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Research article

On geodesic screen generic lightlike submanifolds of locally metallic semi-Riemannian manifolds

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Abstract: This paper presents the concept of screen generic lightlike submanifolds within the context of locally metallic semi-Riemannian manifolds. Our main focus is on investigating the integrability of different distributions and exploring the properties of geodesic screen generic lightlike submanifolds. Additionally, we demonstrate these ideas with an example of a generic lightlike submanifold on a screen.

Keywords: metallic structure; semi-Riemannian manifold; totally geodesic; lightlike submanifolds
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Abbreviations	
GCRL	Generalized Cauchy-Riemannian lightlike
GSG	Geodesic screen generic
LS	Lightlike submanifold
LT	Lightlike transversal
LCC	Levi-Civita connection
LMR	Locally metallic Riemannian
RM	Riemannian manifold
CRL	Cauchy-Riemannian lightlike
SCRL	Screen Cauchy-Riemannian lightlike
SG	Screen generic
SGL	Screen generic lightlike
SRM	Semi-Riemannian manifold
STL	Screen transversal lightlike

1. Introduction

In the field of semi-Riemannian geometry (SRM), the induced metric on a submanifold is not always non-degenerate, which poses challenges when using traditional methods to study the geometric properties of these submanifolds. A key difficulty in analyzing lightlike submanifolds (LS) arises from the complex interplay between their tangent bundle and normal vector bundle. This interaction makes the geometric study of LS within SRM a particularly intriguing area of research. Such studies have important applications in mathematical physics, notably in general relativity.

Duggal and Bejancu [1] pioneered the concept of LS and later expanded it by introducing a non-degenerate screen distribution. This approach helps establish a lightlike transversal (LT) vector bundle that does not overlap with the tangent bundle. Their work was further detailed in a 2010 book by Duggal and Sahin [2], which provides an in-depth exploration of LS differential geometry. The book includes rigorous proofs, novel geometric findings, and discussions on their relevance to mathematical physics. Numerous studies, such as those in [3–5], have also investigated the geometric features of lightlike hypersurfaces and submanifolds.

The notion of generic submanifolds in Kaehler and Sasakian manifolds has evolved as an extension of Cauchy-Riemannian (CR)-submanifolds, as seen in references [6–8]. Since CR-submanifolds encompass both holomorphic and totally real submanifolds, generic submanifolds represent the most comprehensive category. Furthermore, invariant and anti-invariant LS in indefinite Hermitian manifolds can be viewed as special cases of screen Cauchy-Riemann lightlike (SCRL) submanifolds. It was initially thought that SCRL submanifolds would fall under the broader category of general LS in lightlike geometry. However, research in [9] revealed that generic LS do not fully include SCRL submanifolds. As a result, Dogan et al. [10] introduced and analyzed the concept of screen generic lightlike (SGL) submanifolds. Additionally, Sahin [3] proposed the idea of screen transversal lightlike (STL) submanifolds in indefinite Kaehler manifolds to address the absence of true lightlike curves in existing categories like CRL, SCRL, and GCRL submanifolds.

Separately, Crăsăreanu and Hrețcanu [4] developed the concept of a golden structure, rooted in the golden mean and treated as a polynomial structure [11]. They explored submanifolds in Riemannian manifolds (RMs) with this golden structure. Over time, this structure has been extensively studied in various contexts, as documented [12–16]. Building on this foundation, Spinadel [17] generalized the golden mean into the metallic mean family, defined as positive solutions to equations of the form $t^2 - pt - q = 0$, where p and q are positive integers. The metallic mean, denoted $\sigma_{p,q}$ is given by $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$, reducing to the golden mean when $p = q = 1$. Inspired by this, Crâşmăreanu and Hreţcanu [18] introduced metallic structures on Riemannian manifolds, with subsequent research focusing on their properties, including hypersurfaces and submanifolds of LMR manifolds in [5, 19]. Jin and Lee [9] examined generic submanifolds and LS in Sasakian and indefinite Kaehler manifolds, respectively, while Gupta and Sharfuddin [20] studied STL submanifolds in indefinite cosymplectic manifolds. SGL submanifolds in golden SRM has also been explored in [10, 21].

In this study, we concentrate on geodesic SGL submanifolds within locally metallic SRM, drawing on these previous investigations. Section 2 lays out the basic concepts of LS, Section 3 examines the integrability of distributions in SG submanifolds of metallic SRM, and Section 4 explores the properties of totally geodesic SG submanifolds. The final section provides an example of an SG submanifold in a metallic SRM.

2. Preliminaries

Let $(\overline{\aleph}, g)$ be an SRM with $(k+j)$-dimensions, which means that $k, j \geq 1$, and g is a semi-Riemannian metric in $\overline{\aleph}$. We assume that $\overline{\aleph}$ is not a Riemannian manifold (RM) and that q represents a constant index of g.

Consider that $\overline{\aleph}$ has a tensor field ψ of type $(1, 1)$ s.t.

$$\psi^2 = p\psi + qI, \tag{2.1}$$

Here, $\Gamma(\Upsilon\overline{\aleph})$ on I indicates the identity transformation. The structure ψ is commonly known as a metallic structure. A metric g is said to be ψ-compatible if

$$g(\psi\gamma, \zeta) = g(\gamma, \psi\zeta), \tag{2.2}$$

then $(\overline{\aleph}, g, \psi)$ is called MRM. If $\psi\gamma$ is changed to γ in (2.2), then from (2.1) we get

$$g(\psi\gamma, \psi\zeta) = g(p\psi\gamma, \zeta) + g(q\gamma, \zeta) \tag{2.3}$$

for any $\gamma, \zeta \in \Gamma(\Upsilon\overline{\aleph})$.

Given a MRM $(\overline{\aleph}, g, \psi)$ and ψ parallel to the LCC $\overline{\nabla}$ on $\overline{\aleph}$,

$$\overline{\nabla}\psi = 0, \tag{2.4}$$

then the SRM $(\overline{\aleph}, g, \psi)$ is known to have local metallic characteristics.

The LCC is expected to be $\overline{\nabla}$ on $\overline{\aleph}$, we have

$$\overline{\nabla}_\gamma \zeta = -A_h \zeta + \nabla^\perp_\gamma h. \tag{2.5}$$

Using projections, we get

$$\overline{\nabla}_\gamma \zeta = \nabla_\gamma \zeta + h^l(\gamma, \zeta) + h^s(\gamma, \zeta), \tag{2.6}$$

$$\overline{\nabla}_\gamma \aleph = -A_\aleph \gamma + \nabla^l_\gamma \aleph + \lambda^s(\gamma, \aleph), \tag{2.7}$$

$$\overline{\nabla}_\gamma \chi = -A_\chi \gamma + \nabla^s_\gamma + \lambda^l(\gamma, \chi). \tag{2.8}$$

We investigate the distribution of P, which is the projection of $\Upsilon\aleph$ on the screen.

$$\nabla_\gamma P\zeta = \nabla^*_\gamma P\zeta + h^*(\gamma, P\zeta), \tag{2.9}$$

$$\nabla_\gamma \xi = -A^*_\xi \gamma + \nabla^{*t}_\gamma \xi, \tag{2.10}$$

where $\gamma, \zeta \in \Gamma(\Upsilon\aleph)$, $\aleph \in \Gamma(ltr(\Upsilon\aleph))$, $\chi \in \Gamma(S(\Upsilon\aleph^\perp))$, $\xi \in \Gamma(Rad(\Upsilon\aleph))$, $h \in \Gamma(tr(\Upsilon\aleph))$.

3. SGL submanifolds

Definition 3.1. *[10] Let $\aleph$ be a real submanifold of the metallic SRM $\overline{\aleph}$. A submanifold $\aleph$ is classified as an SGL submanifold if it meets the following condition:*

(i) The radical distribution $Rad(\Upsilon\aleph)$ is invariant under the tensor field ψ meaning that

$$\psi(Rad(\Upsilon\aleph)) = Rad(\Upsilon\aleph). \tag{3.1}$$

(ii) A subbundle of $S(\Upsilon M)$, λ_0, exists such that

$$\lambda_0 = \psi(S(\Upsilon M)) \cap S(\Upsilon M), \tag{3.2}$$

where the non-degenerate distribution λ_0 is on $\aleph$.

By considering the notion of a generic LS with respect to a screen, we may deduce that there is a non-degenerate complementary distribution $\lambda' - \lambda_0$ in $S(\Upsilon\aleph)$.

$$S(\Upsilon\aleph) = \lambda_0 \oplus \lambda',$$

where $\psi(\lambda') \nsubseteq S(\Upsilon\aleph)$ and $\psi(\lambda') \nsubseteq S(\Upsilon\aleph^{\perp})$. S_0, S_1, and Q stands for the projection on λ_0, $Rad(\Upsilon\aleph)$ and λ', respectively.

Then we have, for $\forall X \in \Gamma(\Upsilon\aleph)$,

$$\gamma = S_0\gamma + S_1\gamma + Q\gamma = S\gamma + Q\gamma, \tag{3.3}$$

where $\lambda = \lambda_0 \perp Rad(\Upsilon\aleph)$, λ is invariant, and $P\gamma \in \Gamma\lambda$, $Q\gamma \in \Gamma\lambda'$. From (3.3), we get

$$\psi\gamma = \phi\gamma + \omega\gamma, \tag{3.4}$$

where $(\phi\gamma)$ and $(\omega\gamma)$ are tangential and transversal parts of $\psi\gamma$, respectively.

$$\psi\gamma = f\gamma + w\gamma \tag{3.5}$$

for $\gamma, \zeta \in \Gamma(\Upsilon\aleph)$, where $f\gamma = \psi S\gamma$ and $w\gamma = \psi Q\gamma$.

$$\psi\zeta = B\zeta + C\zeta \tag{3.6}$$

for any $\zeta \in \Gamma(tr(\Upsilon\aleph))$, $B\zeta \in \Gamma(\Upsilon\aleph)$ and $C\zeta \in \Gamma(tr(\Upsilon\aleph))$.

A metallic SRM's proper SGL submanifold is defined as $\aleph$ such that λ_0 and λ' are both non-zero. Key properties of a proper SGL submanifolds include the following:

(1) (i) indicates that $dim(Rad(TM)) = 2s \geq 2$.
(2) (i) indicates that $dim(\lambda_0) = 2r \geq 2$.
(3) $dim(\lambda') = 2p \geq 2$. Thus, $dim(\aleph) \geq 6$ and $dim(\overline{\aleph}) \geq 10$.
(4) Any proper 6-dimensional SG-LS must be 2-lightlike.
(5) (i) and metallic SRM $\overline{\aleph}$ imply that $index(\overline{\aleph}) \geq 2$.

Proposition 3.1. *[10] As an SGL submanifold, an SCRL submanifold has a distribution λ' that is completely anti-invariant. In this case, the distribution λ' is in accordance with the equation $S(\Upsilon\aleph^{\perp}) = \omega\lambda' \oplus \mu$, where μ is a non-degenerate invariant distribution.*

Definition 3.2. *[10] Assumed that $\aleph$ is a r-LS of $\overline{\aleph}$, a metallic SRM. Assuming that $S(\Upsilon\aleph)$ of $\aleph$ has a screen distribution such that*

$$\psi(S(\Upsilon\aleph^{\perp})) \subset S(\Upsilon\aleph),$$

where $\aleph$ is a generic r-LS.

Proposition 3.2. *[10] In terms of the screen, a generic r-LS is a LS with* $\mu = 0$.

Theorem 3.1. *Let* $\aleph$ *be an LS of a locally metallic SRM* $\overline{\aleph}$. *The components of* $h^s(\gamma, \psi\zeta)$ *and* $h^s(\gamma, \zeta)$ *in* $\omega\lambda'$ *are zero if* ∇ *is a metric connection. On the other hand, if* $h^s(\gamma, \psi\zeta)$ *and* $h^s(\gamma, \zeta)$ *have no components in* $\omega\lambda'$, *then the induced connection* ∇ *is a metric connection.*

Proof. Consider ∇ as a metric connection. Based on Eq (2.4), we can express the following: $\forall \gamma \in \Gamma(\Upsilon\aleph)$, and $\zeta \in \Gamma(Rad(\Upsilon\aleph))$,

$$\overline{\nabla}_\gamma \zeta = \frac{1}{q}\psi(\overline{\nabla}_\gamma \psi\zeta) - \frac{p}{q}\psi(\overline{\nabla}_\gamma \zeta).$$

Using (2.6), we have

$$\overline{\nabla}_\gamma \zeta = \frac{1}{q}\psi[\nabla_\gamma \psi\zeta + h^l(\gamma, \psi\zeta) + h^s(\gamma, \psi\zeta)] - \frac{p}{q}\psi[\nabla_\gamma \zeta + h^l(\gamma, \zeta) + h^s(\gamma, \zeta)].$$

Using (2.10), we get

$$\begin{aligned} \nabla_\gamma \zeta + h(\gamma, \zeta) &= \frac{1}{q}\psi[-A^*_{\psi\zeta}\gamma + \nabla^*_\gamma \psi\zeta + h^l(\gamma, \psi\zeta) + h^s(\gamma, \psi\zeta)] \\ &\quad + \frac{p}{q}\psi[-A^*_\zeta \gamma + \nabla^*_\gamma \zeta + h^l(\gamma, \zeta) + h^s(\gamma, \zeta)]. \end{aligned}$$

Using (3.4), (3.6) and taking tangential parts, we have

$$\nabla_\gamma \zeta = \frac{1}{q}[-\phi A^*_{\psi\zeta}\gamma + \nabla^*_\gamma \psi\zeta + Bh^s(\gamma, \psi\zeta)] + \frac{p}{q}[-\phi A^*_\zeta \gamma + \nabla^*_\gamma \zeta + Bh^s(\gamma, \zeta)].$$

Since $Rad(\Upsilon\aleph)$ must be parallel to ∇ to be a metric connection, we can conclude that

$$\begin{aligned} g(\nabla_\gamma \zeta, U) &= g(-\frac{1}{q}\phi A^*_{\psi\zeta}\gamma + \frac{1}{q}\nabla^*_\gamma \psi\zeta + \frac{1}{q}Bh^s(\gamma, \psi\zeta) - \frac{p}{q}\phi A^*_\zeta \gamma + \frac{p}{q}\nabla^*_\gamma \zeta + \frac{p}{q}Bh^s(\gamma, \zeta), U), \\ g(\nabla_\gamma \zeta, U) &= g(\frac{1}{q}Bh^s(\gamma, \psi\zeta) + \frac{p}{q}Bh^s(\gamma, \zeta), U). \end{aligned}$$

Since $g(\nabla_X \zeta, U) = 0$,

$$g(\frac{1}{q}h^s(\gamma, \psi\zeta) + \frac{p}{q}h^s(\gamma, \zeta), U) = 0,$$

$\forall \gamma, \zeta \in Rad\Gamma(\Upsilon\aleph)$ and $U \in S\Gamma(\Upsilon\aleph)$.

Hence, $h^s(\gamma, \psi\zeta)$ and $h^s(\gamma, \zeta)$ has no components in $\omega\lambda'$. □

Theorem 3.2. *Examine a locally metallic SGL submanifold* $\aleph$ *in the SRM* $\overline{\aleph}$, *if the following condition is met, the distribution* λ_0 *is integrable.*

$$g(h^*(\gamma, \psi\zeta) - h^*(\zeta, \psi\gamma), \psi N) = g(h^l(\gamma, \zeta) - h^l(\zeta, \gamma), p\psi N).$$

Proof. Since, λ_0 is integrable iff for $\gamma, \zeta \in \Gamma\lambda_0$, $[\gamma, \zeta] \in \lambda_0$, i.e.,

$$g([\gamma, \zeta], \eta) = g([\gamma, \zeta], N) = 0,$$

where $\eta \in \Gamma\lambda'$ and $N \in ltr(\Upsilon\aleph)$.

$$g([\gamma,\zeta],\eta) = \frac{1}{q}[g(\psi\overline{\nabla}\zeta - \overline{\nabla}_{\zeta}\gamma, \psi\eta) - g(\overline{\nabla}\zeta - \overline{\nabla}_{\zeta}\gamma, p\psi\eta)],$$

$$g([\gamma,\zeta],\eta) = \frac{1}{q}[g(\overline{\nabla}_{\gamma}\psi\zeta - \overline{\nabla}_{\zeta}\psi\gamma, \psi\eta) - g(\overline{\nabla}_{\gamma}\zeta - \overline{\nabla}_{\zeta}\gamma, p\psi\eta)].$$

Using (2.6) and (3.4), we have

$$\begin{aligned} g([\gamma,\zeta],\eta) = \ & \frac{1}{q}[g(\nabla_{\gamma}\psi\zeta + h^l(\gamma,\psi\zeta) + h^s(\gamma,\psi\zeta) - \nabla_{\zeta}\psi\gamma - h^l(\zeta,\psi\gamma) \\ & -h^s(\zeta,\psi\gamma), \phi\eta + \omega\eta) - g(\nabla_{\gamma}\zeta + h^l(\gamma,\zeta) + h^s(\gamma,\zeta) \\ & -\nabla_{\zeta}\gamma - h^l(\gamma,\zeta) - h^s(\zeta,\gamma), p(\phi\eta + \omega\eta))]. \end{aligned}$$

Using (2.9), we have

$$\begin{aligned} & g(\nabla^*_{\gamma}\psi\zeta - \nabla^*_{\zeta}\psi\gamma, \phi\eta) + g(\nabla_{\gamma}\zeta - \nabla_{\zeta}\gamma, p\phi\eta) \\ = \ & g(h^s(\gamma,\psi\zeta) - h^s(\gamma,\psi\zeta), \omega\eta) - g(h^s(\gamma,\zeta) - h^s(\zeta,\gamma), p\omega\eta). \end{aligned}$$

Similarly, from (2.3), we have

$$g([\gamma,\zeta],N) = \frac{1}{q}[g(\nabla_{\gamma}\psi\zeta - \nabla_{\zeta}\psi\gamma, \psi N) - g(\nabla_{\gamma}\zeta - \nabla_{\zeta}\gamma, p\psi N)].$$

Using (2.6) and (2.9), we have

$$g(h^*(\gamma,\psi\zeta) - h^*(\zeta,\psi\gamma), \psi N) = g(h^l(\gamma,\zeta) - h^l(\zeta,\gamma), p\psi N).$$

Theorem 3.3. *Let $\aleph$ be an LS of $\overline{\aleph}$, a metallic SRM. If the distribution λ' meets the following criteria, it is integrable.*

$$\begin{aligned} & g(h^l(\eta,\phi\chi) + D^l(\eta,\omega\chi) - h^l(\chi,\phi\eta) - D^l(\chi,\omega\eta), \psi N) \\ = \ & g((h^l(\eta,\phi\chi) + D^l(\eta,\omega\chi) - h^l(\chi,\phi\eta) - D^l(\chi,\omega\eta), pN). \end{aligned}$$

Proof. Given that λ' is integrable, then for $\forall\eta,\chi \in \lambda'$, $\gamma \in \Gamma\lambda_0$ and $N \in \Gamma ltr(\Upsilon\aleph)$, $g([\eta,\chi],\gamma) = g([\eta,\chi],N) = 0$.

$$g([\eta,\chi],\gamma) = \frac{1}{q}[g(\overline{\nabla}_{\eta}\psi V - \overline{\nabla}_{\chi}\psi\eta, \psi\gamma) - g(\overline{\nabla}_{\eta}\psi\chi - \overline{\nabla}_{\chi}\psi\eta,, p\gamma)].$$

Using (2.5) and (3.4), we have

$$g(\nabla_{\eta}\phi\chi - \nabla_{\chi}\phi\eta - A_{\omega\chi}\eta + A_{\omega\eta}\chi, \psi\gamma) = g(\nabla_{\eta}\phi\chi - \nabla_{\chi}\phi\eta - A_{\omega\chi}\eta + A_{\omega\eta}\chi, p\gamma).$$

Similarly,

$$g([\eta,\chi],N) = \frac{1}{q}[g(\overline{\nabla}_{\eta}\psi\chi - \overline{\nabla}_{\chi}\psi\eta, \psi N) - g(\overline{\nabla}_{\eta}\psi\chi - \overline{\nabla}_{\chi}\psi\eta, pN)].$$

Using (2.5) and (3.4), we have

$$\begin{aligned} & g(h^l(\eta,\phi\chi) + D^l(\eta,\omega\chi) - h^l(\chi,\phi\eta) - D^l(\chi,\omega\eta), \psi N) \\ = \ & g((h^l(\eta,\phi\chi) + D^l(\eta,\omega\chi) - h^l(\chi,\phi\eta) - D^l(\chi,\omega\eta), pN). \end{aligned}$$

Theorem 3.4. *Consider $\aleph$ to be an SGL submanifold of a metallic SRM $\overline{\aleph}$. The λ distribution is parallel if*

$$g(h^*(\gamma,\phi\eta)+D^l(\gamma,\omega\eta),\psi\zeta)+g(h^*(\gamma,\phi\eta)+D^l(\gamma,\omega\eta),p\zeta)=0.$$

Proof. If λ is parallel distribution, then, for $\forall\gamma,\zeta\in\lambda$ and $\eta\in\lambda'$, i.e., $g(\overline{\nabla}_\gamma\zeta,\eta)=0$,

$$\begin{aligned} g(\overline{\nabla}_\gamma\zeta,\eta) &= -g(\zeta,\overline{\nabla}_\gamma\eta),\\ g(\overline{\nabla}_\gamma\zeta,\eta) &= -\frac{1}{q}[g(\psi(\overline{\nabla}_\gamma\eta),\psi\zeta)+g(\psi(\overline{\nabla}_\gamma\eta),p\zeta)],\\ g(\overline{\nabla}_\gamma\zeta,\eta) &= -\frac{1}{q}[g((\overline{\nabla}_\gamma\psi\eta),\psi\zeta)+g((\overline{\nabla}_\gamma\psi\eta),p\zeta)]. \end{aligned}$$

Using (2.6), (2.9), and (3.4), we have

$$g(h^*(\gamma,\phi\eta)+D^l(\gamma,\omega\eta),\psi\zeta)+g(h^*(\gamma,\phi\eta)+D^l(\gamma,\omega\eta),p\zeta)=0.$$

Theorem 3.5. *Consider $\aleph$ to be an LS of $\overline{\aleph}$, a locally metallic SRM. λ' distribution is considered parallel if*

$$g(\nabla^*_\eta\phi\chi-A_{\omega\chi}\eta,\psi N) = g(\nabla^*_\eta\phi-A_{\omega\chi}\eta,N), \tag{3.7}$$

for every η and χ in the set $\Gamma\lambda'$.

Proof. If λ is parallel distribution, $\forall\eta,\chi\in\Gamma\lambda'$, $\nabla_\eta\chi\in\Gamma\lambda'$, $\gamma\in\Gamma\lambda_0$ and $N\in\Gamma ltr(\Upsilon\aleph)$, i.e., $g(\nabla_\eta\chi,\gamma)=g(\nabla_\eta\chi,N)=0$, and using (2.8), (2.9), and (3.4), we have

$$g(\nabla^*_\eta\phi\chi+h^*(Z,\phi\chi)-A_{\omega\chi}\eta+D^l(\eta,\omega\chi),\psi X) = g(\nabla_\eta\chi,p\psi\gamma).$$

Similarly, we get

$$g(\nabla_\eta\chi,N) = \frac{1}{q}[g(\psi\nabla_\eta\chi,\psi N)-g(p\psi\nabla_\eta\chi,N)].$$

Using (2.8), (2.9), and (3.4), we have

$$g(\nabla^*_\eta\phi\chi-A_{\omega\chi}\eta,\psi N) = g(\nabla^*_\eta\phi-A_{\omega\chi}\eta,N). \tag{3.8}$$

4. GSG-lightlike submanifold

Proposition 4.1. *The distribution λ of an SGL submanifold $\aleph$ of $\overline{\aleph}$ is a totally geodesic foliation in $\overline{\aleph}$ if and only if $\aleph$ is λ-geodesic and λ is parallel with regard to ∇ on $\aleph$.*

Proof. We assume that λ specifies a completely geodesic foliation in $\overline{\aleph}$, that is, $\overline{\nabla}_\gamma\zeta\in\Gamma\lambda$ for $\gamma,\zeta\in\Gamma\lambda$, then for $\forall\xi\in Rad(\Upsilon\aleph)$, $\eta\in\Gamma\lambda'$ and $\chi\in\Gamma S(\Upsilon\aleph)$,

$$g(\overline{\nabla}_\gamma\zeta,\xi)=g(\overline{\nabla}_\gamma\zeta,\chi)=g(\overline{\nabla}_\gamma\zeta,\eta)=0.$$

Since

$$g(\overline{\nabla}_\gamma\zeta,\xi) = g(h^l(\gamma,\zeta),\xi),\ h^l(\gamma,\zeta) = h^s(\gamma,\zeta) = 0$$

$\implies$ $\aleph$ is a λ-geodesic, and λ is parallel with respect to ∇ on $\aleph$.

Conversely, suppose that

$$h^l(\gamma,\zeta) = h^s(\gamma,\zeta),\quad \forall\gamma,\zeta \in \Gamma\lambda.$$

From (2.6), we have

$$\overline{\nabla}_\gamma\zeta = \nabla_\gamma\zeta \implies \overline{\nabla}_\gamma\zeta \in \Gamma\Upsilon\aleph.$$

Since λ is parallel on $\aleph \implies \nabla_\gamma\zeta \in \lambda$. □

Theorem 4.1. *Let $\aleph$ be an SGL submanifold of a metallic SRM $\overline{\aleph}$. If the following condition applies, then $\aleph$ is mixed geodesic.*

$$g((A_{\omega\zeta})\gamma - \nabla_\gamma\psi\zeta,\phi\chi) = g(h^s(\gamma,\phi Z) + \nabla^s_\gamma\omega\zeta,\omega\chi),$$

$\forall\gamma \in \Gamma\lambda$, $\zeta \in \Gamma\lambda'$ *and* $\chi \in \Gamma S(\Upsilon\aleph^\perp)$.

Proof. If $\aleph$ is mixed geodesic, then using (2.6), we have

$$g(\overline{\nabla}_\gamma\zeta,\xi) = 0.$$

Since $Rad(\Upsilon\aleph)$ is invariant, then

$$g(\overline{\nabla}_\gamma\psi\zeta,\xi) = 0.$$

Using (2.6), (2.8), and (3.4), we get

$$h^l(\gamma,\phi\zeta) = -D^l(\gamma,\omega\zeta).$$

Now,

$$g(\overline{\nabla}_\gamma\psi\zeta,\psi\chi) = 0,$$

using (2.6), (2.8), and (3.4), we get

$$g(A_{\omega\zeta})\gamma - \nabla_\gamma\psi\zeta,\phi\chi) = g(h^s(\gamma,\phi Z) + \nabla^s_\gamma\omega\zeta,\omega\chi),$$

$\forall\gamma \in \Gamma\lambda$, $\zeta \in \Gamma\lambda'$, and $\chi \in \Gamma S(\Upsilon\aleph^\perp)$. □

Proposition 4.2. *Let $\aleph$ be a locally metallic SRM $\overline{\aleph}$, to which $\aleph$ is an SGL submanifold. Then, $\aleph$ is mixed geodesic iff*

$$\frac{1}{q}[\omega Q(\nabla_\gamma\phi\eta) - A_{\omega\eta}\gamma + C(h^l(\gamma,\phi\eta) + D^l(\gamma,\omega\eta)) + C(h^s(\gamma,\phi\eta) + \nabla^s_\gamma\omega\eta)]$$
$$-\frac{p}{q}[h^s(\gamma,\phi\eta) - \nabla^s_\gamma\omega\eta - D^l(\gamma,\omega\eta)] = 0.$$

Proof. Since

$$\overline{\nabla}_\gamma\zeta = \frac{1}{q}[\psi(\overline{\nabla}_\gamma\psi\zeta) - p\psi(\overline{\nabla}_\gamma\zeta)].$$

Now, using (2.6), (2.8), (3.4), and (3.6), we get

$$\begin{aligned} h(\gamma,\eta) &= \overline{\nabla}_\gamma\eta - \nabla_\gamma\eta, \\ h(\gamma,\eta) &= \frac{1}{q}\psi[\nabla_\gamma\phi\eta + h^l(\gamma,\phi\eta) + h^s(\gamma,\phi\eta) - A_{\omega\eta}\gamma + \nabla^s_\gamma\omega\eta + D^l(\gamma,\omega\eta)] \\ &\quad -\frac{p}{q}[\nabla_\gamma\phi\eta + h^l(\gamma,\phi\eta) + h^s(\gamma,\phi\eta) - A_{\omega\eta}\gamma + \nabla^s_\gamma\omega\eta + D^l(\gamma,\omega\eta)] - \nabla_\gamma\eta. \end{aligned}$$

Taking transversal part, we get

$$\begin{aligned} h(\gamma,\eta) &= \frac{1}{q}[\omega Q(\nabla_\gamma\phi\eta - A_{\omega\eta}\gamma) + C(h^l(\gamma,\phi\eta)) + D^l(\gamma,\omega\eta)) + C(h^s(\gamma,\phi\eta) + \nabla^s_\gamma\omega\eta)] \\ &\quad -\frac{p}{q}[h^s(\gamma,\phi\eta) - \nabla^s_\gamma\omega\eta - D^l(\gamma,\omega\eta)]. \end{aligned}$$

Since $h(\gamma,\zeta) = 0$,

$$\begin{aligned} &\tfrac{1}{q}[\omega Q(\nabla_\gamma\phi\eta - A_{\omega\eta}\gamma) + C(h^l(\gamma,\phi\eta) + D^l(\gamma,\omega\eta)) + C(h^s(\gamma,\phi\eta) + \nabla^s_\gamma\omega\eta)] \\ &\quad -\tfrac{p}{q}[h^s(\gamma,\phi\eta) - \nabla^s_\gamma\omega\eta - D^l(\gamma,\omega\eta)] = 0. \end{aligned}$$

Proposition 4.3. *Let $\aleph$ be an SGL submanifold of a metallic SRM $\overline{\aleph}$. Then, for $\forall\gamma \in \Gamma\lambda_0, \eta \in \Gamma\lambda'$, we have*

$$\nabla_\gamma\eta = \frac{1}{q}[\phi\nabla_\gamma\phi\eta - \phi A_{\omega\eta}\gamma + Bh^s(\gamma,\phi\eta) + B\nabla^s_\gamma\omega\eta] + \frac{p}{q}[\phi\nabla_\gamma\eta + Bh^s(\gamma,\eta) - h^s(\gamma,\eta)].$$

Proof. Since

$$\overline{\nabla}_\gamma\zeta = \frac{1}{q}[\psi(\overline{\nabla}_\gamma\psi\zeta) - p\psi(\overline{\nabla}_\gamma\zeta)].$$

Using (2.6), (2.8), (3.4), and (3.6) and taking tangential part, we have

$$\nabla_\gamma\eta = \frac{1}{q}[\phi\nabla_\gamma\phi\eta - \phi A_{\omega\eta}\gamma + Bh^s(\gamma,\phi\eta) + B\nabla^s_\gamma\omega\eta] + \frac{p}{q}[\phi\nabla_\gamma\eta + Bh^s(\gamma,\eta) - h^s(\gamma,\eta)].$$

5. Example

Example 5.1. Suppose $\aleph$ be a submanifold of R^{12}_6 given by

$$\begin{aligned} \gamma &= (0, v_5 cos\alpha, -v_5, -v_6, v_1 cosh\alpha, v_2 cosh\alpha, v_1 sinh\alpha - v_2, v_1 + v_2 sinh\alpha, v_5 sin\alpha, \\ &\quad v_6 sin\alpha, sinv_3 sinv_4, cosv_3 coshv_4). \end{aligned}$$

Examine $\overline{\psi}$, a metallic structure described by

$$\begin{aligned} &\overline{\psi}(\partial\gamma_1,\partial\gamma_2,\partial\gamma_3,\partial\gamma_4,\partial\gamma_5,\partial\gamma_6,\partial\gamma_7,\partial\gamma_8,\partial\gamma_9,\partial\gamma_{10},\partial\gamma_{11},\partial\gamma_{12}) \\ = &(\sigma\partial\gamma_1,\sigma\partial\gamma_2,\sigma\partial\gamma_3,\sigma\partial\gamma_4,\sigma\partial\gamma_5,\sigma\partial\gamma_6,-\sigma\partial\gamma_7,-\sigma\partial x_8,-\sigma\partial\gamma_9,\sigma\partial\gamma_{10},\overline{\sigma}\partial\gamma_{11},-\overline{\sigma}\partial\gamma_{12}). \end{aligned}$$

Then, ΥM is spanned by $\{\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6\}$, where

$$\eta_1 = \partial\gamma_5 + \partial\gamma_7 + \partial\gamma_8 + \partial\gamma_6,\ \eta_2 = \partial\gamma_5 - \partial\gamma_7 - \partial\gamma_8 + \partial\gamma_6,$$

$$\begin{aligned} \eta_3 &= \partial\gamma_{11} + \partial\gamma_{12},\ \eta_4 = \overline{\sigma}\partial\gamma_{11} - \overline{\sigma}\partial\gamma_{12}, \\ \eta_5 &= \partial\gamma_2 + \partial\gamma_3 + \partial\gamma_9,\ \eta_6 = \overline{\sigma}\partial\gamma_3 - \overline{\sigma}\partial\gamma_9, \end{aligned}$$

where $Rad(\Upsilon M) = \{\eta_1, \eta_2\}$ and $\lambda_0 = \{\eta_3, \eta_4\}$. We obtain that $ltr(\Upsilon M)$ is spanned by

$$N_1 = \frac{1}{2}\{\partial\gamma_5 + \partial\gamma_7\},\ \ N_2 = \frac{1}{2}\{\sigma\partial\gamma_5 - \sigma\partial\gamma_7\}.$$

Also, the ST bundle is spanned by

$$\begin{aligned} W_1 &= -\partial\gamma_2 + \partial\gamma_3 + \partial\gamma_9,\ W_2 = \sigma\partial\gamma_2, \\ W_3 &= \partial\gamma_3 + \partial\gamma_9,\ \ W_4 = \sigma\partial\gamma_3 - \sigma\partial\gamma_9. \end{aligned}$$

It is straightforward to show that $\mu = sp\{W_3,\ W_4\}$.

Then, $\lambda' = Sp\{\eta_5,\ \eta_6\}$ and M is a SGL submanifold of R^{12}_6. □

6. Discussion and conclusions

Several characterizations of SGL submanifolds within locally metallic SRM are investigated in this work. We do a detailed analysis of the integrability of distributions associated with these submanifolds. Moreover, a number of SG submanifold features are studied in detail, especially those that have the characteristic of being completely geodesic. To illustrate the notion of SG submanifolds in the context of metallic SRM, an example is given.

This research meticulously examines SGL submanifolds in the setting of locally metallic SRM, emphasizing their defining traits and characteristics. An in-depth analysis of the integrability conditions for related distributions provides valuable perspectives on their geometric framework. Furthermore, we investigate distinct attributes of SG submanifolds, particularly those displaying completely geodesic properties. To reinforce the conceptual foundation, a final example is presented, clearly demonstrating the application of SG submanifolds within metallic SRM. This study advances the comprehension of the geometric and algebraic aspects of these submanifolds in the realm of sophisticated differential geometry.

Author contributions

R. Bossly: Validation, Visualization, Funding acquisition; M. A. Qayyoom: Formal analysis, Investigation, Writing–original draft preparation, Writing–review and editing, Visualization; N. M. Al-Asmari: Formal analysis, Review, Funding acquisition; M. Ahmad: Conceptualization, Methodology, Validation, Investigation, Resources, Writing–review and editing, Visualization, Supervision, Project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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