde{\upsilon}$ such that $\tilde{\upsilon} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q$. Therefore, $\tilde{\lambda}_t^q \in \tilde{\theta}$ since it is the intersection of two soft open sets. Thus, $is_{\tilde{\theta}}(\tilde{\mu})$ is a union of soft open sets. □

The subsequent theorem provides a clear depiction of the correlation between $Cl_{\tilde{\theta}}(\tilde{\mu})$, $is_{\tilde{\theta}}(\tilde{\mu})$, and $D_{\tilde{\theta}}(\tilde{\mu})$ for any soft set $\tilde{\mu} \in S(Q)_T$.

**Theorem 12.** *Let $(Q,\tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in S(Q)_T$. Then,*

$$Cl_{\tilde{\theta}}(\tilde{\mu}) = is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu}).$$

*Proof. Necessity.* Suppose that $\tilde{\lambda}_t^q \tilde{\in} Cl_{\tilde{\theta}}(\tilde{\mu})$. We assume that $\tilde{\lambda}_t^q \tilde{\notin} is_{\tilde{\theta}}(\tilde{\mu})$. Then, we have two cases

- If $\tilde{\lambda}_t^q \tilde{\notin} \tilde{\mu}$, then by Theorem 4, $\tilde{\lambda}_t^q \tilde{\in} D_{\tilde{\theta}}(\tilde{\mu})$ since $Cl_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\mu} \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu})$.
- If $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$, then by Theorem 9, $\tilde{\lambda}_t^q \tilde{\in} D_{\tilde{\theta}}(\tilde{\mu})$.

Hence,

$$Cl_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu}).$$

*Sufficiency.* Suppose that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu})$. Therefore,

- If $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$, then $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$ since $is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} \tilde{\mu}$. Thus, $\tilde{\lambda}_t^q \tilde{\in} Cl_{\tilde{\theta}}(\tilde{\mu})$.
- If $\tilde{\lambda}_t^q \tilde{\in} D_{\tilde{\theta}}(\tilde{\mu})$, then by Theorem 4, $\tilde{\lambda}_t^q \tilde{\in} Cl_{\tilde{\theta}}(\tilde{\mu})$.

Hence,

$$is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} Cl_{\tilde{\theta}}(\tilde{\mu}).$$

This means that $Cl_{\tilde{\theta}}(\tilde{\mu}) = is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu})$. □

**Theorem 13.** *Let $(Q,\tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in S(Q)_T$. We define a soft set*

$$is(\tilde{\mu}) = \{(t, is_{\theta_t}(\mu(t))) : for\ all\ t \in T\},$$

*where $is_{\theta_t}(\mu(t))$ is the set of all isolated points of $\mu(t)$ in $\theta_t$. Then, $is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} is(\tilde{\mu})$.*

*Proof.* Suppose that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$. Therefore, there exists a soft open set $\tilde{\upsilon}$ such that $\tilde{\upsilon} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q$. Then, $\upsilon(t) \cap \mu(t) = \{q\}$. Since $\upsilon(t) \in \theta_t$, then $q \in is_{\theta_t}(\mu(t))$. Thus, $\tilde{\lambda}_t^q \tilde{\in} is(\tilde{\mu})$. □

As exemplified by the next example, the reverse of the above theorem is generally not true.

**Example 5.** *Consider $Q, T, \tilde{\mu}_1$, and $\tilde{\theta}$ as in Example 4. Then,*

$$\theta_{t_1} = \{Q, \emptyset, \{q_1\}, \{q_2\}, \{q_1, q_2\}\} \text{ and } \theta_{t_2} = \{Q, \emptyset, \{q_1\}, \{q_3\}, \{q_1, q_3\}\}.$$

*Therefore,*

$$is(\tilde{\mu}_1) = \{(t_1, \{q_1, q_2\}), (t_2, \{q_3\})\}.$$

*Hence, $is_{\tilde{\theta}}(\tilde{\mu}_1) \tilde{\subset} i(\tilde{\mu}_1)$.*

**Theorem 14.** *Let $(Q_1, \tilde{\theta}_1)_T$ and $(Q_2, \tilde{\theta}_2)_T$ be two disjoint **STS**s. Let $(\underset{i \in I}{\oplus} Q_i, \tilde{\theta})_T$ be a soft topology on $Q = \underset{i \in I}{\cup} Q_i$, where $I = \{1, 2\}$. Assume that $\tilde{\mu}_1 \in S(Q_1)_T$ and $\tilde{\mu}_2 \in S(Q_2)_T$. Then,*

$$is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) = is_{\tilde{\theta}_1}(\tilde{\mu}_1) \tilde{\cup} is_{\tilde{\theta}_2}(\tilde{\mu}_2).$$

*Proof.* Suppose that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2)$. Then, there exists $\tilde{\upsilon} \in \tilde{\theta}$ such that $\tilde{\upsilon} \tilde{\cap} (\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) = \tilde{\lambda}_t^q$. Therefore,

$$\tilde{\upsilon} \tilde{\cap} \tilde{\mu}_1 = \tilde{\lambda}_t^q \text{ or } \tilde{\upsilon} \tilde{\cap} \tilde{\mu}_2 = \tilde{\lambda}_t^q.$$

By Proposition 8, $\tilde{\upsilon} = \{(t, \upsilon(t) \cap Q_i) : \text{for all } t \in T\}$ and it is a soft open set in $\tilde{\theta}_i$, $i \in I$. It follows that

$$(\tilde{\upsilon} \tilde{\cap} \tilde{Q}_1) \tilde{\cap} \tilde{\mu}_1 = \tilde{\lambda}_t^q \text{ or } (\tilde{\upsilon} \tilde{\cap} \tilde{Q}_2) \tilde{\cap} \tilde{\mu}_2 = \tilde{\lambda}_t^q.$$

Since $Q_1$ and $Q_2$ are disjoint sets, there exists $i \in I$ such that $\mu_i(t) \subseteq Q_i$. We assume that

$$(\tilde{\upsilon} \tilde{\cap} \tilde{Q}_1) \tilde{\cap} \tilde{\mu}_1 = \tilde{\lambda}_t^q$$

and $(\tilde{\upsilon} \tilde{\cap} \tilde{Q}_1)$ is a soft open set in $\tilde{\theta}_1$. This means that $\tilde{\lambda}_t^q \in is_{\tilde{\theta}_1}(\tilde{\mu}_1)$. Thus,

$$\tilde{\lambda}_t^q \in is_{\tilde{\theta}_1}(\tilde{\mu}_1) \tilde{\cup} is_{\tilde{\theta}_2}(\tilde{\mu}_2).$$

Now, let $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}_1}(\tilde{\mu}_1) \tilde{\cup} is_{\tilde{\theta}_2}(\tilde{\mu}_2)$. Since $Q_1$ and $Q_2$ are disjoint sets, then $\tilde{\mu}_1$ and $\tilde{\mu}_1$ are soft disjoint sets. So, we assume that $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}_1}(\tilde{\mu}_1)$ and $\tilde{\lambda}_t^q \tilde{\notin} is_{\tilde{\theta}_2}(\tilde{\mu}_2)$. Then, there exists a soft open set $\tilde{\nu} \in \tilde{\theta}_1$ such that $\tilde{\nu} \tilde{\cap} \tilde{\mu}_1 = \tilde{\lambda}_t^q$. Since

$$\tilde{\nu} \tilde{\cap} \tilde{Q}_1 = \tilde{\nu} \in \tilde{\theta}_1 \text{ and } \tilde{\nu} \tilde{\cap} \tilde{Q}_2 = \tilde{\Phi} \in \tilde{\theta}_2,$$

then $\tilde{\nu} \in \tilde{\theta}$. Therefore, $\tilde{\nu} \tilde{\cap} (\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) = \tilde{\lambda}_t^q$. Hence,

$$\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2).$$

Thus, $is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2) = is_{\tilde{\theta}_1}(\tilde{\mu}_1) \tilde{\cup} is_{\tilde{\theta}_2}(\tilde{\mu}_2)$. □

The upcoming results present theorems that are specifically linked to soft isolated sets.

**Proposition 10.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in S(Q)_T$. Then, $\tilde{\mu}$ is a soft isolated set if an only if $\tilde{\mu} \tilde{\cap} D_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\Phi}$.*

*Proof.* If $\tilde{\mu}$ is a soft isolated set, then $is_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\mu}$. Since $is_{\tilde{\theta}}(\tilde{\mu}) \tilde{\subseteq} \tilde{\mu}$, then by Theorem 9, $\tilde{\mu} \tilde{\cap} D_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\Phi}$. □

**Theorem 15.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in S(Q)_T$. If $\tilde{\mu}$ is a soft isolated set, then $\mu(t)$ is an isolated set in $\theta_t$ for all $t \in T$.*

*Proof.* Suppose that $\tilde{\mu}$ is a soft isolated set in $\tilde{\theta}$. Then, $is_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\mu}$. From Theorem 13, $is(\tilde{\mu}) = \tilde{\mu}$. This means that $\mu(t)$ is an isolated set in $\theta_t$ for all $t \in T$. □

The other direction holds in a specific soft topology as stated in the following:

**Theorem 16.** *Let $(Q, \tilde{\theta})_T$ be an extended soft topological space. Let $\tilde{\mu} \in S(Q)_T$ such that $\mu(t)$ is an isolated set in $\theta_t$ for all $t \in T$. Then, $\tilde{\mu}$ is a soft isolated set.*

*Proof.* Suppose that $\mu(t)$ is an isolated set in $\theta_t$ for all $t \in T$. Then, for all $t \in T$, there exists $\upsilon_t \in \theta_t$ such that $\upsilon_t \cap \mu(t) = \{q\}$ for all $q \in \mu(t)$. Since $\tilde{\theta}$ is an extended soft topological space, then $\tilde{\upsilon} = \{(t, \upsilon_t) : \text{for all } t \in T\}$ is a soft open set. Thus, for all $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$, $\tilde{\upsilon} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q$, and it is a soft isolated set. □

**Theorem 17.** *Let $(Q_1, \tilde{\theta}_1)_T$ and $(Q_2, \tilde{\theta}_2)_T$ be two disjoint **STS**s. Let $(\underset{i \in I}{\oplus} Q_i, \tilde{\theta})_T$ be a soft topology on $Q = \underset{i \in I}{\cup} Q_i$, where $I = \{1, 2\}$. Assume that $\tilde{\mu}_1, \in S(Q_1)_T$ and $\tilde{\mu}_2, \in S(Q_2)_T$. Then, $\tilde{\mu}_1$ and $\tilde{\mu}_1$ are soft isolated sets in $\tilde{\theta}_1$ and $\tilde{\theta}_2$, respectively if and only if $\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2$ is a soft isolated set in $\tilde{\theta}$.*

*Proof.* It is obvious from Theorem 14. □

### *4.2. Soft dense sets (spaces) in themselves*

In this subsection, we focus on establishing the definitions of soft dense sets ( spaces ) in themselves. Furthermore, we thoroughly investigate several properties associated with them.

**Definition 24.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in S(Q)_T$. Then, $\tilde{\mu}$ is said to be a soft dense set in itself, symbolized by **SDS**, if $is_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\Phi}$.*

**Remark 5.** *$(Q, \tilde{\theta})_T$ is said to be a soft dense space in itself, symbolized by **SDTS**, if $\tilde{\lambda}_t^q \notin \tilde{\theta}$ for all $\tilde{\lambda}_t^q \in \Omega(Q)_T$.*

**Remark 6.** *For any soft topological space, it is clear that the soft set $\tilde{\lambda}_t^q$ is never to be a soft dense set in itself since there exists at least one soft open set $\tilde{\upsilon}$ such that $\tilde{\upsilon} \tilde{\cap} \tilde{\lambda}_t^q = \tilde{\lambda}_t^q$. Then, $\tilde{\lambda}_t^q \in is_{\tilde{\theta}}(\tilde{\lambda}_t^q)$.*

**Example 6.** *Let $\theta$ be any ordinary topology on $Q$ and $\tilde{\theta}$ be a soft topology generated by soft single points on $Q$ as illustrated in Definition 20. If $\theta$ is a dense space in itself ($\{q\} \notin \theta$) for all $q \in Q$, then $\tilde{\theta}$ is an **SDTS**.*

**Example 7.** *Let $(Q, \tilde{\theta})_T$ be a soft discrete topology on $Q$. It is clear that $(Q, \tilde{\theta})_T$ is not an **SDTS**.*

**Theorem 18.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $Z \subseteq Q$. Then, $\tilde{Z}$ is an **SDS** in $\tilde{\theta}$ if, and only if, $(Z, \tilde{\theta}_Z)_T$ is an **SDTS**.*

*Proof. Necessity.* Suppose that $\tilde{Z}$ is an **SDS** in $\tilde{\theta}$. Then, $is_{\tilde{\theta}}(\tilde{Z}) = \tilde{\Phi}$. We assume that $(Z, \tilde{\theta}_Z)_T$ is not an **SDTS**. Then, there exists at least $\tilde{\lambda}_t^q \in \Omega(Z)_T$ such that $\tilde{\lambda}_t^q \in \tilde{\theta}_Z$. Therefore, there exists a soft open set $\tilde{\upsilon}$ in $\tilde{\theta}$ such that $\tilde{\lambda}_t^q = \tilde{\upsilon} \tilde{\cap} \tilde{Z}$. Thus, $\tilde{\lambda}_t^q \in is_{\tilde{\theta}}(\tilde{Z})$. This is a contradiction. Hence, $(Z, \tilde{\theta}_Z)_T$ is an **SDTS**.

*Sufficiency.* Suppose that $(Z, \tilde{\theta}_Z)_T$ is an **SDTS**. Then, $\tilde{\lambda}_t^q \notin \tilde{\theta}_Z$ for all $\tilde{\lambda}_t^q \in \Omega(Z)_T$. We assume that $\tilde{Z}$ is not a soft dense set in itself in $\tilde{\theta}$ and $\tilde{\lambda}_t^q \in is_{\tilde{\theta}}(\tilde{Z})$. Thus, there exists a soft open set $\tilde{\upsilon}$ such that $\tilde{\lambda}_t^q = \tilde{\upsilon} \tilde{\cap} \tilde{Z}$. However, $\tilde{\upsilon} \tilde{\cap} \tilde{Z} \in \tilde{\theta}_Z$. This is a contradiction. Hence, $\tilde{Z}$ must be an **SDS** in $\tilde{\theta}$. □

**Theorem 19.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\mu} \in S(Q)_T$. Then, $\tilde{\mu} \tilde{\subseteq} D_{\tilde{\theta}}(\tilde{\mu})$ if and only if $\tilde{\mu}$ is an **SDS**.*

*Proof. Necessity.* Suppose that $\tilde{\mu} \tilde{\subseteq} D_{\tilde{\theta}}(\tilde{\mu})$. By Theorem 9, $\tilde{\lambda}_t^q \tilde{\notin} is_{\tilde{\theta}}(\tilde{\mu})$ for all $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$. Hence, $\tilde{\mu}$ is an **SDS**.

*Sufficiency.* Suppose that $\tilde{\lambda}$ is an **SDS**. Therefore, $is_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\Phi}$. By Theorem 9, $\tilde{\lambda}_t^q \tilde{\in} D_{\tilde{\theta}}(\tilde{\mu})$ for all $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$. This means that $\tilde{\mu} \tilde{\subseteq} D_{\tilde{\theta}}(\tilde{\mu})$. □

**Theorem 20.** *Let $(Q, \tilde{\theta})_T$ be an **STS**. Let $\tilde{\lambda}, \tilde{\mu} \in S(Q)_T$. If $\tilde{\lambda}$ and $\tilde{\mu}$ are **SDSs**, then $\tilde{\lambda} \tilde{\cup} \tilde{\mu}$ is an **SDS**.*

*Proof.* Suppose that $\tilde{\lambda}$ and $\tilde{\mu}$ are **SDS**s. Then, $is_{\tilde{\theta}}(\tilde{\lambda}) = \tilde{\Phi}$ and $is_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\Phi}$. By Theorem 8, $is_{\tilde{\theta}}(\tilde{\lambda} \tilde{\cup} \tilde{\mu}) \tilde{\subseteq} \tilde{\Phi}$. Hence, $\tilde{\lambda} \tilde{\cup} \tilde{\mu}$ is an **SDS**. □

It is illustrated by the next instance that the intersection of two soft dense sets in themselves may not yield a soft dense in itself.

**Example 8.** *Consider $Q, T$, and $\tilde{\theta}$ as in Example 4. Let*

$$\tilde{\mu}_1 = \{(t_1, \{q_1\}), (t_2, \{q_3, q_4\})\}, \tilde{\mu}_2 = \{(t_1, \{q_2\}), (t_2, \{q_1, q_4\})\}.$$

*Then,*

$$is_{\tilde{\theta}}(\tilde{\mu}_1) = is_{\theta}(\tilde{\mu}_2) = \tilde{\Phi} \text{ and } is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) = \{(t_1, \emptyset), (t_2, \{q_4\})\} \neq \tilde{\Phi}.$$

**Theorem 21.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\lambda} \in S(Q)_T$. If $\tilde{\lambda}$ is an **SDS**, then $Cl_{\tilde{\theta}}(\tilde{\lambda})$ is also an **SDS**.*

*Proof.* It follows from Theorem 10. □

**Theorem 22.** *Let $(Q, \tilde{\theta})_T$ be an **STS** and $\tilde{\mu}_1, \tilde{\mu}_2 \in S(Q)_T$. If $\tilde{\mu}_1$ is an **SDS** and $\tilde{\mu}_2$ is a soft open set, then $\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2$ is an **SDS**.*

*Proof.* We assume that $\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2$ is not a soft dense set in itself. Let $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2)$. Then, there exists a soft open set $\tilde{\upsilon}$ such that $\tilde{\upsilon} \tilde{\cap} (\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2) = \tilde{\lambda}_t^q$. Therefore, $(\tilde{\upsilon} \tilde{\cap} \tilde{\mu}_2) \tilde{\cap} \tilde{\mu}_1 = \tilde{\lambda}_t^q$, and $\tilde{\upsilon} \tilde{\cap} \tilde{\mu}_2$ is a soft open set. Thus, $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu}_1)$. This contradicts the assumption that $\tilde{\mu}_1$ is an **SDS**. Hence, $\tilde{\mu}_1 \tilde{\cap} \tilde{\mu}_2$ is an **SDS**. □

**Theorem 23.** *Let $(Q, \tilde{\theta})_T$ be an **SDTS**. Then, every soft open set is an **SDS**.*

*Proof.* Suppose that $(Q, \tilde{\theta})_T$ is an **SDTS**. We assume that $\tilde{\mu} \tilde{\in} \tilde{\theta}$ and it is not an **SDS**. Let $\tilde{\lambda}_t^q \tilde{\in} is_{\tilde{\theta}}(\tilde{\mu})$. Then, there exists a soft open set $\tilde{\upsilon}$ such that $\tilde{\upsilon} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q$. Therefore, $\tilde{\lambda}_t^q \tilde{\in} \tilde{\theta}$ since it is an intersection of two soft open sets. This contradicts the assumption that $(Q, \tilde{\theta})_T$ is **SDTS**. Thus, $\tilde{\mu}$ must be an **SDS**. □

**Proposition 11.** *Let $(Q, \tilde{\theta})_T$ be an **SDTS**. Then, every soft dense set in itself is a soft nowhere dense set.*

*Proof.* Suppose that $\tilde{\mu}$ is **SDS**. Then, $is_{\tilde{\theta}}(\tilde{\mu}) = \tilde{\Phi}$. By Theorem 21, $is_{\tilde{\theta}}(Cl_{\tilde{\theta}}(\tilde{\lambda})) = \tilde{\Phi}$. Therefore, $Int_{\tilde{\theta}}(Cl_{\tilde{\theta}}(\tilde{\lambda})) = \tilde{\Phi}$. This means that $\tilde{\mu}$ is a soft nowhere dense set. □

**Theorem 24.** *Let $(Q, \tilde{\theta})_T$ be an **STS**. Then, $(Q, \tilde{\theta})_T$ is an **SDTS** if, and only if, $\theta_t$ is a dense space in itself for all $t \in T$.*

*Proof.* Suppose that $(Q, \tilde{\theta})_T$ is an **SDTS**. Then, $\tilde{\lambda}_t^q \tilde{\notin} \tilde{\theta}$ for all $\tilde{\lambda}_t^q \tilde{\in} \Omega(Q)_T$. Thus, $\{q\} \notin \theta_t$ for all $t \in T$ and for all $q \in Q$. Hence, $\theta_t$ is a space in itself for all $t \in T$. With the same idea, we can prove the other direction. □

**Theorem 25.** *Let $(Q, \tilde{\theta})_T$ be a soft connected topological space. Suppose that $\tilde{\lambda}_t^q \in \Omega(Q)_T$ is a soft closed set for all $q \in Q$. Then, $(Q, \tilde{\theta})_T$ is an **SDTS**.*

*Proof.* Suppose that $(Q, \tilde{\theta})_T$ is a soft connected topological space and $\tilde{\lambda}_t^q \in \Omega(Q)_T$ is a soft closed set for all $q \in Q$. We assume that $(Q, \tilde{\theta})_T$ is not an **SDTS**. Then, there exists at least $q \in Q$ such that $\tilde{\lambda}_t^q \in \tilde{\theta}$. Since $\tilde{\lambda}_t^q$ is a soft closed set, $(\tilde{\lambda}_t^q)^c$ is a soft open set. Therefore, $\tilde{\lambda}_t^q$ and $(\tilde{\lambda}_t^q)^c$ are soft disjoint open sets and their union equals to $\tilde{Q}$. This contradicts the assumption that $(Q, \tilde{\theta})_T$ is a soft connected topological space. Hence, $(Q, \tilde{\theta})_T$ must be an **SDTS**. □

The forthcoming findings demonstrate that density exhibits a hereditary property under certain conditions and is preserved when examining a coarser soft topology.

**Theorem 26.** *Let $(Q, \tilde{\theta})_T$ be an **SDTS**. Let $Z \subseteq Q$, and $\tilde{Z}$ is a soft open set. Then, $(Z, \tilde{\theta}_Z)_T$ is an **SDTS**.*

*Proof.* Suppose that $(Q, \tilde{\theta})_T$ is an **SDTS** and $\tilde{Z}$ is a soft open set. We assume that $(Z, \tilde{\theta}_Z)_T$ is not is an **SDTS**. Then, there exists at least $q \in Z$ such that $\tilde{\lambda}_t^q \in \tilde{\theta}_Z$. It follows that there exists a soft open set $\tilde{\upsilon} \in \tilde{\theta}$ such that $\tilde{\lambda}_t^q = \tilde{\upsilon} \tilde{\cap} \tilde{Z}$. Therefore, $\tilde{\upsilon} \tilde{\cap} \tilde{Z} \in \tilde{\theta}$ since it is an intersection of two soft open sets. This is a contradiction since $(Q, \tilde{\theta})_T$ is an **SDTS**. Hence, $(Z, \tilde{\theta}_Z)_T$ is an **SDTS**. □

**Proposition 12.** *Let $(Q, \tilde{\theta}_1)_T$ and $(Q, \tilde{\theta}_2)_T$ be two soft topological spaces such that $(Q, \tilde{\theta}_1)_T$ is finer or strictly finer than $(Q, \tilde{\theta}_2)_T$. If $(Q, \tilde{\theta}_1)_T$ is an **SDTS**, then $(Q, \tilde{\theta}_2)_T$ must be an **SDTS**.*

*Proof.* It is straightforward. □

The upcoming results demonstrate that both the image and the inverse image of soft dense spaces in themselves can be considered soft dense spaces in themselves, subject to certain conditions.

**Theorem 27.** *Let $(Q, \tilde{\theta}_1)_T$ and $(W, \tilde{\theta}_2)_U$ be two **STSs**. Let $\Upsilon_{H,L} : S(Q)_T \to S(W)_U$ be a soft continuous mapping, where $H : Q \to W$ is a bijective mapping and $L : T \to U$ is a mapping. If $(Q, \tilde{\theta}_1)_T$ is an **SDTS**, then $(W, \tilde{\theta}_2)_U$ is an **SDTS**.*

*Proof.* Suppose that $(Q, \tilde{\theta}_1)_T$ is an **SDTS**. We assume that $(W, \tilde{\theta}_2)_U$ is not an **SDTS**. Then, there exists at least $w \in W$ such that $\tilde{\lambda}_u^w \in \tilde{\theta}_2$. Let $t \in T$ such that $L(t) = u$. By Definition 7,

$$\Upsilon_{H,L}^{-1}(\tilde{\lambda}_u^w) = \{(t, H^{-1}(w)), (r, \emptyset) : \text{for all } r \in T - \{t\}\}.$$

Since $H$ is a bijective mapping, $\Upsilon_{H,L}^{-1}(\tilde{\lambda}_u^w) = \tilde{\lambda}_t^q$. Also, since $\Upsilon_{H,L}$ is a soft continuous mapping, $\Upsilon_{H,L}^{-1}(\tilde{\lambda}_u^w) = \tilde{\lambda}_t^q \in \tilde{\theta}_1$. This contradicts the assumption that $(Q, \tilde{\theta}_1)_T$ is an **SDTS**. Hence, $(W, \tilde{\theta}_2)_U$ is an **SDTS**. □

**Theorem 28.** *Let $(Q, \tilde{\theta}_1)_T$ and $(W, \tilde{\theta}_2)_U$ be two **STSs**. Let $\Upsilon_{H,L} : S(Q)_T \to S(W)_U$ be a soft open mapping, where $H : Q \to W$ is a mapping and $L : T \to U$ is a surjective mapping. If $(W, \tilde{\theta}_2)_U$ is an **SDTS**, then $(Q, \tilde{\theta}_1)_T$ is an **SDTS**.*

*Proof.* Suppose that $(W, \tilde{\theta}_2)_U$ is an **SDTS**. We assume that $(Q, \tilde{\theta}_1)_T$ is not an **SDTS**. Then, there exists $q \in Q$ such that $\tilde{\lambda}_t^q \in \tilde{\theta}_1$. Then, there exists $u \in U$ such that $L(t) = u$. By Definition 7, $\Upsilon_{H,L}(\tilde{\lambda}_t^q) = \tilde{\lambda}_u^w$. Since $\Upsilon_{H,L}$ is a soft open mapping, $\Upsilon_{H,L}(\tilde{\lambda}_t^q) = \tilde{\lambda}_u^w \in \tilde{\theta}_2$, where $H(q) = w$. This contradicts the assumption that $(W, \tilde{\theta}_2)_U$ is an **SDTS**. Hence, $(Q, \tilde{\theta}_1)_T$ is an **SDTS**. □

**Theorem 29.** *Let $(Q_1, \tilde{\theta}_1)_T$ and $(Q_2, \tilde{\theta}_2)_T$ be two disjoint* **STS***s. Let $(\underset{i\in I}{\oplus} Q_i, \tilde{\theta})_T$ be a soft topology on $Q = \underset{i\in I}{\cup} Q_i$, where $I = \{1, 2\}$. Assume that $\tilde{\mu}_1, \in S(Q_1)_T$ and $\tilde{\mu}_2, \in S(Q_2)_T$. Then, $\tilde{\mu}_1$ and $\tilde{\mu}_1$ are* **SDS***s if and only if $\tilde{\mu}_1 \tilde{\cup} \tilde{\mu}_2$ is an* **SDS** *in $\tilde{\theta}$.*

*Proof.* It is obvious from Theorem 14. □

**Theorem 30.** *Let $(Q_i, \tilde{\theta}_i)_T$ be disjoint* **SDTS***s, where $i \in I$. Let $(\underset{i\in I}{\oplus} Q_i, \tilde{\theta})_T$ be a soft topology on $Q = \underset{i\in I}{\cup} Q_i$. Then $(\underset{i\in I}{\oplus} Q_i, \tilde{\theta})_T$ is an* **SDTS***.*

*Proof.* We assume that $\tilde{\theta}$ is not an **SDTS**. Then, there exists at least $q \in Q$ such that $\tilde{\lambda}_t^q \in \tilde{\theta}$. From Proposition 8, $\tilde{\lambda}_t^q \tilde{\cap} \tilde{Q}_i \in \tilde{\theta}_i$ for all $i \in I$. Therefore,

$$\tilde{\lambda}_t^q \tilde{\cap} \tilde{Q}_i = \begin{cases} \tilde{\Phi} & \text{if } q \notin Q_i, \\ \tilde{\lambda}_t^q & \text{if } q \in Q_i. \end{cases}$$

Since $q \in Q$, there exists at least $j \in I$ such that $q \in Q_j$. Then, $\tilde{\lambda}_t^q \tilde{\cap} \tilde{Q}_j = \tilde{\lambda}_t^q \in \tilde{\theta}_j$. This a contradiction since $(Q_i, \tilde{\theta}_i)_T$ are disjoint **SDTS**s for all $i \in I$. □

**Definition 25.** *Let $(Q, \tilde{\theta})_T$ be a soft topological space. A soft set $\tilde{\lambda} \in S(Q)_T$ is said to be a soft perfect set if it is both soft closed and* **SDS***.*

**Example 9.** *Consider $\tilde{\theta}$ as in Example 3. Let $\tilde{\lambda} = \{(t, G) : G$ be a closed set in $\mathcal{U}$ for all $t \in T\} \in S(\mathbb{R})_T$. Then, it is clear that $\tilde{\lambda}$ is a soft perfect set.*

**Example 10.** *The closure of a soft dense set in itself is a soft perfect set.*

**Theorem 31.** *Let $(Q, \tilde{\theta})_T$ be an* **STS** *and $\tilde{\mu} \in S(Q)_T$. Then, $\tilde{\mu} = D_{\tilde{\theta}}(\tilde{\mu})$ if, and only if, $\tilde{\mu}$ is a soft perfect set.*

*Proof.* It follow from Proposition 5 and Theorem 19. □

**Theorem 32.** *Let $(Q, \tilde{\theta})_T$ be an* **STS***. Let $\tilde{\mu} \in S(Q)_T$ be a soft perfect set and $\tilde{\lambda}_t^q \tilde{\in} \tilde{\mu}$. Then, $(\tilde{\upsilon} - \tilde{\lambda}_t^q) \tilde{\cap} \tilde{\mu} \neq \tilde{\Phi}$ for all soft open set $\tilde{\upsilon}$ and $\tilde{\lambda}_t^q \tilde{\in} \tilde{\nu}$.*

*Proof.* Suppose that $\tilde{\varpi}$ is a soft perfect set. We assume that there exists a soft open set $\tilde{\upsilon}$ and $\tilde{\lambda}_t^q \tilde{\in} \tilde{\upsilon}$ such that $(\tilde{\upsilon} - \tilde{\lambda}_t^q) \tilde{\cap} \tilde{\mu} = \tilde{\Phi}$. Therefore, $\tilde{\upsilon} \tilde{\cap} \tilde{\mu} = \tilde{\lambda}_t^q$. Thus, $\tilde{\lambda}_t^q \tilde{\in} i_{\tilde{\theta}}(\tilde{\mu})$. This contradicts the assumption that $\tilde{\mu}$ is a soft perfect set. Hence, $(\tilde{\upsilon} - \tilde{\lambda}_t^q) \tilde{\cap} \tilde{\mu} \neq \tilde{\Phi}$ for all soft open set $\tilde{\upsilon}$ and $\tilde{\lambda}_t^q \tilde{\in} \tilde{\nu}$. □

## 5. Discussion

This study advances the field of soft topological spaces by introducing new concepts and exploring their properties and interrelationships. We introduce the concept of soft isolated points, specifically for soft sets and soft topological spaces. The investigation centers on the relationships among key concepts, particularly the relationship expressed as $Cl_{\tilde{\theta}}(\tilde{\mu}) = i_{\tilde{\theta}}(\tilde{\mu}) \tilde{\cup} D_{\tilde{\theta}}(\tilde{\mu})$. In conventional topology, this relationship is typically represented by equality. Therefore, we seek to answer the question: Does this equality hold in the context of soft topology? By using the soft points defined in Definition 8, we demonstrate that the equality holds, and the results are detailed in Theorem 12.

We also introduce the notion of soft dense sets (spaces) in themselves. A soft topology is defined as soft dense in itself if and only if each of its parametric topologies is classified as a soft dense in itself space. Furthermore, we provide the relationship between the soft dense in itself set and the soft nowhere dense set. Notably, the property of soft density in itself exhibits hereditary characteristics and is preserved under coarser soft topologies and soft continuity, subject to certain conditions.

This research paves the way for many research projects by introducing additional classes of soft sets and soft spaces. It emphasizes the need for ongoing exploration of these soft structures and their relationships with other classes in soft topology. One limitation of this study is the inability to establish a clear relationship between soft dense sets and soft dense sets in themselves, similar to what is found in ordinary topology. Furthermore, we did not identify any connections between soft dense spaces in themselves and soft separation axioms. We advocate for future research to address the identified gaps and to further elucidate the properties and implications of soft topological structures.

## 6. Conclusions

The current study has introduced significant advancements in the field of soft topological spaces through the development of new theorems related to soft limit points and soft isolated points, soft dense sets (spaces) in themselves. Our examination of soft isolated points within soft sets and soft spaces, alongside their properties, has revealed that a soft point qualifies as a soft isolated point if it is not a soft limit point. Furthermore, we have explored the interconnections between these concepts and the notion of soft closure. Our research also encompasses the introduction of soft dense sets and spaces in themselves, establishing that a soft topology is classified as soft dense in itself if and only if each of its parametric topologies is similarly classified. The property of soft density in itself exhibits hereditary characteristics under the open soft subspace topology. Additionally, we have demonstrated that soft dense spaces in themselves are preserved under soft continuity, giving specific conditions. The soft sum of any collection of soft dense spaces in themselves is a soft dense space in itself. We further introduced the concept of a soft perfect set, which is characterized as both soft dense in itself and a closed set.

Further expanding on these ideas, there are additional concepts and results in classical topology that are related to these concepts that can be extended to soft topological spaces, allowing for the development of more theorems in this field. We can leverage these concepts to model and analyze uncertain or fuzzy data, leading to practical applications in decision-making, pattern recognition, and various other fields. Future research could explore these ideas within different topological frameworks, such as fuzzy and rough spaces, which have significant real-world applications.

## Use of Generative-AI tools declaration

The author declare he/she has not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declares no conflict of interest in this paper.

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