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Research article

Bifurcation analysis of an asymmetric three-player three-strategy game with logit dynamics and reverse-cyclic mutations

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Abstract: Rock-Paper-Scissors (RPS) games have been widely studied, while research on asymmetric multiplayer cases has remained relatively limited. In this paper, we constructed a three-player asymmetric RPS model by introducing a novel payoff matrix, which then serves as the basis for developing a reverse-cyclic mutation logit system. Analysis of the interior equilibrium showed that both mutation rate and selection intensity can destabilize the equilibrium and generate oscillatory dynamics through Hopf bifurcations. When the first Lyapunov coefficient vanished, the second Lyapunov coefficient was derived to classify the bifurcation type, enabling us to distinguish supercritical and subcritical cases. Numerical simulations confirmed the theoretical predictions. These results revealed stability conditions and bifurcation mechanisms in asymmetric multiplayer interactions and demonstrated how higher-order Lyapunov coefficients enhance the analysis of complex dynamics.

Keywords: three-player asymmetric RPS game; reverse cyclic mutation; logit system; degenerate Hopf bifurcation
Mathematics Subject Classification: 37G15, 91A22

1. Introduction

Game theory, an essential branch of applied mathematics, primarily investigates the strategic decisions and evolutionary dynamics of players engaged in interactive situations. The seminal 1944 work, Theory of Games and Economic Behavior, authored by Hungarian mathematicians John von Neumann and Oskar Morgenstern [1], is widely recognized as the foundational text of classical game theory. Since then, game theory has demonstrated extensive applicability across disciplines, including biology [2], economics [3], sociology [4], and information science [5]. Within this theoretical framework, players' payoffs are influenced not only by their individual strategic decisions, but also to a large extent by the strategies of the other players involved, resulting in a complex and dynamic interactive network. In the assumptions of classical game theory, the players are regarded as absolutely rational individuals [6]. However, in reality, players always exhibit the characteristic of bounded rationality [7], which has led to the emergence and development of evolutionary game theory. Furthermore, evolutionary game theory explicitly explores how bounded-rational individuals optimize their payoffs through continuous strategic adjustments during repeated interactions. This theoretical approach provides robust analytical tools to examine strategy dynamics within populations [8]. In 1961, Lewontin [9] pioneered the application of game theory concepts to the realm of biological science. On this basis, Smith and Price introduced the concept of Evolutionarily Stable Strategies (ESS) to analyze conflict dynamics among animals, thereby laying a critical foundation for subsequent research in evolutionary game theory [10].

Evolutionary game dynamics, integrating differential equations and game theory, provides a robust framework for studying strategic evolution in complex interactive systems [11,12]. In 1995, economist Jörgen Weibull systematically formalized these principles in his seminal work, "Evolutionary Game Theory", equipping researchers with essential mathematical tools to analyze strategic interactions [13]. Replicator dynamics, as the most extensively studied dynamic models, was first introduced by Taylor [14] and colleagues in ecological evolutionary contexts to describe the relationship between the unit growth rates of different strategic populations and their corresponding fitness within the population. For this framework, mathematician Hofbauer and collaborators performed stability analyses of equilibrium points under Rock-Paper-Scissors (RPS) game payoffs [15]. As a prototypical cyclic dominance game, the RPS game has garnered significant research attention due to its intuitive strategy structure and rich dynamic behaviors [16]. As the number of players and strategic options increases, the resulting dynamics become progressively complex and variable. To gain deeper insights into these dynamics, researchers have continually introduced new elements and parameters into the RPS games. For example, Mobilia [17] incorporated mutation factors into the replicator equations for RPS games, constructing models with global mutations and exploring their effects. His findings indicated that mutations can induce Hopf bifurcations within the system. Additionally, Toupo and Strogatz [18] explored the influences of single mutations, bidirectional mutations, and other mutation types on RPS dynamics using replicator equations. Introducing selective interaction mechanisms in public goods games has also been shown to promote cooperation by lowering critical thresholds [19]. Critical bifurcation thresholds were identified, and numerical simulations to illustrate the evolutionary trajectories and behaviors of these systems were conducted.

As interest in players' bounded rationality and strategic heterogeneity has grown, researchers have increasingly recognized that individuals' strategic choices are often shaped by cognitive biases or random "noise", than being consistently rational. McKelvey and Palfrey (1995) were pioneers in introducing the logit response mechanism (logit dynamics) into game theory, effectively characterizing how bounded rationality and stochastic elements influence strategy adaptation, and laying a theoretical foundation for subsequent studies [20]. Building upon this foundation, Hommes and Ochea [21] demonstrated the presence of stable periodic orbits and multiple stable states within three-strategy best-response dynamics. Following this, Ochea [22] explored the evolutionary dynamics of repeated Prisoner's Dilemma games via logit dynamics, elucidating conditions under which cooperation among boundedly rational individuals could persist or collapse, and identifying the emergence of periodic behaviors and multiple equilibria. Umezuki [23] utilized discrete logit systems to investigate RPS games, discovering Neimark-Sacker bifurcations and determining corresponding bifurcation values.

Beyond logit dynamics, scholars have expanded evolutionary game systems from diverse perspectives. For example, the replicator-mutator framework was extended with time delays by Mittal [24], and it was demonstrated that such delays can induce Hopf bifurcations and stable limit-cycle oscillations, thereby revealing new dynamic behaviors beyond classical replicator dynamics. Mean-field theory (MFT) was applied by Nagatani [25] to RPS games, revealing paradoxical outcomes from backward mutations. Hu [26] examined asymmetric RPS games using imitation dynamics and found that chaotic behaviors can emerge under specific conditions. Additionally, Tamás [27] investigated the dynamic instability of evolutionarily stable strategies (ESS) within the RPS framework under time constraints, demonstrating that in complex ecological and evolutionary contexts, an ESS may no longer represent a dynamically stable equilibrium. Recent eco-evolutionary researchers have also incorporated loner strategies into three-strategy games, demonstrating how additional options beyond the classical RPS cycle can alter coexistence and stability outcomes [28].

While extensive research has been conducted on RPS games involving three or four strategies, demonstrating substantial progress across dynamic models, such as Logit, Replicator, Imitation, and different mutation mechanisms, most researchers have predominantly focused on symmetric structures and two-player scenarios [29, 30]. Relatively limited attention has been devoted to investigating dynamical behaviors in asymmetric and multi-players interactive contexts. Building upon prior studies, the major contributions of this paper are as follows:

(i) We construct a three-player asymmetric payoff matrix for the RPS game. Unlike the symmetric frameworks emphasized in most research, this design captures heterogeneity among players and highlights the role of dominant participants, providing a more realistic representation of strategic interactions.

(ii) We develop a dynamical system that integrates reverse-cyclic mutations with the Logit response mechanism. This approach makes it possible to study the combined influence of mutation and rationality level on system dynamics, thereby extending the scope of classical replicator-mutator models and reflecting decision-making behaviors under bounded rationality.

(iii) In examining equilibrium stability, we explicitly address degenerate Hopf bifurcations. When the first Lyapunov coefficient equals zero, we further derive the second Lyapunov coefficient to classify the bifurcation type. This step offers a more complete characterization of local dynamics near critical thresholds, a situation that has received limited attention in previous work.

Taken together, these aspects contribute to a better understanding of asymmetric multiplayer interactions and complement existing approaches to bifurcation analysis in evolutionary game theory. The remainder of this paper is structured as follows. In Section 2, we introduce the formulation of the proposed three-player asymmetric RPS model together with the construction of the reverse-cyclic mutation Logit system. In this section, we also provide a detailed stability analysis of the interior equilibrium, highlighting the roles of mutation rate and selection intensity in shaping the system's dynamics. In Section 3, we present numerical simulations that illustrate the theoretical predictions and visualize the emergence of oscillatory behaviors under different parameter regimes. In Section 4, we summarize the major findings, discuss their implications for evolutionary game theory, and outline possible directions for future research.

2. Logit dynamics with inverse cyclic mutations

2.1. Model formulation

We consider an asymmetric three-player RPS game model, in which every players simultaneously chooses among rock, paper, and scissors with equal probability. Considering the cyclic dominance structure inherent in three-player RPS games, results remain consistent regardless of which player is the reference subject. Thus, the payoff matrix is constructed from the perspective of player I. Player II is assumed to be a high-profile player, similar to a Stackelberg oligopoly, where a dominant firm leads and smaller firms follow. This assumption captures asymmetry more realistically and highlights how the influence of a dominant player shapes the strategies of others. For clarity, the payoff rules are summarized as follows:

(i) When all players select identical strategies or distinct strategies (e.g., RRR, PPP, SSS, RPS, RSP, PRS, PSR, SRP, and SPR), player I receives a payoff of 0.

(ii) When player I and player II choose the same strategy but lose to player III (e.g., RRP, PPS, and SSR), player I incurs a payoff of -1, indicating a minor loss from aligning with a strong individual's unsuccessful choice.

(iii) When player I and player II adopt the same strategy and prevail against player III (e.g., RRS, PPR, and SSP), player I receives a payoff denoted by $\alpha(\alpha > 1)$, suggesting substantial gains from successfully aligning with the influential player.

(iv) When player I and player II select different strategies and player I loses (e.g., RPR, RPP, PSP, PSS, SRR, and SRS), player I experiences a payoff of $-\alpha$, representing significant losses resulting from decisions contrary to those of a strong player.

(v) When player I selects differently from player II but wins (e.g., RSR, RSS, PRR, PRP, SPP, and SPS), player I's payoff is 1, reflecting limited gains despite winning against the influential player's choice.

Based on this foundation, the corresponding payoff matrix is derived as shown in Table 1.

Table 1. Payoff matrix for player I in the three-player RPS game.

Game Players				III R	III P	III S
I	R	II	R	0	-1	α
			P	$-\alpha$	$-\alpha$	0
			S	1	0	1
	P	II	R	1	1	0
			P	α	0	-1
			S	0	$-\alpha$	$-\alpha$
	S	II	R	$-\alpha$	0	$-\alpha$
			P	0	1	1
			S	-1	α	0

For convenience, given the current population state $x = (x_1, x_2, x_3)$, denote player I's expected payoffs as f_1 for strategy R, f_2 for strategy P, and f_3 for strategy S. Here, x_1, x_2, x_3 represent the proportions of the R, P, and S strategies chosen by player I, respectively, and the state space is denoted by $\Delta^3 = \left\{x \in R^3 : \sum_{i=1}^{3} x_i = 1, x_i \geq 0\right\}$. Thus, When player I adopts strategy k, its expected payoff function is $f_k(x) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{kij}x_ix_j, (k = 1, 2, 3)$, where a_{kij} denotes the instantaneous payoff of player I when player I chooses strategy k, player II chooses strategy i, and player III chooses strategy j. x_i represents the probability that player II selects strategy i, and x_j represents the probability that player III selects strategy j. Expanding these expressions gives:

$$\begin{cases} f_1 = -\alpha x_2^2 + x_3^2 - (1+\alpha)x_1x_2 + (1+\alpha)x_1x_3, \\ f_2 = x_1^2 - \alpha x_3^2 + (1+\alpha)x_1x_2 - (1+\alpha)x_2x_3, \\ f_3 = -\alpha x_1^2 + x_2^2 - (1+\alpha)x_1x_3 + (1+\alpha)x_2x_3. \end{cases} \tag{2.1}$$

We employ the logit dynamic system to describe the temporal evolution of the strategies [20]. Compared with other evolutionary models, such as replicator dynamics, the logit system introduces a rationality parameter $\sigma(0 < \sigma < +\infty)$, which quantifies the degree of rationality exhibited by the players. When $\sigma \approx 0$, the probability of switching between strategies is nearly $\frac{1}{3}$, indicating that players transition among strategies almost randomly, independent of actual payoffs. As σ gradually increases, players become more rational, and the probability of selecting their best-response strategies also increases. When $\sigma = +\infty$, each player selects their optimal response strategy with absolute certainty, eliminating any random deviations ("trembles") and choosing the strategy most advantageous to them with complete rationality.

In biological evolution, mutation is a fundamental evolutionary mechanism [31, 32]. When mutations occur from relatively advantageous strategies to relatively inferior ones at a certain rate $\mu(0 < \mu < 1)$, these mutations are referred to as reverse cyclic mutations, with μ representing the reverse cyclic mutation rate. Specifically, in the RPS game characterized by cyclic dominance among three strategies, reverse cyclic mutations follow the cycle $R \to S \to P \to R$. Consequently, the reverse cyclic mutation logit dynamic system can be constructed as follows:

$$\begin{cases} \dot{x}_1 = \frac{e^{\sigma f_1}}{\sum_{k=1}^{3} e^{\sigma f_k}} - x_1 + u(x_2 - x_1), \\ \dot{x}_2 = \frac{e^{\sigma f_2}}{\sum_{k=1}^{3} e^{\sigma f_k}} - x_2 + u(x_3 - x_2), \\ \dot{x}_3 = \frac{e^{\sigma f_3}}{\sum_{k=1}^{3} e^{\sigma f_k}} - x_2 + u(x_1 - x_3). \end{cases} \tag{2.2}$$

To further simplify the analysis, let $x_1 = x, x_2 = y, x_3 = 1 - x - y$, to reduce the dimensionality of system (2.2). Substituting the expected payoff functions f_k into the system and the two-dimensional strategy space:

$$S \equiv \left\{(x, y, 1 - x - y) \in R^2 : (x, y, 1 - x - y) \in \Delta^3\right\}, \tag{2.3}$$

the system (2.2) simplifies as follows:

$$\begin{cases} \dot{x} = \frac{e^{[30(x+y-1)^2 - 30\alpha y^2 - 30xy(\alpha+1) - 30x(\alpha+1)(x+y-1)]}}{A} - \text{x} - \mu(\text{x} - \text{y}), \\ \dot{y} = \frac{e^{[30x^2 - 30\alpha(x+y-1)^2 + 30xy(\alpha+1) + 30y(\alpha+1)(x+y-1)]}}{A} - \text{y} - \mu(\text{x} + 2\text{y} - 1), \end{cases} \tag{2.4}$$

where

$$A = e^{[30x^2-30\alpha(x+y-1)^2+30xy(\alpha+1)+30y(\alpha+1)(x+y-1)]} + e^{[30y^2-30\alpha x^2+30x(\alpha+1)(x+y-1)-30y(\alpha+1)(x+y-1)]} + e^{[30(x+y-1)^2-30\alpha y^2-30xy(\alpha+1)-30x(\alpha+1)(x+y-1)]}. \tag{2.5}$$

Finally, the system possesses an interior equilibrium point $x^* = (\frac{1}{3}, \frac{1}{3})$, which also serves as the Nash equilibrium of the game. The stability of this equilibrium point will be analyzed in the subsequent discussion.

2.2. Equilibrium stability and Hopf bifurcation analysis

In order to investigate how mutation μ affects the evolutionary dynamics of the game, we assume the selection strength parameter σ to be constant and obtain the following theorem:

Theorem 2.1. *When $\alpha > 1$, let $\mu_0 = \frac{2\sigma(\alpha-1)}{27} - \frac{2}{3}$. The reverse cyclic mutation affects system (2.4) as follows:*
Case 1. *If $\mu > \mu_0$, the equilibrium x^* is asymptotically stable. If $\mu < \mu_0$, the equilibrium x^* becomes unstable, and an unstable periodic solution bifurcates from it.*
Case 2. *When $\mu = \mu_0$, denote the corresponding quantity as $E = 630\alpha + 13\sigma - 147\alpha\sigma - 273\alpha^2\sigma - 169\alpha^3\sigma + 414\alpha^2 + 468$. If $E < 0$, the system undergoes a supercritical Hopf bifurcation. If $E \geq 0$, a subcritical Hopf bifurcation occurs.*

Proof. Linearizing the nonlinear system (2.4) at the equilibrium point x^*, we obtain the Jacobian matrix of the system at x^* as follows:

$$J(\frac{1}{3}, \frac{1}{3}) = \begin{bmatrix} -\mu-\frac{\sigma}{3}-\frac{\alpha\sigma}{9}-1 & \mu-\frac{4\sigma}{9}-\frac{4\alpha\sigma}{9} \\ \frac{4\sigma}{9}-\mu+\frac{4\alpha\sigma}{9} & \frac{\sigma}{9}-2\mu+\frac{\alpha\sigma}{3}-1 \end{bmatrix}, \tag{2.6}$$

the characteristic equation of $J(1/3, 1/3)$ is given by:

$$\lambda^2 + \lambda(3\mu + \frac{2\sigma}{9} - \frac{2\alpha\sigma}{9} + 2) + 1 + 3\mu + \frac{2(\sigma-\alpha)}{9} - \frac{\mu\sigma}{3} + \frac{22\alpha\sigma^2}{81} + 3\mu^2 + \frac{13\sigma^2(1+\alpha^2)}{81} - \alpha\mu\sigma = 0, \tag{2.7}$$

the corresponding eigenvalues are a pair of complex conjugate roots:

$$\lambda_{1,2} = \gamma \pm i\beta, \tag{2.8}$$

where

$$\gamma = \frac{(\alpha-1)\sigma}{9} - \frac{3}{2}\mu - 1, \ \beta = \frac{\sqrt{3}}{2}\mu - \frac{2\sqrt{3}\sigma(1+\alpha)}{9}, \tag{2.9}$$

when $\alpha > 1$, setting $\gamma = 0$, we can obtain $\mu_0 = \frac{2\sigma(\alpha-1)}{27} - \frac{2}{3}$. If $\mu > \mu_0$, it follows that $\gamma < 0$, and thus the equilibrium x^* is asymptotically stable. Conversely, if $\mu < \mu_0$, we have $\gamma > 0$, the equilibrium x^* becomes unstable, the system undergoes a Hopf bifurcation at $\mu = \mu_0$.

When $\mu = \mu_0, \beta(\mu_0) = \frac{-\sqrt{3}(7\sigma+5\alpha\sigma+9)}{27} = \beta_0$, the system (2.4) can be simplified to:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\beta_0 \\ \beta_0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g(x_1, x_2, \mu_0) \\ h(x_1, x_2, \mu_0) \end{pmatrix}, \tag{2.10}$$

where $g(x_1, x_2, \mu_0)$ and $h(x_1, x_2, \mu_0)$ are given by:

$$\begin{aligned} g(x_1, x_2, \mu_0) &= \frac{e^{[-\sigma(\alpha y^2-(x+y-1)^2+xy(\alpha+1)+x(\alpha+1)(x+y-1))]}}{F} - (x-y)(\frac{2\sigma(\alpha-1)}{27} - \frac{2}{3}) - x - \frac{\sqrt{3}y(7\sigma+5\alpha\sigma+9)}{27}, \\ h(x_1, x_2, \mu_0) &= \frac{e^{[\sigma(x^2-\alpha(x+y-1)^2+xy(\alpha+1)+y(\alpha+1)(x+y-1))]}}{F} - (\frac{2\sigma(\alpha-1)}{27} - \frac{2}{3})(x+2y-1) - y + \frac{\sqrt{3}x(7\sigma+5\alpha\sigma+9)}{27}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} F = {} & e^{[-\sigma(\alpha y^2-(x+y-1)^2+xy(\alpha+1)+x(\alpha+1)(x+y-1))]} + e^{[-\sigma(\alpha x^2-y^2-x(\alpha+1)(x+y-1)+y(\alpha+1)(x+y-1))]} \\ & + e^{[\sigma(x^2-\alpha(x+y-1)^2+xy(\alpha+1)+y(\alpha+1)(x+y-1))]}. \end{aligned}$$

In polar coordinates, system (2.10) can be transformed into the following form:

$$\begin{cases} \dot{r} = r(d\mu_0 + ar^2), \\ \dot{\theta} = \beta_0 + c\mu_0 + br^2, \end{cases} \tag{2.12}$$

consider the first Lyapunov coefficient of the system at the equilibrium point $(\frac{1}{3}, \frac{1}{3}, \mu_0)$,

$$\begin{aligned} a = {} & \frac{1}{16}(g_{x_1x_1x_1} + h_{x_1x_2x_2} + h_{x_2x_2x_2}) + \frac{1}{16\omega}[g_{x_1x_2}(g_{x_1x_1} + g_{x_2x_2}) \\ & - h_{x_1x_2}(h_{x_1x_1} + h_{x_2x_2}) - g_{x_1x_1}h_{x_1x_1} + g_{x_2x_2}h_{x_2x_2}], \end{aligned} \tag{2.13}$$

after calculation, the result is:

$$\begin{aligned} a &= \frac{\sigma^2}{3888}(630\alpha + 13\sigma - 147\alpha\sigma - 273\alpha^2\sigma - 169\alpha^3\sigma + 414\alpha^2 + 468), \\ d &= \frac{\partial}{\partial\mu}(\gamma(\mu)) = -\frac{3}{2} < 0. \end{aligned} \tag{2.14}$$

Setting $E = 630\alpha + 13\sigma - 147\alpha\sigma - 273\alpha^2\sigma - 169\alpha^3\sigma + 414\alpha^2 + 468$. If $E < 0, a < 0$, the system undergoes a supercritical Hopf bifurcation. If $E > 0, a > 0$, the system undergoes a subcritical Hopf bifurcation. When $E = 0$, i.e., $\sigma = -\frac{630\alpha+414\alpha^2+468}{13-147\alpha-273\alpha^2-169\alpha^3}$, it follows that $a = 0$. According to the center manifold theory, the system near the Hopf bifurcation point can be reduced to the standard form expressed in terms of a one-dimensional complex coordinate z:

$$\dot{z} = iw_0z + l_1z|z|^2 + l_2z|z|^4 + O(|z|^6), \tag{2.15}$$

linearizing the nonlinear system (2.4) at the equilibrium point, we obtain the Jacobian matrix as follows:

$$J_0(\frac{1}{3}, \frac{1}{3}, \mu_0, \sigma) = \begin{pmatrix} -m & -n \\ n & m \end{pmatrix}, \tag{2.16}$$

where

$$m = \frac{133\alpha^3+315\alpha^2+299\alpha+117}{169\alpha^3+273\alpha^2+147\alpha-13}, n = \frac{266\alpha^3+630\alpha^2+598\alpha+234}{169\alpha^3+273\alpha^2+147\alpha-13},$$

the corresponding eigenvalues are a pair of purely imaginary roots: $\pm iw$, with

$$w = \frac{\sqrt{3}(\alpha+1)(133\alpha^2+182\alpha+117)}{(13\alpha-1)(13\alpha^2+22\alpha+13)}.$$

According to

$$\begin{cases} J_0q = iw_0q, \\ p^TJ_0 = iw_0p^T, \end{cases} \tag{2.17}$$

and p, q satisfying

$$\langle p, q\rangle = 1, \tag{2.18}$$

we derive

$$q = \begin{pmatrix} (\sqrt{6}i - \sqrt{2})/4 \\ \sqrt{2}/2 \end{pmatrix}, p = \begin{pmatrix} -\sqrt{6}/3 \\ (\sqrt{2} - \sqrt{6}i)/6 \end{pmatrix},$$

substituting into the formula for the second Lyapunov coefficient,

$$l_2 = \frac{1}{2w_0}\mathrm{Re}[\langle p, C(q, q, \overline{q})\rangle - 2\left\langle p, B(q, J_0^{-1}B(q, \overline{q}))\right\rangle + \left\langle p, B(\overline{q}, (J_0 - 2iw_0 I)^{-1}B(q, q))\right\rangle], \tag{2.19}$$

we obtain:

$$l_2 = \frac{63(5\alpha^2 + 6\alpha + 5)(23\alpha^2 + 35\alpha + 26)^2}{2(13\alpha - 1)^2(1729\alpha^4 + 5292\alpha^3 + 7254\alpha^2 + 4940\alpha + 1521)}, \tag{2.20}$$

since $\alpha > 1$, it follows that $l_2 > 0$, and thus the system undergoes a subcritical Hopf bifurcation. □

From the theorem above, it can be seen that mutations significantly influence the dynamics of the system. Specifically, when $\alpha > 1$, if the mutation rate exceeds a certain critical value μ_0 but remains less than 1, the three strategies coexist stably in the long run. Conversely, if the mutation rate is below this critical value μ_0 but greater than 0, the three strategies exhibit periodic oscillations.

Next, considering μ as a constant, we examine how the selection intensity σ affects the game dynamics, leading to the following theorem:

Theorem 2.2. *When $\alpha > 1$, let $\sigma_0 = \frac{9}{2}\frac{3\mu+2}{\alpha-1}$. If $\sigma < \sigma_0$, the equilibrium x^* is asymptotically stable. If $\sigma > \sigma_0$, the equilibrium x^* becomes unstable, and an unstable periodic solution bifurcates from it. In other words, the system undergoes a supercritical Hopf bifurcation at $\sigma = \sigma_0$.*

Proof. When $\alpha > 1$, setting $\gamma = 0$, we obtain $\sigma_0 = \frac{9}{2}\frac{3\mu+2}{\alpha-1}$. If $\sigma < \sigma_0$, it follows that $\gamma < 0$, and thus the equilibrium x^* is asymptotically stable. Conversely, if $\sigma > \sigma_0$, we have $\gamma > 0$, the equilibrium x^* becomes unstable, the system undergos a Hopf bifurcation at $\sigma = \sigma_0$.

When $\sigma = \sigma_0, \beta(\sigma_0) = -\frac{\sqrt{3}(4\alpha+7\mu+5\alpha\mu+4)}{2(\alpha-1)} = \beta_1$, system (2.4) can be expressed as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\beta_1 \\ \beta_1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g(x_1, x_2, \sigma_0) \\ h(x_1, x_2, \sigma_0) \end{pmatrix}, \tag{2.21}$$

where $g(x_1, x_2, \sigma_0)$ and $h(x_1, x_2, \sigma_0)$ are given by:

$$\begin{aligned} g(x_1, x_2, \sigma_0) = & \frac{e^{[-(27\mu/2+9)(\alpha y^2-(x+y-1)^2+xy(\alpha+1)+x(\alpha+1)(x+y-1))/(\alpha-1)]}}{M} - x - \mu(x - y) \\ & - \sqrt{3}y(4\alpha + 7\mu + 5\alpha\mu + 4)/(2(\alpha - 1)), \\ h(x_1, x_2, \sigma_0) = & \frac{e^{[(27\mu/2+9)(x^2-\alpha(x+y-1)^2+xy(\alpha+1)+y(\alpha+1)(x+y-1))/(\alpha-1)]}}{M} - \mu(x + 2y - 1) \\ & - y + \sqrt{3}x(4\alpha + 7\mu + 5\alpha\mu + 4)/(2(\alpha - 1)), \end{aligned} \tag{2.22}$$

with

$$\begin{aligned} M = & e^{[-((27\mu/2+9)(\alpha y^2-(x+y-1)^2+xy(\alpha+1)+x(\alpha+1)(x+y-1)))/(\alpha-1)]} \\ & + e^{[-((27\mu/2+9)(\alpha x^2-y^2-x(\alpha+1)(x+y-1)+y(\alpha+1)(x+y-1)))/(\alpha-1)]} \\ & + e^{[(27\mu/2+9)(x^2-\alpha(x+y-1)^2+xy(\alpha+1)+y(\alpha+1)(x+y-1))/(\alpha-1)]}. \end{aligned}$$

In polar coordinates, system (2.21) can be transformed into the following form:

$$\begin{cases} \dot{r} = r(d\sigma_0 + ar^2), \\ \dot{\theta} = \beta_0 + c\sigma_0 + br^2, \end{cases} \tag{2.23}$$

analogous to the theorem above, calculations yield the first Lyapunov coefficient in this case as follows:

$$\begin{aligned} a &= -\frac{9(3\mu+2)^2(110\alpha-13\mu+147\alpha\mu+273\alpha^2\mu+169\alpha^3\mu+166\alpha^2+82\alpha^3+26)}{128(\alpha-1)^3} < 0, \\ d &= \frac{\partial}{\partial\sigma}(\gamma(\sigma)) = \frac{\alpha-1}{9} > 0, \end{aligned} \tag{2.24}$$

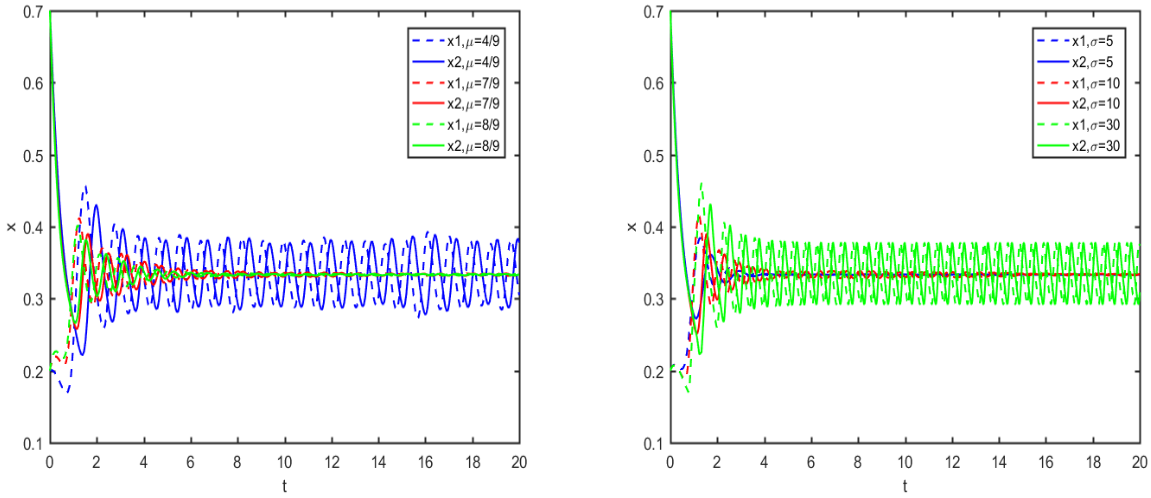
in this case, the system undergoes a supercritical Hopf bifurcation. □

During the evolutionary game process, when the game is structured as a "high-risk, high-reward" scenario, the selection intensity has a significant impact on the evolutionary outcomes. Specifically, if the selection intensity is relatively low, the three strategies can coexist stably over the long term. However, if the selection intensity exceeds a certain critical threshold, the three strategies exhibit periodic oscillations. In summary, according to Theorems 2.1 and 2.2, although both the mutation rate and selection intensity can induce Hopf bifurcations in the system, under certain conditions, the directions of the bifurcations caused by these two factors may differ.

3. Numerical simulation

In the previous section, we explored the effects the impact of selection intensity and mutation rate on the interior equilibrium of a three-player asymmetric RPS game. To further understand the dynamic behaviors of this game model, we employ numerical simulations for validation and deeper exploration. Specifically, MATLAB software is used to generate phase-space trajectories and time-series diagrams, visually illustrating the system's dynamic behavior under various parameter conditions.

Figure 1 summarizes the time series plots of system (2.4) under different values of the mutation rate μ (panel (a)) and selection intensity σ (panel (b)), with initial conditions set as $(0.2, 0.7)$. Specifically, in panel (a), we set $\alpha = 4, \sigma = 5$, from which it can be observed that the equilibrium $(\frac{1}{3}, \frac{1}{3})$ is asymptotically stable when $\mu > \mu_0 = \frac{4}{9}$, whereas periodic oscillations appear when $\mu < \mu_0$. This illustrates that higher mutation levels stabilize coexistence among strategies. In panel (b), we set $\alpha = 2, \mu = 0.6$, demonstrating that the equilibrium $(\frac{1}{3}, \frac{1}{3})$ is asymptotically stable when $\sigma < \sigma_0 = 17.1$, but increasing σ beyond this threshold destabilizes the equilibrium, giving rise to sustained oscillations around it. These observations are consistent with Theorems 2.1 and 2.2, providing numerical evidence of Hopf bifurcations and highlighting how mutation and selection jointly shape the dynamic outcomes of the asymmetric three-player RPS game.



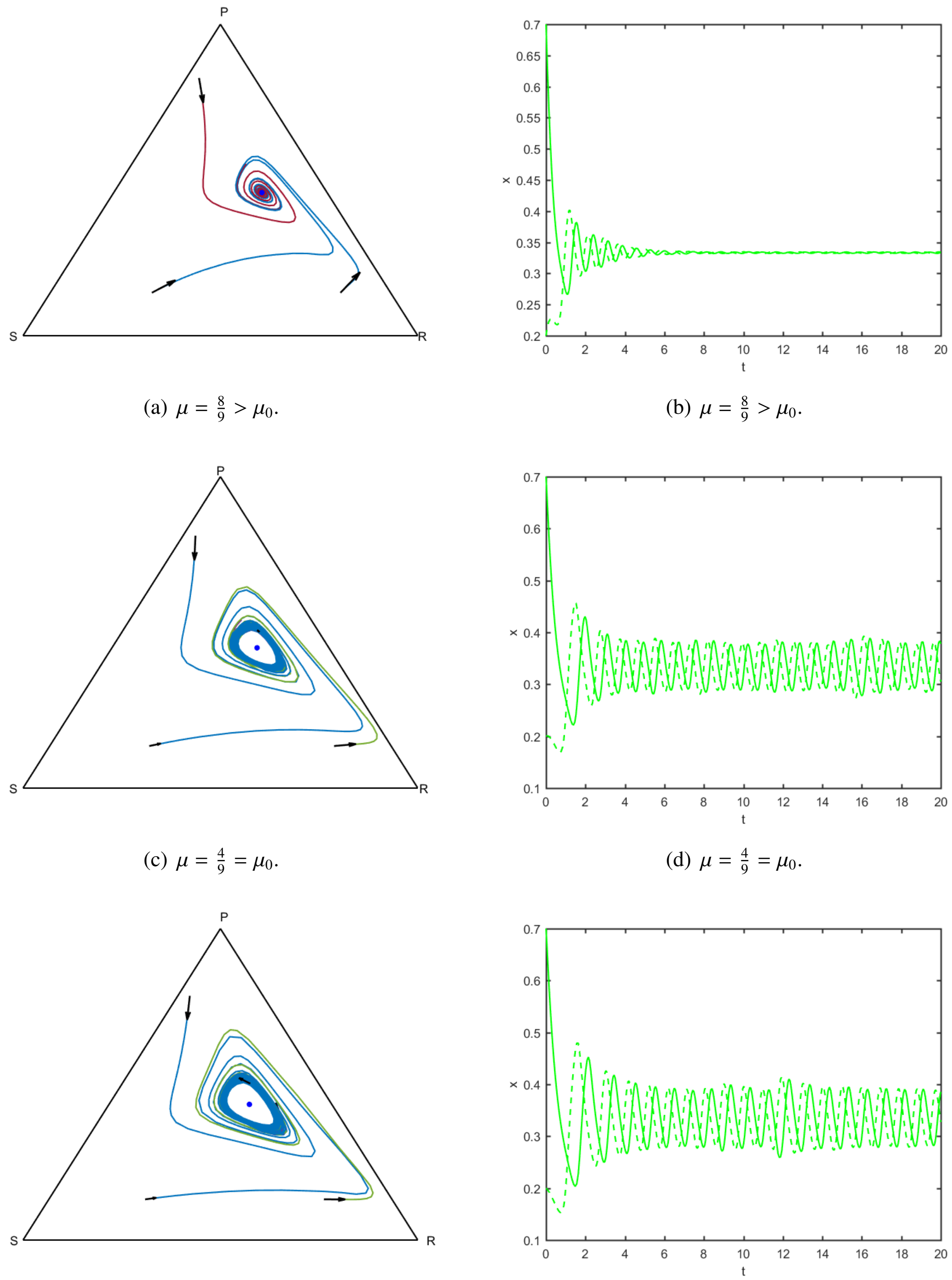
(a) Different values of μ.

(b) Different values of σ.

Figure 1. Time series of system (2.4) with initial condition (0.2, 0.7), the equilibrium $(\frac{1}{3}, \frac{1}{3})$ is asymptotically stable when $\mu > \mu_0$ or $\sigma < \sigma_0$, while periodic oscillations emerge otherwise, indicating a Hopf bifurcation.

Figure 2 illustrates the effects of varying values of μ on the reverse cyclic mutation logit system under a fixed selection intensity of σ. Both phase portraits and time series plots are presented. Selecting the initial conditions as (0.2, 0.7), when $\alpha = 4, \sigma = 5$, the corresponding value is $\mu_0 = \frac{4}{9}$. As shown in panels (a) and (b), when $\mu = \frac{8}{9} > \mu_0$, the trajectories converge, directly to the interior equilibrium $(\frac{1}{3}, \frac{1}{3})$, indicating asymptotic stability. Panels (c) and (d) depict the system at a critical state when $\mu = \frac{4}{9} = \mu_0$, where the trajectories circle around the equilibrium in a marginally stable manner, reflecting the onset of a Hopf bifurcation. Finally, panels (e) and (f) demonstrate that when $\mu = \frac{3}{9} < \mu_0$, the system exhibits a limit cycle around the equilibrium point, as evidenced by the closed orbits in the phase portrait and the sustained oscillations in the time series. Taken together, Figure 2 highlights the full transition from stability to oscillation and visualizes the formation of periodic orbits, thereby providing clear numerical evidence of Hopf bifurcation in the asymmetric three-player RPS game.

Figure 3 illustrates the influence of different σ values on the reverse-cyclic mutation logit system under a fixed mutation rate μ. Both phase diagrams and time series plots are provided. Setting initial conditions as (0.2, 0.7), we consider the scenario when $\alpha = 2, \mu = 0.6$, corresponding to the case $\sigma_0 = 17.1$. Panels (a) and (b) show that when $\sigma = 5 < \sigma_0$, the trajectories stabilize over time, converging smoothly to the interior equilibrium $(\frac{1}{3}, \frac{1}{3})$, indicating asymptotic stability. In panels (c) and (d), the system is at a critical state when $\sigma = \sigma_0 = 17.1$, where the system oscillates around the equilibrium without settling, signaling the onset of a Hopf bifurcation. Last, panels (e) and (f) illustrate that the system exhibits a limit cycle around the equilibrium point when $\sigma = 30 > \sigma_0$, as evidenced by the closed orbit in the phase portrait and the sustained oscillations in the time series. Compared with Figure 2, which emphasizes the role of the mutation rate, Figure 3 highlights how increasing rationality (higher σ) destabilizes the equilibrium, demonstrating that mutation and selection can independently induce Hopf bifurcations in the asymmetric three-player RPS game.

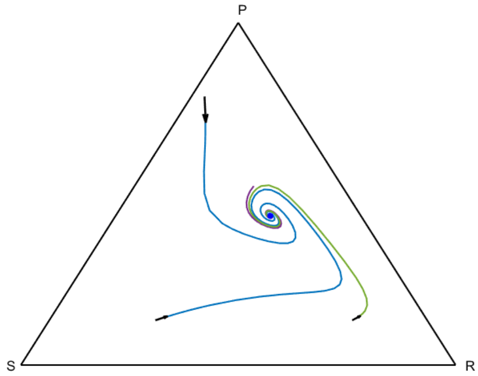


(a) $\mu = \frac{8}{9} > \mu_0$. (b) $\mu = \frac{8}{9} > \mu_0$.

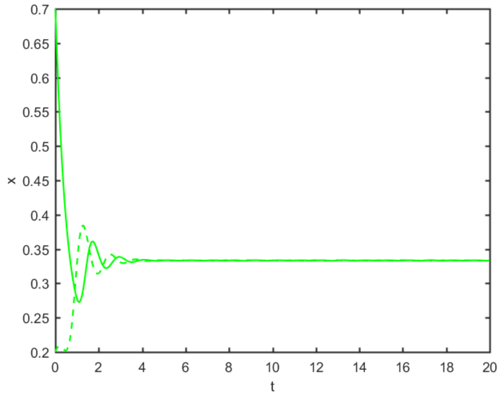
(c) $\mu = \frac{4}{9} = \mu_0$. (d) $\mu = \frac{4}{9} = \mu_0$.

(e) $\mu = \frac{3}{9} < \mu_0$. (f) $\mu = \frac{3}{9} < \mu_0$.

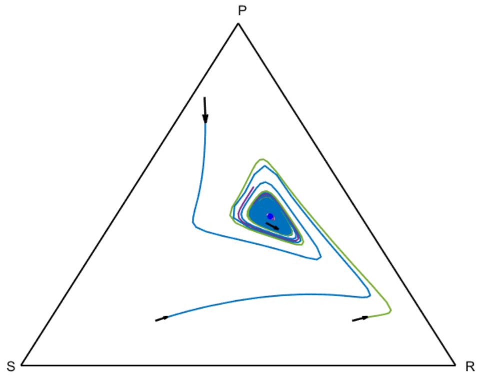
Figure 2. Phase portraits and time series of system (2.4) with $\alpha = 4$, $\sigma = 5$, and initial condition $(0.2, 0.7)$. The equilibrium $(\frac{1}{3}, \frac{1}{3})$ is asymptotically stable when $\mu > \mu_0 = \frac{4}{9}$ and becomes marginally stable at $\mu = \mu_0$, and loses stability with a stable limit cycle when $\mu < \mu_0$.



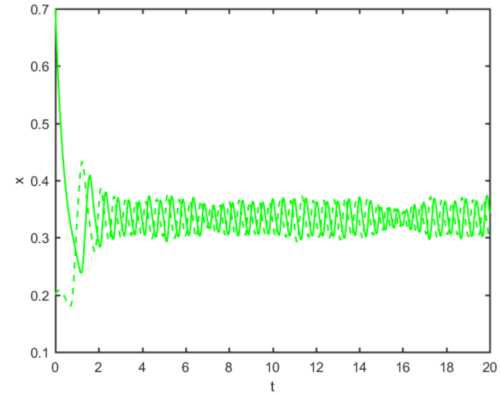
(a) $\sigma = 5 < \sigma_0$.



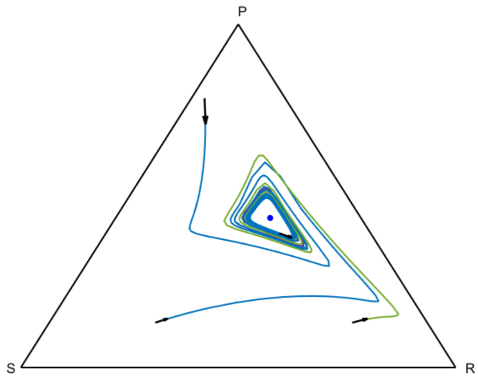
(b) $\sigma = 5 < \sigma_0$.



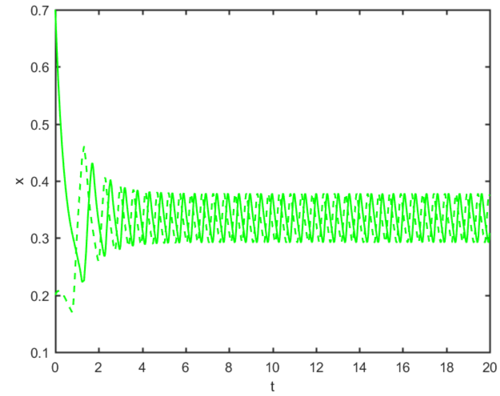
(c) $\sigma = \sigma_0 = 17.1$.



(d) $\sigma = \sigma_0 = 17.1$.



(e) $\sigma = 30 > \sigma_0$.



(f) $\sigma = 30 > \sigma_0$.

Figure 3. Phase portraits and time series of system (2.4) with $\alpha = 2$, $\mu = 0.6$, and initial condition $(0.2, 0.7)$. The equilibrium $(\frac{1}{3}, \frac{1}{3})$ is asymptotically stable when $\sigma < \sigma_0 = 17.1$, becomes marginally stable at $\sigma = \sigma_0$ and loses stability with a stable limit cycle when $\sigma > \sigma_0$, showing the occurrence of a Hopf bifurcation.

4. Conclusions

In this paper, we focus on a three-player asymmetric RPS game by constructing an asymmetric payoff matrix and subsequently establishing a reverse cyclic mutation logit system. The interior equilibrium points of the system are then analyzed in depth. First, we investigate the effects of varying values of μ on the dynamics of the system. It is revealed that the equilibrium x^* is asymptotically stable when μ exceeds the critical value μ_0 but loses stability once μ drops below μ_0, giving rise to an unstable periodic orbit via a Hopf bifurcation. During the computations, when the first Lyapunov coefficient is nonzero, the bifurcation type can be directly determined. However, if the first Lyapunov coefficient equals zero, the system may experience a degenerate Hopf bifurcation. In this case, further analysis is required, and the second Lyapunov coefficient is computed to determine the precise bifurcation type: If $E < 0$, the system undergoes a supercritical Hopf bifurcation; and if $E \geq 0$, a subcritical Hopf bifurcation occurs. Furthermore, we explore how varying values of σ influence the system. The results indicate that when $\sigma < \sigma_0$, the equilibrium point x^* is asymptotically stable. Conversely, when $\sigma > \sigma_0$, this equilibrium loses stability, and a stable periodic solution emerges around it, indicating that the system undergoes a supercritical Hopf bifurcation at $\sigma = \sigma_0$. Finally, numerical simulations are performed to validate the theoretical findings, and the results are consistent with the theoretical analysis, thereby confirming the accuracy of the conclusions drawn in this paper. Beyond these results, future work could extend the present framework to other forms of asymmetric multiplayer games, incorporate additional mechanisms such as time delays or stochastic perturbations, and explore applications in socio-economic and biological systems where asymmetric strategic interactions are prevalent.

Author contributions

Jie Liu: Writing-original draft, methodology, software; Wenjun Hu: Writing-review & editing, methodology, funding acquisition, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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