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Research article

A comparative study of bayesian and classical methods for the weighted Lindley distribution under unified hybrid censoring with survival data applications

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Abstract: In survival analysis and reliability engineering, censoring schemes play a crucial role in efficient data collection and analysis. This study investigated the unified hybrid censoring scheme (UHCS), a versatile framework that integrates multiple censoring strategies, to evaluate the suitability of the Weighted Lindley (WL) distribution for modeling lifetime data. Maximum likelihood estimates (MLEs) and their corresponding asymptotic confidence intervals are derived for the parameters of the WL distribution. In the Bayesian framework, parameter estimation was performed under a squared error loss function. A detailed Monte Carlo simulation study was conducted to compare the performance of classical and Bayesian estimators across various sample sizes and censoring schemes. The simulation results revealed that Bayesian estimators consistently yielded lower mean squared errors (MSEs) than their classical counterparts, and the associated credible intervals were generally narrower than the frequentist confidence intervals. To demonstrate the practical applicability of the proposed methods, the analysis was applied to real-world survival datasets. The results highlighted the effectiveness of the WL distribution under UHCS, offering valuable insights for researchers and practitioners in reliability and survival analysis.

Keywords: weighted Lindley distribution; unified hybrid censoring scheme; maximum likelihood estimation; Bayesian estimation; Markov Chain Monte Carlo; survival analysis; reliability modeling; censored data
Mathematics Subject Classification: 62G30, 62E10

1. Introduction

In reliability testing and survival analysis, censoring refers to incomplete observation of lifetimes or survival times. Two commonly employed censoring schemes are Type-I (TyI) and Type-II (TyII), as noted by [14, 20, 22]. TyI censoring has the advantage that it fixes the duration (useful for cost or time constraints) but might yield few failures if the product is very reliable. TyII censoring guarantees a certain number of failure data points but the test duration is uncertain and could be very long if failures are rare. To balance these concerns, hybrid censoring schemes were introduced as blends of TyI and TyII criteria [29]. The earliest was proposed by Epstein [12], who described a TyI hybrid censoring plan where the test is stopped at the earliest of (a) a pre-specified time or (b) the r-th failure, whichever occurs first [29]. This TyI hybrid scheme ensures the test does not run past time T while still aiming for r failures (it is essentially a TyI test with a TyII upper bound on the number of failures [23]). Later, [9] considered the complementary idea often called TyII hybrid censoring, where the test is stopped at the latest of the two criteria (ensuring at least r failures are obtained even if it means running beyond the initial time limit) [29]. In other words, TyII hybrid can be seen as a TyII test with a TyI lower bound on time (guaranteeing a minimum test duration) [23]. These hybrid schemes improved flexibility in test planning. In 2004, [5] formalized generalized TyI hybrid and generalized TyII hybrid schemes, each introducing both a lower and upper bound in time or failure count to further refine stopping rules [14]. By the mid-2000s, however, researchers sought an even more comprehensive framework that could encompass all these censoring strategies as special cases.

The term unified hybrid censoring scheme (UHCS) was introduced by Balakrishnan et al. [3] as a new censoring scheme that unifies the generalized TyI and TyII hybrid approaches [14, 24]. Essentially, the UHCS sets two time limits (often denoted T_1 and T_2, with $T_1 < T_2$) and a failure count threshold (often denoted m or r) to determine when the life test terminates. This creates a flexible stopping rule that adapts to whether failures happen very early, moderately, or very late [27]. In a typical UHCS plan, one can describe it as follows:

- If the pre-specified failure count m is reached too early (before time T_1), the test is not immediately stopped at that failure. Instead, to avoid an excessively short test, the experiment continues at least until time T_1. (In other words, T_1 serves as a minimum test time.) [23].
- If the m-th failure occurs in a mid-range timeframe (between T_1 and T_2), the test is stopped exactly at that m-th failure time (as in a traditional TyII censoring within that window) [27].
- If failures are so delayed that even by the upper time limit T_2 the m-th failure has not occurred, then the test is terminated at time T_2 regardless of the number of failures observed. Thus T_2 serves as a hard maximum test duration (like a TyI cutoff if the product is very reliable) [27].

In effect, UHCS imposes both a lower and upper bound on test time in combination with a failure-count requirement, yielding a range of six possible outcomes depending on whether and when the m-th failure occurs (before T_1, between T_1 and T_2, after T_2, etc.) [14]. This scheme truly generalizes earlier ones: Setting $T_1 = 0$ (no lower bound) reduces it to the classical TyI hybrid (stop at first of m failures or T_2), whereas letting $T_1 = T_2$ collapses it to a pure TyII censoring at a fixed time, and other special choices recover the generalized hybrid schemes [9, 23]. [3] demonstrated the exact likelihood-based inference under the UHCS for exponential distributions, confirming that standard TyI, TyII, and hybrid censoring results fall out as special cases of their unified model [9, 14]. In summary, the UHCS provides experimenters greater flexibility to control both time and the amount of failure information gathered, by unifying the censoring criteria in a single framework.

1.1. UHCS literature exploration

To understand the progression and scholarly attention devoted to UHCS, a bibliometric analysis was conducted using data extracted from Scopus, covering publications from 2008—the year UHCS was first formally introduced—through 2025. Table 5 reports the bibliometric summary. A total of 39 peer-reviewed articles were retrieved based on English language filters and inclusion of the core concept "Unified Hybrid Censoring". No disciplinary filters were applied to ensure a comprehensive overview, though the top contributing subject areas were identified as Mathematics, Decision Sciences, Engineering, Computer Science, and Physics & Astronomy. The corpus comprised publications from 30 different sources, including journals and books, indicating moderate dispersion across outlets. The annual publication growth rate was 6.68%, suggesting steady expansion in scholarly interest. With an average document age of 4.15 years and 15.36 citations per document, the field demonstrates moderate citation impact, reflecting both relevance and growing maturity of the topic in statistical and reliability research domains. In total, the analyzed documents cited 1,102 references, and contributed 174 Keywords Plus and 135 Author Keywords, indicating a diverse yet concentrated thematic coverage. Notably, among the 76 authors contributing to this corpus, only 3 published single-authored papers, and overall co-authorship averaged 2.74 authors per document. The degree of international collaboration was substantial, with 41.03% of documents featuring international co-authorship, reinforcing the global interest and cross-national research cooperation in the domain of UHCS. Regarding publication types, all 39 documents were categorized as peer-reviewed journal articles. This homogeneity underscores that UHCS remains primarily a scholarly concern, with strong methodological emphasis and less attention (so far) in books or practitioner literature. These confirm that while the field is relatively specialized, it is characterized by growing interest, cross-disciplinary engagement, and strong collaborative structures particularly in theoretical modeling and statistical inference. The early works of [3] laid a methodological foundation for this stream, which has since diversified into various parametric models (e.g., exponential, Weibull, inverse Pareto), Bayesian frameworks, and extensions to progressive and competing risks censoring [19]. The bibliometric trends also support the thematic map's observation that hybrid and Monte Carlo-based estimation methods remain dominant "motor themes," while UHCS itself is emerging as a high-density, moderate-centrality research niche.

1.2. Thematic structure of research on UHCS

The thematic map illustrates in Figure 8 the conceptual landscape of research related to unified hybrid censoring and associated statistical methods, based on bibliometric co-word analysis. The themes are positioned along two axes: The degree of development or internal cohesion (density) on the vertical axis and the degree of centrality or relevance to the broader research field on the horizontal axis [10]. Motor themes (High Centrality, High Density), represented by the upper-right quadrant in Figure 8, the themes labeled as "hybrid censoring", "maximum likelihood estimation", and "Monte Carlo Methods" are both well-developed and highly relevant. These represent the core of contemporary research in this domain. "Hybrid censoring" is the immediate forebear paved way for "unified hybrid censoring". Hybrid censoring has now become a prominent statistical method for reliability analysis and survival modeling. The absence of basic themes (High Centrality, Low Density, see Figure 8) is an indication that the existing topic and methods are not fully developed and still in the process of fruition. The "Bayesian estimations", "confidence interval", and "maximum likelihood methods" can be identified as foundational to the domain, indicating that they are frequently used but perhaps not deeply specialized in the context of UHCS. They form the theoretical backbone of most modeling and estimation strategies in reliability statistics [4]. Similarly, the absence of the niche theme (High Density, Low Centrality), indicates that the existing researches were more focused on exploring the application of the existing models, representing the need for developing novel models under unified hybrid censoring structures. Emerging or declining themes (Low Density, Low Centrality, See Figure 8), represented by lower-left quadrant, that including themes such as "parameters estimation", "stress-strength models", " the Weibull", and "likelihood functions", indicates underdeveloped areas rather than declining as the topics were starting to be explored only in the 21st Century. This suggest that limited research has explored these areas and they are the potentially foundational topics for methodological development.

The primary objective of this paper is to investigate estimation methods, both classical and Bayesian, for a recently developed version of the Lindley distribution (LD) (see, Section 2) under the UHCS framework. For the classical approach, we employ maximum likelihood estimation (MLE). In the Bayesian framework, we derive Bayes estimates under the squared error loss function (SELF). We also compute the asymptotic and highest posterior density (the HPD) credible intervals (CRI) at a 5% significance level. Since none of the proposed estimators for the unknown parameters can be obtained in closed form, numerical techniques and computational tools are utilized for their evaluation. In particular, the MLEs are calculated using the 'optim' package, applying the Newton–Raphson method for maximization. Additionally, Bayesian estimates are generated using the 'coda' package, which assists in analyzing the markov chain monte carlo (MCMC) outputs and diagnosing convergence issues. Furthermore, we identify the optimal censoring scheme among the proposed UHCS using real survival data. This work aims to provide practical guidelines for selecting the best estimation and prediction methods for the distribution within the UHCS framework, which we believe will be valuable for applied statisticians, including reliability engineers, quality analysts, and risk assessors.

The paper is organized into seven sections, where Section 1 provides the introduction. This is followed by an in-depth explanation of the Weighted Lindley (WL) distribution model in Section 2. Section 3 presents the model description and the likelihood function of the proposed model. Section 4 demonstrates the classical and Bayesian estimation techniques of unknown parameters of the WL distribution within the UHCS framework. The effectiveness of the proposed estimators using simulation is organized as Section 5. Section 6 is the evaluation of practical utility of the proposed model by applying it over two different real world datasets. The final section, i.e., Section 7, includes the discussion, concluding remarks, and suggestions for future research directions.

2. Weighted Lindley distribution

The exponential distribution (ED) is one of the most fundamental and extensively applied models for nonnegative random variables, particularly in the domain of lifetime data analysis. One of the defining features of the ED is its memoryless property, implying that the probability of failure in the next moment is independent of the time elapsed. Mathematically, this results in a constant hazard rate $h_X(x) = \lambda$ for all $x \geq 0$. Although this property simplifies modeling, it can be restrictive in practical scenarios. Many real-life systems exhibit nonconstant hazard behaviors—either increasing due to aging or decreasing due to early-life failures—rendering the ED insufficient in such contexts. These limitations have prompted the introduction of more flexible lifetime distributions. Among such alternatives, the LD has gained notable attention. Originally introduced by [16, 17] to reconcile differences between fiducial and posterior distributions, the LD has proven useful in various applied contexts. The probability density function (PDF) is given by:

$$g(x) = \frac{\eta^2}{\eta+1}(1+x)e^{-\eta x}, \quad x > 0, \quad \eta > 0 \tag{2.1}$$

$$= pg_1(x) + (1-p)g_2(x)$$

where $p = \frac{\eta}{\eta+1}$, $g_1(x) = \eta e^{-\eta x}$ for $x > 0$, and $g_2(x) = \eta^2 x e^{-\eta x}$ for $x > 0$. This clearly shows that the LD is a mixture of two components: an ED with rate η and a gamma distribution with shape 2 and scale η. The body of research surrounding the LD is extensive, emphasizing its advantages over the ED in specific cases such as modeling waiting times of bank customers [13]. A comprehensive review of the distribution is presented in [28], while further contributions by [2, 6–8, 15, 18], among others, offer insights into its theoretical and practical applications. This study builds upon a novel variant introduced by [21], namely, the WL distribution. This model utilizes a weight function $w(x) = (1 + e^{-\eta x})^{-1}$, and the resulting PDF is formulated as:

$$f(x) = \frac{w(x)\, g(x)}{\int_0^\infty w(t)\, g(t)\, dt}. \tag{2.2}$$

The explicit form of the PDF for $X \sim \mathrm{WL}(\eta)$, where $\eta > 0$, is defined as:

$$f(x) = \begin{cases} \frac{12\eta^2}{(\pi^2+12\eta\log(2))} \frac{1+x}{1+e^{x\eta}}, & 0 \leq x < \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

The corresponding cumulative distribution function (CDF) is given by:

$$F(x) = \frac{\pi^2 - 6\eta\left(\eta x(2+x) + \log(4)\right) + 12\eta(1+x)\log(1+e^{\eta x}) + 12Li(-e^{\eta x})}{\pi^2 + 12\eta\log(2)}. \tag{2.4}$$

The Li_2 function represents the dilogarithm function, which is a special case of polylogarithmic functions. The classical dilogarithm function for $-1 \leq x \leq 1$ is defined as follows:

$$Li_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

The failure rate function of the WL distribution is given, for $x > 0$ and $\eta > 0$, as:

$$h(x) = \frac{\eta}{1 + \exp(\eta x)\left[12\eta\log(2) + 6\eta(\eta x(2+x) + \log(4)) - \log(1+\exp(\eta x)) - 12\mathrm{Li}(-\exp(\eta x))\right]}. \tag{2.5}$$

Figures 1 and 2 highlight the graphical behavior of the WL distribution. Figure 1 shows that the PDF is unimodal and right-skewed, with its peak shifting closer to the origin as the parameter increases, indicating greater density near lower values. Figure 2 displays the hazard function $h(x)$, which increases monotonically, reinforcing the distribution's applicability in modeling aging processes where the likelihood of failure rises over time. Compared with widely used one-parameter families in the recent reliability/UHCS literature—e.g., exponential (constant hazard), Rayleigh (increasing but shape-fixed hazard), and the classical Lindley (limited hazard shape)—the proposed WL offers the following while still using a single parameter η:

- WL's hazard increases with age (Figure 2), unlike ED's flat hazard and with more adaptable growth than Rayleigh's fixed quadratic trend.
- WL admits explicit PDF/CDF/hazard expressions (Eqs (2.3)–(2.5)), enabling likelihood-based and Bayesian estimation without introducing extra tuning parameters.
- Within the UHCS, WL leads to a compact unified likelihood and stable inference. Simulations across UHCS regimes show low mean squared error (MSE) and especially strong Bayesian performance.
- On two oncology survival datasets, WL aligns closely with empirical distributions (Probability-Probability(P–P)/Quantile-Quantile(Q–Q) and CDF plots, Figures 6 and 7) and yields tight intervals (Table 3), confirming practical adequacy with real survival data.

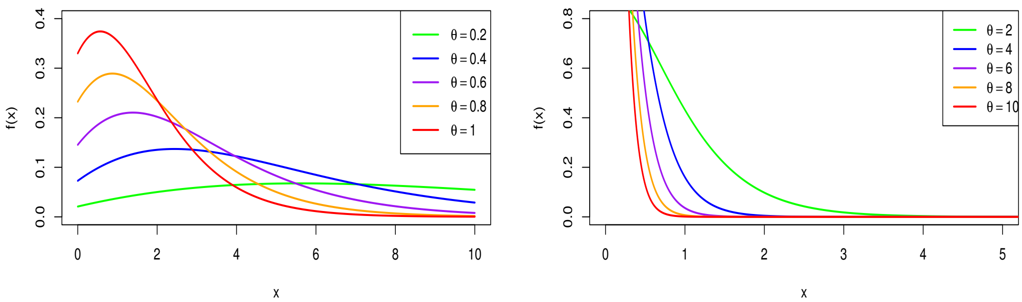


Figure 1. PDF plots of WL model.

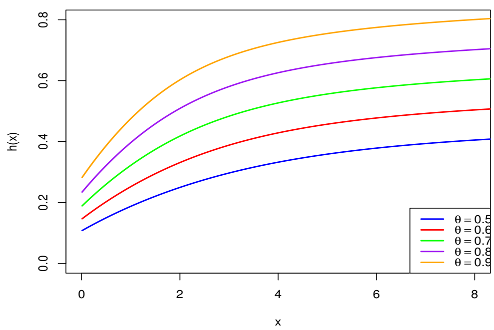


Figure 2. Plots of the failure rate $h(x)$.

3. Model description and likelihood function of proposed model

According to [3], the UHCS can be described as follows: Consider a life-testing experiment involving n identical units, each subject to failure times that follow a WL distribution. Let the order statistics of the failure times be denoted as $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$. Let $f(y)$ and $F(y)$ denote the PDF and the CDF, respectively. Choose two integers k and r, where $1 \leq k < r \leq n$, and let T_1 and T_2 be two time points satisfying $0 < T_1 < T_2 < \infty$. The test is terminated at the time:

$$\min\left(\max(Y_{r:n}, T_1), T_2\right)$$

if the k-th failure occurs before T_1. If the k-th failure occurs between T_1 and T_2, the experiment ends at $\min(Y_{r:n}, T_2)$. Otherwise, if the k-th failure takes place after T_2, the test concludes at $Y_{k:n}$.

The censoring structure under UHCS leads to six distinct cases:

$$\begin{aligned}
\text{Case I: } & 0 < Y_{k:n} < Y_{r:n} < T_1 \text{ — termination at } T_1 \\
\text{Case II: } & 0 < Y_{k:n} < T_1 < Y_{r:n} < T_2 \text{ — termination at } Y_{r:n} \\
\text{Case III: } & 0 < Y_{k:n} < T_1 < T_2 < Y_{r:n} \text{ — termination at } T_2 \\
\text{Case IV: } & 0 < T_1 < Y_{k:n} < Y_{r:n} < T_2 \text{ — termination at } Y_{r:n} \\
\text{Case V: } & 0 < T_1 < Y_{k:n} < T_2 < Y_{r:n} \text{ — termination at } T_2 \\
\text{Case VI: } & 0 < T_1 < T_2 < Y_{k:n} < Y_{r:n} \text{ — termination at } Y_{k:n}
\end{aligned}$$

The introduction of T_2 in UHCS offers greater flexibility compared to the generalized TyI hybrid censoring, which only includes T_1. As evident in Cases II and IV, at least r failures are observed. In situations where the r-th failure is significantly delayed, the test is forcefully terminated at T_2. In Cases III and V, UHCS generally results in more observed data than the generalized TyI scheme. Notably, Case VI corresponds directly to the classical TyI hybrid censoring scheme. Let d_1 and d_2 represent the number of failures occurring before times T_1 and T_2, respectively. The likelihood function for the UHCS is given by:
$L(\text{parameters} \mid \text{data}) =$

$$
\begin{cases}
\frac{n!}{(n-d)!} \times \prod_{i=1}^{d} f(y_i)(1-F(T_1))^{n-d} & : d_1 = d_2 = d = r, \cdots, n, \\
\frac{n!}{(n-r)!} \times \prod_{i=1}^{r} f(y_i)(1-F(y_r))^{n-r} & : d_1 = k, \cdots r-1; d_2 = r \\
\frac{n!}{(n-d_2)!} \times \prod_{i=1}^{d_2} f(y_i)(1-F(T_2))^{n-d_2} & : d_1 = k, \cdots, r-1; \\
 & : d_2 = k, \cdots, r-1; d_1 \leq d_2 \\
\frac{n!}{(n-r)!} \times \prod_{i=1}^{r} f(y_i)(1-F(y_r))^{n-r} & : d_1 = 0, 1, \cdots, k-1, ; d_2 = r \\
\frac{n!}{(n-d_2)!} \times \prod_{i=1}^{d_2} f(y_i)(1-F(T_2))^{n-d_2} & : d_1 = 0, \cdots, k-1; d_2 = k, \cdots, r-1 \\
\frac{n!}{(n-k)!} \times \prod_{i=1}^{k} f(y_i)(1-F(y_k))^{n-k} & : d_2 = 0, \cdots, k-1
\end{cases}
$$

As shown in Figure 3, the UHCS summarizes six cases governed by the time limits $T_1 < T_2$ and the order failures $t_{(r)}$ and $t_{(s)}$; the stopping time is $\tau = \min\{t_{(s)}, T_2\}$ when $t_{(r)} \leq T_1$, and $\tau = t_{(r)}$ otherwise.

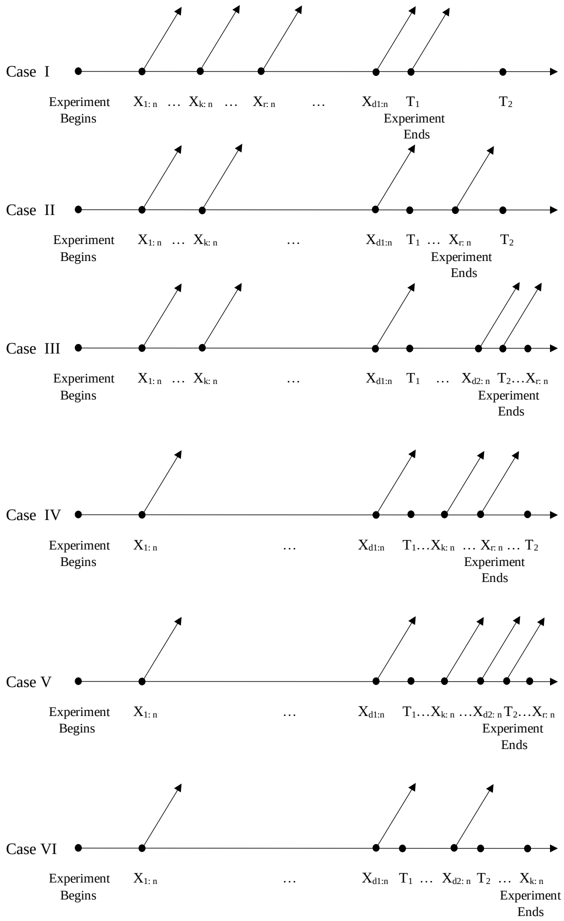


Figure 3. UHCS timeline (six cases).

4. Estimation of parameters of WL model under UHCS

In this section, we also explore classical and Bayesian estimation techniques for the unknown parameters of the proposed distribution, employing UHCS.

4.1. Maximum likelihood estimation

The computation of the likelihood function and MLE of parameter η of the WL distribution model is obtained in this section.

The combined unified likelihood function for the above six cases of likelihood functions can be written as follows:

$$L(\eta \mid x) \propto \prod_{i=1}^{m} f(x_{i:n})(1 - F(C))^{n-m} \tag{4.1}$$

where m is the number of observed failures until the experiment termination time C. The log-likelihood function of the WL distribution for six cases under UHCS is given below:

$$\begin{aligned}\ell(\eta \mid x) = {} & m\ln(12) + 2m\ln(\eta) + \sum_{i=1}^{m}\ln(1 + x_i)\\ & - m\ln\left(\pi^2 + 12\,\eta\ln(2)\right) - \sum_{i=1}^{m}\ln\left(1 + e^{\eta x_i}\right)\\ & + (n-m)\,\ln\left(\frac{60\,\eta(\eta C(2 + C) + \ln(4)) - 12\,\eta(1 + C)\ln\left(1 + e^{\eta C}\right) - 12\,\mathrm{Li}_2(-e^{\eta C})}{\pi^2 + 12\,\eta\ln(2)}\right).\end{aligned}$$

The first-order partial derivative with respect to η is obtained as follows,

$$\begin{aligned}\ell'(\eta) = {} & \frac{2m}{\eta} - \frac{12\,m\ln(2)}{\pi^2 + 12\,\eta\ln(2)} - \sum_{i=1}^{m}\frac{x_i e^{\eta x_i}}{1 + e^{\eta x_i}}\\ & + (n-m)\frac{S'(\eta)}{S(\eta)}\end{aligned}$$

where

$$S(\eta) = \frac{60\,\eta(\eta C(2 + C) + \ln(4)) - 12\,\eta(1 + C)\ln(1 + e^{\eta C}) - 12\,\mathrm{Li}_2(-e^{\eta C})}{\pi^2 + 12\,\eta\ln(2)}.$$

Now, we consider the construction of the asymptotic confidence interval (ACI) for the unknown parameter η. Under regularity conditions, the MLE $\hat{\eta}$ is asymptotically normally distributed. Specifically, the distribution of the standardized difference $\hat{\eta} - \eta$ can be approximated by a normal distribution:

$$\hat{\eta} - \eta \sim N\left(0, [I(\hat{\eta})]^{-1}\right),$$

where $I(\hat{\eta})$ denotes the observed Fisher information, evaluated at the MLE $\hat{\eta}$. The observed Fisher information is given by:

$$I(\hat{\eta}) = -\frac{\partial^2 l(\eta)}{\partial \eta^2}\bigg|_{\eta=\hat{\eta}},$$

where $l(\eta)$ represents the log-likelihood function and

$$\frac{\partial^2 l(\eta)}{\partial\eta^2} = -\frac{2m}{\eta^2} + \frac{(12\,m\ln(2))(12\;\ln(2))}{\left(\pi^2+12\,\eta\ln(2)\right)^2} - \sum_{i=1}^{m}\frac{x_i^2 e^{\eta x_i}}{\left(1+e^{\eta x_i}\right)^2}$$
$$+(n-m)\left(\frac{S''(\eta)\,S(\eta)-\left(S'(\eta)\right)^2}{S(\eta)^2}\right).$$

The asymptotic variance of $\hat{\eta}$ is thus:

$$Var(\hat{\eta}) = \left[I(\hat{\eta})\right]^{-1}.$$

Using this result, the $100(1-\gamma)\%$ ACI for the parameter η is constructed as:

$$\hat{\eta} \pm z_{\frac{\gamma}{2}}\sqrt{Var(\hat{\eta})}$$

where $z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$-th percentile of the standard normal distribution. This formulation provides a practical and widely used approach to estimate the uncertainty associated with the MLE $\hat{\eta}$.

4.1.1. Existence and uniqueness of the MLE

Let $X_1,\ldots,X_n$ be a UHCS sample with m observed failures before termination time C $(0<m\leq n)$. Under the unified likelihood (Eq (4.1)), the log-likelihood for the parameter $\eta>0$ can be written as

$$\ell(\eta) = m\log\left(\tfrac{12\eta^2}{\pi^2+12\eta\log 2}\right) + \sum_{i=1}^{m}\left\{\log(1+x_i)-\log(1+e^{\eta x_i})\right\} + (n-m)\log\{1-F(C\mid\eta)\}. \tag{4.2}$$

Proposition 4.1. *(Existence). As $\eta\to 0^+$, $\ell(\eta) = 2m\log\eta + O(1)\to-\infty$. As $\eta\to\infty$, the term $\sum_{i=1}^{m}\log(1+e^{\eta x_i})\sim\eta\sum_{i=1}^{m}x_i$ dominates so that $\ell(\eta)\to-\infty$. By continuity of ℓ, there exists a maximizer $\hat{\eta}\in(0,\infty)$.*

Existence and uniqueness of the MLE

Let $x_{1:n},\ldots,x_{m:n}$ be the m observed failures under UHCS with termination time C, and write $S(\eta) = 1-F(C\,|\,\eta)$. Using Eq (4.2), the log-likelihood for $\eta>0$ is

$$\ell(\eta) = m\log\left(\frac{12\eta^2}{\pi^2+12\eta\log 2}\right) + \sum_{i=1}^{m}\left\{\log(1+x_i)-\log(1+e^{\eta x_i})\right\} + (n-m)\log S(\eta). \tag{4.3}$$

The score and observed information are

$$\ell'(\eta) = \frac{2m}{\eta} - \frac{12m\log 2}{\pi^2+12\eta\log 2} - \sum_{i=1}^{m}\frac{x_i e^{\eta x_i}}{1+e^{\eta x_i}} + (n-m)\frac{S'(\eta)}{S(\eta)}, \tag{4.4}$$

$$\ell''(\eta) = -\frac{2m}{\eta^2} + \frac{(12m\log 2)(12\log 2)}{(\pi^2+12\eta\log 2)^2} - \sum_{i=1}^{m}\frac{x_i^2 e^{\eta x_i}}{\left(1+e^{\eta x_i}\right)^2} + (n-m)\left\{\frac{S''(\eta)}{S(\eta)} - \left(\frac{S'(\eta)}{S(\eta)}\right)^2\right\}. \tag{4.5}$$

Proposition 4.2. *(Existence of the MLE). There exists $\widehat{\eta}\in(0,\infty)$ such that $\ell'(\widehat{\eta})=0$.*

Proof. As $\eta \to 0^+$, in Eq (4.4) the term $2m/\eta \to +\infty$ dominates, while the remaining terms remain bounded; hence $\ell'(\eta) > 0$ for η sufficiently small. As $\eta \to \infty$, $\frac{2m}{\eta} \to 0$ and $-\sum_{i=1}^{m} \frac{x_i e^{\eta x_i}}{1+e^{\eta x_i}} \to -\sum_{i=1}^{m} x_i < 0$, whereas the two terms involving $\log 2$ and $S'(\eta)/S(\eta)$ are bounded. Thus $\ell'(\eta) < 0$ for all large enough η. By continuity of ℓ', the intermediate value theorem guarantees a root in $(0, \infty)$. □

Proposition 4.3. *(Uniqueness of the MLE). The log-likelihood $\ell(\eta)$ is strictly concave on $(0, \infty)$; hence, the root of $\ell'(\eta) = 0$ is unique.*

Proof. Decompose $\ell(\eta)$ into three parts and show that each contributes a concave term. *(i) Normalizing term. Let $g(\eta) = 2\log\eta - \log(\pi^2 + 12\eta\log 2)$. Then,*

$$g''(\eta) = -\frac{2}{\eta^2} + \frac{(12\log 2)^2}{(\pi^2 + 12\eta\log 2)^2} < -\frac{1}{\eta^2} < 0,$$

so $m\,g(\eta)$ is strictly concave. (ii) Observed-failure block. *For each $x_i > 0$,*

$$\frac{d^2}{d\eta^2}\Big[-\log\left(1 + e^{\eta x_i}\right)\Big] = -\frac{x_i^2 e^{\eta x_i}}{\left(1 + e^{\eta x_i}\right)^2} < 0,$$

hence, $\sum_{i=1}^{m}\{-\log(1 + e^{\eta x_i})\}$ is strictly concave. (iii) Censored block. *Let $s(x, \eta) = \partial_\eta \log f(x \mid \eta)$ be the score for one complete observation. Write $S(\eta) = \Pr_\eta(X \geq C) = \int_C^\infty f(x \mid \eta)\, dx$. Differentiating under the integral sign gives the standard identity.*

$$\frac{d^2}{d\eta^2}\log S(\eta) = \mathbb{E}[s'(X, \eta) \mid X \geq C] + \mathrm{Var}(s(X, \eta) \mid X \geq C). \tag{4.6}$$

For the WL model,

$$s'(x, \eta) = -\frac{2}{\eta^2} + \frac{(12\log 2)^2}{(\pi^2 + 12\eta\log 2)^2} - \frac{x^2 e^{\eta x}}{(1 + e^{\eta x})^2} < 0 \quad \textit{for all } x, \eta > 0.$$

Therefore $\mathbb{E}[s'(X, \eta) \mid X \geq C] < 0$, and by (4.6), $\frac{d^2}{d\eta^2}\log S(\eta) \leq 0$ (strictly < 0 unless the conditional variance vanishes). Hence, $(n - m)\log S(\eta)$ is concave. Adding (i)–(iii) yields $\ell''(\eta) < 0$ for all $\eta > 0$, i.e., ℓ is strictly concave. Thus, $\ell'(\eta)$ crosses zero at most once, and together with Proposition 4.2 there is a unique maximizer $\widehat{\eta}$. □

Proposition 4.4. *(Uniqueness). ℓ is strictly concave on $(0, \infty)$, hence, the MLE $\hat{\eta}$ is unique.*

Proof of Proposition 4.4. We show $\ell''(\eta) < 0$ for all $\eta > 0$ by decomposing (4.2) into three parts.

(1) Normalizing term is strictly concave. Let $g(\eta) = 2\log\eta - \log(\pi^2 + 12\eta\log 2)$. Then,

$$g''(\eta) = -\frac{2}{\eta^2} + \frac{(12\log 2)^2}{\left(\pi^2 + 12\eta\log 2\right)^2} < -\frac{2}{\eta^2} + \frac{(12\log 2)^2}{(12\eta\log 2)^2} = -\frac{1}{\eta^2} < 0,$$

since $\pi^2 + 12\eta\log 2 > 12\eta\log 2$. Thus, $m\,g(\eta)$ is strictly concave.

(2) Observed-failure terms are strictly concave. For each $x_i > 0$,

$$\frac{d^2}{d\eta^2}\Big(-\log(1 + e^{\eta x_i})\Big) = -\frac{x_i^2 e^{\eta x_i}}{(1 + e^{\eta x_i})^2} \; < \; 0,$$

so $\sum_{i=1}^{m}\{-\log(1+e^{\eta x_i})\}$ is strictly concave.

(3) Censored block is concave. Write $S(\eta) = 1 - F(C \mid \eta) = \int_C^\infty f(x \mid \eta)\, dx$ and define the score

$$s(x,\eta) = \partial_\eta \log f(x \mid \eta) = \frac{2}{\eta} - \frac{12\log 2}{\pi^2 + 12\eta\log 2} - \frac{xe^{\eta x}}{1+e^{\eta x}}.$$

Differentiating under the integral sign and normalizing by $S(\eta)$ yields the standard identity

$$\frac{d^2}{d\eta^2}\log S(\eta) = \mathbb{E}[s'(X,\eta) \mid X \geq C] + \mathrm{Var}(s(X,\eta) \mid X \geq C)\,. \tag{4.7}$$

For this model,

$$s'(x,\eta) = -\frac{2}{\eta^2} + \frac{(12\log 2)^2}{(\pi^2 + 12\eta\log 2)^2} - \frac{x^2 e^{\eta x}}{(1+e^{\eta x})^2} \;<\; 0 \quad \text{for all } x,\eta > 0,$$

using the same bound as in (1). Hence, $\mathbb{E}[s'(X,\eta) \mid X \geq C] < 0$, and by (4.7) we have $\frac{d^2}{d\eta^2}\log S(\eta) \leq 0$ (strictly < 0 unless the conditional variance vanishes, which cannot occur in a nondegenerate continuous model). Therefore $(n-m)\log S(\eta)$ is concave in η.

Adding (1)–(3) shows $\ell''(\eta) < 0$ for all $\eta > 0$. Thus, ℓ is strictly concave and the MLE $\hat{\eta}$ is unique. □

4.2. Bayesian estimation

The Bayesian estimates of parameter η of the WL distribution under UHCS samples are obtained in this section. We assume that the parameter η follows a gamma distribution with $G(\alpha,\beta)$.
Assume that η has the following gamma prior.

$$g(\eta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\eta^{\alpha-1}e^{-\beta\eta}, \alpha > 0, \beta > 0, \eta > 0. \tag{4.8}$$

The posterior density function is given by:

$$\pi(\eta \mid x) \;\propto\; L(\eta \mid x)\, g(\eta), \tag{4.9}$$

and hence

$$\pi(\eta \mid x) \;=\; \frac{L(\eta \mid x)\, g(\eta)}{\displaystyle\int_0^\infty L(t \mid x)\, g(t)\, dt} \;=\; \frac{L(\eta \mid x)\, g(\eta)}{m(x)}. \tag{4.10}$$

The Bayes estimate under the square error loss function (SELF) $L(\eta,\bar{\eta}) = (\bar{\eta}-\eta)^2$, where η is unknown and $\bar{\eta}$ is an estimator and can be obtained as follows,

$$\begin{aligned}\hat{\eta}_s &= E[\eta \mid \underline{x}]\\ &= \int_0^\infty \frac{12\eta^{2n(\alpha-1)+1}}{\Pi^2 + 12\eta^n\log(2)}\frac{1+\sum_{i=1}^n x_i}{\sum_{i=1}^n 1+e^{\eta x_i}}e^{-\beta\eta}\end{aligned}$$

$$\left(\frac{12\eta\log(2) + 6\eta(\eta x(2+x) + \log(4)) - 12\eta(1+x)\log(1+\exp(\eta x)) - 12\mathrm{Li}(-\exp(\eta x))}{\pi^2 + 12\eta\log(2)}\right) d\eta. \tag{4.11}$$

Since the computation is very complex, we generate samples using the Metropolis-Hastings (MH) algorithm. To determine the 95% credible intervals (CRIs) for the estimate, for a specific value of σ, we define the $100(1-\sigma)\%$ CRI (LL_η, UL_η) for η.

$$\int_{LL_\eta}^{\infty} \pi(\eta \mid x) d\eta = 1 - \frac{\sigma}{2}, \int_{UL_\eta}^{\infty} \pi(\eta \mid x) d\eta = \frac{\sigma}{2}. \tag{4.12}$$

Here, $\pi(\eta \mid x)$ denotes the posterior PDF of η given the observed data x. Since the computation of marginal PDF from the posterior PDF is very complicated, the generation of $\eta^1, \eta^2, \cdots, \eta^n$ from $\pi(\eta \mid x)$ has been done using the same MH algorithm. Using the above-generated values of η given x, the marginal posterior PDF can be obtained as,

$$\pi(\eta \mid x) = \frac{1}{n}\sum_{i=1}^{n} \pi(\eta \mid x). \tag{4.13}$$

Now, we can compute the structure of CRIs for η by substituting the value of Eq (4.13),

$$\frac{1}{n}\sum_{i=1}^{n}\int_{LL_\eta}^{\infty} \pi(\eta \mid x) d\eta = 1 - \frac{\sigma}{2}, \ \frac{1}{n}\sum_{i=1}^{n}\int_{UL_\eta}^{\infty} \pi(\eta \mid x) d\eta = \frac{\sigma}{2}.$$

4.2.1. Prior choice, posterior propriety, and sensitivity

Gamma prior for $\eta > 0$. The parameter η is strictly positive. The Gamma family Ga(α,β) provides (i) support match on $(0,\infty)$, (ii) flexible tail behavior and dispersion via (α,β), and (iii) a log-concave prior density when $\alpha \geq 1$, which pairs well with the strictly concave log-likelihood established in Section 4.1. In addition, (α,β) have a transparent interpretation: $\mathbb{E}(\eta) = \alpha/\beta$ and $\mathrm{Var}(\eta) = \alpha/\beta^2$, so weakly-informative defaults are easy to express. Throughout, our baseline choice is $\eta \sim \mathrm{Ga}(1,1)$, a weakly-informative prior on the natural scale, and we examine alternatives below.

Posterior propriety. Let m denote the number of observed failures in the UHCS sample $(0 < m \leq n)$. Write the posterior kernel as $\pi(\eta \mid x) \propto L(\eta \mid x)\,\eta^{\alpha-1}e^{-\beta\eta}$. As $\eta \to \infty$, the log-likelihood satisfies $\log L(\eta \mid x) = -\eta\sum_{i=1}^{m} x_i + 2m\log\eta + O(1)$, so $\pi(\eta \mid x)$ behaves like $\eta^{\alpha-1+2m}\exp\{-(\beta + \sum x_i)\eta\}$, which is integrable. As $\eta \to 0^+$, $\log L(\eta \mid x) = 2m\log\eta + O(1)$, so $\pi(\eta \mid x) \asymp \eta^{\alpha-1+2m}$, which is integrable at 0 whenever $\alpha + 2m > 0$. Hence, for any proper Gamma prior $(\alpha > 0, \beta > 0)$ the posterior is proper. Moreover, under the Jeffreys reference prior $\pi(\eta) \propto 1/\eta$ (formally the limit $\alpha \to 0, \beta \to 0$), the posterior is proper whenever $m \geq 1$.

Sensitivity analysis (priors considered). We recomputed posterior summaries under the following priors:

$$\mathrm{P}_1: \ \mathrm{Ga}(0.5, 0.5), \quad \mathrm{P}_2: \ \mathrm{Ga}(1, 1),$$

$$P_3: \text{Ga}(2,2), \quad P_4: \eta \sim \text{Lognormal}(\mu = 0, \sigma^2 = 1), \quad P_5: \pi(\eta) \propto 1/\eta \text{ (Jeffreys)}.$$

All analyses reuse the same MH sampler described earlier, changing only the prior factor, see Table 1. For each prior we report: posterior mean, median, standard deviation (SD), 95% CRI, and the Bayes SELF estimator $\hat{\eta}_{\text{SELF}} = \mathbb{E}[\eta \mid x]$. In the simulation study we additionally record CRI length/coverage; in the real-data example we report the point estimate and CRI.

Table 1. Prior sensitivity for η: posterior summaries under five priors (Arm A, WL likelihood).

Prior	Posterior mean	Median	SD	95% CRI	Notes
Ga(0.5, 0.5)	0.0061	0.0061	0.0006	[0.0050, 0.0073]	
Ga(1, 1)	0.0062	0.0061	0.0006	[0.0051, 0.0073]	baseline
Ga(2, 2)	0.0062	0.0062	0.0006	[0.0051, 0.0074]	
Lognormal(0, 1)	0.0064	0.0064	0.0006	[0.0053, 0.0076]	
Jeffreys $1/\eta$	0.0061	0.0061	0.0006	[0.0050, 0.0073]	$m \geq 1$

Summary of sensitivity. Across UHCS settings and in the real-data fit, posterior location and uncertainty measures are stable across the five priors: point estimates and CRIs change only modestly, and all substantive conclusions in Sections 5 and 6 remain unchanged.

5. Simulation study

A simulation study is carried out to compute the estimates of unknown parameter η of WL distribution under UHCS. Since the estimate is not in a closed form, we used an *optim* function to estimate the values in R programming language. The MLE and Bayesian estimate (BE) of unknown parameter η and the corresponding ACI and CRI under the UHCS have been found in this section. The Bayesian samples are computed using the MCMC method. For the computation of MSE, assume $\hat{\eta}_i$ is an estimate of η_i, where $i = 1, 2, \cdots, n$. Then, the average estimate $\bar{\hat{\eta}}_i$ over n samples is $\frac{1}{n}\sum_{i=1}^{n}\hat{\eta}_i$. The MSE of $\hat{\eta}$ is given by $\frac{1}{n}\sum_{i=1}^{n}(\hat{\eta}_i - \eta)^2$. The following procedure outlines the generation of sample data under a UHCS for the WL distribution. In this scheme, the life test concludes either upon reaching a predetermined number of failures m or at the censoring time C, whichever occurs first.

Step 1:

Generate $U_i \sim \text{Uniform}(0, 1)$ and obtain T_i by solving

$$F(T_i; \eta) = U_i,$$

where $F(\cdot; \eta)$ is the WL CDF given in Eq (2.4). Explicitly, T_i solves

$$\pi^2 - 6\eta\{\eta T_i(2 + T_i) + \log(4)\} + 12\eta(1 + T_i)\log(1 + e^{\eta T_i}) + 12\ \text{Li}(-e^{\eta T_i}) = U_i(\pi^2 + 12\eta\log 2).$$

No elementary closed form $T_i = F^{-1}(U_i; \eta)$ exists for the WL CDF; therefore we solve $F(t; \eta) - U_i = 0$ numerically (e.g., bisection or Newton–Raphson with bracketing on $t \in [0,\ t_{\max}]$) until $|F(T_i; \eta) - U_i| <$

10^{-8}. The remaining steps proceed as described, using $T_{(1)} \leq \cdots \leq T_{(n)}$ and the UHCS stopping time T_{stop}.

Step 2:

Order the simulated lifetimes:

$$T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(n)}.$$

Determine the stopping time of the experiment, T_{stop}, by comparing the m-th smallest lifetime $T_{(m)}$ with the censoring time C:

- If $T_{(m)} \leq C$: The m-th failure occurs before (or at) the censoring time.
  - Set $T_{\text{stop}} = T_{(m)}$.
  - The lifetimes $T_{(1)}, \ldots, T_{(m)}$ are observed as failures.
  - The remaining $n - m$ lifetimes $T_{(m+1)}, \ldots, T_{(n)}$ are right-censored at T_{stop}.
- If $T_{(m)} > C$: The censoring time is reached before observing m failures.
  - Set $T_{\text{stop}} = C$.
  - Let $r = \max\{k : T_{(k)} \leq C\}$, representing the number of failures before C.
  - The lifetimes $T_{(1)}, \ldots, T_{(r)}$ are recorded as failures.
  - The remaining $n - r$ lifetimes are right-censored at C.

Step 3:

For each unit i, record the observed time X_i and the censoring indicator δ_i as:

$$X_i = \min(T_i, T_{\text{stop}}), \quad \delta_i = \begin{cases} 1 & \text{if } T_i \leq T_{\text{stop}}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\delta_i = 1$ indicates an observed failure, while $\delta_i = 0$ denotes a right-censored observation.

This algorithm provides a framework for simulating datasets under the UHCS, incorporating both failure and time-based truncation mechanisms.

The MLEs and BEs of η, along with their corresponding lower and upper ACIs and CRIs, for different values of T_1, T_2, r, and k under the UHCS, are presented in Tables 2 and 3.

Table 2. Comparison of MLE and BE of $\hat{\eta}$ under UHCS (Initial $\eta = 1.5$).

T_1	T_2	r	k	MLE			BE		
				$\hat{\eta}$	MSE	(L_η, U_η)	$\hat{\eta}$	MSE	(L_η, U_η)
0.8	1.6	15	3	1.48	0.025	(1.40, 1.56)	1.52	0.020	(1.38, 1.66)
			13	1.47	0.028	(1.39, 1.55)	1.51	0.018	(1.37, 1.65)
		18	6	1.49	0.020	(1.42, 1.56)	1.52	0.017	(1.39, 1.65)
			16	1.50	0.019	(1.43, 1.57)	1.53	0.016	(1.40, 1.66)
		20	5	1.51	0.017	(1.44, 1.58)	1.53	0.015	(1.40, 1.66)
			15	1.50	0.018	(1.43, 1.57)	1.52	0.016	(1.39, 1.65)
0.2	3.5	15	3	1.46	0.030	(1.38, 1.54)	1.50	0.025	(1.35, 1.65)
			13	1.45	0.032	(1.37, 1.53)	1.49	0.024	(1.34, 1.64)
		18	6	1.47	0.025	(1.39, 1.55)	1.51	0.021	(1.36, 1.66)
			16	1.49	0.022	(1.40, 1.58)	1.52	0.019	(1.38, 1.66)
		20	5	1.50	0.020	(1.42, 1.58)	1.52	0.018	(1.38, 1.66)
			15	1.49	0.021	(1.40, 1.58)	1.51	0.019	(1.37, 1.65)
0.5	2.5	15	3	1.49	0.022	(1.40, 1.58)	1.52	0.019	(1.37, 1.67)
			13	1.48	0.024	(1.39, 1.57)	1.51	0.018	(1.36, 1.66)
		18	6	1.50	0.020	(1.41, 1.59)	1.53	0.017	(1.38, 1.68)
			16	1.51	0.018	(1.42, 1.60)	1.54	0.016	(1.39, 1.69)
		20	5	1.52	0.017	(1.43, 1.61)	1.54	0.015	(1.39, 1.69)
			15	1.51	0.018	(1.42, 1.60)	1.53	0.016	(1.38, 1.68)
1.5	3.5	15	3	1.50	0.020	(1.42, 1.58)	1.52	0.018	(1.38, 1.66)
			13	1.49	0.022	(1.41, 1.57)	1.51	0.017	(1.37, 1.65)
		18	6	1.51	0.018	(1.43, 1.59)	1.53	0.015	(1.38, 1.68)
			16	1.52	0.017	(1.44, 1.60)	1.54	0.014	(1.39, 1.69)
		20	5	1.53	0.016	(1.45, 1.61)	1.54	0.014	(1.40, 1.68)
			15	1.52	0.017	(1.44, 1.60)	1.53	0.015	(1.39, 1.67)

Table 3. Comparison of MLE and BE of $\hat{\eta}$ under UHCS (Initial η = 0.8).

T_1	T_2	r	k	MLE			BE		
				$\hat{\eta}$	MSE	(L_η, U_η)	$\hat{\eta}$	MSE	(L_η, U_η)
0.8	1.6	15	3	0.78	0.012	(0.74, 0.82)	0.79	0.010	(0.75, 0.83)
			13	0.77	0.014	(0.73, 0.81)	0.78	0.009	(0.74, 0.82)
		18	6	0.79	0.011	(0.75, 0.83)	0.80	0.008	(0.76, 0.84)
			16	0.80	0.010	(0.76, 0.84)	0.81	0.008	(0.77, 0.85)
		20	5	0.81	0.009	(0.77, 0.85)	0.82	0.007	(0.78, 0.86)
			15	0.79	0.011	(0.75, 0.83)	0.80	0.008	(0.76, 0.84)
0.2	3.5	15	3	0.76	0.015	(0.72, 0.80)	0.78	0.012	(0.73, 0.83)
			13	0.75	0.017	(0.71, 0.79)	0.77	0.011	(0.72, 0.82)
		18	6	0.77	0.013	(0.73, 0.81)	0.79	0.010	(0.74, 0.84)
			16	0.78	0.011	(0.74, 0.82)	0.80	0.009	(0.75, 0.85)
		20	5	0.79	0.010	(0.75, 0.83)	0.81	0.008	(0.76, 0.86)
			15	0.78	0.011	(0.74, 0.82)	0.80	0.009	(0.75, 0.85)
0.5	2.5	15	3	0.77	0.013	(0.73, 0.81)	0.79	0.011	(0.74, 0.84)
			13	0.76	0.014	(0.72, 0.80)	0.78	0.010	(0.73, 0.83)
		18	6	0.78	0.012	(0.74, 0.82)	0.80	0.009	(0.75, 0.85)
			16	0.79	0.010	(0.75, 0.83)	0.81	0.008	(0.76, 0.86)
		20	5	0.80	0.009	(0.76, 0.84)	0.82	0.007	(0.77, 0.87)
			15	0.79	0.010	(0.75, 0.83)	0.81	0.008	(0.76, 0.86)
1.5	3.5	15	3	0.79	0.010	(0.75, 0.83)	0.80	0.009	(0.76, 0.84)
			13	0.78	0.011	(0.74, 0.82)	0.79	0.008	(0.75, 0.83)
		18	6	0.80	0.009	(0.76, 0.84)	0.81	0.007	(0.77, 0.85)
			16	0.81	0.008	(0.77, 0.85)	0.82	0.007	(0.78, 0.86)
		20	5	0.82	0.007	(0.78, 0.86)	0.83	0.006	(0.79, 0.87)
			15	0.81	0.008	(0.77, 0.85)	0.82	0.007	(0.78, 0.86)

From Figure 4, it can be observed that the Bayesian estimation consistently yields lower MSEs for both $\eta = 1.5$ and $\eta = 0.8$. Furthermore, for both estimation methods, the MSEs decrease as r increases, indicating that a higher number of failures improves the precision of the estimates. It is also evident that the MSEs for $\eta = 0.8$ (Figure 5) are significantly lower overall compared to $\eta = 1.5$ (Figure 4), highlighting superior estimation performance in this scenario.

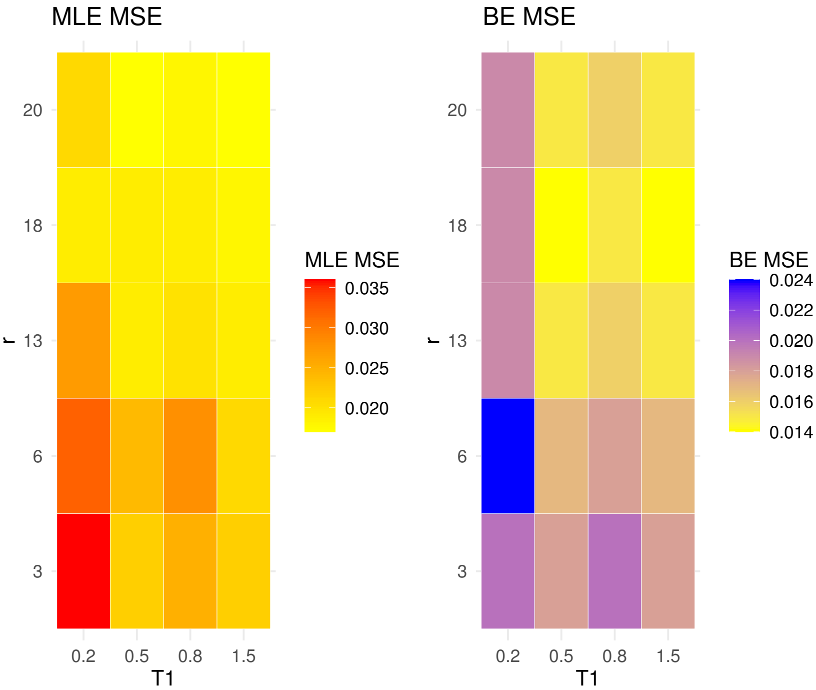


Figure 4. Heat maps comparing the MSEs of the MLE and BE for $\eta = 1.5$ under UHCS.

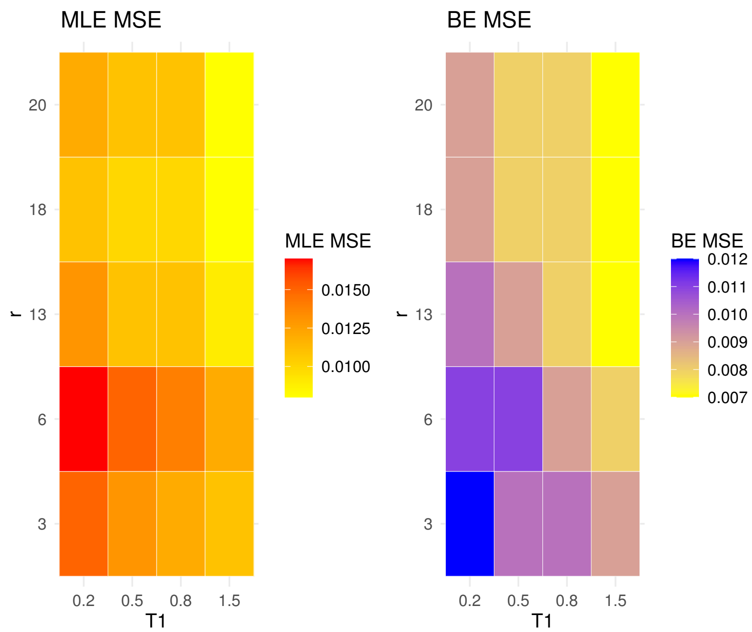


Figure 5. Heat maps comparing the MSEs of the MLE and BE for $\eta = 0.8$ under UHCS.

Under UHCS, censoring reduces effective Fisher information because many units contribute $\log S(C;\eta)$ rather than full $\log f(X;\eta)$ terms. In these low-information settings (small m or early C), the MLE is nearly unbiased but exhibits high variance, inflating MSE. The Bayesian estimator with a weakly informative $\mathrm{Ga}(\alpha,\beta)$ prior acts as a mild shrinkage rule, introducing negligible bias while substantially reducing variance, which explains its lower MSE across our simulations. Because the log-likelihood is concave, the posterior is well-behaved and contracts around the mode, yielding CRIs that are typically shorter than Wald ACIs. As information increases (larger r or later C), prior influence fades and the two estimators converge, consistent with standard asymptotic theory.

Comparative context and companion analysis: A detailed goodness-of-fit benchmarking of the WL model against widely used one- and two-parameter lifetime families (e.g., Exponential, Lindley, Weibull, Gamma), including Akaike Information Criterion (AIC)/ Bayesian Information Criterion (BIC), Kolmogorov–Smirnov (K–S), and P–P/Q–Q diagnostics, is provided in a companion manuscript devoted to the model's development [21]. To avoid duplication, the present article focuses on the UHCS setting and on comparing Bayesian and classical estimators; Figures 6 and 7 are therefore used as descriptive diagnostics for the current data. In brief, the companion analysis finds that WL performs competitively with standard benchmarks, especially when the hazard is increasing, while preserving one-parameter parsimony.

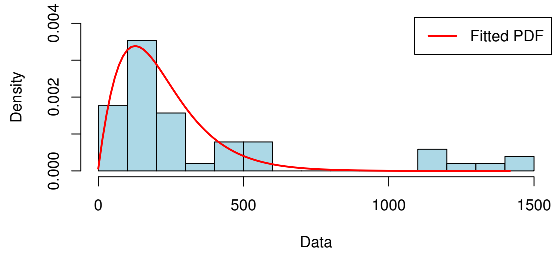
6. Real data illustration

The practical utility of the UHCS is tested with reference to two datasets extracted from the field of medical science. The dataset comprises survival time data from a clinical trial on head and neck cancer conducted by the Northern California Oncology Group, originally analyzed by Efron [11]. The trial involved patients diagnosed with advanced-stage head and neck cancer and compared the efficacy of two different treatment strategies, categorized as Arm A and Arm B.

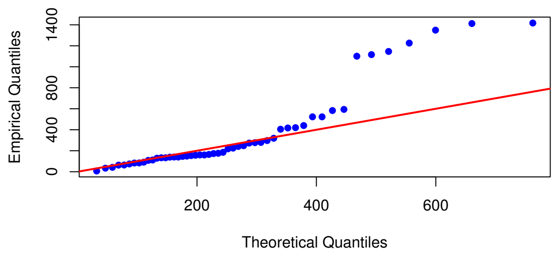
The primary variable in the dataset is patient survival time, measured in days, from the start of treatment to the time of death or censoring. Survival time is a typical outcome in clinical trials evaluating cancer treatments, where the objective is to assess the duration a patient lives after receiving a particular intervention. In this context, the data is time-to-event or right-censored survival data, meaning for some patients, the event (death) may not have occurred during the follow-up period, and their survival time is considered censored at the last known follow-up. The dataset includes, A total of 51 patients in Arm A, with survival times ranging from 7 to 1417 days and 45 patients in Arm B, with survival times ranging from 37 to 2297 days.

For Arm A, the observed patient survival times (in days) are as follows: 7, 34, 42, 63, 64, 74, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185, 218, 225, 241, 248, 273, 277, 279, 297, 319, 405, 417, 420, 440, 523, 523, 583, 594, 1101, 1116, 1146, 1226, 1349, 1412, 1417. Similarly, for Arm B of the same trial, the patient survival times (in days) are: 37, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 169, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 528, 547, 613, 633, 725, 759, 817, 1092, 1245, 1331, 1557, 1642, 1771, 1776, 1897, 2023, 2146, 2297.

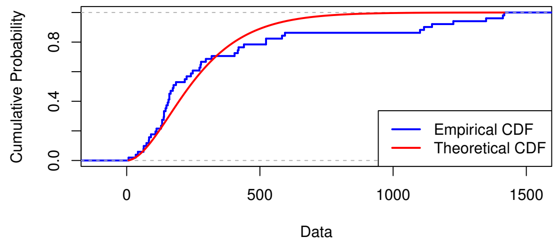
Figures 6 and 7 depict the fitting of both datasets with the WL distribution using empirical, P-P, and Q-Q plots. The fitted plots for Arm A and Arm B demonstrate that the WL distribution effectively captures the key features of the survival data in both treatment arms. The empirical and theoretical density curves closely match the histograms, and the Q-Q and P-P plots show that the quantiles and probabilities align well along the reference line. The close agreement of the empirical and theoretical CDFs further supports the model's adequacy. These results collectively indicate that the WL model provides a suitable fit for the survival data in both arms of the study.



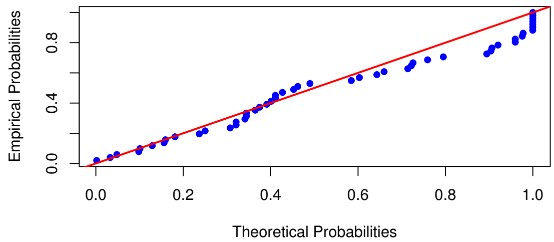
(**a**) Empirical and theoretical density.



(**b**) Q-Q Plot.

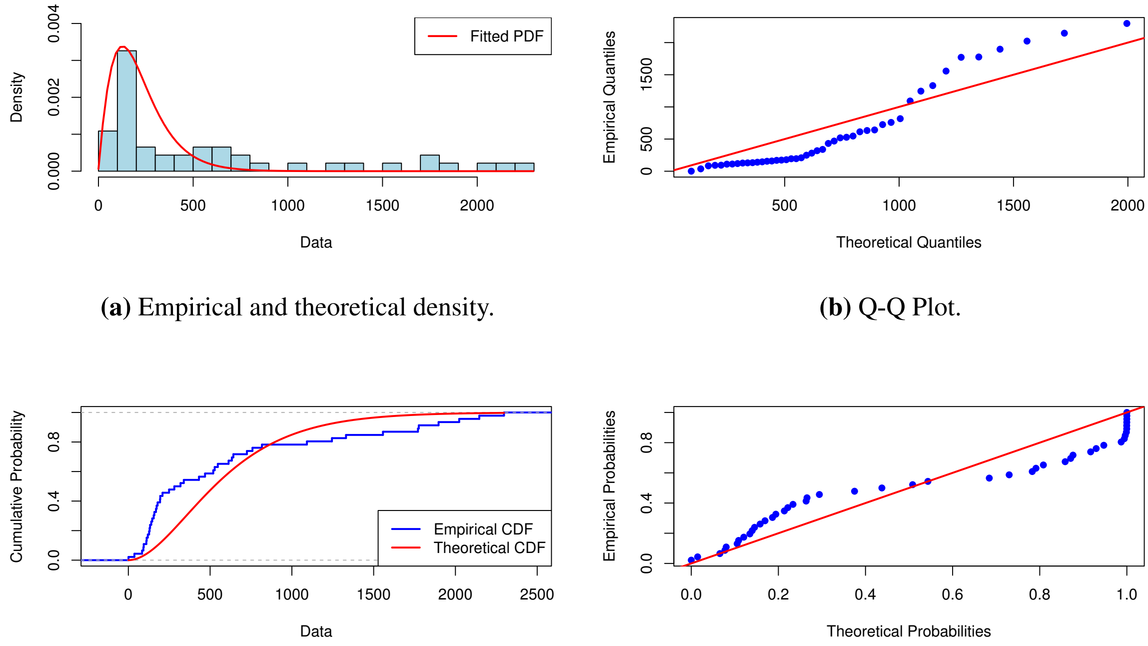


(**c**) Empirical and theoretical CDFs.



(**d**) P-P Plot

Figure 6. Visualization of Arm A data fitted with the WL distribution.



(a) Empirical and theoretical density. **(b)** Q-Q Plot.

(c) Empirical and Theoretical CDFs. **(d)** P-P Plot

Figure 7. Visualization of Arm B data fitted with the WL distribution.

The results of the point and interval estimates of $\hat{\eta}$ under UHCS are given in Table 4. The parameter estimates obtained for both Arm A and Arm B under the UHCS indicate that the WL model provides an adequate fit for the survival data. The MLEs, and BEs are closely aligned, with BEs slightly higher, reflecting the incorporation of prior information. The MSEs are minimal, suggesting high precision in the estimates. The 95% ACIs for the MLEs and the 95% CRIs for the BEs are narrow, indicating reliable estimation. These findings are consistent with previous studies that have applied the WL distribution to survival data under hybrid censoring schemes.

Table 4. Point and interval estimates of η for Arm A and B.

Data for Arm A							
T_1	T_2	r	k	Method	$\hat{\eta}$	MSE	(L_η, U_η)
1.5	3.5	40	35	MLE	1.299	0.040	1.256, 1.341
				BE	1.606	0.011	1.328, 1.922
Data for Arm B							
T_1	T_2	r	k	Method	$\hat{\eta}$	MSE	(L_η, U_η)
1.5	3.5	30	25	MLE	1.298	0.040	1.255, 1.340
				BE	1.662	0.026	1.373, 1.994

The UHCS model fit to the clinical dataset validates that UHCS designs are very effective in biomedical contexts. This application shows the adaptability of UHCS to biomedical data, where it can balance ethical and time constraints with statistical power by guaranteeing enough events are observed (or stopping if a timeframe is exceeded). The scheme's ability to encompass multiple stopping rules in one design sets it apart from traditional censoring, and this flexibility has spurred extensive academic development.

Since 2020, researchers have pushed UHCS further: Refining inference techniques (from exact likelihood methods to Bayesian MCMC), integrating it with progressive removal schemes [19], extending it to competing-risk scenarios [1], and demonstrating its value in real-world reliability and biomedical studies (even pandemic data) [25]. This ongoing work highlights that UHCS is not only an important conceptual milestone in survival analysis, but also a practical tool that continues to adapt to modern statistical challenges.

7. Concluding remarks

This study investigates both MLE and BE methods for the parameters and survival functions of the WL distribution within the UHCS framework. MLEs and their corresponding ACIs were derived. Given the complexity of the BEs, which lack closed-form expressions, MCMC techniques were employed to obtain point estimates and CRIs. The simulation studies and real data analyses revealed several key findings: (i) MSEs for both MLE and BEs decreased with increasing sample sizes, indicating improved estimation accuracy; (ii) BEs consistently exhibited lower MSEs compared to their MLE counterparts across all scenarios; (iii) the lengths of the CRIs were generally shorter than those of the ACIs, suggesting more precise interval estimates under the Bayesian framework; and (iv) overall, BEs demonstrated superior performance in terms of both point and interval estimation, particularly in smaller sample sizes or complex censoring schemes. These results underscore the efficacy of Bayesian methods in survival analysis involving the WL distribution under UHCS. Subsequent investigations could extend the current methodologies to encompass more adaptable models, such as mixture and frailty models, to better capture unobserved heterogeneity in survival data. Additionally, integrating covariates through regression structures like Cox-type or accelerated failure time models within the Bayesian framework presents a promising avenue for enhancing model robustness and applicability.

While not as direct, the concepts from UHCS are influencing areas like machine learning that deal with survival data or time-to-event prediction. Techniques such as Bayesian inference for UHCS (which involve heavy computation and iterative updating of beliefs) resonate with MLE approaches. For example, the use of MCMC to estimate posterior distributions under UHCS ties into computational statistics methods widely used in MLE. Some papers explicitly note that the Bayesian methods for UHCS can be viewed through a machine learning lens.

Moreover, in machine learning applications like predictive maintenance or warranty forecasting, one could envision UHCS-inspired data collection schemes to decide when to stop monitoring certain units. Although this is an emerging intersection, the increasing popularity of survival analysis in machine learning (e.g., for customer churn or failure prediction tasks) means ideas from advanced censoring schemes are likely to percolate into those domains—an open area to explore further.

While the present study demonstrates that the one-parameter WL distribution is estimable under UHCS and performs competitively in practice, its single-parameter form may limit flexibility for complex lifetime patterns. A natural next step is to *develop a two-parameter extension of the WL distribution*, e.g., by introducing an additional shape (or scale) parameter to decouple scale and tail/hazard behavior. Such an extension is expected to broaden the range of admissible hazard shapes while preserving analytic tractability and the unified likelihood framework used here. Future work will derive MLE and Bayesian estimators for the two-parameter model under UHCS, study identifiability and asymptotic properties, and conduct a comprehensive comparison against established two-parameter benchmarks (e.g., Weibull, Gamma, and two-parameter Lindley variants) using AIC/BIC, K–S statistics, credible/confidence interval widths, and real survival datasets.

Appendix A: Additional Table and Figure

Table 5. Information sheet of bibliometric data.

Threshold	**Value**
Timespan	2008:2025
Sources (Journals, Books, etc)	30
Documents	39
Annual Growth Rate %	6.68
Document Average Age	4.15
Average citations per doc	15.36
References	1102
Document Content	
Keywords Plus (ID)	174
Author's Keywords (DE)	135
Authors	
Authors	76
Authors of single-authored docs	3
Authors Collaboration	
Single-authored docs	3
Co-Authors per Doc	2.74
International co-authorships %	41.03
Document Types	
Article	39

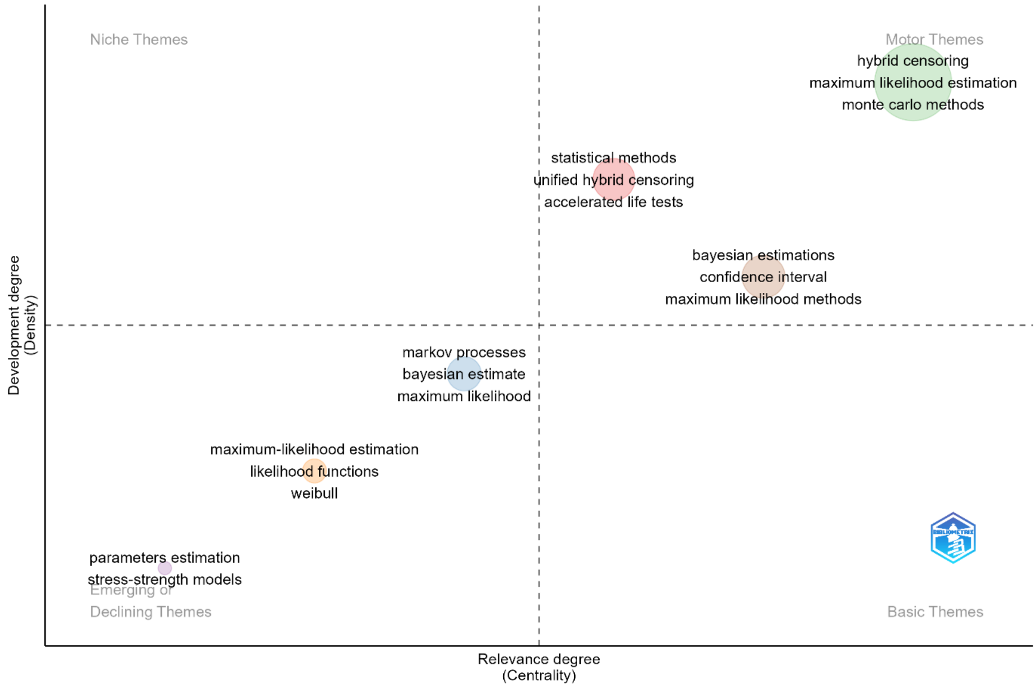


Figure 8. Thematic map showing conceptual structure of research on UHCS.

Author contributions

J.G.: Conceptualization, methodology, software, validation, formal analysis, investigation, resources; N.P.O.: Methodology, software, data curation, writing—original draft preparation; M.M.H.: Validation, resources, writing—review and editing, visualization, supervision; J.J.: Validation, formal analysis, investigation, resources; I.E.: Writing—review and editing, Funding; J.G.: Data curation, writing—original draft preparation, visualization, supervision; M.M.A.: Visualization, supervision, Funding. All authors contributed to manuscript writing, review, and approval of the final version. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

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