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*Research article*

# Well-posedness and stability of fractional stochastic integro-differential equations with general memory effects

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**Abstract:** Most existing research on well-posedness and stability has focused on fractional stochastic differential equations, with relatively fewer studies addressing fractional stochastic integro-differential equations (FSIDEs). In this work, we address this gap by establishing theoretical results on the well-posedness of FSIDEs. In particular, we derive a generalized Grönwall inequality and present results on Ulam-Hyers stability (UHS). Moreover, we extend existing findings by incorporating both the Φ-Caputo fractional derivative and the pth moment, thereby unifying and generalizing current results in the literature.

**Keywords:** qualitative analysis; delay term; fractional calculus; stochastic models; existence and uniqueness; stability
**Mathematics Subject Classification:** 34A07, 34A08, 60G22

## 1. Introduction

The Φ-Caputo fractional derivative (Cap-FD) is a generalization of the classical Cap-FD that incorporates an additional weight function Φ. This extension allows for more flexibility in modeling non-local phenomena with variable memory or scaling properties, making it useful in applications where standard fractional derivatives are too restrictive.

Numerous researchers have established various results regarding the Φ-Cap-FD. For instance, Chefnaj et al. [1] investigated the existence and uniqueness (Ex-Un) of solutions for fractional-order systems with impulsive effects, employing the contraction mapping theorem (CMT) and the Φ-Cap-FD. In [2], the author investigated the stability and Ex-Un of integro-differential equations involving the Φ-Cap-FD. Taqbibt et al. [3] analyzed the Ex-Un of fractional differential equations (FDEs) with Φ-Cap-FD using Picard's method. Selvam et al. [4] presented results on the controllability of FDEs involving Φ-Cap-FD. Ma et al. [5] studied the stability of such equations and established several important theorems for Φ-Cap-FD. In [6], Almeida presented various generalized results for FDEs involving Φ-Cap-FD. Yang et al. [7] established significant results concerning mild solutions of stochastic FDEs with Φ-Cap-FD. Zhu et al. [8] explored the convergence analysis of FDEs involving Φ-Cap-FD. Additionally, in [9], authors have presented results on stability for fractional systems of FDEs involving Φ-Cap-FD.

The Φ-Cap-FD is defined for $q : [c, v] \to \Re$ as follows [10]:

$$D_c^{\alpha,\Phi} q(t) = \frac{1}{\Gamma(\mathfrak{n}-\alpha)} \int_c^t (\Phi(t)-\Phi(s))^{\mathfrak{n}-1-\alpha} \Phi'(s) \left(\frac{1}{\Phi(s)} \frac{d}{ds}\right)^{\mathfrak{n}} q(s)ds, \tag{1.1}$$

where $\mathfrak{n}-1 < \alpha < \mathfrak{n} \in \mathbb{Z}^+$ and $\Phi \in C^1[c, v]$ with $\Phi'(t) > 0$, $\forall t \in [c, v]$.

**Remark 1.1.** *When $\Phi(t) = t$, the Φ-Cap-FD reduces to the Cap-FD; when $\Phi(t) = \ln t$, it becomes the Hadamard fractional derivatives (HFD); and when $\Phi(t) = t^x$, with $x > 0$, it corresponds to the Katugampola fractional derivative (KFD) [11]. Figure 1 demonstrates this phenomenon.*

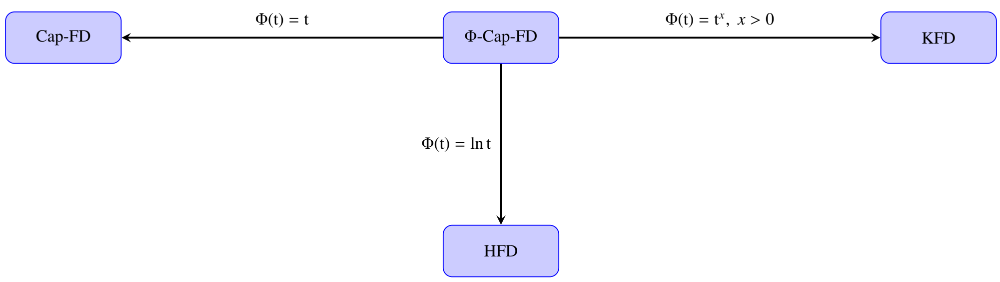


**Figure 1.** Behavior of the Φ-Cap-FD for different forms of Φ(t).

The Φ-Caputo fractional integral is defined as:

$$I_c^{\alpha,\Phi} q(t) = \frac{1}{\Gamma(\alpha)} \int_c^t \frac{\Phi'(s)}{(\Phi(t)-\Phi(s))^{1-\alpha}} q(s)ds. \tag{1.2}$$

Table 1 summarizes various definitions of fractional-order derivatives along with their formulations, distinctive features, and advantages.

**Table 1.** A comparative overview of different definitions of fractional derivatives, highlighting their formulations, limitations, and main applications.

| Definition | Formulation | Key Features and Advantages |
|---|---|---|
| **Riemann–Liouville** | $\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$ $(n-1<\alpha<n)$ | • Foundational and widely used definition.<br>• Initial conditions are non-local (involving fractional integrals).<br>• Employs a singular power-law kernel; mainly of theoretical importance. |
| **Caputo** | ${}^{C}\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$ $(n-1<\alpha<n)$ | • Uses standard integer-order initial conditions.<br>• Employs a singular power-law kernel.<br>• Preferred for physical and engineering applications. |
| **Grünwald–Letnikov** | $\mathrm{D}_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\lfloor t/h \rfloor} (-1)^k \binom{\alpha}{k} f(t-kh)$ | • Discrete, limit-based definition.<br>• Naturally suited for numerical approximation schemes.<br>• Provides the foundation for finite difference methods. |
| **Hadamard** | $\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_0^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1} \frac{f(\tau)}{\tau} d\tau$ $(n-1<\alpha<n)$ | • Defined on the half-line and involves a logarithmic kernel.<br>• Is scale-invariant: $\mathrm{D}^{\alpha}[f(ct)] = c^{\alpha}(\mathrm{D}^{\alpha} f)(ct)$.<br>• Suitable for problems with logarithmic scaling behavior. |
| **Caputo–Fabrizio** | ${}^{CF}\mathrm{D}_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(\tau) \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) d\tau$ $(0<\alpha<1)$ | • Employs a non-singular exponential kernel.<br>• Effective for modeling processes with fading memory.<br>• Avoids the singularities of classical definitions. |
| **Caputo–Katugampola** | ${}^{C}\mathrm{D}_{0^+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t^{\rho}-\tau^{\rho})^{\alpha-n+1}} \tau^{\rho-1} d\tau$ $(n-1<\alpha<n,\ \rho>0)$ | • Generalization using a positive parameter $\rho$.<br>• Interpolates between Caputo ($\rho=1$) and Hadamard ($\rho\to 0^+$).<br>• Provides flexibility for modeling time-graded or anisotropic processes. |

FSIDEs serve as powerful mathematical models that incorporate fractional derivatives, integral operators, and stochastic processes to describe system dynamics influenced by memory, hereditary effects, and randomness. Fractional stochastic differential equations (FSDEs) and FSIDEs have found extensive applications in modeling complex systems that exhibit both memory effects and random fluctuations. In finance, these models are used to represent long-range dependence in asset prices, volatility clustering, and sudden market jumps. In physics and engineering, they describe anomalous diffusion, viscoelastic materials, and signal processing where hereditary properties play a key role. In biology and medicine, FSDEs are employed to study population dynamics under random environmental influences, neuronal activity with memory effects, and the spread of infectious diseases subject to stochastic perturbations. In climate science and hydrology, they are applied to model long-memory rainfall patterns, temperature variations, and groundwater dynamics. Overall, FSIDEs and FSDEs provide a comprehensive framework that unifies fractional calculus with stochastic processes, enabling realistic representations of natural and engineered systems shaped by both randomness and memory.

Establishing that a solution to the FSIDE is well-posed is fundamental to ensuring the reliability of the model. This requires meeting three essential criteria: the existence of a solution, its uniqueness within an appropriate function space (commonly an $L^2$ Hilbert space or a related Sobolev space), and its continuous dependence on the initial data and system parameters. Existence is typically demonstrated using fixed-point principles, such as the CMT, the Schauder fixed-point theorem, or Picard's iterative method. These approaches are applied to an operator constructed from the integrated form of the equation, ensuring that, under suitable conditions, a unique or at least one solution can be obtained.

This formulation naturally incorporates a singular kernel that captures memory effects, characteristic of fractional operators involving, for example, Mittag-Leffler functions, together with a stochastic integration scheme (e.g., Itô calculus or its fractional extensions) driven by processes such as Wiener noise. Proving such results requires a rigorous probabilistic framework, employing estimates based on inequalities such as Hölder's, Grönwall-Bellman's, and Burkholder-Davis-Gundy's to control solution growth and address inherent singularities.

Uniqueness is most often established by imposing Lipschitz continuity on the coefficients of the equation and then showing that the norm of the difference between any two candidate solutions converges to zero. Finally, continuous dependence verifies the model's predictive utility and physical relevance, as it ensures that small variations in the inputs or initial states lead to proportionally small deviations in the solution path. This property is typically proven by estimating the discrepancy between solutions under perturbed conditions and demonstrating that it vanishes as the perturbation diminishes.

The Banach fixed-point theorem is a fundamental tool for establishing the Ex-Un of solutions to FSDEs and FSIDEs. In this framework, the FSIDE is first reformulated as an equivalent integral equation, typically involving fractional operators. By defining an appropriate operator on a Banach space of stochastic processes, one can demonstrate that it is a contraction under suitable conditions, such as Lipschitz continuity and linear growth constraints on the drift and diffusion terms. The Banach fixed-point theorem then guarantees the existence of a unique fixed point, which corresponds to the unique solution of the original FSIDE. This approach is widely used in mathematical analysis because it provides a constructive method for verifying well-posedness, ensuring both existence and uniqueness. Consequently, it forms a theoretical foundation for studying stability, numerical schemes, and applications of FSIDEs in finance, physics, engineering, and biology.

Numerous significant results have been established for FSDEs and FSIDEs. For instance, one group of researchers proved the stability of FSDEs [12]. The Ex-Un of solutions has been a central focus; some authors demonstrated this property for coupled systems of FSDEs using Picard's iteration method [13], while others investigated it for models involving the Φ-Hilfer fractional derivative by employing sectorial operators [14]. The study of mild solutions has also been advanced, with research in Hilbert spaces incorporating optimal control approaches [15]. Several important findings concerning solutions under the Caputo derivative have likewise been reported [16]. For further details, see [17–20].

Further work has demonstrated the Ex-Un for Caputo-Fabrizio FSDEs driven by multiplicative white noise, also verifying the convergence of the Euler-Maruyama numerical approach [21]. Other studies have derived results concerning the controllability of Caputo-Hadamard FSDEs, with the Ex-Un confirmed using the contraction mapping theorem [22]. A qualitative analysis of variable-order FSDEs has been conducted, obtaining solutions via Picard iterations and introducing new sufficient conditions [23].

Applications have also been explored, such as a novel financial chaotic model formulated as an FSDE with the Atangana-Baleanu operator, which was solved numerically with graphical results provided [24]. The Ex-Un for FSDEs with impulses has been explored [25], and the finite-time stability of a class of FSDEs has been analyzed [26].

Regarding FSIDEs, the Ex-Un of mild solutions involving the Hilfer operator has been studied [27]. Controllability results for FSIDEs with the Cap-FD have been established [28], and stability results for such equations have been presented [29]. The use of the conformable fractional derivative operator for stochastic problems has also been investigated [30]. Finally, the Ex-Un of mild solutions for FSIDEs with Cap-FD has been examined [31].

Our research provides the following key advancements:

1. To the best of our understanding, this work presents comprehensive results on the Ex-Un of solutions to FSIDEs that continuously depend on the initial conditions, along with the UHS concerning the Φ-Cap-FD in the pth moment.
2. We establish a generalized Grönwall inequality (GGI) involving the Φ-Cap-FD.
3. All findings are validated for the Φ-Cap-FD, which encompasses various types of fractional derivatives.
4. Most existing results concerning well-posedness and UHS for FSIDEs have been derived within the mean square; in contrast, we formulate and establish these results in the more general pth moment.

We consider the following FSIDEs:

$$\begin{cases} \mathrm{D}_0^{\alpha,\Phi}\left(\mathrm{q}(\mathrm{t}) - \sum_{\mathrm{r}=1}^{\mathrm{w}} \mathrm{I}_{0^+}^{\mathfrak{m}_\mathrm{r},\Phi} \ell_\mathrm{r}(\mathrm{t},\mathrm{q}(\mathrm{t}))\right) = \lambda(\mathrm{t},\mathrm{q}(\mathrm{t}),\mathrm{q}(\mathrm{t}-\kappa)) \\ \qquad +\eta(\mathrm{t},\mathrm{q}(\mathrm{t}),\mathrm{q}(\mathrm{t}-\kappa))\frac{\mathrm{dB(t)}}{\mathrm{dt}}, \quad \mathrm{t} \in [0,\mathrm{T}], \\ \mathrm{q}(0) = \vartheta, \end{cases} \tag{1.3}$$

where $\mathrm{D}_0^{\alpha,\Phi}$ and $\mathrm{I}_0^{\mathfrak{m}_\mathrm{r},\Phi}$ denote the Φ-Cap-FD and integral, respectively, with $\alpha \in (\frac{1}{2},1)$, $\frac{1}{2} \leq \mathfrak{m}_\mathrm{r} \leq 1$, $1 \leq \mathrm{r} \leq \mathrm{w}$, and $\kappa \in \Re$ is delay, the $\lambda : [0,\mathrm{T}] \times \Re^\mathrm{z} \times \Re^\mathrm{z} \to \Re^\mathrm{z}$ and $\eta : [0,\mathrm{T}] \times \Re^\mathrm{z} \times \Re^\mathrm{z} \to \Re^{\mathrm{z}\times\mathrm{y}}$ are measurable and continuous mappings. $(B_t)_{t\in[0,\infty)}$ denotes a y-dimensional Brownian motion defined on a complete filtered probability space $(\Omega, F, \mathbb{F} = (F_t)_{t\in[0,\infty)}, P)$.

This study includes several important elements. Section 2 presents the foundational concepts that form the cornerstone of our analysis. The main results for FSIDEs involving the Φ-Cap-FD are established in Sections 3 and 4. A numerical example is provided in Section 5 to illustrate the theoretical findings. Section 6 discusses the conclusions, and finally, we outline directions for future work.

## 2. Preliminaries

Here, we present essential results for deriving key findings.

**Definition 2.1.** *Suppose* $\mathcal{V}_t^p = \mathscr{L}^p(\Omega, F_t, P)$ *represents the* $F_t$*-measurable and* $p^{th}$ *integrable functions* $q = (q_1, q_2, \cdots, q_m)^{\mathbb{T}}$*, then* $\Omega \to \Re^m$ *satisfies*

$$\|q\|_p = \left(\sum_{\iota=1}^{\mathfrak{b}} E|q_\iota|^p\right)^{\frac{1}{p}}.$$

A process satisfying the measurability condition $q : [0, T] \to \mathscr{L}^p(\Omega, F_t, P)$ is termed $\mathbb{F}$-adapted provided that $q(t) \in \mathcal{V}_t^p$ for all $t \geq 0$. The solution of (1.3) satisfying the initial condition $q(0) = \vartheta$ is expressed as follows:

$$\begin{aligned} q(t) = \\ \vartheta + \sum_{r=1}^{w} \frac{1}{\Gamma(\mathfrak{m}_r)} \int_0^t (\Phi(t) - \Phi(s))^{\mathfrak{m}_r - 1} \Phi'(s) \ell_r(s, q(s)) ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (\Phi(t) - \Phi(s))^{\alpha-1} \Phi'(s) \lambda(s, q(s), q(s-\kappa)) ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (\Phi(t) - \Phi(s))^{\alpha-1} \Phi'(s) \eta(s, q(s), q(s-\kappa)) dB(s). \end{aligned} \tag{2.1}$$

For $\lambda$ and $\eta$, assume the following:

1. $(\hbar_1)$ $\forall \gamma_1, \gamma_2, \beta_1, \beta_2 \in \Re^{\varkappa}$, there is $\mu$ such as

$$\begin{aligned} &\|\ell_r(t, \gamma_1) - \ell_r(t, \beta_1)\|_p + \|\lambda(t, \gamma_1, \gamma_2) - \lambda(t, \beta_1, \beta_2)\|_p \\ &+ \|\eta(t, \gamma_1, \gamma_2) - \eta(t, \beta_1, \beta_2)\|_p \\ &\leq \mu(\|\gamma_1 - \beta_1\|_p + \|\gamma_2 - \beta_2\|_p), \ r = 1, 2, \cdots, w. \end{aligned}$$

2. $(\hbar_2)$ The $\ell_r(t, .)$, $r = 1, 2, \cdots, w$, $\lambda(t, ., .)$ and $\eta(t, ., .)$ satisfy

$$\begin{aligned} &\underset{t\in[0,T]}{esssup}\|\ell_r(t, 0)\|_p < \mathcal{U}, \ \underset{t\in[0,T]}{esssup}\|\lambda(t, 0, 0)\|_p < \mathcal{U}, \\ &\underset{t\in[0,T]}{esssup}\|\eta(t, 0, 0)\|_p < \mathcal{U}. \end{aligned}$$

## 3. Well-posedness

Within the framework of the pth moment, we obtain extended results concerning the well-posedness of FSIDE solutions.

We denote $\mathcal{H}^p([0,T])$ as the space of all measurable and $\mathbb{F}_T$-adapted processes q such that the norm $\|q\|\mathcal{H}^p = \underset{t\in[0,T]}{esssup}\|q(t)\|_p$ is finite. It is straightforward to verify that $(\mathcal{H}^p([0,T]), \|\cdot\|_{\mathcal{H}^p})$ forms a complete normed vector space.

We proceed by defining the mapping $\mathcal{Z}_\vartheta : \mathcal{H}^p([0,T]) \to \mathcal{H}^p([0,T])$, with the property that $\mathcal{Z}_\vartheta(q(0)) = \vartheta$, and

$$\begin{aligned}\mathcal{Z}_\vartheta(q(t)) = \vartheta &+ \sum_{r=1}^{w} \frac{1}{\Gamma(\mathfrak{m}_r)} \int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Phi'(s)\ell_r(s,q(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)\lambda(s,q(s),q(s-\kappa))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)\eta(s,q(s),q(s-\kappa))dB(s). \end{aligned} \tag{3.1}$$

The lemma presented below significantly contributes to the proof of various theorems.

$$\left\|q_1+q_2\right\|_p^p \le 2^{p-1}(\|q_1\|_p^p + (\|q_2\|_p^p),\ \forall q_1,q_2 \in \mathfrak{R}^{\varkappa}. \tag{3.2}$$

**Lemma 3.1.** *Let* $(\hbar_1)$ *and* $(\hbar_2)$ *hold. Then,* $\mathcal{Z}_\vartheta$ *is well-defined.*

*Proof.* Suppose $q(t) \in \mathcal{H}^p[0,T]$ and $\forall t \in [0,T]$. From (3.1) and (3.2), we have

$$\begin{aligned}&\left\|\mathcal{Z}_\vartheta(q(t))\right\|_p^p \le 2^{2p-2}\|\vartheta\|_p^p \\ &+2^{(w+2)p-(w+2)} \sum_{r=1}^{w} \left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \left\| \int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Phi'(s)\ell_r(s,q(s))ds\right\|_p^p \\ &+\frac{2^{2p-2}}{(\Gamma(\alpha))^p}\left\| \int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\lambda(s,q(s),q(s-\kappa))\Phi'(s)ds\right\|_p^p \\ &+\frac{2^{2p-2}}{(\Gamma(\alpha))^p}\left\| \int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\eta(s,q(s),q(s-\kappa))\Phi'(s)dB(s)\right\|_p^p. \end{aligned} \tag{3.3}$$

Invoking Hölder's inequality (H-I), we get:

$$\begin{aligned}&2^{(w+2)p-(w+2)} \sum_{r=1}^{w} \left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \left\| \int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_\iota-1}\Phi'(s)\ell_r(s,q(s))ds\right\|_p^p \\ &= 2^{(w+2)p-(w+2)} \sum_{r=1}^{w} \left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \sum_{\iota=1}^{\mathfrak{b}} E\left( \int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\left|\ell_{\iota,r}(s,q(s))\right|\Phi'(s)ds\right)^p \\ &\le 2^{(w+2)p-(w+2)} \sum_{r=1}^{w} \left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \sum_{\iota=1}^{\mathfrak{b}} E\left(\left( \int_0^t (\Phi(t)-\Phi(s))^{\frac{(\mathfrak{m}_r-1)p}{(p-1)}}(\Phi'(s))^{\frac{p}{p-1}}ds\right)^{p-1} \int_0^t \left|\ell_{\iota,r}(s,q(s))\right|^p ds\right)\end{aligned}$$

$$\leq 2^{(w+2)p-(w+2)} \sum_{r=1}^{w} \left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \sum_{\iota=1}^{\mathfrak{b}} E\Bigg(\Big(\sup_{0<s\leq t} (\Phi'(s))^{\frac{1}{p-1}}\Big)^{p-1}\Big(\int_0^t (\Phi(t)-\Phi(s))^{\frac{(\mathfrak{m}_r-1)p}{(p-1)}}\Phi'(s)ds\Big)^{p-1}$$
$$\int_0^t \left|\ell_{\iota,r}(s,q(s))\right|^p ds\Bigg)$$
$$\leq 2^{(w+2)p-(w+2)} \sum_{r=1}^{w} \left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \mathbb{Y}^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\mathfrak{m}_r p-1}{p-1}}\Big)^{p-1}\left(\frac{p-1}{\mathfrak{m}_r p-1}\right)^{p-1}\int_0^t \left\|\ell_r(s,q(s))\right\|_p^p ds, \tag{3.4}$$

where $\mathbb{Y} = \sup_{0<s\leq t}(\Phi'(s))^{\frac{1}{p-1}}$.

In accordance with $(\hbar_1)$, we have:

$$\begin{aligned}\left\|\ell_r(s,q(s))\right\|_p^p \leq & 2^{p-1}\Big(\left\|\ell_r(s,q(s))-\ell_r(s,0)\right\|_p^p + \left\|\ell_r(s,0)\right\|_p^p\Big)\\ \leq & 2^{p-1}\Big(\mu^p\left\|q(s)\right\|_p^p + \left\|\ell_r(s,0)\right\|_p^p\Big).\end{aligned} \tag{3.5}$$

Hence, we establish

$$\begin{aligned}\int_0^t \left\|\ell_r(s,q(s))\right\|_p^p ds \leq & 2^{p-1}\mu^p \operatorname*{esssup}_{s\in[1,T]}\|q(s)\|_p^p \int_0^t 1ds + 2^{p-1}\int_0^t \left\|\ell_r(s,0)\right\|_p^p ds\\ \leq & 2^{p-1}T\mu^p\left\|q(s)\right\|_{\mathcal{H}^p}^p + 2^{p-1}\int_0^t \left\|\ell_r(s,0)\right\|_p^p ds.\end{aligned} \tag{3.6}$$

As a result, we accomplish

$$2^{(w+2)p-(w+2)} \sum_{r=1}^{w} \left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \left\|\int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_\iota-1}\Phi'(s)\ell_r(s,q(s))ds\right\|_p^p \leq 2^{(w+2)p-(w+2)} \sum_{r=1}^{w} \left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p$$
$$\left(\mathbb{Y}^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\mathfrak{m}_r p-1}{p-1}}\Big)^{p-1}\left(\frac{p-1}{\mathfrak{m}_r p-1}\right)^{p-1}\Big(2^{p-1}T\mu^p\left\|q(s)\right\|_{\mathcal{H}^p}^p + 2^{p-1}\int_0^t \left\|\ell_r(s,0)\right\|_p^p ds\Big)\right). \tag{3.7}$$

We apply H-I to obtain:

$$\left\|\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\lambda(s,q(s),q(s-\kappa))\Phi'(s)ds\right\|_p^p =$$
$$\sum_{\iota=1}^{\mathfrak{b}} E\Big(\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\left|\lambda_\iota(s,q(s),q(s-\kappa))\right|\Phi'(s)ds\Big)^p$$
$$\leq \sum_{\iota=1}^{\mathfrak{b}} E\Bigg(\Big(\int_0^t (\Phi(t)-\Phi(s))^{\frac{(\alpha-1)p}{(p-1)}}(\Phi'(s))^{\frac{p}{p-1}}ds\Big)^{p-1}\int_0^t \left|\lambda_\iota(s,q(s),q(s-\kappa))\right|^p ds\Bigg)$$
$$\leq \sum_{\iota=1}^{\mathfrak{b}} E\Bigg(\Big(\sup_{0<s\leq t}(\Phi'(s))^{\frac{1}{p-1}}\Big)^{p-1}\Big(\int_0^t (\Phi(t)-\Phi(s))^{\frac{(\alpha-1)p}{(p-1)}}\Phi'(s)ds\Big)^{p-1}$$
$$\int_0^t \left|\lambda_\iota(s,q(s),q(s-\kappa))\right|^p ds\Bigg)$$

$$\leq \mathbb{Y}^{p-1}\left((\Phi(t)-\Phi(0))^{\frac{p\alpha-1}{p-1}}\right)^{p-1}\left(\frac{p-1}{\alpha p-1}\right)^{p-1}\int_0^t \left\|\lambda(s,q(s),q(s-\kappa))\right\|_p^p ds. \tag{3.8}$$

Based on ($\hbar_1$), we derive

$$\begin{aligned}\left\|\lambda(s,q(s),q(s-\kappa))\right\|_p^p \leq & 2^{p-1}\left(\left\|\lambda(s,q(s),q(s-\kappa))-\lambda(s,0,0)\right\|_p^p+\left\|\lambda(s,0,0)\right\|_p^p\right)\\ \leq & 2^{p-1}\left(2^{p-1}\mu^p\left(\left\|q(s)\right\|_p^p+\left\|q(s-\kappa)\right\|_p^p\right)+\left\|\lambda(s,0,0)\right\|_p^p\right).\end{aligned} \tag{3.9}$$

Consequently, we achieve the following:

$$\begin{aligned}\int_0^t \left\|\lambda(s,q(s),q(s-\kappa))\right\|_p^p ds \leq & 2^{2p-2}\mu^p\left(\left(\underset{s\in[0,T]}{esssup}\|q(s)\|_p\right)^p+\left(\underset{s\in[0,T]}{esssup}\left\|q(s-\kappa)\right\|_p\right)^p\right)\\ & \int_0^t 1ds+2^{p-1}\int_0^t \left\|\lambda(s,0,0)\right\|_p^p ds\\ \leq & 2^{2p-2}T\mu^p\left(\left\|q(s)\right\|_{\mathcal{H}^p}^p+\left\|q(s-\kappa)\right\|_{\mathcal{H}^p}^p\right)+2^{p-1}\int_0^t \left\|\lambda(s,0,0)\right\|_p^p ds.\end{aligned} \tag{3.10}$$

In light of (3.8) and (3.10), we determine

$$\begin{aligned}&\left\|\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\lambda(s,q(s),q(s-\kappa))\Phi'(s)ds\right\|_p^p \leq \mathbb{Y}^{p-1}\left((\Phi(t)-\Phi(0))^{\frac{(\alpha p-1)}{p-1}}\right)^{p-1}\\ &\left(\frac{p-1}{\alpha p-1}\right)^{p-1}2^{p-1}\left(2^{p-1}\mu^p T\left(\left\|q(s)\right\|_{\mathcal{H}^p}^p+\left\|q(s-\kappa)\right\|_{\mathcal{H}^p}^p\right)+\int_0^t \left\|\lambda(s,0,0)\right\|_p^p ds\right).\end{aligned} \tag{3.11}$$

Based on ($\hbar_2$), we derive from (3.11) that

$$\begin{aligned}&\left\|\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\lambda(s,q(s),q(s-\kappa))\Phi'(s)ds\right\|_p^p \leq \mathbb{Y}^{p-1}\left((\Phi(t)-\Phi(0))^{\frac{(\alpha p-1)}{p-1}}\right)^{p-1}\\ &\left(\frac{p-1}{\alpha p-1}\right)^{p-1}2^{p-1}\left(2^{p-1}\mu^p T\left(\left\|q(s)\right\|_{\mathcal{H}^p}^p+\left\|q(s-\kappa)\right\|_{\mathcal{H}^p}^p\right)+T\mathcal{U}^p\right).\end{aligned} \tag{3.12}$$

To evaluate the third component of (3.3), we utilize the Burkholder-Davis-Gundy inequality (B-D-G-I) and H-I, from which we infer:

$$\begin{aligned}&\left\|\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\eta(s,q(s),q(s-\kappa))\Phi'(s)dB(s)\right\|_p^p\\ &\quad=\sum_{\iota=1}^{\mathfrak{b}} E\left|\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\eta_\iota(s,q(s),q(s-\kappa))\Phi'(s)dB(s)\right|^p\\ &\quad\leq \sum_{\iota=1}^{\mathfrak{b}} C_p E\left|\int_0^t (\Phi(t)-\Phi(s))^{2\alpha-2}\left|\eta_\iota(s,q(s),q(s-\kappa))\right|^2(\Phi'(s))^2 ds\right|^{\frac{p}{2}}\end{aligned}$$

$$\leq \sum_{\imath=1}^{\mathfrak{b}} C_{\rm p}\mathrm{E}\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}\Big|\eta_\imath({\rm s},{\rm q}({\rm s}),{\rm q}({\rm s}-\kappa))\Big|^{\rm p}(\Phi'({\rm s}))^2{\rm ds}$$
$$\Big(\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}(\Phi'({\rm s}))^2{\rm ds}\Big)^{\frac{\rm p-2}{2}}$$
$$\leq \sum_{\imath=1}^{\mathfrak{b}} C_{\rm p}\mathrm{E}\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}\Big|\eta_\imath({\rm s},{\rm q}({\rm s}),{\rm q}({\rm s}-\kappa))\Big|^{\rm p}(\Phi'({\rm s}))^2{\rm ds}$$
$$\Big(\sup_{0<{\rm s}\leq{\rm t}}\Phi'({\rm s})\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}\Phi'({\rm s}){\rm ds}\Big)^{\frac{\rm p-2}{2}}$$
$$\leq \aleph^{\frac{\rm p-2}{2}}C_{\rm p}\Big(\frac{(\Phi({\rm t})-\Phi(0))^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{\rm p-2}{2}}\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}\Big\|\eta({\rm s},{\rm q}({\rm s},{\rm q}({\rm s}-\kappa))\Big\|_{\rm p}^{\rm p}(\Phi'({\rm s}))^2{\rm ds}, \quad (3.13)$$

where $\aleph = \sup_{0<{\rm s}\leq{\rm t}}\Phi'({\rm s})$ and $C_{\rm p}=\Big(\frac{\rm p^{p+1}}{2(\rm p-1)^{p-1}}\Big)^{\frac{\rm p}{2}}$.

Utilizing $(\hbar_1)$ and $(\hbar_2)$, we conclude

$$\begin{aligned}\Big\|\eta({\rm s},{\rm q}({\rm s}),{\rm q}({\rm s}-\kappa))\Big\|_{\rm p}^{\rm p} &\leq 2^{2\rm p-2}\mu^{\rm p}\Big(\Big\|{\rm q}({\rm s})\Big\|_{\rm p}^{\rm p}+\Big\|{\rm q}({\rm s}-\kappa)\Big\|_{\rm p}^{\rm p}\Big)+2^{\rm p-1}\Big\|\eta({\rm s},0,0)\Big\|_{\rm p}^{\rm p}\\ &\leq 2^{2\rm p-2}\mu^{\rm p}\Big(\Big\|{\rm q}({\rm s})\Big\|_{\rm p}^{\rm p}+\Big\|{\rm q}({\rm s}-\kappa)\Big\|_{\rm p}^{\rm p}\Big)+2^{\rm p-1}\mathcal{U}^{\rm p}. \end{aligned}\quad (3.14)$$

Accordingly, we get

$$\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}\Big\|\eta({\rm s},{\rm q}({\rm s}),{\rm q}({\rm s}-\kappa))\Big\|_{\rm p}^{\rm p}(\Phi'({\rm s}))^2{\rm ds}\leq 2^{2\rm p-2}\mu^{\rm p}$$
$$\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}\Big(\Big(\mathop{esssup}_{{\rm s}\in[0,{\rm T}]}\Big\|{\rm q}({\rm s})\Big\|_{\rm p}\Big)^{\rm p}+\Big(\mathop{esssup}_{{\rm s}\in[0,{\rm T}]}\Big\|{\rm q}({\rm s}-\kappa)\Big\|_{\rm p}\Big)^{\rm p}\Big)(\Phi'({\rm s}))^2{\rm ds}$$
$$+2^{\rm p-1}\mathcal{U}^{\rm p}\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}(\Phi'({\rm s}))^2{\rm ds}$$
$$\leq 2^{2\rm p-2}\mu^{\rm p}\sup_{0<{\rm s}\leq{\rm t}}\Phi'({\rm s})\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}\Big(\Big(\mathop{esssup}_{{\rm s}\in[0,{\rm T}]}\Big\|{\rm q}({\rm s})\Big\|_{\rm p}\Big)^{\rm p}$$
$$+\Big(\mathop{esssup}_{{\rm s}\in[0,{\rm T}]}\Big\|{\rm q}({\rm s}-\kappa)\Big\|_{\rm p}\Big)^{\rm p}\Big)\Phi'({\rm s}){\rm ds}+2^{\rm p-1}\mathcal{U}^{\rm p}\sup_{0<{\rm s}\leq{\rm t}}\Phi'({\rm s})\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}\Phi'({\rm s}){\rm ds}$$
$$=\aleph\frac{2^{\rm p-1}(\Phi({\rm t})-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\Big(2^{\rm p-1}\mu^{\rm p}\Big(||{\rm q}({\rm s})||_{\mathcal{H}_{\rm p}}^{\rm p}+||{\rm q}({\rm s}-\kappa)||_{\mathcal{H}_{\rm p}}^{\rm p}\Big)+\mathcal{U}^{\rm p}\Big). \quad (3.15)$$

Hence, the preceding leads to

$$\begin{aligned}\int_0^{\rm t}(\Phi({\rm t})-\Phi({\rm s}))^{2\alpha-2}\Big\|\eta({\rm s},{\rm q}({\rm s}),{\rm q}({\rm s}-\kappa))\Big\|_{\rm p}^{\rm p}(\Phi'({\rm s}))^2{\rm ds} &\leq \frac{2^{\rm p-1}(\Phi({\rm t})-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\aleph\\ &\Big(2^{\rm p-1}\mu^{\rm p}\Big(||{\rm q}({\rm s})||_{\mathcal{H}_{\rm p}}^{\rm p}+||{\rm q}({\rm s}-\kappa)||_{\mathcal{H}_{\rm p}}^{\rm p}\Big)+\mathcal{U}^{\rm p}\Big). \end{aligned}\quad (3.16)$$

From (3.13) and (3.16), we conclude

$$\Big\| \int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\eta(s,q(s),q(s-\kappa))\Phi'(s)dB(s)\Big\|_p^p \leq \aleph^{\frac{p-2}{2}}$$
$$C_p\Big(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\Big)^{\frac{p-2}{2}}\frac{2^{p-1}(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}$$
$$\aleph\Big(2^{p-1}\mu^p\Big(\|q(s)\|_{\mathcal{H}_p}^p+\|q(s-\kappa)\|_{\mathcal{H}_p}^p\Big)+\mathcal{U}^p\Big). \tag{3.17}$$

Hence, we conclude that $\|\mathcal{Z}(q(t))\|_{\mathcal{H}^p}$ remains bounded, thereby confirming the desired outcome.

**Lemma 3.2.** *Assume* $\alpha, \varpi > 0$*, and*
*Remathrmt* $\in [0,T]$*, then*

$$I_{0^+}^{\alpha,\Phi}\exp(\varpi(\Phi(t)-\Phi(0))) \leq \frac{\exp(\varpi(\Phi(t)-\Phi(0)))}{\varpi^\alpha}.$$

*Proof.* It can be inferred from (1.2) that:

$$I_{0^+}^{\alpha,\Phi}\exp(\varpi(\Phi(t)-\Phi(0))) = \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\exp(\varpi(\Phi(s)-\Phi(0)))\Phi'(s)ds.$$

By $\mathbb{K}=\Phi(t)-\Phi(s)$,

$$I_{0^+}^{\alpha,\Phi}\exp(\varpi(\Phi(t)-\Phi(0))) = \frac{\exp(\varpi(\Phi(t)-\Phi(0)))}{\Gamma(\alpha)}\int_0^{\Phi(t)-\Phi(0)}\mathbb{K}^{\alpha-1}\exp(-\varpi\mathbb{K})d\mathbb{K}. \tag{3.18}$$

By applying $\mathcal{G}=\varpi\mathbb{K}$ in (3.18), we obtain

$$\begin{aligned}
I_{0^+}^{\alpha,\Phi}\exp(\varpi(\Phi(t)-\Phi(0))) &= \frac{\exp(\varpi(\Phi(t)-\Phi(0)))}{\varpi^\alpha\Gamma(\alpha)}\int_0^{\varpi\big(\Phi(t)-\Phi(0)\big)}\mathcal{G}^{\alpha-1}\exp(-\mathcal{G})d\mathcal{G} \\
&\leq \frac{\exp(\varpi(\Phi(t)-\Phi(0)))}{\varpi^\alpha\Gamma(\alpha)}\int_0^{\infty}\mathcal{G}^{\alpha-1}\exp(-\mathcal{G})d\mathcal{G} \\
&= \frac{\exp(\varpi(\Phi(t)-\Phi(0)))}{\varpi^\alpha}.
\end{aligned}$$

This implies that

$$\frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\exp(\varpi(\Phi(t)-\Phi(0)))\Phi'(s)ds \leq \frac{\exp(\varpi(\Phi(t)-\Phi(0)))}{\varpi^\alpha}. \tag{3.19}$$

**Theorem 3.1.** *Under conditions* $(\hbar_1)$ *and* $(\hbar_2)$*, the delayed FSIDE* (1.3) *ensures the existence of a unique solution.*

*Proof.* With respect to $\|\cdot\|_\varpi$, it follows that:

$$\|q(t)\|_\varpi = \underset{t\in[0,T]}{esssup}\Big(\frac{\|q(t)\|_p^p}{\Lambda(t)}\Big)^{\frac{1}{p}},\ \varpi>0, \tag{3.20}$$

where $\Lambda(t)=\exp(\varpi(\Phi(t)-\Phi(0)))$.

Since the $\|\cdot\|_{\mathcal{H}^p}$ and $\|\cdot\|_{\varpi}$ are equivalent, it follows that $(\mathcal{H}^p([0,t]),\|\cdot\|_{\varpi})$ is complete.

Note the following:

$$\zeta=\Bigg(2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}\frac{\Gamma(2\mathfrak{m}_r-1)}{\varpi^{2\mathfrak{m}_r-1}}$$
$$+\frac{2^{2p-2}}{(\Gamma(\alpha))^p}\Bigg(2^{p-1}\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}+2^{p-1}\aleph^{\frac{p-2}{2}}\left(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\right)^{\frac{p-2}{2}}\mu^pC_p\aleph\Bigg)$$
$$\frac{2\Phi\Gamma(2\alpha-1)}{\varpi^{2\alpha-1}}\Bigg)<1. \tag{3.21}$$

For $q(t),q^*(t)\in\mathcal{H}^p([0,T])$, we obtain

$$\|\mathcal{Z}_\vartheta(q(t))-\mathcal{Z}_\vartheta(q^*(t))\|_p^p\le 2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p$$
$$\left\|\int_0^t(\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Big(\ell_r(s,q(s))-\ell_r(s,q^*(s))\Big)\Phi'(s)ds\right\|_p^p$$
$$+\frac{2^{2p-2}}{(\Gamma(\alpha))^p}\left\|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\lambda(s,q(s),q(s-\kappa))-\lambda(s,q^*(s),q^*(s-\kappa))\Big)\Phi'(s)ds\right\|_p^p+$$
$$+\frac{2^{2p-2}}{(\Gamma(\alpha))^p}\left\|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\eta(s,q(s),q(s-\kappa))-\eta(s,q^*(s),q^*(s-\kappa))\Big)\Phi'(s)dB(s)\right\|_p^p. \tag{3.22}$$

Combining H-I with $(\hbar_1)$, we establish:

$$2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\left\|\int_0^t(\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Big(\ell_r(s,q(s))-\ell_r(s,q^*(s))\Big)\Phi'(s)ds\right\|_p^p$$
$$=2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\sum_{\iota=1}^{\mathfrak{b}}E\left(\int_0^t(\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Big(\ell_{\iota,r}(s,q(s))-\ell_{\iota,r}(s,q^*(s))\Big)\Phi'(s)ds\right)^p$$
$$\le2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\sum_{\iota=1}^{\mathfrak{b}}E\Bigg(\left(\int_0^t(\Phi(t)-\Phi(s))^{\frac{(\mathfrak{m}_r-1)(p-2)}{p-1}}(\Phi'(s))^{\frac{p-2}{p-1}}ds\right)^{p-1}$$
$$\left(\int_0^t(\Phi(t)-\Phi(s))^{2\mathfrak{m}_r-2}\Big|\ell_{\iota,r}(s,q(s))-\ell_{\iota,r}(s,q^*(s))\Big|\Phi'(s)ds\right)\Bigg)$$
$$\le2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\sum_{\iota=1}^{\mathfrak{b}}E\Bigg(\left(\sup_{0<s\le t}(\Phi'(s))^{\frac{1}{1-p}}\int_0^t(\Phi(t)-\Phi(s))^{\frac{(\mathfrak{m}_r-1)(p-2)}{p-1}}\Phi'(s)ds\right)^{p-1}$$
$$\left(\int_0^t(\Phi(t)-\Phi(s))^{2\mathfrak{m}_r-2}\Big|\ell_{\iota,r}(s,q(s))-\ell_{\iota,r}(s,q^*(s))\Big|(\Phi'(s))^2ds\right)\Bigg)$$
$$\le2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}$$
$$\sup_{0<s\le t}\Phi'(s)\int_0^t(\Phi(t)-\Phi(s))^{2\mathfrak{m}_r-2}\Big(\|q(s)-q^*(s)\|_p^p\Big)\Phi'(s)ds, \tag{3.23}$$

where $\mathfrak{n} = \sup\limits_{0<s\leq t}(\Phi'(s))^{\frac{1}{1-p}}$.
Accordingly, we find

$$2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\left\|\int_0^t(\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}(\ell_r(s,q(s))-\ell_r(s,q^*(s)))\Phi'(s)ds\right\|_p^p$$
$$\leq 2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}$$
$$\int_0^t(\Phi(t)-\Phi(s))^{2\mathfrak{m}_r-2}\underset{t\in[0,T]}{esssup}\left(\frac{\left\|q(s)-q^*(s)\right\|_p^p}{\exp\left(\varpi(\Phi(s)-\Phi(0))\right)}\right)\exp(\varpi(\Phi(s)-\Phi(0))\Phi'(s)ds. \quad (3.24)$$

Following (3.24), we deduce:

$$2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\left\|\int_0^t(\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}(\ell_r(s,q(s))-\ell_r(s,q^*(s)))\Phi'(s)ds\right\|_p^p$$
$$\leq 2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}$$
$$\int_0^t(\Phi(t)-\Phi(s))^{2\mathfrak{m}_r-2}\left\|q(s)-q^*(s)\right\|_\varpi^p\exp\left(\varpi(\Phi(s)-\Phi(0))\right)\Phi'(s)ds$$
$$\leq 2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}\left\|q(s)-q^*(s)\right\|_\varpi^p$$
$$\frac{\Gamma(2\mathfrak{m}_r-1)\exp(\varpi(\Phi(t)-\Phi(0)))}{\varpi^{2\mathfrak{m}_r-1}}. \quad (3.25)$$

For the second term in (3.22), applying H-I and ($\hbar_1$) yields:

$$\left\|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\lambda(s,q(s),q(s-\kappa))-\lambda(s,q^*(s),q^*(s-\kappa))\Big)\Phi'(s)ds\right\|_p^p$$
$$=\sum_{\iota=1}^{\mathfrak{b}}E\left|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\lambda_\iota(s,q(s),q(s-\kappa))-\lambda_\iota(s,q^*(s),q^*(s-\kappa))\Big)\Phi'(s)ds\right|^p$$
$$\leq\sum_{\iota=1}^{\mathfrak{b}}E\Bigg(\left(\int_0^t(\Phi(t)-\Phi(s))^{\frac{(\alpha-1)(p-2)}{p-1}}(\Phi'(s))^{\frac{p-2}{p-1}}ds\right)^{p-1}$$
$$\left(\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\left|\lambda_\iota(s,q(s),q(s-\kappa))-\lambda_\iota(s,q^*(s),q^*(s-\kappa))\right|^p(\Phi'(s))^2ds\right)\Bigg)$$
$$\leq\sum_{\iota=1}^{\mathfrak{b}}E\Bigg(\left(\sup_{0<s\leq t}(\Phi'(s))^{\frac{1}{1-p}}\int_0^t(\Phi(t)-\Phi(s))^{\frac{(\alpha-1)(p-2)}{p-1}}\Phi'(s)ds\right)^{p-1}$$
$$\left(\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\left|\lambda_\iota(s,q(s),q(s-\kappa))-\lambda_\iota(s,q^*(s),q^*(s-\kappa))\right|^p(\Phi'(s))^2ds\right)\Bigg)$$
$$\leq 2^{p-1}\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}$$

$$\sup_{0<s\leq t}\Phi'(s)\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big(\big\|q(s)-q^*(s)\big\|_p^p+\big\|q(s-\kappa)-q^*(s-\kappa)\big\|_p^p\Big)\Phi'(s)ds. \tag{3.26}$$

It follows that

$$\begin{aligned}&\Big\|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\lambda(s,q(s),q(s-\kappa))-\lambda(s,q^*(s),q^*(s-\kappa))\Big)\Phi'(s)ds\Big\|_p^p\\&\leq 2^{p-1}\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}\\&\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big(\big\|q(s)-q^*(s)\big\|_p^p+\big\|q(s-\kappa)-q^*(s-\kappa)\big\|_p^p\Big)\Phi'(s)ds.\end{aligned} \tag{3.27}$$

By B-D-G-I and $(\hbar_1)$, we obtain:

$$\begin{aligned}&\Big\|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\eta(s,q(s),q(s-\kappa))-\eta(s,q^*(s),q^*(s-\kappa))\Big)\Phi'(s)dB(s)\Big\|_p^p\\&=\sum_{\iota=1}^{\mathfrak{b}}E\Big|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\eta_\iota(s,q(s),q(s-\kappa))-\eta_\iota(s,q^*(s),q^*(s-\kappa))\Big)\Phi'(s)dB(s)\Big|^p\\&\leq\sum_{\iota=1}^{\mathfrak{b}}C_pE\Big|\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big|\eta_\iota(s,q(s),q(s-\kappa))-\eta_\iota(s,q^*(s),q^*(s-\kappa))\Big|^2(\Phi'(s))^2ds\Big|^{\frac{p}{2}}\\&\leq\sum_{\iota=1}^{\mathfrak{b}}C_pE\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big|\eta_\iota(s,q(s),q(s-\kappa))-\eta_\iota(s,q^*(s),q^*(s-\kappa))\Big|^p(\Phi'(s))^2ds\\&\Big(\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}(\Phi'(s))^2ds\Big)^{\frac{p-2}{2}}\\&\leq\sum_{\iota=1}^{\mathfrak{b}}C_pE\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big|\eta_\iota(s,q(s),q(s-\kappa))-\eta_\iota(s,q^*(s),q^*(s-\kappa))\Big|^p(\Phi'(s))^2ds\\&\Big(\sup_{0<s\leq t}(\Phi'(s))\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Phi'(s)ds\Big)^{\frac{p-2}{2}}\\&\leq 2^{p-1}\aleph^{\frac{p-2}{2}}\Big(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\Big)^{\frac{p-2}{2}}\mu^pC_p\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big(\|q(s)-q^*(s)\|_p^p+\|q(s-\kappa)-q^*(s-\kappa)\|_p^p\Big)\\&(\Phi'(s))^2ds\\&\leq 2^{p-1}\aleph^{\frac{p-2}{2}}\Big(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\Big)^{\frac{p-2}{2}}\mu^pC_p\sup_{0<s\leq t}\Phi'(s)\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\\&\Big(\|q(s)-q^*(s)\|_p^p+\|q(s-\kappa)-q^*(s-\kappa)\|_p^p\Big)\Phi'(s)ds.\end{aligned}$$

Accordingly, we derive

$$\Big\|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\eta(s,q(s),q(s-\kappa))-\eta(s,q^*(s),q^*(s-\kappa))\Big)\Phi'(s)dB(s)\Big\|_p^p$$

$$\leq 2^{p-1}\aleph^{\frac{p-2}{2}}\left(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\right)^{\frac{p-2}{2}}\mu^{p}C_{p}\aleph\int_{0}^{t}(\Phi(t)-\Phi(s))^{2\alpha-2} \tag{3.28}$$
$$\left(\left\|q(s)-q^{*}(s)\right\|_{p}^{p}+\left\|q(s-\kappa)-q^{*}(s-\kappa)\right\|_{p}^{p}\right)\Phi'(s)ds.$$

From (3.22), we arrive at:

$$\left\|\mathcal{Z}_{\vartheta}(q(t))-\mathcal{Z}_{\vartheta}(q^{*}(t))\right\|_{p}^{p}$$
$$\leq 2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_{r})}\right)^{p}\aleph\mathfrak{n}^{p-1}\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_{r}-2\mathfrak{m}_{r}+1)}(p-1)^{p-1}}{(p\mathfrak{m}_{r}-2\mathfrak{m}_{r}+1)^{p-1}}\left\|q(s)-q^{*}(s)\right\|_{\varpi}^{p} \tag{3.29}$$
$$\frac{\Gamma(2\mathfrak{m}_{r}-1)\exp\left(\varpi(\Phi(t)-\Phi(0))\right)}{\varpi^{2\mathfrak{m}_{r}-1}}$$
$$+\frac{2^{2p-2}}{(\Gamma(\alpha))^{p}}\left(2^{p-1}\aleph\mathfrak{n}^{p-1}\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}+2^{p-1}\aleph^{\frac{p-2}{2}}\left(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\right)^{\frac{p-2}{2}}\mu^{p}C_{p}\aleph\right)$$
$$\int_{0}^{t}\left(\left\|q(s)-q^{*}(s)\right\|_{p}^{p}+\left\|q(s-\kappa)-q^{*}(s-\kappa)\right\|_{p}^{p}\right)(\Phi(t)-\Phi(s))^{2\alpha-2}\Phi'(s)ds. \tag{3.30}$$

Following (3.29), we establish:

$$\frac{\left\|\mathcal{Z}_{\vartheta}(q(t))-\mathcal{Z}_{\vartheta}(q^{*}(t))\right\|_{p}^{p}}{\exp\left(\varpi(\Phi(t)-\Phi(0))\right)}\leq\frac{1}{\exp\left(\varpi(\Phi(t)-\Phi(0))\right)}2^{(w+1)p-(w+1)}$$
$$\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_{r})}\right)^{p}\aleph\mathfrak{n}^{p-1}\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_{r}-2\mathfrak{m}_{r}+1)}(p-1)^{p-1}}{(p\mathfrak{m}_{r}-2\mathfrak{m}_{r}+1)^{p-1}}\left\|q(s)-q^{*}(s)\right\|_{\varpi}^{p}\frac{\Gamma(2\mathfrak{m}_{r}-1)\exp\left(\varpi(\Phi(t)-\Phi(0))\right)}{\varpi^{2\mathfrak{m}_{r}-1}}$$
$$+\frac{2^{2p-2}}{(\Gamma(\alpha))^{p}}\frac{1}{\exp\left(\varpi(\Phi(t)-\Phi(0))\right)}$$
$$\left(2^{p-1}\aleph\mathfrak{n}^{p-1}\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}+2^{p-1}\aleph^{\frac{p-2}{2}}\left(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\right)^{\frac{p-2}{2}}\mu^{p}C_{p}\aleph\right)$$
$$\int_{0}^{t}\left(\frac{\left\|q(s)-q^{*}(s)\right\|_{p}^{p}}{\exp\left(\varpi(\Phi(s)-\Phi(0))\right)}\exp\left(\varpi(\Phi(s)-\Phi(0))\right)+\frac{\left\|q(s-\kappa)-q^{*}(s-\kappa)\right\|_{p}^{p}}{\exp\left(\varpi(\Phi(s-\kappa)-\Phi(-\kappa))\right)}\exp\left(\varpi(\Phi(s-\kappa)-\Phi(-\kappa))\right)\right)$$
$$(\Phi(t)-\Phi(s))^{2\alpha-2}\Phi'(s)ds$$
$$\leq 2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_{r})}\right)^{p}\aleph\mathfrak{n}^{p-1}\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_{r}-2\mathfrak{m}_{r}+1)}(p-1)^{p-1}}{(p\mathfrak{m}_{r}-2\mathfrak{m}_{r}+1)^{p-1}}\left\|q(s)-q^{*}(s)\right\|_{\varpi}^{p}\frac{\Gamma(2\mathfrak{m}_{r}-1)}{\varpi^{2\mathfrak{m}_{r}-1}}$$
$$+\frac{2^{2p-2}}{(\Gamma(\alpha))^{p}}\frac{1}{\exp\left(\varpi(\Phi(t)-\Phi(0))\right)}$$
$$\left(2^{p-1}\aleph\mathfrak{n}^{p-1}\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}+2^{p-1}\aleph^{\frac{p-2}{2}}\left(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\right)^{\frac{p-2}{2}}\mu^{p}C_{p}\aleph\right)$$
$$\int_{0}^{t}\left(\exp\left(\varpi(\Phi(s)-\Phi(0))\right)esssup_{s\in[0,T]}\left(\frac{\left\|q(s)-q^{*}(s)\right\|_{p}^{p}}{\exp\left(\varpi(\Phi(s)-\Phi(0))\right)}\right)+\exp(\varpi(\Phi(s-\kappa)-\Phi(-\kappa)))\right.$$

$$\underset{s\in[0,T]}{esssup}\Big(\frac{\left\|q(s-\kappa)-q^*(s-\kappa)\right\|_p^p}{\exp(\varpi(\Phi(s-\kappa)-\Phi(-\kappa)))}\Big)\Big)(\Phi(t)-\Phi(s))^{2\alpha-2}\Phi'(s)ds$$

$$\leq 2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}\left\|q(s)-q^*(s)\right\|_\varpi^p$$

$$\frac{\Gamma(2\mathfrak{m}_r-1)\exp(\varpi(\Phi(t)-\Phi(0)))}{\varpi^{2\mathfrak{m}_r-1}}+\frac{2^{2p-2}}{(\Gamma(\alpha))^p}\frac{1}{\exp(\varpi(\Phi(t)-\Phi(0)))}$$

$$\left(2^{p-1}\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}+2^{p-1}\aleph^{\frac{p-2}{2}}\left(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\right)^{\frac{p-2}{2}}\mu^p C_p\aleph\right)$$

$$\left\|q(s)-q^*(s)\right\|_\varpi^p\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big(\exp(\varpi(\Phi(s)-\Phi(0)))+\exp(\varpi(\Phi(s-\kappa)-\Phi(-\kappa)))\Big)\Phi'(s)ds$$

$$\leq 2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}\left\|q(s)-q^*(s)\right\|_\varpi^p\frac{\Gamma(2\mathfrak{m}_r-1)}{\varpi^{2\mathfrak{m}_r-1}}$$

$$+\frac{2^{2p-2}}{(\Gamma(\alpha))^p}\frac{1}{\exp\left(\varpi(\Phi(t)-\Phi(0))\right)}$$

$$\left(2^{p-1}\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}+2^{p-1}\aleph^{\frac{p-2}{2}}\left(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\right)^{\frac{p-2}{2}}\mu^p C_p\aleph\right)$$

$$2\left\|q(s)-q^*(s)\right\|_\varpi^p\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Phi'(s)\exp\left(\varpi(\Phi(s)-\Phi(0))\right)ds$$

$$\leq 2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}\left\|q(s)-q^*(s)\right\|_\varpi^p\frac{\Gamma(2\mathfrak{m}_r-1)}{\varpi^{2\mathfrak{m}_r-1}}$$

$$+\frac{2^{2p-2}}{(\Gamma(\alpha))^p}\left(2^{p-1}\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}+2^{p-1}\aleph^{\frac{p-2}{2}}\left(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\right)^{\frac{p-2}{2}}\mu^p C_p\aleph\right)$$

$$2\left\|q(s)-q^*(s)\right\|_\varpi^p\frac{\Gamma(2\alpha-1)}{\varpi^{2\alpha-1}}. \tag{3.31}$$

Accordingly, from (3.31), we get:

$$\|\mathcal{Z}_\vartheta(q(t))-\mathcal{Z}_\vartheta(q^*(t))\|_\varpi^p\leq\Bigg(2^{(w+1)p-(w+1)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^p \aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}$$

$$\frac{\Gamma(2\mathfrak{m}_r-1)}{\varpi^{2\mathfrak{m}_r-1}}+\frac{2^{2p-2}}{(\Gamma(\alpha))^p}\left(2^{p-1}\aleph\mathfrak{n}^{p-1}\frac{\mu^p(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}+2^{p-1}\aleph^{\frac{p-2}{2}}\left(\frac{(\Phi(t)-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\right)^{\frac{p-2}{2}}\right.$$

$$\left.\mu^p C_p\aleph\right)\frac{2\Phi\Gamma(2\alpha-1)}{\varpi^{2\alpha-1}}\Bigg)\left\|q(s)-q^*(s)\right\|_\varpi^p. \tag{3.32}$$

Consequently, we acquire

$$\|\mathcal{Z}_\vartheta(q(t))-\mathcal{Z}_\vartheta(q^*(t))\|_\varpi\leq\zeta^{\frac{1}{p}}\|q(s)-q^*(s)\|_\varpi. \tag{3.33}$$

According to (3.21), we find that $\zeta<1$. Hence, the required conclusion is confirmed.

**Theorem 3.2.** *The following assertion is valid for all $\vartheta$ and $\vartheta'$:*

$$\|\daleth_\alpha(\mathrm{t},\vartheta)-\daleth_\alpha(\mathrm{t},\vartheta')|_\mathrm{p}\leq\mu\|\vartheta-\vartheta'\|_\mathrm{p},\ \forall\ \mathrm{t}\in[0,\ \mathrm{T}]. \tag{3.34}$$

*Proof.* Therefore, it follows that

$$\begin{aligned}
&\daleth_\alpha(\mathrm{t},\vartheta)-\daleth_\alpha(\mathrm{t},\vartheta')=\vartheta-\vartheta'\\
&+\sum\nolimits_{\mathrm{r}=1}^{\mathrm{w}}\frac{1}{\Gamma(\mathfrak{m}_\mathrm{r})}\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\mathfrak{m}_\mathrm{r}-1}\Big(\ell_\mathrm{r}(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta))-\ell_\mathrm{r}(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta'))\Big)\Phi'(\mathrm{s})\mathrm{ds}\\
&+\frac{1}{\Gamma(\alpha)}\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\alpha-1}\Big(\lambda(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta))-\lambda(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta'),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta'))\Big)\Phi'(\mathrm{s})\mathrm{ds}\\
&+\frac{1}{\Gamma(\alpha)}\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\alpha-1}(\eta(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta))-\eta(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta'),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta')))\Phi'(\mathrm{s})\mathrm{dB}(\mathrm{s}).
\end{aligned}\tag{3.35}$$

Inserting (3.2) into (3.35) yields

$$\begin{aligned}
&\Big\|\daleth_\alpha(\mathrm{t},\vartheta)-\daleth_\alpha(\mathrm{t},\vartheta')\Big\|_\mathrm{p}^\mathrm{p}\leq 2^{2\mathrm{p}-2}\Big\|\vartheta-\vartheta'\Big\|_\mathrm{p}^\mathrm{p}+2^{(\mathrm{w}+2)\mathrm{p}-(\mathrm{w}+2)}\sum\nolimits_{\mathrm{r}=1}^{\mathrm{w}}\left(\frac{1}{\Gamma(\mathfrak{m}_\mathrm{r})}\right)^\mathrm{p}\\
&\Big\|\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\mathfrak{m}_\mathrm{r}-1}\Big(\ell_\mathrm{r}(\mathrm{s},\daleth(\mathrm{s},\vartheta))-\ell_\mathrm{r}(\mathrm{s},\daleth(\mathrm{s},\vartheta'))\Big)\Phi'(\mathrm{s})\mathrm{ds}\Big\|_\mathrm{p}^\mathrm{p}\\
&+\frac{2^{2\mathrm{p}-2}}{(\Gamma(\alpha))^\mathrm{p}}\Big\|\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\alpha-1}\Big(\lambda(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta))-\lambda(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta'),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta'))\Big)\Phi'(\mathrm{s})\mathrm{ds}\Big\|_\mathrm{p}^\mathrm{p}\\
&+\frac{2^{2\mathrm{p}-2}}{(\Gamma(\alpha))^\mathrm{p}}\Big\|\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\alpha-1}\Big(\eta(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta)))-\eta(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta))\Big)\Phi'(\mathrm{s})\mathrm{dB}(\mathrm{s})\Big\|_\mathrm{p}^\mathrm{p}.
\end{aligned}\tag{3.36}$$

By H-I and ($\hbar_1$), we deduce that

$$\begin{aligned}
&2^{(\mathrm{w}+2)\mathrm{p}-(\mathrm{w}+2)}\sum\nolimits_{\mathrm{r}=1}^{\mathrm{w}}\left(\frac{1}{\Gamma(\mathfrak{m}_\mathrm{r})}\right)^\mathrm{p}\Big\|\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\mathfrak{m}_\mathrm{r}-1}\Big(\ell_\mathrm{r}(\mathrm{s},\daleth(\mathrm{s},\vartheta))-\ell_\mathrm{r}(\mathrm{s},\daleth(\mathrm{s},\vartheta'))\Big)\Phi'(\mathrm{s})\mathrm{ds}\Big\|_\mathrm{p}^\mathrm{p}\\
&=2^{(\mathrm{w}+2)\mathrm{p}-(\mathrm{w}+2)}\sum\nolimits_{\mathrm{r}=1}^{\mathrm{w}}\left(\frac{1}{\Gamma(\mathfrak{m}_\mathrm{r})}\right)^\mathrm{p}\sum\nolimits_{\iota=1}^{\mathfrak{b}}\mathrm{E}\Big(\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\mathfrak{m}_\mathrm{r}-1}\Big(\ell_{\iota,\mathrm{r}}(\mathrm{s},\daleth(\mathrm{s},\vartheta))-\ell_{\iota,\mathrm{r}}(\mathrm{s},\daleth(\mathrm{s},\vartheta'))\Big)\Phi'(\mathrm{s})\mathrm{ds}\Big)^\mathrm{p}\\
&\leq 2^{(\mathrm{w}+2)\mathrm{p}-(\mathrm{w}+2)}\sum\nolimits_{\mathrm{r}=1}^{\mathrm{w}}\left(\frac{1}{\Gamma(\mathfrak{m}_\mathrm{r})}\right)^\mathrm{p}\sum\nolimits_{\iota=1}^{\mathfrak{b}}\mathrm{E}\Big(\Big(\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\frac{(\mathfrak{m}_\mathrm{r}-1)(\mathrm{p}-2)}{\mathrm{p}-1}}(\Phi'(\mathrm{s}))^{\frac{\mathrm{p}-2}{\mathrm{p}-1}}\mathrm{ds}\Big)^{\mathrm{p}-1}\\
&\Big(\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\mathfrak{m}_\mathrm{r}-2}\Big|\ell_{\iota,\mathrm{r}}(\mathrm{s},\daleth(\mathrm{s},\vartheta))-\ell_{\iota,\mathrm{r}}(\mathrm{s},\daleth(\mathrm{s},\vartheta'))\Big|\Phi'(\mathrm{s})\mathrm{ds}\Big)\Big)\\
&\leq 2^{(\mathrm{w}+2)\mathrm{p}-(\mathrm{w}+2)}\sum\nolimits_{\mathrm{r}=1}^{\mathrm{w}}\left(\frac{1}{\Gamma(\mathfrak{m}_\mathrm{r})}\right)^\mathrm{p}\sum\nolimits_{\iota=1}^{\mathfrak{b}}\mathrm{E}\Big(\Big(\sup_{0<\mathrm{s}\leq\mathrm{t}}(\Phi'(\mathrm{s}))^{\frac{1}{1-\mathrm{p}}}\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\frac{(\mathfrak{m}_\mathrm{r}-1)(\mathrm{p}-2)}{\mathrm{p}-1}}\Phi'(\mathrm{s})\mathrm{ds}\Big)^{\mathrm{p}-1}\\
&\Big(\int_0^\mathrm{t}(\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\mathfrak{m}_\mathrm{r}-2}\Big|\ell_{\iota,\mathrm{r}}(\mathrm{s},\daleth(\mathrm{s},\vartheta))-\ell_{\iota,\mathrm{r}}(\mathrm{s},\daleth(\mathrm{s},\vartheta'))\Big|(\Phi'(\mathrm{s}))^2\mathrm{ds}\Big)\Big)
\end{aligned}$$

$$\leq 2^{(w+2)p-(w+2)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^{p}\mathfrak{n}^{p-1}\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}$$
$$\sup_{0<s\leq t}\Phi'(s)\int_0^t(\Phi(t)-\Phi(s))^{2\mathfrak{m}_r-2}\left(\left\|\beth(s,\vartheta)-\beth(s,\vartheta')\right\|_p^p\right)\Phi'(s)ds. \quad (3.37)$$

Consequently, we derive

$$2^{(w+2)p-(w+2)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^{p}\left\|\int_0^t(\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}(\ell_r(s,\beth_\alpha(s,\vartheta))-\ell_r(s,\beth(s,\vartheta')))\Phi'(s)ds\right\|_p^p$$
$$\leq 2^{(w+2)p-(w+2)}\sum_{r=1}^{w}\left(\frac{1}{\Gamma(\mathfrak{m}_r)}\right)^{p}\aleph\mathfrak{n}^{p-1}\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)}(p-1)^{p-1}}{(p\mathfrak{m}_r-2\mathfrak{m}_r+1)^{p-1}}$$
$$\int_0^t(\Phi(t)-\Phi(s))^{2\mathfrak{m}_r-2}\left\|\beth(s,\vartheta)-\beth(s,\vartheta')\right\|_p^p\Phi'(s)ds. \quad (3.38)$$

Invoking H-I and $(\hbar_1)$ on the second term of (3.36), we obtain

$$\left\|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\lambda(s,\beth_\alpha(s,\vartheta),\beth_\alpha(s-\kappa,\vartheta))-\lambda(s,\beth_\alpha(s,\vartheta'),\beth_\alpha(s-\kappa,\vartheta'))\Big)\Phi'(s)ds\right\|_p^p$$
$$=\sum_{\iota=1}^{\mathfrak{b}}E\left(\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\lambda_\iota(s,\beth_\alpha(s,\vartheta),\beth_\alpha(s-\kappa,\vartheta))-\lambda_\iota(s,\beth_\alpha(s,\vartheta'),\beth_\alpha(s-\kappa,\vartheta'))\Big)\Phi'(s)ds\right)^p$$
$$\leq\sum_{\iota=1}^{\mathfrak{b}}E\Bigg(\left(\int_0^t(\Phi(t)-\Phi(s))^{\frac{(\alpha-1)(p-2)}{p-1}}(\Phi'(s))^{\frac{p-2}{p-1}}ds\right)^{p-1}$$
$$\left(\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big|\lambda_\iota(s,\beth_\alpha(s,\vartheta),\beth_\alpha(s-\kappa,\vartheta))-\lambda_\iota(s,\beth_\alpha(s,\vartheta'),\beth_\alpha(s-\kappa,\vartheta'))\Big|(\Phi'(s))^2ds\right)\Bigg)$$
$$\leq\sum_{\iota=1}^{\mathfrak{b}}E\Bigg(\left(\sup_{0<s\leq t}(\Phi'(s))^{\frac{1}{1-p}}\int_0^t(\Phi(t)-\Phi(s))^{\frac{(\alpha-1)(p-2)}{p-1}}\Phi'(s)ds\right)^{p-1}$$
$$\left(\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big|\lambda_\iota(s,\beth_\alpha(s,\vartheta),\beth_\alpha(s-\kappa,\vartheta))-\lambda_\iota(s,\beth_\alpha(s,\vartheta'),\beth_\alpha(s-\kappa,\vartheta'))\Big|(\Phi'(s))^2ds\right)\Bigg)$$
$$\leq 2^{p-1}\mathfrak{n}^{p-1}\left(\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}\right)\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\left(\left\|\beth_\alpha(s,\vartheta)-\beth_\alpha(s,\vartheta')\right\|_p^p\right)(\Phi'(s))^2ds.$$
$$\leq 2^{p-1}\mathfrak{n}^{p-1}\left(\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}\right)\sup_{0<s\leq t}\Phi'(s)\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\left(\left\|\beth_\alpha(s,\vartheta)-\beth_\alpha(s,\vartheta')\right\|_p^p\right)$$
$$\Phi'(s)ds$$
$$=2^{p-1}\mathfrak{n}^{p-1}\left(\frac{\mu^{p}(\Phi(t)-\Phi(0))^{(p\alpha-2\alpha+1)}(p-1)^{p-1}}{(p\alpha-2\alpha+1)^{p-1}}\right)\aleph\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\left(\left\|\beth_\alpha(s,\vartheta)-\beth_\alpha(s,\vartheta')\right\|_p^p\right)\Phi'(s)ds. \quad (3.39)$$

The application of the B-D-G-I and condition $(\hbar_1)$ to the third term of (3.36) yields the result that

$$\left\|\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Big(\eta(s,\beth_\alpha(s,\vartheta),\beth_\alpha(s-\kappa,\vartheta))-\eta(s,\beth_\alpha(s,\vartheta'),\beth_\alpha(s-\kappa,\vartheta'))\Big)\Phi'(s)dB(s)\right\|_p^p$$

$$
\begin{aligned}
&= \sum_{\imath=1}^{\mathfrak{b}} \mathrm{E}\Big| \int_0^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\alpha-1}\Big(\eta_\imath(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta)) - \eta_\imath(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta'),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta'))\Big)\Phi'(\mathrm{s})\mathrm{dB}(\mathrm{s})\Big|^{\mathrm{p}} \\
&\le \sum_{\imath=1}^{\mathfrak{b}} \mathcal{C}_{\mathrm{p}}\mathrm{E}\Big| \int_0^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\alpha-2}\Big|\eta_\imath(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta)) - \eta_\imath(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta'),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta'))\Big|^2(\Phi'(\mathrm{s}))^2\mathrm{ds}\Big|^{\frac{\mathrm{p}}{2}} \\
&\le \sum_{\imath=1}^{\mathfrak{b}} \mathcal{C}_{\mathrm{p}}\mathrm{E} \int_0^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\alpha-2}\Big|\eta_\imath(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta)) - \eta_\imath(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta'),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta'))\Big|^{\mathrm{p}}(\Phi'(\mathrm{s}))^2\mathrm{ds} \\
&\quad \Big(\int_0^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\alpha-2}(\Phi'(\mathrm{s}))^2\mathrm{ds}\Big)^{\frac{\mathrm{p}-2}{2}} \\
&\le \sum_{\imath=1}^{\mathfrak{b}} \mathcal{C}_{\mathrm{p}}\mathrm{E} \int_0^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\alpha-2}\Big|\eta_\imath(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta)) - \eta_\imath(\mathrm{s},\daleth_\alpha(\mathrm{s},\vartheta'),\daleth_\alpha(\mathrm{s}-\kappa,\vartheta'))\Big|^{\mathrm{p}}(\Phi'(\mathrm{s}))^2\mathrm{ds} \\
&\quad \Big(\sup_{0<\mathrm{s}\le\mathrm{t}}\Phi'(\mathrm{s}) \int_0^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\alpha-2}\Phi'(\mathrm{s})\mathrm{ds}\Big)^{\frac{\mathrm{p}-2}{2}} \\
&\le 2^{\mathrm{p}-1}\aleph^{\frac{\mathrm{p}-2}{2}}\mu^{\mathrm{p}}\mathcal{C}_{\mathrm{p}}\Big(\frac{(\Phi(\mathrm{t})-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\Big)^{\frac{\mathrm{p}-2}{2}} \int_0^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\alpha-2}\Big(\Big\|\daleth_\alpha(\mathrm{s},\vartheta)-\daleth_\alpha(\mathrm{s},\vartheta')\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big)(\Phi'(\mathrm{s}))^2\mathrm{ds}. \\
&\le 2^{\mathrm{p}-1}\aleph^{\frac{\mathrm{p}-2}{2}}\mu^{\mathrm{p}}\mathcal{C}_{\mathrm{p}}\Big(\frac{(\Phi(\mathrm{t})-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\Big)^{\frac{\mathrm{p}-2}{2}} \sup_{0<\mathrm{s}\le\mathrm{t}}\Phi'(\mathrm{s}) \int_0^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\alpha-2}\Big(\Big\|\daleth_\alpha(\mathrm{s},\vartheta)-\daleth_\alpha(\mathrm{s},\vartheta')\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big)\Phi'(\mathrm{s})\mathrm{ds}. \\
&= 2^{\mathrm{p}-1}\aleph^{\frac{\mathrm{p}-2}{2}}\mu^{\mathrm{p}}\mathcal{C}_{\mathrm{p}}\Big(\frac{(\Phi(\mathrm{t})-\Phi(0))^{(2\alpha-1)}}{(2\alpha-1)}\Big)^{\frac{\mathrm{p}-2}{2}} \aleph \int_0^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{2\alpha-2}\Big(\Big\|\daleth_\alpha(\mathrm{s},\vartheta)-\daleth_\alpha(\mathrm{s},\vartheta')\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big)\Phi'(\mathrm{s})\mathrm{ds}. \qquad (3.40)
\end{aligned}
$$

By considering (3.36), (3.38)–(3.40), we establish

$$
\begin{aligned}
\Big\|\daleth_\alpha(\mathrm{t},\vartheta)-\daleth_\alpha(\mathrm{t},\vartheta')\Big\|_{\varpi}^{\mathrm{p}} \le{}& \frac{2^{2\mathrm{p}-2}}{\exp(\varpi(\Phi(\mathrm{t})-\Phi(0)))}\|\vartheta-\vartheta'\|_{\mathrm{p}}^{\mathrm{p}} \\
&+ \zeta\Big\|\daleth_\alpha(\mathrm{s},\vartheta)-\daleth_\alpha(\mathrm{s},\vartheta')\Big\|_{\varpi}^{\mathrm{p}}. \qquad (3.41)
\end{aligned}
$$

Accordingly, we conclude

$$
\Big\|\daleth_\alpha(\mathrm{t},\vartheta)-\daleth_\alpha(\mathrm{t},\vartheta')\Big\|_{\varpi}^{\mathrm{p}}(1-\zeta) \le \frac{2^{2\mathrm{p}-2}}{\exp(\varpi(\Phi(\mathrm{t})-\Phi(0)))}\|\vartheta-\vartheta'\|_{\mathrm{p}}^{\mathrm{p}}.
$$

Consequently, we arrive at the required result:

$$
\lim_{\vartheta\to\vartheta'}\Big\|\daleth_\alpha(\mathrm{t},\vartheta)-\daleth_\alpha(\mathrm{t},\vartheta')\Big\|_{\varpi}^{\mathrm{p}} = 0.
$$

## 4. Stability results

Initially, GGI involving the Φ-Cap-FD is presented. Subsequently, we verify the UHS of FSIDEs in the pth moment.

**Lemma 4.1.** *Assume* $\mathbb{M}(\mathrm{t}) > 0$ *and* $\rho(\mathrm{t}) > 0$ *are the functions on the interval* $\mathrm{t} \in [\mathfrak{m},\mathrm{T})$*, where* $\mathfrak{m} \ge 0$ *and* $\mathrm{T} \le +\infty$*. Subsequently, address* $\mathbb{Z} : [\mathfrak{m},\mathrm{T}) \to [0,\theta]$*, which is continuous and nondecreasing, with* $\theta$ *a positive real number.*
*If following hold:*

$$
\rho(\mathrm{t}) \le \mathbb{M}(\mathrm{t}) + \frac{\mathbb{Z}(\mathrm{t})}{\Gamma(\alpha)}\int_{\mathfrak{m}}^{\mathrm{t}} (\Phi(\mathrm{t})-\Phi(\mathrm{s}))^{\alpha-1}\Phi'(\mathrm{s})\rho(\mathrm{s})\mathrm{ds},\ \ \mathrm{t}\in[\mathfrak{m},\mathrm{T}),
$$

*then*

$$\rho(\mathrm{t}) \leq \mathbb{M}(\mathrm{t}) + \int_{\mathfrak{m}}^{\mathrm{t}} \Big( \sum_{\jmath=0}^{\infty} \frac{(\mathbb{Z}(\mathrm{t}))^{\jmath}}{\Gamma(\jmath\alpha)} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\jmath\alpha-1} \mathbb{M}(\mathrm{s}) \Big) \Phi'(\mathrm{s}) d\mathrm{s}, \ \mathrm{t} \in [\mathfrak{m}, \mathrm{T}).$$

*Proof.* For locally integrable function $F$, assume

$$\phi F(\mathrm{t}) = \frac{\mathbb{Z}(\mathrm{t})}{\Gamma(\alpha)} \int_{\mathfrak{m}}^{\mathrm{t}} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\alpha-1} \Phi'(\mathrm{s}) F(\mathrm{s}) d\mathrm{s}.$$

So, we get

$$\rho(\mathrm{t}) \leq \mathbb{M}(\mathrm{t}) + \phi\rho(\mathrm{t}),$$

therefore,

$$\rho(\mathrm{t}) \leq \sum_{\iota=0}^{\jmath-1} \phi^{\iota} \mathbb{M}(\mathrm{t}) + \phi\rho^{\jmath}(\mathrm{t}).$$

We need to prove

$$\phi^{\jmath}\rho(\mathrm{t}) \leq \int_0^{\mathrm{t}} \frac{(\mathbb{Z})^{\alpha}}{\Gamma(\jmath\alpha)} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\jmath\alpha-1} \Phi'(\mathrm{s}) \rho(\mathrm{s}) d\mathrm{s} \tag{4.1}$$

and $\phi^{\jmath}\rho(\mathrm{t}) \to 0$ when $\jmath \to \infty$ $\forall \mathrm{t} \in [\mathfrak{m}, \mathrm{T})$.

The case $\jmath = 1$ satisfies (4.1). **Induction hypothesis:** Assume the inequality is valid for $\jmath = \iota$ where $\iota \geq 1$. **Inductive step ($\jmath = \iota + 1$):** We now demonstrate that (4.1) holds for $\jmath = \iota + 1$, which completes the proof by induction.

$$\begin{aligned} \phi^{\iota+1}\rho(\mathrm{t}) =& \phi(\phi^{\iota}\rho(\mathrm{t})) \leq \frac{\mathbb{Z}(\mathrm{t})}{\Gamma(\alpha)} \int_0^{\mathrm{t}} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\alpha-1} \Phi'(\mathrm{s}) \\ & \Big( \int_0^{\mathrm{s}} \frac{(\mathbb{Z}(\mathrm{s}))^{\iota}}{\Gamma(\iota\alpha)} (\Phi(\mathrm{s}) - \Phi(\xi))^{\iota\alpha-1} \Phi'(\xi) \rho(\xi) d\xi \Big) d\mathrm{s} \\ & \leq \frac{(\mathbb{Z}(\mathrm{t}))^{\iota+1}}{\Gamma(\alpha)} \int_0^{\mathrm{t}} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\alpha-1} \Phi'(\mathrm{s}) \\ & \Big( \int_0^{\mathrm{s}} \frac{1}{\Gamma(\iota\alpha)} (\Phi(\mathrm{s}) - \Phi(\xi))^{\iota\alpha-1} \Phi'(\xi) \rho(\xi) d\xi \Big) d\mathrm{s}. \end{aligned} \tag{4.2}$$

It follows that we have the following:

$$\phi^{\iota+1}\rho(\mathrm{t}) \leq \int_0^{\mathrm{t}} \frac{(\mathbb{Z}(\mathrm{t}))^{\iota+1}}{\Gamma((\iota+1)\alpha)} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{(\iota+1)\alpha-1} \Phi'(\mathrm{s}) \rho(\mathrm{s}) d\mathrm{s}.$$

Therefore, inequality (4.1) is verified. Proceeding, we use inequality (4.1) to reach the following:

$$\phi^{\jmath}\rho(\mathrm{t}) \leq \int_0^{\mathrm{t}} \frac{\mathrm{T}^{\jmath}}{\Gamma(\jmath\alpha)} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\jmath\alpha-1} \Phi'(\mathrm{s}) \rho(\mathrm{s}) d\mathrm{s} \to 0.$$

As $\jmath \to +\infty$ for $\mathrm{t} \in [\mathfrak{m}, \mathrm{T})$, the proof is completed.

Under the condition $\mathbb{Z}(\mathrm{t}) = \mathfrak{r}$ as stated in Lemma 4.1, the following inequality follows.

**Corollary 4.1.** *Presume* $\mathfrak{r}$ *and* $\alpha$ *are positive, and that* $\mathbb{M}(\mathrm{t})$ *is a nonnegative, locally integrable mapping for* $0 \leq \mathrm{t} < \mathrm{T}$*, where* $\mathrm{T} \leq +\infty$*. Additionally, let* $\rho(\mathrm{t})$ *satisfy the same conditions. Then, we obtain:*

$$\rho(\mathrm{t}) \leq \mathbb{M}(\mathrm{t}) + \mathfrak{r}\int_0^{\mathrm{t}} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\alpha-1}\Phi'(\mathrm{s})\rho(\mathrm{s})\mathrm{ds},$$

*then*

$$\rho(\mathrm{t}) \leq \mathbb{M}(\mathrm{t}) + \int_0^{\mathrm{t}} \Big(\sum_{\jmath=0}^{\infty} \frac{(\mathfrak{r}\Gamma(\alpha))^{\jmath}}{\Gamma(\jmath\alpha)}(\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\jmath\alpha-1}\mathbb{M}(\mathrm{s})\Big)\Phi'(\mathrm{s})\mathrm{ds},\ 0 \leq \mathrm{t} < \mathrm{T}.$$

**Corollary 4.2.** *Assuming the conditions of Lemma 4.1 hold, and that* $\mathbb{M}(\mathrm{t})$ *is nondecreasing over* $[0, \mathrm{T})$*, we derive the following:*

$$\rho(\mathrm{t}) \leq \mathbb{M}(\mathrm{t})\mathbb{E}_{\alpha}(\mathbb{Z}(\mathrm{t})\Gamma(\alpha)(\Phi(\mathrm{t}) - \Phi(0))^{\alpha}),$$

*here* $\mathbb{E}_{\alpha}(\mathrm{t}) = \sum_{\iota=0}^{\infty} \frac{\mathrm{t}^{\iota}}{\Gamma(\iota\alpha+1)}$.

*Proof.* By the hypotheses

$$\begin{aligned}
\rho(\mathrm{t}) &\leq \mathbb{M}(\mathrm{t})\Big(1 + \int_0^{\mathrm{t}} \sum_{\jmath=1}^{\infty} \frac{(\mathbb{Z}(\mathrm{t})\Gamma(\alpha))^{\jmath}}{\Gamma(\jmath\alpha}(\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\jmath\alpha-1}\Phi'(\mathrm{s})\mathrm{ds}\Big)\\
&= \mathbb{M}(\mathrm{t})\sum_{\jmath=0}^{\infty} \frac{(\mathbb{Z}(\mathrm{t})\Gamma(\alpha))^{\jmath}}{\Gamma(\jmath\alpha + 1)}\\
&= \mathbb{M}(\mathrm{t})\mathbb{E}_{\alpha}(\mathbb{Z}(\mathrm{t})\Gamma(\alpha)(\Phi(\mathrm{t}) - \Phi(0))).
\end{aligned}$$

This concludes the proof.

**Definition 4.1.** *We define system* (1.3) *as UHS for* $\varepsilon$ *if a positive constant* $\mathbb{V}$ *exists such that* $\varepsilon > 0$ *and any function* $\chi \in \mathcal{H}^{\mathrm{p}}(0, \mathrm{T})$ *satisfying* $\chi(0) = \chi_0$*,*

$$\begin{aligned}
&\mathrm{E}\Big(\Big\|\chi(\mathrm{t}) - \chi(0) - \Big(\sum\nolimits_{\mathrm{r}=1}^{\mathrm{w}} \frac{1}{\Gamma(\mathfrak{m}_{\mathrm{r}})}\int_0^{\mathrm{t}} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\mathfrak{m}_{\mathrm{r}}-1}\Phi'(\mathrm{s})\ell_{\mathrm{r}}(\mathrm{s}, \chi(\mathrm{s}))\mathrm{ds}\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\alpha-1}\Phi'(\mathrm{s})\lambda(\mathrm{s}, \chi(\mathrm{s}), \chi(\mathrm{s} - \kappa))\mathrm{ds}\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{t}} (\Phi(\mathrm{t}) - \Phi(\mathrm{s}))^{\alpha-1}\Phi'(\mathrm{s})\eta(\mathrm{s}, \chi(\mathrm{s}), \chi(\mathrm{s} - \kappa))\mathrm{dB}(\mathrm{s})\Big)\Big\|_{\mathrm{p}}^{\mathrm{p}}\Big) < \varepsilon,\ \mathrm{t} \in [0,\ \mathrm{T}], \qquad (4.3)
\end{aligned}$$

*we establish the existence of a solution* $\mathbb{U} \in \mathcal{H}^{\mathrm{p}}(0, \mathrm{T})$ *to* (1.3)*, satisfying the initial condition* $\mathbb{U}(\mathrm{t}) = \chi_0$ *for* $\mathrm{t} \in [0, \mathrm{T}]$*, and the following holds:*

$$\mathrm{E}(\|\chi(\mathrm{t}) - \mathbb{U}(\mathrm{t})\|_{\mathrm{p}}^{\mathrm{p}}) \leq \mathbb{V}\varepsilon,\ \forall \mathrm{t} \in [0, \mathrm{T}].$$

The upcoming theorem provides a proof of UHS for the class of delayed FSIDEs.

**Theorem 4.1.** *If conditions* $(\hbar_1)$ *and* $(\hbar_2)$ *are met, then system* (1.3) *exhibits UHS over* $[0, \mathrm{T}]$.

*Proof.* Let $\varepsilon > 0$, and consider $\chi \in \mathcal{H}^{p}(0,T)$ as the solution to Eq (4.3), while $\mathbb{U} \in \mathcal{H}^{p}(0,T)$ corresponds to the solution of Eq (1.3) with the initial condition $\mathbb{U}(0) = \chi_0$. Based on this setup, the following result is derived:

$$\begin{aligned}\mathbb{U}(t) = \chi(0) &+ \sum_{r=1}^{w} \frac{1}{\Gamma(\mathfrak{m}_r)} \int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Phi'(s)\ell_r(s,\mathbb{U}(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)\lambda(s,\mathbb{U}(s),\mathbb{U}(s-\kappa))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)\eta(s,\mathbb{U}(s),\mathbb{U}(s-\kappa))dB(s). \end{aligned} \tag{4.4}$$

Consequently, we arrive at

$$\begin{aligned}\chi(t)-\mathbb{U}(t) = \chi(t)-\chi(0) &- \Big(\sum_{r=1}^{w} \frac{1}{\Gamma(\mathfrak{m}_r)} \int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Phi'(s)\ell_r(s,\chi(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)\lambda(s,\chi(s),\chi(s-\kappa))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)\eta(s,\chi(s),\chi(s-\kappa))dB(s)\Big) \\ &+ \sum_{r=1}^{w} \frac{1}{\Gamma(\mathfrak{m}_r)} \int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Phi'(s)(\ell_r(s,\chi(s))-\ell_r(s,\mathbb{U}(s)))ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)(\lambda(s,\chi(s),\chi(s-\kappa))-\lambda(s,\mathbb{U}(s),\mathbb{U}(s-\kappa)))ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)(\eta(s,\chi(s),\chi(s-\kappa))-\eta(s,\mathbb{U}(s),\mathbb{U}(s-\kappa)))dB(s). \end{aligned} \tag{4.5}$$

Applying Jensen's inequality, H-I, and B-D-G-I, we conclude the following:

$$\begin{aligned}E(\|\chi(t)-\mathbb{U}(t)\|_p^p) \le 2^{p-1}E\Big(\Big\|\chi(t)-\chi(0) &- \Big(\sum_{r=1}^{w} \frac{1}{\Gamma(\mathfrak{m}_r)} \int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Phi'(s)\ell_r(s,\chi(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)\lambda(s,\chi(s),\chi(s-\kappa))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)\eta(s,\chi(s),\chi(s-\kappa))dB(s)\Big)\Big\|_p^p\Big) \\ &+ 2^{p-1}E\Big(\Big\|\sum_{r=1}^{w} \frac{1}{\Gamma(\mathfrak{m}_r)} \int_0^t (\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Phi'(s)(\ell_r(s,\chi(s))-\ell_r(s,\mathbb{U}(s)))ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)(\lambda(s,\chi(s),\chi(s-\kappa))-\lambda(s,\mathbb{U}(s),\mathbb{U}(s-\kappa)))ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)(\eta(s,\chi(s),\chi(s-\kappa))-\eta(s,\mathbb{U}(s),\mathbb{U}(s-\kappa)))dB(s)\Big)\Big\|_p^p \end{aligned}$$

$$\leq 2^{p-1}\varepsilon + 2^{(w+2)p-(w+2)}E\Big(\Big\|\sum_{r=1}^{w}\frac{1}{\Gamma(\mathfrak{m}_r)}\int_0^t(\Phi(t)-\Phi(s))^{\mathfrak{m}_r-1}\Phi'(s)(\ell_r(s,\chi(s))-\ell_r(s,\mathbb{U}(s)))ds\Big\|_p^p\Big)$$
$$+6^{p-1}E\Big(\Big\|\frac{1}{\Gamma(\alpha)}\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)(\lambda(s,\chi(s),\chi(s-\kappa))-\lambda(s,\mathbb{U}(s),\mathbb{U}(s-\kappa)))ds\Big\|_p^p\Big)$$
$$+6^{p-1}E\Big(\Big\|\frac{1}{\Gamma(\alpha)}\int_0^t(\Phi(t)-\Phi(s))^{\alpha-1}\Phi'(s)(\eta(s,\chi(s),\chi(s-\kappa))-\eta(s,\mathbb{U}(s),\mathbb{U}(s-\kappa)))dB(s)\Big\|_p^p\Big)$$
$$\leq 2^{p-1}\varepsilon + 2^{(w+2)p-(w+2)}\sum_{r=1}^{w}\Big(\frac{1}{\Gamma(\mathfrak{m}_r)}\Big)^p\mathbb{Y}^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\mathfrak{m}_r p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\mathfrak{m}_r p-1}\Big)^{p-1}$$
$$\int_0^t\big\|\ell_r(s,\chi(s)-\ell_r(s,\mathbb{U}(s))\big\|_p^p ds$$
$$+6^{p-1}\varphi^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\alpha p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\alpha p-1}\Big)^{p-1}\int_0^t\big\|\lambda(s,\chi(s),\chi(s-\kappa))-\lambda(s,\mathbb{U}(s),\mathbb{U}(s-\kappa))\big\|_p^p ds$$
$$+6^{p-1}\aleph^{\frac{p-2}{2}}C_p\Big(\frac{(\Phi(t)-\Phi(0))^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{p-2}{2}}$$
$$\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\big\|\eta(s,\chi(s,\chi(s-\kappa))-\eta(s,\mathbb{U}(s,\mathbb{U}(s-\kappa))\big\|_p^p(\Phi'(s))^2 ds$$
$$\leq 2^{p-1}\varepsilon + 2^{(w+2)p-(w+2)}\mu^p\sum_{r=1}^{w}\Big(\frac{1}{\Gamma(\mathfrak{m}_r)}\Big)^p\mathbb{Y}^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\mathfrak{m}_r p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\mathfrak{m}_r p-1}\Big)^{p-1}\int_0^t\big\|\chi(s)-\mathbb{U}(s)\big\|_p^p ds$$
$$+12^{p-1}\mu^p\varphi^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\alpha p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\alpha p-1}\Big)^{p-1}\int_0^t\Big(\big\|\chi(s)-\mathbb{U}(s)\big\|_p^p+\big\|\mathbb{U}(s-\kappa)-\mathbb{U}(s-\kappa)\big\|_p^p\Big)ds$$
$$+12^{p-1}\mu^p\aleph^{\frac{p-2}{2}}C_p\Big(\frac{(\Phi(t)-\Phi(0))^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{p-2}{2}}$$
$$\sup_{0<s\leq t}\Phi'(s)\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Big(\big\|\chi(s)-\mathbb{U}(s)\big\|_p^p+\big\|\mathbb{U}(s-\kappa)-\mathbb{U}(s-\kappa)\big\|_p^p\Big)\Phi'(s)ds. \tag{4.6}$$

Assume that:

$$\Theta(t) = \underset{t\in[0,T]}{esssup}E(\|\chi(t)-\mathbb{U}(t)\|_p^p),\ t\in[0,T],$$

we get $E(\|\chi(t)-\mathbb{U}(t)\|_p^p)\leq\Theta(t)$ and $E(\|\chi(t-\kappa)-\mathbb{U}(t-\kappa)\|_p^p)\leq\Theta(t)$ when $t\in[0,T]$. This yields:

$$E(\|\chi(t)-\mathbb{U}(t)\|_p^p)$$
$$\leq 2^{p-1}\varepsilon + 2^{(w+2)p-(w+2)}\mu^p\sum_{r=1}^{w}\Big(\frac{1}{\Gamma(\mathfrak{m}_r)}\Big)^p\mathbb{Y}^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\mathfrak{m}_r p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\mathfrak{m}_r p-1}\Big)^{p-1}\int_0^t\Theta(s)ds$$
$$+12^{p-1}\mu^p\varphi^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\alpha p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\alpha p-1}\Big)^{p-1}2\int_0^t\Theta(s)ds$$
$$+12^{p-1}\mu^p\aleph^{\frac{p-2}{2}}C_p\Big(\frac{(\Phi(t)-\Phi(0))^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{p-2}{2}}$$
$$2\mathbf{G}\int_0^t(\Phi(t)-\Phi(s))^{2\alpha-2}\Theta(s)\Phi'(s)ds. \tag{4.7}$$

From (4.7), we derive the following:

$$\Theta(t) \leq \Big(2^{p-1}\varepsilon + 2^{(w+2)p-(w+2)}\mu^p \sum\nolimits_{r=1}^{w} \Big(\frac{1}{\Gamma(\mathfrak{m}_r)}\Big)^p \mathbb{Y}^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\mathfrak{m}_r p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\mathfrak{m}_r p-1}\Big)^{p-1}$$
$$+ 12^{p-1}\mu^p\varphi^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\alpha p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\alpha p-1}\Big)^{p-1}2\Big)\int_0^t \Theta(s)ds$$
$$+ 12^{p-1}\mu^p\aleph^{\frac{p-2}{2}}C_p\Big(\frac{(\Phi(t)-\Phi(0))^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{p-2}{2}} 2\mathbf{G}\int_0^t (\Phi(t)-\Phi(s))^{2\alpha-2}\Theta(s)\Phi'(s)ds. \tag{4.8}$$

Utilizing the GGI, we obtain from (4.8)

$$\Theta(t) \leq \Big\{2^{p-1}\varepsilon + \Big(2^{(w+2)p-(w+2)}\mu^p \sum\nolimits_{r=1}^{w} \Big(\frac{1}{\Gamma(\mathfrak{m}_r)}\Big)^p \mathbb{Y}^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\mathfrak{m}_r p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\mathfrak{m}_r p-1}\Big)^{p-1}$$
$$+ 12^{p-1}\mu^p\varphi^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\alpha p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\alpha p-1}\Big)^{p-1}2\Big)\int_0^t \Theta(s)ds\Big\}$$
$$\times \mathbb{E}_{2\alpha-1,1}\Big(12^{p-1}\mu^p\aleph^{\frac{p-2}{2}}C_p\Big(\frac{(\Phi(t)-\Phi(0))^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{p-2}{2}} 2\mathbf{G}\Gamma(2\alpha-1)(\Phi(t)-\Phi(0))^{(2\alpha-1)}\Big)$$
$$= M_1\varepsilon + M_2\int_0^t \Theta(s)ds, \ t \in [0, T],$$

where

$$M_1 = 2^{p-1}\mathbb{E}_{2\alpha-1,1}\Big(12^{p-1}\mu^p\aleph^{\frac{p-2}{2}}C_p\Big(\frac{(\Phi(t)-\Phi(0))^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{p-2}{2}} 2\mathbf{G}\Gamma(2\alpha-1)(\Phi(t)-\Phi(0))^{(2\alpha-1)}\Big),$$

and

$$M_2 = \Big(12^{p-1}\mu^p\varphi^{p-1}\Big((\Phi(t)-\Phi(0))^{\frac{\alpha p-1}{p-1}}\Big)^{p-1}\Big(\frac{p-1}{\alpha p-1}\Big)^{p-1}2\Big)\Big)$$
$$\times \mathbb{E}_{2\alpha-1,1}\Big(12^{p-1}\mu^p\aleph^{\frac{p-2}{2}}C_p\Big(\frac{(\Phi(t)-\Phi(0))^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{p-2}{2}} 2\mathbf{G}\Gamma(2\alpha-1)(\Phi(t)-\Phi(0))^{(2\alpha-1)}\Big).$$

We arrive at

$$\begin{aligned}\Theta(t) &\leq M_1\varepsilon\exp(M_2 t)\\ &\leq M_1\varepsilon\exp(M_2 T)\\ &= \mathbb{V}\varepsilon.\end{aligned}$$

Consequently, the final expression is

$$E(\|\chi(t) - \mathbb{U}(t)\|_p^p) \leq \mathbb{V}\varepsilon, \ \ t \in [0, T].$$

This indicates that (1.3) exhibits UHS with respect to $\varepsilon$.

## 5. Example

To demonstrate the theoretical results, we provide an example in this section.

**Example 1.** Analyze the following:

$$\begin{cases} D_{0^+}^{\alpha,\Phi}(q(t) - I_{0^+}^{\mathfrak{m}_1,\Phi}5t\sin(q(t)) - I_{0^+}^{\mathfrak{m}_2,\Phi}4\cos(q(t))) = 3\cos(q(t))\sin(q(t-\frac{1}{3}))+ \\ 4\sin(q(t))\cos(q(t-\frac{1}{3}))\frac{dB(t)}{dt},\ t \in [0,6], \\ q(0) = \vartheta, \end{cases} \tag{5.1}$$

where $\Phi = t^{\frac{1}{2}}$, $\kappa = \frac{1}{3}$ w = 2, T = 6, $\ell_1(t, q(t)) = 6t\sin(q(t))$, $\ell_2(t, q(t)) = 5\cos(q(t))$, $\lambda(t, q(t), q(t-\kappa)) = 4\cos(q(t))\sin(q(t-\frac{1}{5}))$, $\eta(t, q(t), q(t-\kappa)) = 5\sin(q(t))$ $\cos(q(t-\frac{1}{5}))$.

We compute the following:

$$\|\ell_1(t, \Psi_1'(t)) - \ell_1(t, \Psi_2'(t))\| \le 6\|\Psi_1'(t)) - \Psi_2'(t))\|,$$

$$\|\ell_2(t, \Psi_1'(t)) - \ell_2(t, \Psi_2'(t))\| \le 5\|\Psi_1'(t)) - \Psi_2'(t))\|,$$

$$\|\lambda(t, \Psi_1'(t), \Psi_1'(t-\kappa)) - \lambda(t, \Psi_2'(t), \Psi_2'(t-\kappa))\| \le 4(\|\Psi_1'(t)) - \Psi_2'(t))\| + \|\Psi_1'(t-\kappa) - \Psi_2'(t-\kappa)\|),$$

$$\|\eta(t, \Psi_1'(t), \Psi_1'(t-\kappa)) - \eta(t, \Psi_2'(t), \Psi_2'(t-\kappa))\| \le 5(\|\Psi_1'(t)) - \Psi_2'(t))\| + \|\Psi_1'(t-\kappa) - \Psi_2'(t-\kappa)\|).$$

Hence, condition $(\hbar_1)$ holds with $\mu = 6$. Similarly, for r = 1, 2, we find that $\underset{t\in[0,6]}{esssup}\|\ell_r(t,0)\|_p < 6$, $\underset{t\in[0,6]}{esssup}\|\lambda(t,0,0)\|_p < 6$, and $\underset{t\in[0,6]}{esssup}\|\eta(t,0,0)\|_p < 6$. Therefore, Theorem 3.1 ensures the Ex-Un of the solution to system (5.1). Moreover, applying Theorem 4.1, we conclude that system (5.1) possesses UHS, as conditions $(\hbar_1)$ and $(\hbar_2)$ are fulfilled.

**Example 2.** Consider the following:

$$\begin{cases} D_{0^+}^{\alpha,\Phi}(q(t) - I_{0^+}^{\mathfrak{m}_1,\Phi}a_1\sin(t)) = a_2 + a_3\frac{dB(t)}{dt},\ t \in [0,10], \\ q(0) = 100. \end{cases} \tag{5.2}$$

The FSIDE is characterized by six key parameters governing its dynamic behavior. The initial condition is set to q(0) = 100, establishing the starting value for the state variable q(t). The sinusoidal forcing amplitude $a_1 = 0.1$ determines the intensity of the periodic external influence, represented by the term a1 sin(t). The constant drift coefficient $a_2 = 0.05$ controls the deterministic trend, providing a steady directional push to the system's dynamics. The noise intensity parameter $a_3 = 0.1$ quantifies the magnitude of stochastic fluctuations introduced via the Brownian motion increments dB(t), capturing random environmental influences.

The fractional order $\alpha = 0.8$ (with $\Phi = t$) governs the memory properties and anomalous diffusion characteristics in the drift and stochastic components, while $\mathfrak{m}_1 = 0.6$ specifically controls the fractional dynamics of the sinusoidal forcing term. The simulation was conducted over a time horizon of T = 10 with a resolution of $\Delta t = 0.01$, resulting in 1000 time steps. This ensures numerical stability and accuracy in approximating the fractional integrals through Riemann sum discretization. This parameter configuration represents a balanced system exhibiting deterministic trends, periodic forcing, and stochastic perturbations with intermediate memory effects, which are characteristic of fractional-order systems.

Therefore, Theorem 3.1 ensures the Ex-Un of the solution to system (5.2). Moreover, applying Theorem 4.1, we conclude that system (5.2) possesses UHS, as conditions ($\hbar_1$) and ($\hbar_2$) are fulfilled.

The numerical solution of the FSIDE is obtained using an Euler-Maruyama type discretization scheme adapted for fractional integrals. The method discretizes the fractional integrals using Riemann sum approximations, where the time domain $[0, T]$ is partitioned into $N = 1000$ equally spaced intervals with step size $\Delta t = 0.01$. For each time step $t_n$, the fractional integral terms are approximated by summing contributions from all previous time points $t_j$ weighted by the kernel functions $(t_n - t_j)^{\alpha-1}$ and $(t_n - t_j)^{\mathfrak{m}_1-1}$, which capture the memory effects characteristic of fractional calculus. The stochastic component is incorporated through Brownian motion increments $\Delta B_j$ generated as independent normal random variables with mean zero and variance $\Delta t$. This numerical scheme maintains the non-Markovian nature of the system through the persistent memory kernels while efficiently handling the stochastic perturbations via the Euler-Maruyama approach, providing a computationally feasible method for solving this class of complex fractional stochastic equations with both deterministic forcing and random fluctuations.

Figure 2 shows the solution of system (5.2) obtained using the Euler-Maruyama method.

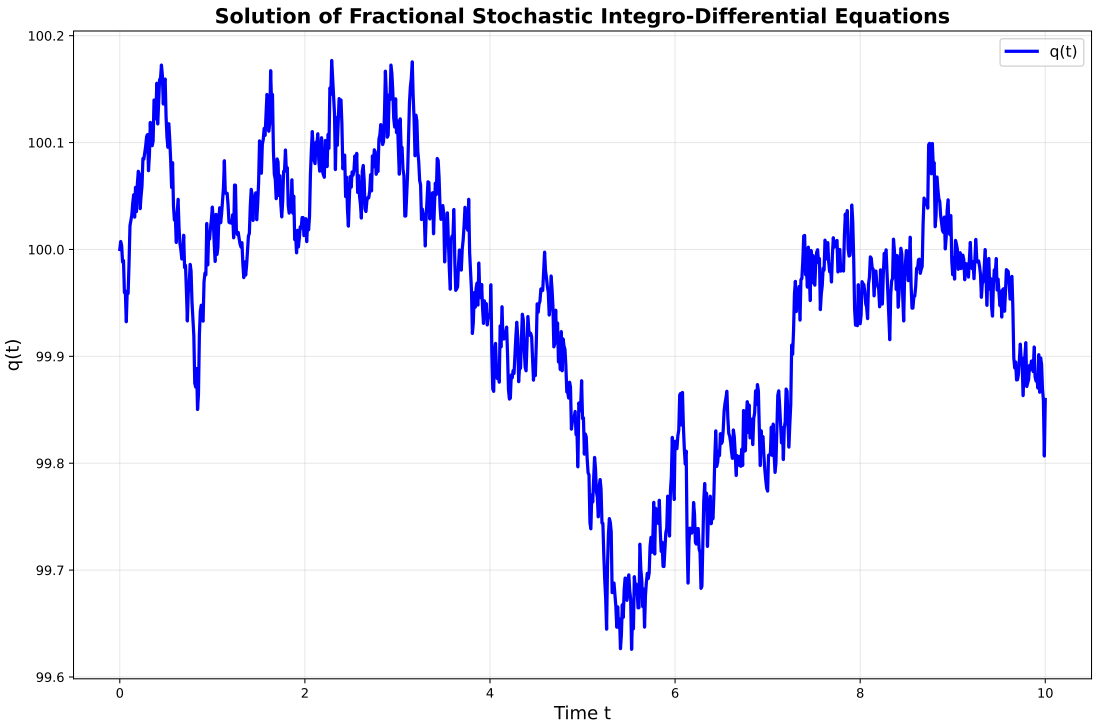


**Figure 2.** Parameters: initial condition $q(0) = 100$; fractional orders $\rho = 0.8$, $\mathfrak{m}_1 = 0.6$ (with $\Phi = t$); coefficients $a_1 = 0.05$ (drift), $a_2 = 0.1$ (volatility), $a_3 = 0.1$ (forcing amplitude); and total time $T = 10$.

## 6. Conclusions

We first demonstrated the results on Ex-Un using the CMP and proved the result concerning continuous dependence. Next, we presented GGI and established UHS based on it.

We offered three significant contributions toward establishing the well-posedness and UHS results:

First, by deriving results involving the pth moment, we extended the findings specifically for the case p = 2. Second, we developed results within the framework of the Φ-Cap-FD. Third, we addressed FSIDEs, which encompassed a broader class of FSDEs.

## 7. Future directions

We plan to develop a financial model based on FSIDEs in the future.

## Author contributions

E. S. Aly, M. I. Liaqat, S. Alshammari, and M. El-Morshedy: Conceptualization, methodology, software, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing, visualization; E. S. Aly and M. El-Morshedy: Resources, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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