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Research article

On the general sum-connectivity index of hypergraphs

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Abstract: Given a non-zero real number α, the general sum-connectivity index χ_α for graph G is given by the sum $\Sigma_{xy\in E(G)}(d_x + d_y)^\alpha$. Here, d_x denotes the degree of a vertex x in graph G, and $E(G)$ is the edge set of G. A hypertree is a connected hypergraph without cycles. In a k-uniform hypergraph, every hyperedge contains exactly k vertices, and a hypergraph is termed linear if any two distinct hyperedges share at most one vertex. This study addresses analytical challenges inherent in hypergraphs by concentrating on key subclasses and introducing innovative perturbation methods to solve fundamental extremal problems within these frameworks. Through this approach, we aim to broaden the scope and utility of graph-theoretic techniques. Specifically, within the class of uniform linear hypergraphs, we characterize the extremal hypergraphs with respect to the χ_α operation. Furthermore, we investigate extremal problems related to χ_α in both general hypergraphs and bipartite hypergraphs, yielding new insights into their previously unexplored structural properties.

Keywords: k-uniform hypergraph; bipartite hypergraphs; general sum-connectivity index; extremal problem
Mathematics Subject Classification: 05C35, 05C50, 05C90

1. Introduction

Topological indices are a class of mathematical descriptors utilized to characterize and predict physical and chemical properties of chemical compounds. Among topological indices applied in molecular studies, the Randić index [26] stands out as one of the most thoroughly investigated. Notably, the methodology, which was proposed in 1975, evaluates the extremal characteristics of carbon skeleton branching patterns. The mathematical formulation of the Randić index for a given graph G is as follows:

$$R(G) = \sum_{uv \in E(G)} (d_u d_v)^{-\frac{1}{2}},$$

where uv represents an edge linking u and v, and d_u represents to the degree of vertex u. Reference [20] provides extensive coverage of the mathematical and chemical applications of these indices, complemented by discussions in recent works [1, 9, 12–14, 17, 19, 21, 24, 28] and other pertinent literature. In 2009, Zhou and Trinajstić [34] proposed a refinement of the Randić index and subsequently termed the sum-connectivity index:

$$\chi(G) = \sum_{uv \in E(G)} (d_u + d_v)^{-\frac{1}{2}}.$$

Further applications of the index are elaborated upon in reference [22]. This concept was later generalized to the general sum-connectivity index [35], and the definition of the general sum-connectivity index for graph G was provided as follows:

$$\chi_\alpha(G) = \sum_{uv \in E(G)} (d_u + d_v)^{\alpha},$$

where α represents a non-zero real number. Here, $\chi_{-\frac{1}{2}}(G)$ denotes the sum-connectivity index when $\alpha = -\frac{1}{2}$. Notably, $\chi_1(G)$ corresponds to the first Zagreb index, which is a cornerstone among topological indices [7, 8]. Evidently, the general sum-connectivity index generalizes both the sum-connectivity and first Zagreb indices. Significant research interest has focused on the general sum-connectivity index, reflected in studies such as [2–5, 10, 18, 25] and related works. References [15, 30, 34] present significant findings on the extremal properties of $\chi_\alpha(G)$. Specific graph classes have been analyzed, including the following: unicyclic graphs in [16]; bicyclic graphs in [4, 5]; and graphs defined by cyclomatic complexity in [6]. Although topological indices have broad applications, certain aspects of their study remain comparatively underexplored. Despite its importance, the vertex degree-based index in hypergraphs has received relatively little attention, apart from a few recent breakthroughs.

Prior work has investigated indices such as the Sombor index [29], the general Randić index [31], the Wiener index, and the eccentricity-based index, alongside analyses of spectral radii [11, 23, 27, 32, 33].

Graph-theoretic indices generalization to hypergraphs remains relatively underdeveloped. This is particularly due to the combinatorial complexity and structural diversity inherent to hypergraphs. Motivated by the need to develop systematic tools to analyze connectivity-based indices in such structures, this study contributes to a topic with significant implications across network science, computational chemistry, and combinatorics. Building upon and extending previous work on extremal problems related to topological indices, we specifically focus on k-uniform linear hypertrees, general hypergraphs, and bipartite hypergraphs. A central aim of this work is to deepen the understanding of how connectivity properties—as captured by the general sum-connectivity index-vary under structural constraints, and to characterize hypergraph configurations that lead to extreme values. These insights are not only theoretically valuable, but also applicable in fields such as the chemical graph theory, network design, and combinatorial optimization, where topological indices serve as important descriptors of structural features and system stability.

The paper is organized as follows: Section 2 provides definitions of key hypergraph concepts; Section 3 describes two hypergraph operations and perturbations for the generalized sum-connectivity index, where graph transformations are used to identify the maximal and minimal indices among k-uniform linear hypertrees; and Section 4 derives bounds for the generalized sum-connectivity index in hypergraphs and bipartite hypergraphs.

2. Preliminaries

In this section, we define the essential hypergraph terminology. A hypergraph $H = (V, E)$ consists of a finite vertex set V and a set E of hyperedges, where each hyperedge $e \in E$ is a non-empty subset of V. The degree of a vertex v, denoted d_v, is the number of hyperedges containing v. The size of a hyperedge e is its cardinality (i.e., the number of vertices it contains). A hypergraph is connected if there exists a path between any two vertices, where a path is a sequence $(v_0, e_1, v_2, \cdots, e_t, v_t)$ such that $v_{i-1}v_i \in e_i$ for $1 \leq i \leq t$. A hypertree is a connected hypergraph that contains no cycles. A hypergraph is linear if any two distinct hyperedges share at most one vertex. A vertex v is said to be pendant if $d(v) = 1$. A hyperedge e is pendant if it contains at least one pendant vertex and is incident to exactly one non-pendant vertex. A hypergraph is bipartite if its vertex set V can be partitioned into two non-empty subsets, V_1 and V_2, such that every hyperedge has a non-empty intersection with both V_1 and V_2.

The complete hypergraph H_{K_n} contains every possible subset of its vertex set V as a hyperedge. Removing all hyperedges of size one from H_{K_n} yields the hypergraph K_n^*. Similarly, a complete bipartite hypergraph $H_{K_{p,q}}$ includes all possible hyperedges that contain at least one vertex from each of the designated subsets V_1 and V_2. A hyperpath is a hypertree in which each vertex has a degree of at most 2 and each hyperedge is incident to at most two other hyperedges. A hyperstar is a hypertree in which all hyperedges share a common vertex. Throughout this paper, we only consider simple hypergraphs, that is, hypergraphs without repeated hyperedges.

Regarding the generalized sum-connectivity index $\chi_\alpha(G)$, the cases $\alpha > 0$ and $\alpha < 0$ exhibit opposing behaviors and are derivable from each other. Consequently, a subsequent analysis will exclusively address $\alpha > 0$.

To enhance the readability, we provide the following summary of key notations used throughout the paper, see Table 1.

Table 1. Notation summary.

Notation	Description
$H = (V, E)$	Hypergraph with vertex set V and edge set E
$d(v)$	Degree of vertex v
$\|e\|$	Size of hyperedge e
k-uniform	Hypergraph where all edges have size k
$\chi_\alpha(H)$	Generalized sum-connectivity index
H_K	Complete hypergraph on vertex set V
K_n^*	H_K without hyperedges of size one
$H_{K_{p,q}}$	Complete bipartite hypergraph with parts of sizes p, q
Hypertree	Connected, acyclic hypergraph
Hyperpath	Hypertree with max vertex degree 2 and linear edge-incidence

3. k-Uniform linear hypertrees with extremal generalized sum-connectivity index

This section investigates the perturbation of a k-uniform linear hypertree using the general edge-moving operation described in [23]. Figure 1 shows an example of a 3-uniform hypergraph. A k-uniform linear hypertree is a specific type of hypergraph that combines the constraints of being k-uniform and linear, which together give it a tree-like structure. Next, we apply a similar operation to compute the sum-connectivity index $\chi_\alpha(G)$ for k-uniform linear hypertrees.

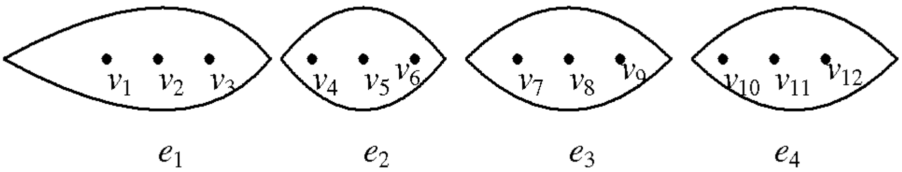

Figure 1. 3-Uniform hypergraph with 12 vertices v_1,v_2,$\cdots$,v_{12} and four hyperedges $e_1 = v_1v_2v_3$, $e_2 = v_4v_5v_6$,$e_3 = v_7v_8v_9$, and $e_4 = v_{11}v_{12}v_{13}$.

Definition 1 ([23]). *Consider a hypergraph $G = (V, E)$. Let $u \in V$, and let $e_1, \cdots, e_r$ $(r \geq 1)$ be distinct hyperedges in E. For each hyperedge e_i, select a vertex $v_i \in e_i$. Construct a new hyperedge $e_i^{'}$ by replacing vertex v_i within e_i with vertex u, for each $i = 1, 2, \cdots, r$. Then, the hypergraph $G^{'} = (V, E^{'})$ is formed by removing the original hyperedges $(e_1, \cdots, e_r)$ from E and adding the new hyperedges $\{e_1^{'}, e_2^{'}, \cdots, e_r^{'}\}$. We say $G^{'}$ results from moving the hyperedges $(e_1, \cdots, e_r)$ away from vertices $(v_1, \cdots, v_r)$ to vertex u.*

Lemma 3.1. *Let G be a connected hypergraph. Suppose $G^{'}$ is obtained by transferring hyperedges $e_1, e_2, \cdots, e_r$ from vertices $v_1, \cdots, v_r$ to vertex u and that $G^{'}$ contains no parallel hyperedges. If $d_u > max_{1\leq i\leq r} d_{v_i}$, where $r \geq 1$, then $\chi_\alpha(G^{'}) > \chi_\alpha(G)$ for $\alpha > 0$.*

Proof. Define the modified edges e'_i as $e'_i = (e_i \setminus \{v_i\} \cup u$ for $i = 1, \cdots, r$. Note that G and G' only differ in their hyperedges. Analyzing these modified edges and applying the definition of χ_α yields the following:

$$\begin{aligned}
&\chi_\alpha(G^{'}) - \chi_\alpha(G)\\
&= \sum_{e_i^{'}\in E(G^{'})} (\sum_{v\in e_i^{'}} d_v)^\alpha - \sum_{e_i\in E(G)} (\sum_{v\in e_i} d_v)^\alpha\\
&= \sum_{k=1}^{r} (\sum_{w_k\in e_k\setminus\{v_k\}} d_G(w_k) + d_G(u))^\alpha - \sum_{k=1}^{r} (\sum_{w_k\in e_k\setminus\{v_k\}} d_G(w_k) + d_G(v_k))^\alpha\\
&= \sum_{k=1}^{r} [(\sum_{w_k\in e_k\setminus\{v_k\}} d_G(w_k) + d_G(u))^\alpha - (\sum_{w_k\in e_k\setminus\{v_k\}} d_G(w_k) + d_G(v_k))^\alpha]\\
&> 0,
\end{aligned}$$

since $\alpha > 0$, and each term in the sum is strictly positive under the given condition $d_u > \max_{1\le i\le r} d_{v_i}$. This completes the proof. □

Definition 2 ([23]). *Let G be a k-uniform linear hypergraph, e be a non-pendant edge of G, and $u \in e$. Let $E = \{e_1, e_2, \cdots, e_r\}$ be the set of all edges adjacent to e that do not contain u. Moreover, denote the common vertex of e_i and e by v_i for $i = 1, \cdots, r$. The edge-releasing of e at u transforms G into a hypergraph $G^{'}$ by moving the edges $e_1, \cdots, e_r$ from vertices $v_1, \cdots, v_r$ to u.*

Remark. Clearly, according to Definition 1, edge-releasing constitutes a special case of edge-moving. Lemma 3.2 states that the $\chi_\alpha(G)$ index strictly increases under the edge-releasing operation in a connected k-uniform linear hypertree.

Lemma 3.2. *Let G be a connected k-uniform linear hypertree of order n, and e be a non-pendent edge of G. Let $G^{'}$ be the hypergraph obtained from G by releasing the edge e at vetex u. Then, $G^{'}$ is also a hypertree and $\chi_\alpha(G^{'}) > \chi_\alpha(G)$.*

Proof. First, we show that the transformed graph $G^{'}$ remains connected and is still a hypertree. As G is connected, $G^{'}$ is clearly also connected. Since the edge-releasing transformation preserves the number of hyperedges, $G^{'}$ and G have the same number of edges. The edge numbers in both hypergraphs are $\frac{n-1}{k-1}$, as this operation does not introduce cycles, and $G^{'}$ remains acyclic and is consequently also a hypertree. The argument establishing the inequality

$$\chi_\alpha(G^{'}) > \chi_\alpha(G)$$

follows analogously to the proof of Lemma 3.1. □

Now, we introduce a novel operation termed total grafting, which is designed to reduce the general sum-connectivity index.

To establish the groundwork, let $P = (v_0, e_1, v_1, \cdots, v_{p-1}, e_p, v_p)$ denote a path in a k-uniform hypergraph H. A pendant path originating at v_0 is defined as follows: $d_{v_0} \ge 2$, all intermediate vertices $v_1, \cdots, v_{p-1}$ have a degree of two, the terminal vertex v_p has a degree of one, and every vertex in $e_i\setminus\{v_{i-1}, v_i\}$ for $(i = 1, 2, \cdots, p)$ has a degree of one in H.

Definition 3 ([23]). *Let G be a k-uniform linear hypergraph with a vertex v. The hypergraph $G(v; p, q)$ is constructed by attaching two pendant paths $P = (v, e_1, v_1, \cdots, v_{p-1}, e_p, v_p)$ and $Q =$*

$(v, e_{p+1}, v_{p+1}, \cdots, e_{p+q}, v_{p+q})$ at v, where $V(P) \bigcap V(Q) = \{v\}$. Alternatively, $G(v; p+q, 0)$ is formed by adding a single pendant path $P_1 = (v, e_1, v_1, \cdots, v_{p-1}, e_p, v_p, e_{p+1}, \cdots, e_{p+q}, v_{p+q})$ at v. The total grafting operation at v transforms $G(v; p, q)$ into $G(v; p+q, 0)$.

A loose k-uniform path of length m, denoted $P_{m,k}$, is a k-uniform hypertree with exactly $m-1$ vertices of degree two. In such a path, any two edges share at most one vertex, and all other vertices have a degree of one. For clarity, we define $H = G(v; p, q)$ and $H' = G(v; p+q, 0)$ as follows. The hypergraphs of H and H' are shown in Figure 2.

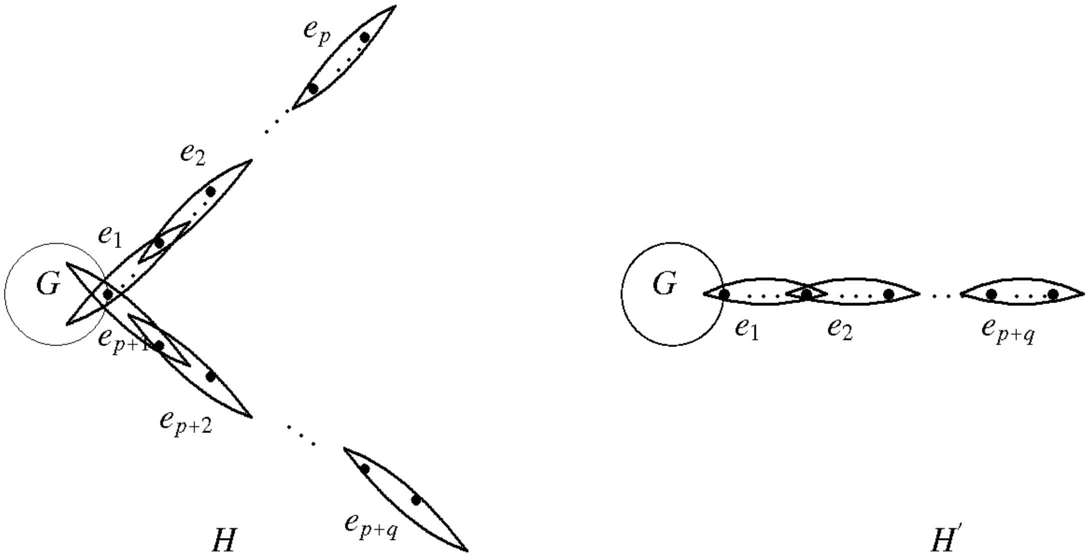

Figure 2. The hypergraph H and H'.

Lemma 3.3. *Let G be a k-uniform linear hypertree with $|E(G)| \geq 1$, and let $v \in V(G)$. Consider a k-uniform loose path P with $p+q$ edges, which is represented as $P = (v_0, e_1, v_1, \cdots, v_{p+q-1}, e_{p+1}, v_{p+q})$. Let H and H' be the k-uniform linear hypertrees obtained by merging the vertex v of G with the vertex v_p of P and v_0 of P, respectively, where $p \geq q \geq 1$. Then, $\chi_\alpha(H) > \chi_\alpha(H')$.*

Proof. From the structural characteristics of H' and H during the total grafting process, it is clear that vertices with varying degrees are primarily located along the edges connecting vertex v and e_{p+1}. Let E_v denote the hyperedges incident to v in H, and E_v' the corresponding hyperedges in H'. An analysis of the graph structure yields the following key observations:
(1) The intersection $E(G) \cap E_v$ equals $E(G) \cap E_v'$.
(2) The degree of v satisfies $d_H(v) > d_{H'}(v)$ and $d_H(v) > d_{H'}(v_p)$, respectively.
(3) The sum of degrees for vertices in $e_{p+1} \backslash \{v\}$ equals that for vertices in $e_{p+1} \backslash \{v_p\}$ (i.e., $\sum_{x \in e_{p+1} \backslash \{v\}} d_x = \sum_{y \in e_{p+1} \backslash \{v_p\}} d_y$).

$$
\begin{aligned}
\chi_\alpha(H) - \chi_\alpha(H') &\geq \sum_{e_i \in E_v} (\sum_{x \in e_i} d_x)^\alpha - \sum_{e_i \in E_v'} (\sum_{y \in e_i} d_y)^\alpha \\
&= \sum_{e_i \in E(G) \cap E_v} (\sum_{x \in e_i} d_x)^\alpha - \sum_{e_i \in E(G) \cap E_v'} (\sum_{y \in e_i} d_y)^\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad (\sum_{x\in e_1} d_x)^\alpha + (\sum_{x\in e_{p+1}} d_x)^\alpha - (\sum_{y\in e_1'} d_y)^\alpha - (\sum_{y\in e_{p+1}'} d_y)^\alpha \\
&= \quad \sum_{e_i\in E(G)\cap E_v} (\sum_{x\in e_i\setminus\{v\}} d_x + d_H(v))^\alpha - \sum_{e_i\in E(G)\cap E_v'} (\sum_{y\in e_i\setminus\{v\}} d_y + d_{H'}(v))^\alpha \\
&+ \quad (\sum_{x\in e_1\setminus\{v\}} d_x + d_H(v))^\alpha + (\sum_{x\in e_{p+1}\setminus\{v\}} d_x + d_H(v))^\alpha \\
&- \quad (\sum_{y\in e_1'\setminus\{v\}} d_y + d_{H'}(v))^\alpha - (\sum_{x\in e_{p+1}\setminus\{v_p\}} d_x + d_{H'}(v_p))^\alpha \\
&= \quad \sum_{e_i\in E(G)\cap E_v} [(\sum_{x\in e_i\setminus\{v\}} d_x + d_H(v))^\alpha - (\sum_{y\in e_i\setminus\{v\}} d_y + d_{H'}(v))^\alpha] \\
&+ \quad [(\sum_{x\in e_1\setminus\{v\}} d_x + d_H(v))^\alpha - (\sum_{y\in e_1'\setminus\{v\}} d_y + d_{H'}(v))^\alpha] \\
&+ \quad [(\sum_{x\in e_{p+1}\setminus\{v\}} d_x + d_H(v))^\alpha - (\sum_{x\in e_{p+1}\setminus\{v_p\}} d_x + d_{H'}(v_p))^\alpha] \\
&> \quad 0.
\end{aligned}
$$

Consequently,

$$\chi_\alpha(H) > \chi_\alpha(H').$$

The verification process is finished. □

The k-uniform hyperstar $S_{m,k}$ is defined as a k-uniform linear hypertree where all hyperedges are pendant. In the subsequent sections, we prove theorem 3.1, which identifies the k-uniform linear hypertrees that achieve the maximum and minimum values of $\chi_\alpha(G)$.

Theorem 3.1. *Let H be a k-uniform linear supertree with m edges. Then, the following inequalities hold:*

$$2(k+1)^\alpha + (m-2)(k+2)^\alpha \le \chi_\alpha(H) \le m(m+k-1)^\alpha,$$

The upper bound is attained if and only if $H \cong S_{m,k}$, and the lower bound is attained if and only if $H \cong P_{m,k}$.

Proof. First, we consider the upper bound. Let $N_2(H)$ denote the number of non-pendant vertices in H. We comprehensively analyze $N_2(H)$. If $N_2(H) = 1$, then $H \cong S_{m,k}$. The result holds trivially if $N_2(H) < k$. Suppose $N_2(H) = k \ge 2$; let w and u be non-pendant vertices. There exists at least one non-pendant hyperedge e on the path connecting w to u. Construct a new hypertree H' from H by edge-releasing e. By Lemma 3.2, $\chi_\alpha(H) < \chi_\alpha(H')$. Furthermore, $N_2(H') = N_2(H) - 1 < N_2(H)$. By induction, $\chi_\alpha(H') \le \chi_\alpha(S_{m,k})$. Combining these inequalities yields the following:

$$\chi_\alpha(H) \le \chi_\alpha(S_{m,k}).$$

Now, we establish the lower bound for $\chi_\alpha(H)$. Let $N_3(H)$ denote the number of vertices in H with a degree of at least 3. If H is not isomorphic to a loose path, then it must contain at least one branching vertex, which implies that $N_3(H) \ge 1$. We proceed by induction on $N_3(H)$. The result is immediate for either $N_3(H) = 0$ or $N_3(H) = 1$. Assume the conclusion holds for $N_3(H) = k - 1$. Consider H with

$N_3(H) = k$. Let v be a vertex of H, and let u be the branching vertex farthest from v (i.e., $d_H(u) \geq 3$). At least $d_H(u) - 1$ pendant paths emanate from u. Performing multiple grafting operations at u on these pendant paths yields a supertree H' of order n with a reduced number of branches; it is clear that the vertex u will have a degree of 2 in the resulting graph H'. We achieve the desired result by applying induction on the supertree. Next, we compute the values for the extremal hypergraphs. Direct calculations yield:

$$\chi_\alpha(P_{m,k}) = 2(k-1+2)^\alpha + (m-2)(k-2+4)^\alpha = 2(k+1)^\alpha + (m-2)(k+2)^\alpha,$$

and

$$\chi_\alpha(S_{m,k}) = m(m+k-1)^\alpha.$$

This completes the proof of the theorem. □

4. Hypergraphs and bipartite hypergraphs

This section examines the general sum-connectivity index for both general hypergraphs and bipartite hypergraphs.

Theorem 4.1. *Let H be a connected hypergraph consisting of n vertices ($n \geq 2$) and no isolated hyperedges. The following inequalities hold:*

$$n^\alpha \leq \chi_\alpha(H) \leq (2^{n-1} - 1)^\alpha \sum_{i=2}^{n} \binom{n}{i} i^\alpha,$$

where the lower bound is attained by the hypergraph consisting of a single hyperedge containing all vertices, and the upper bound is achieved by the hypergraph $H^*_{K_n}$.

Proof. Since H is connected, the hypergraph that achieves the lower bound must consist of a single hyperedge that encompasses all vertices, thus establishing $\chi_\alpha(H) \geq n^\alpha$. For the upper bound, the extremal hypergraph must maximize both the number of hyperedges and their degrees. This is realized by $H^*_{K_n}$, which contains no empty edges, parallel edges, or hyperedges of a degree of one. Instead, it includes $\binom{n}{i}$ hyperedges of a degree i for $2 \leq i \leq n$. As $H^*_{K_n}$ is regular, the result follows.

Since the degree of each vertex is uniform, choose an arbitrary vertex $u \in V(H^*_{K_n})$. Its degree is given by the following:

$$d_u = \sum_{k=1}^{n-1} \binom{n-1}{k} = 2^{n-1} - 1.$$

Using the definition of the general sum-connectivity index $\chi_\alpha(H)$, we derive the following:

$$\chi_\alpha(H^*_{K_n}) = \sum_{e_i \in E} (\sum_{u \in e_i} d_u)^\alpha = \sum_{e_i \in E} (|e_i| d_u)^\alpha,$$

since $d_v = d_u$ for all v, and $|e_i|$ denotes the hyperedge size. Grouping hyperedges by cardinality yields the following:

$$\chi_\alpha(H^*_{K_n}) = \sum_{i=2}^{n} \sum_{\substack{e_i \in E \\ |e_i| = i}} (id_u)^\alpha = \sum_{i=2}^{n} \binom{n}{i} (id_u)^\alpha.$$

Substituting $d_u = 2^{n-1} - 1$ gives the final form as follows:

$$\chi_\alpha(H^*_{K_n}) = (2^{n-1} - 1)^\alpha \sum_{i=2}^{n} \binom{n}{i} i^\alpha.$$

This completes the derivation. □

A direct application of Theorem 4.1 yields the following:

Corollary 4.1. *Let H be a connected hypergraph on $n \geq 1$ vertices. If H contains at least one single-vertex hyperedge, then*

$$\chi_\alpha(H) \leq (2^{n-1})^\alpha \sum_{i=1}^{n} \binom{n}{i} i^\alpha,$$

with equality if and only if H is the complete hypergraph H_{K_n}.

Let $H^{(k)}_{K_n}$ denote the complete k-uniform hypergraph on n vertices.

Theorem 4.2. *Let H be a connected k-uniform hypergraph on $n \geq 2$ vertices. Then,*

$$\chi_\alpha(H) \leq \binom{n}{k} k^\alpha \binom{n-1}{k-1}^\alpha,$$

where the equality holds if and only if $H \cong H^{(k)}_{K_n}$.

Proof. Employing reasoning analogous to Theorem 4.1, it follows that the k-uniform hypergraph attaining the upper bound for $\chi_\alpha(H)$ must be $H^{(k)}_{K_n}$. By definition of a k-uniform hypergraph, each hyperedge has a cardinality k. Denoting the degree of the vertex u in H by d_u, we have $d_u = \binom{n-1}{k-1}$. From the definition of the general sum-connectivity index, we obtain the following:

$$\chi_\alpha(H) \leq \binom{n}{k} (kd_u)^\alpha = \binom{n}{k} k^\alpha \binom{n-1}{k-1}^\alpha.$$

This concludes the proof. □

Now, we investigate the generalized sum-connectivity index of bipartite hypergraphs and characterize their extremal structures.

Theorem 4.3. *Let $H = (V_1 \cup V_2, E)$ be a connected bipartite hypergraph with $n = p + q$ vertices, where $|V_1| = p$ and $|V_2| = q$. Then,*

$$n^\alpha \leq \chi_\alpha(H) \leq \sum_{i=1}^{j-1} \binom{p}{i} \binom{q}{j-i} (id_u + (j-i)d_v)^\alpha,$$

where $u \in V_1$ and $v \in V_2$, and $d_u = 2^{p-1}(2^q - 1)$, and $d_v = 2^{q-1}(2^p - 1)$. The lower bound is achieved by the hypergraph with an edge set $E = \{V\}$ (the single hyperedge containing all vertices). The bipartite hypergraph attaining the upper bound is $H_{K_{p,q}}$.

Proof. Let V_1 and V_2 partition the vertex set V of the bipartite hypergraph H, with $|V_1| = p$ and $|V_2| = q$. The hypergraph achieving the lower bound is trivial: it consists of a single hyperedge containing all vertices, giving each vertex a degree of one. Thus, $\chi_\alpha(H) = n^\alpha$ holds for this case. Conversely, hypergraphs attaining the upper bound require numerous hyperedges with high vertex degrees within them. Consequently, such extremal bipartite hypergraphs must be isomorphic to $H_{K_{p,q}}$.

Now, we compute the degrees of vertices in the two parts. For any vertex $u \in V_1$, we have the following:

$$d_u = \sum_{i=0}^{p-1} \binom{p-1}{i}(2^q - 1) = (2^q - 1)\sum_{i=0}^{p-1} \binom{p-1}{i} = 2^{p-1}(2^q - 1).$$

Similarly, $d_v = 2^{q-1}(2^p - 1)$, for $v \in V_2$.

Let E denote the edge set of the complete bipartite hypergraph. Applying the definition of the generalized sum-connectivity index, we obtain the following results:

$$\begin{aligned}
\chi_\alpha(H_{K_{p,q}}) &= \sum_{e_i \in E}(\sum_{w \in e_i} d_w)^\alpha \\
&= \sum_{\substack{e_i \in E \\ |e_i|=2}}(\sum_{w \in e_i} d_w)^\alpha + \sum_{\substack{e_i \in E \\ |e_i|=3}}(\sum_{w \in e_i} d_w)^\alpha + \cdots + \sum_{\substack{e_i \in E \\ |e_i|=n}}(\sum_{w \in e_i} d_w)^\alpha.
\end{aligned}$$

Specifically,

$$\sum_{\substack{e_i \in E \\ |e_i|=2}}(\sum_{w \in e_i} d_w)^\alpha = \sum_{\substack{e_i \in E \\ |e_i|=2}}(d_u + d_v)^\alpha = pq(d_u + d_v)^\alpha,$$

where, $u \in V_1$ and $v \in V_2$. For hyperedges of size 3, the expression becomes the following:

$$\begin{aligned}
\sum_{\substack{e_i \in E \\ |e_i|=3}}(\sum_{w \in e_i} d_w)^\alpha &= \binom{q}{1}\binom{p}{2}(2d_u + d_v)^\alpha + \binom{p}{1}\binom{q}{2}(d_u + 2d_v)^\alpha \\
&= \sum_{i=1}^{2} \binom{p}{i}\binom{q}{3-i}(id_u + (3-i)d_v)^\alpha.
\end{aligned}$$

Extending this to hyperedges of an arbitrary size j ($2 \le j \le n$), we derive the following:

$$\sum_{\substack{e_i \in E \\ |e_i|=j}}(\sum_{w \in e_i} d_w)^\alpha = \sum_{i=1}^{j-1} \binom{p}{i}\binom{q}{j-i}(id_u + (j-i)d_v)^\alpha,$$

with the convention that $\binom{t}{s} = 0$ when $s > t$. Combining these results with the definition of the generalized sum-connectivity index, we establish the upper bound as follows:

$$\chi_\alpha(H) \le \sum_{j=2}^{n}\sum_{i=1}^{j-1} \binom{p}{i}\binom{q}{j-i}(id_u + (j-i)d_v)^\alpha.$$

This completes the proof. □

Next, we investigate k-uniform bipartite hypergraphs and characterize their extremal structures. Let $H^{(k)}_{K_{p,q}}$ represent a bipartite hypergraph with the vertex set $V = V_1 \cup V_2$. The edge set of this hypergraph consists of all possible subsets of V under the constraint that each hyperedge must include at least one vertex from both V_1 and V_2.

Theorem 4.4. *Let $H = (V_1 \cup V_2, E)$ be a connected k-uniform bipartite hypergraph with $n = p + q$ vertices, where $|V_1| = p$ and $|V_2| = q$. If $s > t$, then there is $\binom{t}{s} = 0$ and $\binom{r}{2} = 0$ (implying $r < 2$), where $r = min\{k-1, p, q\}$; then, the generalized sum-connectivity index $\chi_\alpha(H)$ satisfies the following:*

$$\chi_\alpha(H) \leq \sum_{j=1}^{r} \binom{p}{j}\binom{q}{k-j}(j\mathcal{F}_1 + (k-j)\mathcal{F}_2)^{\alpha},$$

where $\mathcal{F}_1 = \sum_{i=0}^{r} \binom{p-1}{i}\binom{q}{k-1-i}$, and $\mathcal{F}_2 = \sum_{i=0}^{r} \binom{q-1}{i}\binom{p}{k-1-i}$. The equality holds for hypergraphs isomorphic to $H^{(k)}_{K_{p,q}}$.

Proof. For any vertex $u \in V_1$, its degree is given by the following:

$$d_u = \mathcal{F}_1 = \sum_{i=0}^{r} \binom{p-1}{i}\binom{q}{k-1-i}.$$

Similarly, for any vertex $v \in V_2$, its degree is as follows:

$$d_v = \mathcal{F}_2 = \sum_{i=1}^{r} \binom{q-1}{i}\binom{p}{k-1-i}.$$

Applying the definition of the generalized sum-connectivity index $\chi_\alpha(H)$ and leveraging the properties of k-uniform bipartite hypergraph, we derive the stated upper bound as follows:

$$\chi_\alpha(H) \leq \sum_{j=1}^{r} \binom{p}{j}\binom{q}{k-j}(j\mathcal{F}_1 + (k-j)\mathcal{F}_2)^{\alpha}.$$

This completes the proof. □

5. Conclusions

In this study, we established rigorous lower and upper bounds for the general sum-connectivity index χ_α of k-uniform linear supertrees and broader classes of hypergraphs, as detailed in Theorems 3.1, 4.1, and 4.2. These results extend the framework of classical graph indices to hypergraphs and introduce a systematic methodology for analyzing extremal structures, particularly within connected bipartite hypergraphs. By developing edge-moving operations and analyzing their effects on χ_α, we identified extremal hypergraphs that optimize this index under structural perturbations. This contributes to a deeper understanding of how specific hypergraph configurations influence algebraic-topological invariants.

Our findings carry significant theoretical and practical implications. Theoretically, they bridge a notable gap between graph and hypergraph invariants, thus enriching hypergraph theory and providing a foundation for further comparative analyses. Practically, the extremal hypergraphs identified here may inform the design of efficient networks—such as communication, biological, or social networks—where optimized connectivity is crucial. In the chemical graph theory, these results could aid in predicting molecular behavior or designing compounds with desired properties by modeling molecular structures as hypergraphs and evaluating their connectivity indices.

Despite these contributions, our study has limitations. The analysis primarily focused on k-uniform and bipartite hypergraphs; future work could address non-uniform or directed hypergraphs. Additionally, the current operations are limited to edge-moving; other hypergraph transformations could yield further insights. Future research directions include extending this work to dynamic hypergraphs, exploring correlations between χ_α and other topological indices, and adapting our methodology to hypergraph classes with additional constraints. Such efforts would not only broaden the theoretical scope, but also enhance applicable insights in areas such as machine learning on hypernetworks and complex system modeling.

Author contributions

Hongzhuan Wang: Conceptualization, Investigation, Writing–review and editing; Piaoyang Yin and Yan Li: Investigation, Writing–original draft, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors of this paper declare that they have no conflicts of interest.

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