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Research article

A multivariate model for the Wiener process range, with statistical properties under stochastic volatility

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Abstract: In this paper, we presented a mathematical model to analyze financial instruments that are sensitive to the difference between the highest and lowest prices of n independent stocks in a random volatility environment. This model relies on the multivariate distribution of the ranges of n independent Wiener processes, describing the difference between the highest and lowest stock prices for a known time period. In addition to deriving the statistical characteristics of this distribution and its truncated version, including reliability properties, moments, the stress–strength parameter, and order statistics; we considered Bonferroni and Lorenz curves and the Gini index of the proposed model, as well as assessed its robustness in turbulent market environments. The proposed distribution enhances the modeling of range-based financial products to enable the construction of more efficient risk management and hedging strategies. Simulations with real financial data also confirmed its effectiveness in modeling range-based products and reducing volatility in markets.

Keywords: multivariate distribution; Wiener process range; truncated distribution; stock price; stress-strength parameter
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1. Introduction

The use of range-based models, which study the distribution of the difference between maximum and minimum asset prices, has gained increasing attention in financial mathematics in recent years. These models are particularly relevant for analyzing financial instruments such as range options, hedge options, and volatility-linked derivatives. These instruments are constructed on the price range of the stock during a specified time interval rather than against its mere terminal value. This modeling method is based on some mathematical properties of a process called the Wiener process $\{W(t), t \in R^+\}$, where R^+ is the set of real numbers. It is one of the very few important mathematical tools available for quantifying random variations in prices, especially when calculating the price of exotic financial products. This process is also used to model the behavior of financial assets, assuming that instantaneous price changes follow Gaussian distributions with time independence.

In recent decades, financial models have evolved from relying solely on the terminal value of the stock to more complex models that take into account its behavior over the entire time period. Among several modeling approaches for the analysis of financial risk and derivative instrument pricing, it is the range-based models that are considered the most efficient in analyzing financial risk and pricing derivative instruments, especially in highly volatile markets. Here, the range refers to the difference between the minimum and maximum values that a stock may reach over a time interval $(0,T)$, denoted as range $\overline{R}(T) = \sup_{(0,T)} W(t) - \inf_{(0,T)} W(t)$. This interval, in the case of Brownian motion, is intrinsically related to the extrema of the Wiener process. Feller [1] obtained the distribution of this range by using the method of images. He obtained the probability density function of this distribution, which was later used by [2] to derive the cumulative distribution function and the associated quantile estimators.

Most previous studies focused on the unitary properties of this distribution, without considering its behavior under high random fluctuations or over multiple time periods. In this regard, [3] presented a probability distribution for the range of the Wiener process when restricted to two specific values using the truncation method, which is considered more practical. They obtained the probability density function of the new distribution and analyzed its statistical properties, including survival and hazard functions, statistical moments, the stress-strength parameter, ordered statistics, and econometric skewness. The distribution's effectiveness in representing the behavior of range-bound stock prices was also verified by applying it to real-world data. The results showed that the distribution provides an accurate and appropriate model for describing prices in financial environments with bounded fluctuations. As an extension of this type of distribution, El-Hadidy and Alfreedi [4] exploited the concept of internal truncation of distributions introduced in [5] to obtain a new distribution for the range of the Wiener process, aimed at excluding periods of minor price fluctuations. The probabilities of the removed periods were distributed equally or in different proportions to the periods with non-removed random fluctuations. In addition, El-Hadidy and Alfreedi [4] presented the previous statistical properties of this distribution, where the results showed that the new distribution accurately reflected the statistical effect resulting from excluding periods of low volatility, making it suitable for financial applications including stock price modeling. El-Hadidy [6] also presented a discrete distribution for the range of the Wiener process using a special transformation based on the number of observed points. The presented discrete distribution showed good agreement with financial data and is an effective alternative to traditional continuous distributions when dealing with limited or discrete data.

Multivariate distributions are a foundation in financial modeling as they are used to describe the interdependence and mutual influence of different financial assets under the presence of uncertain circumstances. In an earlier work, Alexander [7] and McNeil et al. [8] showed that the multivariate models provide a reasonable description of the covariance of stocks, an element fundamental in risk management, portfolio analysis, and derivative pricing. For example, distributions such as the multivariate normal distribution and copula-based distributions in [9] are used to examine nonlinear correlations between returns, estimate systematic risk, and determine optimal portfolio diversification. The multivariate models presented in [10,11] have also demonstrated stunning superiority in value-at-risk applications and in modeling the dynamics of highly correlated assets during financial crises. Portions of the literature in [12,13] have demonstrated that uses of such models are not only limited to forecasting improved projections but also to facilitating strategic decision-making in dynamic investment horizons, which are impacted by a series of related variables. El-Hadidy and Alfreedi [14] were able to develop a multivariate distribution for the multidimensional Wiener process and analyzed its properties using probability range vectors, which can be applied to describe the temporal relationships between multiple fluctuations in financial markets. El-Hadidy and Alraddadi [15] also presented an accurate maximum decline distribution for the Wiener process as a model for measuring financial market risk, which included calculating the expected value of risk and extending it to the multidimensional case with applications to investment portfolio management.

The objective here is to obtain a multivariate distribution of n independent Wiener process ranges that represent the price differences of n stocks in the time interval $(0,T)$ and its truncated version under the stochastic volatility constraints. We aim to derive the classical statistical properties of this distribution, such as reliability properties, moments, the stress-strength parameter, and order statistics. This study focuses on applying the multivariate structure to evaluate the effectiveness of range-based stock pricing models. In addition, we study in detail the bivariate case of this distribution and its truncated version. In doing so, we explore the influence of the combined behavior of maximum and minimum price values for independent and different stocks in the time period $(0,T)$, influenced by varying stochastic volatility, on the accuracy of stock price valuation. Although the current distribution assumes independence of Wiener processes, it can be extended to handle non-independent cases using copula functions, thereby allowing the representation of cross-sectional dependencies among financial assets. This tractability enables explicit derivation of joint distributions, moments, and reliability measures.

This research includes the following sections: Section 2 presents the mathematical formulation of the multivariate distribution of n independent Wiener process ranges that give the price differences of n stocks in the time interval $(0,T)$ and its truncated version under stochastic volatility. Section 3 examines the statistical properties of the bivariate case of this distribution and its truncated version. Section 4 presents a numerical application that simulates restricted markets using real financial data. The dynamics of the common price range are analyzed, and then statistical fit tests and performance comparisons with existing models are studied. Section 5 concludes with the most important findings and future recommendations.

2. *n*-dimensional Wiener range distribution

The Wiener process range probability density function is a fundamental component in the study of this process due to its broad applications in physics, statistics, and financial modeling. This function has found concrete applications in the study of volatility in financial markets, particularly in risk models and the estimation of extreme price movements. Within the framework of the stochastic model of price movement that is assumed to proceed as a Wiener process, the amplitude of this movement over a given time period is a quantitative indicator of the overall market volatility. One of the most prominent practical applications is the use of the range distribution to estimate value-at-risk limits and analyze extreme scenarios in investment portfolio management. For example, the expected range of the Wiener process is used to establish upper and lower confidence limits for short-term price forecasts, without needing to know the exact parameters of the price distribution, taking advantage of the Gaussian and skewed nature of the process. The range distribution is used in hypothesis testing of the stability of financial data in which the computed range is compared with theoretical values for the purpose of establishing the existence of non-random behavior in the time series. Several studies have demonstrated the relevance of the Wiener range in this context, including those by Feller [1], Withers and Nadarajah [2], and Teamah et al. [3], who provided both theoretical expansions and practical insights into its implementation in finance. The range of the Wiener process $G(T)$ on the time interval $(0,T)$ can be defined as the difference between the stock's highest and lowest prices over a given time period. So, it can be written as $G(T)=\sup_{(0,T)} W(t)-\inf_{(0,T)} W(t)$. Applying the image method, which was used in [1], one can obtain the probability density function of $G(T)$ by a series formula as follows:

$$f_{G(T)}(g)=\frac{4\sqrt{T}}{g^2}\sum_{k=1}^{\infty}\exp\left[-\frac{4C_kT}{g_i^2}\right], \tag{2.1}$$

where $C_k=\frac{(2k-1)^2\pi^2}{8}$, $0<g<\infty$, and $T>0$.

The use of multivariate distributions not only enhances the accuracy of forecasts of future returns, but also contributes to supporting risk management strategies and diversifying investment portfolios in highly volatile markets. Therefore, we consider (2.1) to obtain the multivariate distribution of price differences for n independent stocks (ranges of independent Wiener processes) in the time interval $(0,T)$, which is given by

$$\left[G_1(T),G_2(T),...,G_n(T)\right]=\left[\sup_{(0,T)} W_1(t)-\inf_{(0,T)} W_1(t),\sup_{(0,T)} W_2(t)-\inf_{(0,T)} W_2(t),...,\sup_{(0,T)} W_n(t)-\inf_{(0,T)} W_n(t)\right],$$

where $\left[W_1(t),W_2(t),...,W_n(t)\right]$ is a vector of n independent Wiener process. The assumption of independence between Wiener processes is theoretically and practically crucial since it is mathematically tractable and enables explicit calculation of joint distributions, moments, and reliability measures, which would be unattainable in the case of dependence. Moreover, this assumption provides a solid base for multivariate modeling, from which dependence structures such as copulas or correlation matrices can later be added without affecting the key range properties. Therefore, a joint probability density function of the multivariate distribution of independent random variables $G_i, i=1,2,...,n$ is given by

$$f_{G_1(T),G_2(T),...,G_n(T)}(g_1,g_2,...,g_n;t)=\prod_{i=1}^{n}\frac{4\sqrt{T}}{g_i^2}\sum_{k=1}^{\infty}\exp\left[-\frac{4C_kT}{g_i^2}\right]. \tag{2.2}$$

On the other hand, the cumulative distribution function of this distribution is given by

$$F_{G_1(T),G_2(T),...,G_n(T)}(g_1,g_2,...,g_n;t)=\prod_{i=1}^{n}\sum_{k=1}^{\infty}\left(\frac{g_i^2+8C_kT}{C_kg_i^2}\right)\exp\left[-\frac{8C_kT}{g_i^2}\right]. \tag{2.3}$$

Therefore, its survival function is obtained from

$$S_{G_1(T),G_2(T),...,G_n(T)}(g_1,g_2,...,g_n;t) = 1 - F_{G_1(T),G_2(T),...,G_n(T)}(g_1,g_2,...,g_n;t)$$
$$= 1 - \prod_{i=1}^{n}\sum_{k=1}^{\infty}\left(\frac{g_i^2 + 8C_kT}{C_k g_i^2}\right)\exp\left[-\frac{8C_kT}{g_i^2}\right]. \qquad (2.4)$$

2.1. Double truncated version

The use of a truncated multivariate probability distribution is an effective tool for improving the accuracy of statistical estimation of stock price fluctuations in financial markets with high random volatility. The concept of truncation for one- and two-sided probability distributions was introduced in earlier works, such as [16–18]. El-Hadidy [5] also introduced a more general and comprehensive concept of truncation, which allows truncating *N* intervals of the probability distribution. More recently, El-Hadidy and Alraddadi [19] were able to introduce a new concept of truncating *N* intervals within a multivariate distribution. These types of truncated distributions were able to provide a clearer picture of the distribution of data within specific time periods, such as the works in [20,21], as they allow us to ignore outliers or extreme values that often distort classical measures such as variance and correlation, especially in markets that experience exceptional events or price spikes. By putting limits on spreads, either as regulatory market limits or by using prior knowledge modeling, the truncated distribution can then be utilized to provide a more realistic and accurate description of the probabilistic structure of return vectors. This approach also assists in enhancing the stability of financial models, particularly when used to estimate risk of a portfolio or optimize performance of machine learning models when dealing with financial data with non-Gaussian features. Furthermore, the distribution of price becomes closer to the empirical observation under this type of distribution, supporting its evidence for applications in terms of financial derivatives pricing and systemic risk management.

Since $G_i, i = 1,2,...,n$, are independent random variables, and applying the truncated definition in [16], we can get the joint double truncated probability density function of the new vector of independent random variables $[L_1(T), L_2(T),..., L_n(T)]$. For all $i = 1,2,...,n,$ we let $x_i = l_i / 2\sqrt{T}$, $\bar{a}_i = a_i / 2\sqrt{T}$, and $\bar{b}_i = b_i / 2\sqrt{T}$, and then (2.1) becomes

$$\phi_T(x_i) = \sum_{k=1}^{\infty}\left[\left(-4x_i^{-3}\exp\left[-C_k x_i^{-2}\right]\right) + \left(2C_k x_i^{-3}\exp\left[-C_k x_i^{-2}\right]\right)\left(C_k^{-1} + 2x_i^{-2}\right)\right].$$

Also, its cumulative distribution function becomes $\Phi_T(x_i) = \sum_{k=1}^{\infty}\left(C_k^{-1} + 2\bar{a}_i^{-2}\right)\exp\left[-C_k x_i^{-2}\right]$, which gives the values $\Phi_T(\bar{a}_i) = \sum_{k=1}^{\infty}\left(C_k^{-1} + 2\bar{a}_i^{-2}\right)\exp\left[-C_k \bar{a}_i^{-2}\right]$ and $\Phi_T(\bar{b}_i) = \sum_{k=1}^{\infty}\left(C_k^{-1} + 2\bar{b}_i^{-2}\right)\exp\left[-C_k \bar{b}_i^{-2}\right]$. If $\bar{a}_i \le x_i \le \bar{b}_i$, then the double truncated probability density function of $x_i, i = 1,2,...,n,$ can take the form

$$\bar{\phi}_T(x_i) = \frac{\phi_T(x_i)}{\Phi_T(\bar{b}_i) - \Phi_T(\bar{a}_i)} = \frac{\sum_{k=1}^{\infty}\left[\left(-4x_i^{-3}\exp\left[-C_k x_i^{-2}\right]\right) + \left(2C_k x_i^{-3}\exp\left[-C_k x_i^{-2}\right]\right)\left(C_k^{-1} + 2x_i^{-2}\right)\right]}{\sum_{k=1}^{\infty}\left(C_k^{-1} + 2\bar{b}_i^{-2}\right)\exp\left[-C_k \bar{b}_i^{-2}\right] - \sum_{k=1}^{\infty}\left(C_k^{-1} + 2\bar{a}_i^{-2}\right)\exp\left[-C_k \bar{a}_i^{-2}\right]}.$$

By restoring values $x_i = l_i / 2\sqrt{T}$, $\bar{a}_i = a_i / 2\sqrt{T}$, and $\bar{b}_i = b_i / 2\sqrt{T}$ to their original values, one can get the joint double truncated probability density function $[L_1(T), L_2(T),..., L_n(T)]$ by

$$u_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)=\prod_{i=1}^{n}\frac{f_{G_i(T)}(l_i;t)}{F_{G_i(T)}(b_i)-F_{G_i(T)}(a_i)}$$

$$=\prod_{i=1}^{n}\frac{\left|\frac{T^{-\frac{1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{8T}{l_i^2}\right)\exp\left[-\frac{4C_kT}{l_i^2}\right]}{\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{2T}{b_i^2}\right)\exp\left[-\frac{C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{2T}{a_i^2}\right)\exp\left[-\frac{C_kT}{a_i^2}\right]}, \tag{2.5}$$

where $a_i < l_i < b_i, i=1,2,\dots,n$.

Proposition 1. The joint double truncated cumulative distribution function of $[L_1(T),L_2(T),\dots,L_n(T)]$ is given as

$$U_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)=\prod_{i=1}^{n}\left[\frac{\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}C_k^{-1}\left(-a_i\exp\left[-\frac{4C_kT}{a_i^2}\right]+l_i\exp\left[-\frac{4C_kT}{l_i^2}\right]\right)}{\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{8T}{b_i^2}\right)\exp\left[-\frac{4C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{8T}{a_i^2}\right)\exp\left[-\frac{4C_kT}{a_i^2}\right]}\right]. \tag{2.6}$$

Proof. Since the vector $[L_1(T),L_2(T),\dots,L_n(T)]$ consists of independent random variables, the joint distribution can be written as the product of the marginal distributions. Each marginal distribution is obtained by integrating its density over the truncated interval $[a_i,b_i], i=1,2,\dots,n$. Specifically,

$$U_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)=\prod_{i=1}^{n}\int_{a_i}^{l_i}\frac{f_{G_i(T)}(l_i;t)}{F_{G_i(T)}(b_i)-F_{G_i(T)}(a_i)}dl_i\ .$$

Evaluating the integral for each i requires applying integration by parts, which reduces the expression into the compact exponential form shown in (2.6). This completes the proof.

Proposition 2. The joint double truncated survival function of $[L_1(T),L_2(T),\dots,L_n(T)]$ is given by

$$\tilde{S}_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)=1-\prod_{i=1}^{n}\left[\frac{\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}C_k^{-1}\left(-a_i\exp\left[-\frac{4C_kT}{a_i^2}\right]+l_i\exp\left[-\frac{4C_kT}{l_i^2}\right]\right)}{\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{8T}{b_i^2}\right)\exp\left[-\frac{4C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{8T}{a_i^2}\right)\exp\left[-\frac{4C_kT}{a_i^2}\right]}\right]. \tag{2.7}$$

Proof. By definition, the survival function is the complement of the cumulative distribution function, i.e., $\tilde{S}_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)=1-U_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)$. Hence, for each truncated variable $L_i(T)$, its survival function is obtained by integrating the tail probability from the upper truncation point to infinity. Because the variables are independent, the joint survival function is simply the product of these individual survival functions:

$$\tilde{S}_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)=\prod_{i=1}^{n}\int_{b_i}^{\infty}\frac{f_{G_i(T)}(l_i;t)}{F_{G_i(T)}(b_i)-F_{G_i(T)}(a_i)}dl_i\ .$$

Evaluating each integral involves repeated use of the exponential expansion and summation properties of the truncated density. After simplification, this leads to the compact series–exponential form reported in (2.7). This completes the proof.

2.1.1. Moments

Statistical moments are the basic instruments for explaining and exploring the probability distribution of financial stock returns, particularly in multivariate models used to study the correlation between stock price changes. The first moment (mean) is used to estimate the expected return, while the second moment gauges the variance and covariance that enable the derivation of the covariance matrix necessary to estimate joint risks in investment portfolios as in [22]. The importance of moments extends beyond this, as the higher moments such as skewness (third moment) and kurtosis (fourth moment) are exact measures of the asymmetry of distribution and "tailedness" attributes that deviate from classical assumptions of normal distribution, as found in [23], which investigated the stylized properties of financial markets. These moments are also used in approximate expansions, such as the Edgeworth expansion, to refine the approximation of actual return distributions, which effectively affects the accuracy in estimating risk measures such as value at risk and conditional probability of loss, as in [24]. Thus, moment analysis of multivariate distributions is not just a statistical description; it is a basic tool of financial performance measurement and risk management.

Theorem 1. Let $\left[L_1(T), L_2(T),..., L_n(T)\right]$ be a vector of independent Wiener process range, and then its joint moment generating function is given by

$$M_{L_1(T),L_2(T),...,L_n(T)}(l_1,l_2,...,l_n;t)=\prod_{i=1}^{n}\frac{\left|\frac{T^{-\frac{1}{2}}}{2}\right| I_i(a_i,b_i,t,T)}{F_{G_i(T)}(b_i)-F_{G_i(T)}(a_i)}, \tag{2.8}$$

where $I_i(a_i,b_i,t,T)=\int_{a_i}^{b_i}\sum_{k=1}^{\infty}\left(\alpha_k+\frac{\hat{a}}{l_i^2}\right)\exp\left[tl_i-\frac{\beta_k}{l_i^2}\right]dl_i$.

Proof. Use (2.5) and the independence principle of $L_i, i=1,2,...,n$, to get the joint moment generating function (m.g.f.):

$$\begin{aligned}
M_{L_1(T),L_2(T),...,L_n(T)}(l_1,l_2,...,l_n;t) &= \prod_{i=1}^{n}E(e^{tl_i})=\prod_{i=1}^{n}\int_{a_i}^{b_i}e^{tl_i}u_{L_i(T)}(l_i;t)dl_i \\
&= \prod_{i=1}^{n}\frac{\left|\frac{T^{-\frac{1}{2}}}{2}\right|\int_{a_i}^{b_i}e^{tl_i}\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{8T}{l_i^2}\right)\exp\left[-\frac{4C_kT}{l_i^2}\right]dl_i}{\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{2T}{b_i^2}\right)\exp\left[-\frac{C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{2T}{a_i^2}\right)\exp\left[-\frac{C_kT}{a_i^2}\right]} \\
&= \prod_{i=1}^{n}\frac{\left|\frac{T^{-\frac{1}{2}}}{2}\right| I_i(a_i,b_i,t,T)}{\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{2T}{b_i^2}\right)\exp\left[-\frac{C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{2T}{a_i^2}\right)\exp\left[-\frac{C_kT}{a_i^2}\right]},
\end{aligned}$$

where $\hat{a}=8T$, $\alpha_k=\dfrac{1}{C_k}$ (it depends on the number k), and $\beta_k=4C_kT$ depends on k and the moment T. Since the exponential expansion is valid for all $l_i\in(-\infty,\infty)$, we can interchange summation and integration. To justify this interchange, note that on any compact interval $[a_i,b_i]$, each term of the series is of the form $\exp\left[tl_i-\beta_k l_i^{-2}\right]$ with $\beta_k\to\infty$ in k. Hence, $\left|\exp\left[tl_i-\beta_k l_i^{-2}\right]\right|\le\exp\left[|t|\max|l_i|-\beta_k a_i^{-2}\right]$ and the series $\sum_k\exp\left[-\beta_k l_i^{-2}\right]$ converges uniformly. Therefore, by the Weierstrass M-test [25], the interchange of summation and integration is justified. Thus,

$$\exp\left[tl_i-\frac{\beta_k}{l_i^2}\right]=\sum_{m=0}^{\infty}\frac{\left(\exp\left[tl_i-\frac{\beta_k}{l_i^2}\right]\right)^m}{m!}=1+tl_i-\frac{\beta_k}{l_i^2}+\sum_{m=2}^{\infty}\frac{\left(\exp\left[tl_i-\frac{\beta_k}{l_i^2}\right]\right)^m}{m!}=1+tl_i-\frac{\beta_k}{l_i^2}+\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\frac{t^{m-\mu}}{m!}\binom{m}{\mu}(-\beta_k)^{\mu}l_i^{m-3\mu}.$$

Now, we let $I_i(a_i,b_i,t,T)=I_{i1}(a_i,b_i,t,T)+I_{i2}(a_i,b_i,t,T)$, where

$$I_{i1}(a_i,b_i,t,T)=\sum_{k=1}^{\infty}\int_{a_i}^{b_i}\alpha_k\exp\left[tl_i-\frac{\beta_k}{l_i^2}\right]dl_i=\sum_{k=1}^{\infty}\int_{a_i}^{b_i}\alpha_k\left(1+l_ir-\frac{\beta_k}{l_i^2}+\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\frac{t^{m-\mu}}{m!}\binom{m}{\mu}(-\beta_k)^{\mu}l_i^{m-3\mu}\right)dl_i\,, \tag{2.9}$$

and

$$I_{i2}(a_i,b_i,t,T)=\sum_{k=1}^{\infty}\int_{a_i}^{b_i}\frac{\hat{a}}{l_i^2}\exp\left[tl_i-\frac{\beta_k}{l_i^2}\right]dl_i=\hat{a}\sum_{k=1}^{\infty}\int_{a_i}^{b_i}\left(1+tl_i-\frac{\beta_k}{l_i^2}+\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\frac{t^{m-\mu}}{m!}\binom{m}{\mu}(-\beta_k)^{\mu}l_i^{m-2-3\mu}\right)dl_i. \tag{2.10}$$

As in [26], we can solve the following equations:

$$m-3\mu=-1, \tag{2.11}$$

$$m-2-3\mu=-1, \tag{2.12}$$

by using the Diophantine equations, where the set of the solutions of (2.11) is

$$\Omega_1=\{(m,\mu):m=3\mu-1,\mu\in N\}, \tag{2.13}$$

and of (2.12) is

$$\Omega_2=\{(m,\mu):m=3\mu+1,\mu\in N\}. \tag{2.14}$$

Therefore, $I_{i1}(a_i,b_i,t,T)$ and $I_{i2}(a_i,b_i,t,T)$ can be written as follows:

$$\begin{aligned}I_{i1}(a_i,b_i,t,T)=&\sum_{k=1}^{\infty}\alpha_k\left[(b_i-a_i)+\frac{t}{2}(b_i^2-a_i^2)-\beta_k\left(\frac{1}{a_i}-\frac{1}{b_i}\right)\right]+\sum_{k=1}^{\infty}\alpha_k\overbrace{\left[\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}t^{m-\mu}}{m!(m+1-3\mu)}\left(b_i^{m+1-3\mu}-a_i^{m+1-3\mu}\right)\right]}^{\forall\ m,(\mu\neq\mu_{S_1}\wedge m=m_{S_1})}\\&+\sum_{k=1}^{\infty}\alpha_k\left(\sum_{(m,\mu)\in S_1}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}}{m!}t^{m-\mu}\ln\frac{b_i}{a_i}\right),\end{aligned} \tag{2.15}$$

$$I_{i2}(a_i,b_i,t,T)=\hat{a}\sum_{k=1}^{\infty}\left[\left(\frac{1}{a_i}-\frac{1}{b_i}\right)+t\ln\frac{b_i}{a_i}-\frac{\beta_k}{3}\left(\frac{1}{a_i^3}-\frac{1}{b_i^3}\right)\right]+\hat{a}\sum_{k=1}^{\infty}\overbrace{\left[\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}t^{m-\mu}}{m!(m-1-3\mu)}\left(b_i^{m-1-3\mu}-a_i^{m-1-3\mu}\right)\right]}^{\forall\ m,(\mu\neq\mu_{S_2}\wedge m=m_{S_2})}$$
$$+\hat{a}\sum_{k=1}^{\infty}\left(\sum_{(m,\mu)\in S_2}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}}{m!}t^{m-\mu}\ln\frac{b_i}{a_i}\right). \tag{2.16}$$

As a result, we can get the value of $I_i(a_i,b_i,t,T)=\int_{a_i}^{b_i}\sum_{k=1}^{\infty}\left(\alpha_k+\frac{\hat{a}}{l_i^2}\right)\exp\left[tl_i-\frac{\beta_k}{l_i^2}\right]dl_i$, and then $M_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)$ will be obtained.

Central moments are likely the most useful analytical tool in studying stock price volatility, especially when comparing data over a limited time frame. Some studies, for instance, Cont's [23] research on statistical properties of financial asset returns, have demonstrated that using statistical moments around zero can better reflect market behavior and risk analysis than traditional models based on normal assumptions. Recent works also highlight the role of higher-order central moments in capturing volatility clustering and tail risks in financial markets [46].

Theorem 2. If $\xi_i, i=1,2,\dots,n$, is the set of exponential integral functions, then the joint central moments of $L_i, i=1,2,\dots,n$, can be obtained from

$$\tilde{M}_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)=\prod_{i=1}^{n}E(l_i^q)=\prod_{i=1}^{n}\left(\frac{2^qT^{\frac{q}{2}}\sum_{k=1}^{\infty}\left(a_i^{-2+q}\xi_i\left[\frac{q}{2},\frac{C_k}{a_i^2}\right]-b_i^{-2+q}\xi_i\left[\frac{q}{2},\frac{C_k}{b_i^2}\right]+\lambda_k\right)}{\sum_{k=1}^{\infty}\left(\alpha_k+8Tb_i^{-2}\right)\exp\left[-\frac{4C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\alpha_k+8Ta_i^{-2}\right)\exp\left[-\frac{4C_kT}{a_i^2}\right]}\right), \tag{2.17}$$

where $\lambda_k=\frac{C_k}{2}\left[a_i^{-4+q}\xi_i\left[-1+\frac{q}{2},\frac{C_k}{a_i^2}\right]+b_i^{-4+q}\xi_i\left[-1+\frac{q}{2},\frac{C_k}{b_i^2}\right]\right]$.

Proof. From the independence principle of $L_i, i=1,2,\dots,n$, the joint central moments are given by $\tilde{M}_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)=\prod_{i=1}^{n}E(l_i^q)=\prod_{i=1}^{n}\int_{a_i}^{b_i}l_i^q u_{L_i(T)}(l_i;t)dl_i$. By applying the change of variables $x_i=l_i/2\sqrt{T}$, then we have $\bar{a}_i=a_i/2\sqrt{T}$ and $\bar{b}_i=b_i/2\sqrt{T}$. These truncation limits become functions of the time horizon T. This formulation allows the central moments to adapt more flexibly to stochastic volatility. Consequently, $\tilde{M}_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)$ becomes

$$\Psi_{iT}(x_i)=\frac{1}{\wp_i}\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left[(-4x_i^{-3}\exp(-C_kx_i^{-2}))+(2C_kx_i^{-3}\exp(-C_kx_i^{-2}))(\alpha_k+2x_i^{-2})\right], \tag{2.18}$$

where $\wp_i=\sum_{k=1}^{\infty}\left(\alpha_k+8T\bar{b}_i^{-2}\right)\exp\left[-\frac{4C_kT}{\bar{b}_i^2}\right]-\sum_{k=1}^{\infty}\left(\alpha_k+8T\bar{a}_i^{-2}\right)\exp\left[-\frac{4C_kT}{\bar{a}_i^2}\right]$. From the convergence of the series $\sum_{k=1}^{\infty}\left[(-4x_i^{-3}\exp(-C_kx_i^{-2}))+(2C_kx_i^{-3}\exp(-C_kx_i^{-2}))(\alpha_k+2x_i^{-2})\right]$, one can get

$$E(x_i^q)=\prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{\infty}\int_{a_i}^{b_i}x_i^q\left[(-4x_i^{-3}\exp(-C_kx_i^{-2}))+(2C_kx_i^{-3}\exp(-C_kx_i^{-2}))(\alpha_k+2x_i^{-2})\right]dx_i}{\sum_{k=1}^{\infty}\left(\alpha_k+8T\bar{b}_i^{-2}\right)\exp\left[-\frac{4C_kT}{\bar{b}_i^2}\right]-\sum_{k=1}^{\infty}\left(\alpha_k+8T\bar{a}_i^{-2}\right)\exp\left[-\frac{4C_kT}{\bar{a}_i^2}\right]}\right)$$

$$=\prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{\infty}\left(\bar{a}_i^{-2+q}\xi_i\left[\frac{q}{2},\frac{C_k}{\bar{a}_i^2}\right]-\bar{b}_i^{-2+q}\xi_i\left[\frac{q}{2},\frac{C_k}{\bar{b}_i^2}\right]+2C_k\left[\bar{a}_i^{-4+q}\xi_i\left[-1+\frac{q}{2},\frac{C_k}{\bar{a}_i^2}\right]+\bar{b}_i^{-4+q}\xi_i\left[-1+\frac{q}{2},\frac{C_k}{\bar{b}_i^2}\right]\right]\right)}{\sum_{k=1}^{\infty}\left(\alpha_k+8T\bar{b}_i^{-2}\right)\exp\left[-\frac{4C_kT}{\bar{b}_i^2}\right]-\sum_{k=1}^{\infty}\left(\alpha_k+8T\bar{a}_i^{-2}\right)\exp\left[-\frac{4C_kT}{\bar{a}_i^2}\right]}\right). \quad (2.19)$$

This completes the proof.

Note that, since the distribution is double-truncated, all central moments are finite. The truncation ensures the existence of moments by eliminating potential divergences from the tail behavior of the untruncated range distribution.

The characteristic function is a central mathematical tool in characterizing the probability distributions of stock returns, especially in models that go beyond the classical assumptions of normal distribution, such as jump and price explosion models. Its usefulness in studying stock price volatility over a limited time range is highlighted by its ability to represent the entire distribution of asset returns, even in cases where it is difficult to explicitly specify the probability density function. This function also allows for Fourier analysis to evaluate the theoretical prices of financial derivatives and to estimate the risk coefficients associated with thick-tailed distributions. Carr and Madan [28] demonstrated how the impedance function can be used with price options under non-normal distributions, opening the way for more precise applications in modeling financial market volatility.

Theorem 3. If $L_i, i=1,2,...,n$, are independent random variables, then their joint characteristic function is

$$\hat{M}_{L_1(T),L_2(T),...,L_n(T)}(l_1,l_2,...,l_n;t)=\prod_{i=1}^{n}\frac{\left|\frac{T^{\frac{-1}{2}}}{2}\right|\left(\tilde{I}_{i1}(a_i,b_i,t,T)+\tilde{I}_{i2}(a_i,b_i,t,T)\right)}{\sum_{k=1}^{\infty}\left(\alpha_k+8Tb_i^{-2}\right)\exp\left[-\frac{4C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\alpha_k+8Ta_i^{-2}\right)\exp\left[-\frac{4C_kT}{a_i^2}\right]}, \quad (2.20)$$

where

$$\tilde{I}_{i1}(a_i,b_i,t,T)=\sum_{k=1}^{\infty}\alpha_k\left[(b_i-a_i)+\frac{\sqrt{i^2}t}{2}(b_i^2-a_i^2)-\beta_k\left(\frac{1}{a_i}-\frac{1}{b_i}\right)\right]+\sum_{k=1}^{\infty}\alpha_k\overbrace{\left[\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}(\sqrt{i^2}t)^{m-\mu}}{m!(m+1-3\mu)}\left(b_i^{m+1-3\mu}-a_i^{m+1-3\mu}\right)\right]}^{\forall\ m,(\mu\neq\mu_{S_1}\wedge m=m_{S_1})}$$
$$+\sum_{k=1}^{\infty}\alpha_k\left(\sum_{(m,\mu)\in S_1}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}}{m!}(\sqrt{i^2}t)^{m-\mu}\ln\frac{b_i}{a_i}\right),$$

$$\widetilde{I}_{i2}(a_i,b_i,t,T)=\hat{a}\sum_{k=1}^{\infty}\left[\left(\frac{1}{a_i}-\frac{1}{b_i}\right)+\sqrt{i^2}\,t\ln\frac{b_i}{a_i}-\frac{\beta_k}{3}\left(\frac{1}{a_i^2}-\frac{1}{b_i^2}\right)\right]+\hat{a}\sum_{k=1}^{\infty}\overbrace{\left[\sum_{m=2}^{\infty}\sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}(\sqrt{i^2}t)^{m-\mu}}{m!(m-1-3\mu)}\left(b_i^{m-1-3\mu}-a_i^{m-1-3\mu}\right)\right]}^{\forall\ m,(\mu\neq\mu_{S_2}\wedge m=m_{S_2})}$$
$$+\hat{a}\sum_{k=1}^{\infty}\left(\sum_{(m,\mu)\in S_2}\binom{m}{\mu}\frac{(-\beta_k)^{\mu}}{m!}(\sqrt{i^2}t)^{m-\mu}\ln\frac{b_i}{a_i}\right),$$

and $i^2=-1$.

Proof. Since the joint characteristic function of independent random variables is the product of their marginal characteristic functions,

$$\hat{M}_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)=\prod_{i=1}^{n}E(e^{\sqrt{i^2}tl_i})=\prod_{i=1}^{n}\int_{a_i}^{b_i}e^{\sqrt{i^2}tl_i}\frac{\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left(\alpha_k+8l_i^{-2}T\right)\exp\left[-\frac{4C_kT}{l_i^2}\right]}{\sum_{k=1}^{\infty}\left(\alpha_k+8Tb_i^{-2}\right)\exp\left[-\frac{4C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\alpha_k+8Ta_i^{-2}\right)\exp\left[-\frac{4C_kT}{a_i^2}\right]}dl_i$$

and $\left|e^{\sqrt{i^2}tl_i}\right|=1$, the same majorant used in Theorem 1, bounds every term uniformly on $[a_i,b_i]$; hence, dominated convergence (or the same M-test bound) justifies term-wise integration for the joint characteristic function, as well. This completes the proof.

2.1.2. Stress-strength parameter

The stress-strength parameter in a multivariate framework represents a powerful extension of the risk assessment associated with multiple interrelated financial assets. It is used to measure the joint probability that positive market forces (such as returns) will outweigh negative pressures (such as volatility or shocks) across more than one stock or index over a limited time period. Here, the coefficient takes the form of $P(\boldsymbol{X}>\boldsymbol{Y})$, where $\boldsymbol{X}=\lfloor X_1,X_2,\dots,X_n\rfloor$ and $\boldsymbol{Y}=\lfloor Y_1,Y_2,\dots,Y_n\rfloor$ are random vectors of stock performance and market stress forces, respectively. Studies such as in [29, 30] have shown the effectiveness of this coefficient in comparing two probability distributions, and thus it is suitable for modeling stock dynamics under market volatility and investigating a financial asset's ability to withstand short-run shocks. The method is particularly valuable in emerging markets or during periods of crisis, where stock interactions play a significant role in collective volatility.

In this section, we consider the probability vector $\boldsymbol{Y}=\prod_{i=1}^{n}P(G_i(T_2)<G_i(T_1))$, where $G_i(T_1)$ and $G_i(T_2)$, $i=1,2,\dots,n$, are independent random variables, each distributed according to the specification in Eq (2.2), with respective parameters T_1, and T_2. In statistical literature, the quantity Y_i, $i=1,2,\dots,n$, is commonly referred to as the stress-strength reliability parameter. It quantifies the likelihood that a system, modeled here as a difference in stock prices at different times, will operate successfully under uncertain conditions. Specifically, $G_i(T_1)$ represents the random strength (e.g., resistance of the difference in the price of stock number $i=1,2,\dots,n$), and $G_i(T_2)$ represents the random stress (e.g., external market pressure in the price of stock number $i=1,2,\dots,n$). A change in the price of stock number $i=1,2,\dots,n$ occurs when the applied stress $G_i(T_2)$ exceeds the available strength $G_i(T_1)$. The

system is deemed functional whenever $G_i(T_2) > G_i(T_1)$; for more details, see [31]. For the range distribution under consideration for n independent stocks, the stress-strength probability can be represented as $\boldsymbol{Y} = \prod_{i=1}^{n} \int_0^{\infty} F_{G_i(T)}(g_i;t).f_{G_i(T)}(g_i;t)dg_i$.

Analyzing the distribution of the difference in the prices for n independent stocks within a bounded range is more effective and essential for assessing performance probabilities. Thus, we use the truncated distribution (2.5) to obtain the stress-strength reliability parameter, defined as $P(\tilde{\boldsymbol{X}} > \tilde{\boldsymbol{Y}})$, which provides a probabilistic measure of a stock's return vector $\tilde{\boldsymbol{X}} = \lfloor \tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n \rfloor$ exceeding others' $\tilde{\boldsymbol{Y}} = \lfloor \tilde{Y}_1, \tilde{Y}_2, ..., \tilde{Y}_n \rfloor$, where $\tilde{X}_i \in [a_i, b_i]$ and $\tilde{Y}_i \in [a_i, b_i]$ give the stock performance and market stress force for each $i = 1,2,...,n$, respectively. This metric is widely applied in financial risk assessment and portfolio optimization to quantify relative strength under uncertainty. Thus, we have $\tilde{\boldsymbol{Y}} = \prod_{i=1}^{n} P(L_i(T_2) < L_i(T_1)) = \prod_{i=1}^{n} \int_0^{\infty} U_{L_i(T)}(l_i;t).u_{L_i(T)}(l_i;t)dl_i$.

Assumption 1. We assume that stress and strength are obtained from independent Wiener processes.

Assumption 2. We let

$$Q_{i1} = \sum_{k=1}^{\infty} \left(C_k^{-1} + 8T_1 b_i^{-2}\right) \exp\left[-4C_k T_1 b_i^{-2}\right] - \sum_{k=1}^{\infty} \left(C_k^{-1} + 8T_1 a_i^{-2}\right) \exp\left[-4C_k T_1 a_i^{-2}\right],$$

$$Q_{i2} = \sum_{k=1}^{\infty} \left(C_k^{-1} + 8T_2 b_i^{-2}\right) \exp\left[-4C_k T_2 b_i^{-2}\right] - \sum_{k=1}^{\infty} \left(C_k^{-1} + 8T_2 a_i^{-2}\right) \exp\left[-4C_k T_2 a_i^{-2}\right],$$

where $\tilde{C}_{ik} = \frac{-8a_i}{(\pi - 2k\pi)^2}$, $B_k = C_k^{-1}$, $N_k = \frac{(1-2k)^2 \pi^2}{2}$, $D_\Lambda = \frac{8}{(2\Lambda - 1)^2 \pi^2}$, and $F_\Lambda = \frac{(2\Lambda - 1)^2 \pi^2}{2}$ for all $i = 1,2,...,n$.

Assumption 3. We consider the series $\sum_k Q_{i1}$ and $\sum_k Q_{i2}$ to converge absolutely (i.e., $\sum_k Q_{i1} < \infty$ and $\sum_k Q_{i2} < \infty$) which follows automatically from the assumption $a_i > 0$ (i.e., the truncation bounds are strictly positive and bounded away from zero) together with the exponential decay in k.

Theorem 4. If the conditions of integrability and convergence hold, then the stress–strength reliability parameter is

$$\tilde{\boldsymbol{Y}} = \prod_{i=1}^{n} \left(\frac{1}{4\left|\sqrt{T_1 T_2}\right| Q_{i1} Q_{i2}} \sum_{k=1}^{\infty} \sum_{L=1}^{\infty} \int_{a_i}^{b_i} \left(\tilde{C}_{ik} \exp\left[-N_k a_i^{-2} T_2\right] + B_k l_i \exp\left[N_k l_i^{-2} T_2\right]\right)\left(D_\Lambda \exp\left[-F_\Lambda l_i^{-2} T_1\right] + 8 l_i^{-2} T_1 \exp\left[-F_\Lambda l_i^{-2} T_1\right]\right) dl_i \right).$$

Proof. We let

$$J_i(a_i, b_i, T_1, T_2) = \int_{a_i}^{b_i} \left(\tilde{C}_{ik} \exp\left[-N_k a_i^{-2} T_2\right] + B_k l_i \exp\left[-N_k l_i^{-2} T_2\right]\right)\left(D_\Lambda \exp\left[-F_\Lambda l_i^{-2} T_1\right] + 8 l_i^{-2} T_1 \exp\left[-F_\Lambda l_i^{-2} T_1\right]\right) dl_i$$

$$= \int_{a_i}^{b_i} \left(\left(D_\Lambda \tilde{C}_{ik} + 8\tilde{C}_{ik} l_i^{-2} T_1\right) \exp\left[-\left(N_k a_i^{-2} T_2 + F_\Lambda l_i^{-2} T_1\right)\right] + \left(D_\Lambda B_k l_i + 8 l_i^{-1} T_1 B_k\right) \exp\left[-l_i^{-2}\left(N_k T_2 + F_\Lambda T_1\right)\right]\right) dl_i$$

$$= \tilde{C}_{ik} D_\Lambda \exp\left[-N_k a_i^{-2} T_2\right]\int_{a_i}^{b_i} \exp\left[-F_\Lambda l_i^{-2} T_1\right] dl_i + 8T_1 \tilde{C}_{ik} \exp\left[-N_k a_i^{-2} T_2\right]\int_{a_i}^{b_i} l_i^{-2} \exp\left[-N_k l_i^{-2} T_1\right] dl_i$$

$$+ B_k D_\Lambda \int_{a_i}^{b_i} l_i \exp\left[-l_i^{-2}\left(N_k T_2 - F_\Lambda T_1\right)\right] dl_i + B_k 8T_1 \int_{a_i}^{b_i} l_i^{-1} \exp\left[-l_i^{-2}\left(N_k T_2 - F_\Lambda T_1\right)\right] dl_i$$

$$= \tilde{C}_{ik} \exp[-N_k a_i^{-2} T_2]\left(D_\Lambda J_{i1}(a_i, b_i, T_1, T_2) + 8T_1 J_{i2}(a_i, b_i, T_1, T_2)\right) + B_k D_\Lambda J_{i3}(a_i, b_i, T_1, T_2) + B_k 8T_1 J_{i4}(a_i, b_i, T_1, T_2),$$

where

$$J_{i1}(a_i, b_i, T_1, T_2) = \int_{a_i}^{b_i} \exp[-F_\Lambda l_i^{-2} T_1] dl_i = -a_i \exp\left[-\frac{F_\Lambda T_1}{a_i^2}\right] + b_i \exp\left[-\frac{F_\Lambda T_1}{b_i^2}\right] + \sqrt{\pi F_\Lambda T_1}\left(-Erf\left(\frac{\sqrt{F_\Lambda T_1}}{a_i}\right) + Erf\left(\frac{\sqrt{F_\Lambda T_1}}{b_i}\right)\right);$$

$$J_{i2}(a_i, b_i, T_1, T_2) = \int_{a_i}^{b_i} l_i^{-2} \exp[-F_\Lambda l_i^{-2} T_1] dl_i = \frac{\sqrt{\pi}\left(Erf\left(\frac{\sqrt{F_\Lambda T_1}}{a_i}\right) - Erf\left(\frac{\sqrt{F_\Lambda T_1}}{b_i}\right)\right)}{2\sqrt{F_\Lambda T_1}};$$

$$J_{i3}(a_i, b_i, T_1, T_2) = \int_{a_i}^{b_i} l_i \exp[-N_k l_i^{-2} T_2] \exp[-F_\Lambda l_i^{-2} T_1] dl_i = \frac{1}{2} F_\Lambda T_1\left(\Gamma\left(0, \frac{F_\Lambda T_1 + N_K T_2}{a_i^2}\right) - \Gamma\left(0, \frac{F_\Lambda T_1 + N_K T_2}{b_i^2}\right)\right)$$

$$+ \frac{1}{2}\left(-a_i^2 \exp\left[-\frac{F_\Lambda T_1 + N_K T_2}{a_i^2}\right] + b_i^2 \exp\left[-\frac{F_\Lambda T_1 + N_K T_2}{b_i^2}\right] + N_K T_2\left(\Gamma\left(0, \frac{F_\Lambda T_1 + N_K T_2}{a_i^2}\right) - \Gamma\left(0, \frac{F_\Lambda T_1 + N_K T_2}{b_i^2}\right)\right)\right);$$

and

$$J_{i4}(a_i, b_i,, T_1, T_2) = \int_{a_i}^{b_i} l_i^{-1} \exp[-N_k l_i^{-2} T_2] \exp[-F_\Lambda l_i^{-2} T_1] dl_i = \frac{1}{2}\left(-\Gamma\left(0, \frac{F_\Lambda T_1 + N_K T_2}{a_i^2}\right) + \Gamma\left(0, \frac{F_\Lambda T_1 + N_K T_2}{b_i^2}\right)\right).$$

Since the conditions of integrability and convergence are satisfied, then

$$\tilde{\boldsymbol{Y}} = \prod_{i=1}^{n} \frac{1}{4\left|\sqrt{T_1 T_2}\right| Q_{i1} Q_{i2}} \sum_{k=1}^{\infty}\sum_{L=1}^{\infty} J_i(a_i, b_i, T_1, T_2)$$

$$= \prod_{i=1}^{n} \frac{1}{4\left|\sqrt{T_1 T_2}\right| Q_{i1} Q_{i2}} \sum_{k=1}^{\infty}\sum_{L=1}^{\infty}\left(\tilde{C}_{ik} \exp[-N_k a_i^{-2} T_2]\left(D_\Lambda J_{i1}(a_i, b_i, T_1, T_2) + 8T_1 J_{i2}(a_i, b_i, T_1, T_2)\right)\right. \quad (2.21)$$

$$\left. + B_k D_\Lambda J_{i3}(a_i, b_i, T_1, T_2) + B_k 8T_1 J_{i4}(a_i, b_i, T_1, T_2)\right).$$

This completed the proof.

In a similar way, one can get $\boldsymbol{Y} = \prod_{i=1}^{n}\int_{0}^{\infty} F_{G_i(T)}(g_i; t) . f_{G_i(T)}(g_i; t) dg_i$.

2.1.3. Order statistics

In the context of pricing options based on the time path of the price, such as Asian mean options, rank statistics and multivariate probability distributions of the Wiener process are of great importance. Although closed-form formulas are available for pricing options based on the geometric mean under the assumption that the stock price follows a lognormal distribution, as in [32], extracting a similar closed-form formula for the arithmetic mean remains mathematically infeasible in the general case, due to the lack of an accurate probability distribution for the arithmetic mean of lognormal variables [33]. This issue has led to the application of rank statistics as an analytical tool that helps approximate and understand the behavior of extreme values and averages on the price path. Multivariate distributions provide a precise mathematical framework for measuring the intertemporal relationship between stock prices at different points in time. For example, logarithmic price returns can be modeled using a multivariate normal distribution, as in [34], which allows for the analysis of covariance and correlation between prices, which are essential elements for understanding the behavior of price changes and time lags in financial assets. These models become essential for evaluating the probabilistic performance of mean-based or extreme/minimal price options. The integration of ranking statistics and multivariate probability distributions enhances the ability to build efficient numerical pricing algorithms and compensates for the lack of precise analytical solutions. Accordingly, these tools are a theoretical pillar in modern financial modeling of complex mean-based or extreme-based options [35]. Thus, we consider the random vectors $\lfloor L_{1(1:\tilde{n})}, L_{1(2:\tilde{n})},..., L_{1(\tilde{n}:\tilde{n})} \rfloor \le \lfloor L_{2(1:\tilde{n})}, L_{2(2:\tilde{n})},..., L_{2(\tilde{n}:\tilde{n})} \rfloor \le ... \le \lfloor L_{n(1:\tilde{n})}, L_{n(2:\tilde{n})},..., L_{n(\tilde{n}:\tilde{n})} \rfloor$, which denote the order statistics of a random sample $L_{i1}, L_{i2},..., L_{i\tilde{n}}, i = 1,2,..., n$, from a distribution which has joint double truncated probability density function (2.5). Consequently, the joint probability density function of the p_i^{th} order statistic for each $L_{i(p_i:\tilde{n})}$, $i = 1,2,..., n$, is

$$\tilde{u}_{L_1(T),L_2(T),...,L_n(T)(p_1:\tilde{n},p_2:\tilde{n},...,p_n:\tilde{n})}(l_1,l_2,...,l_n;t) = \prod_{i=1}^{n} \frac{\tilde{n}!}{(p_i-1)!(\tilde{n}-p_i)!} \left(U_{L_i(T)}(l_i)\right)^{p_i-1} \left(1-U_{L_i(T)}(l_i)\right)^{\tilde{n}-p_i} u_{L_i(T)}(l_i),$$

$$= \prod_{i=1}^{n} \frac{\tilde{n}!}{(p_i-1)!(\tilde{n}-p_i)!} \left(\frac{\left|\frac{T^{\frac{-1}{2}}}{2}\right| \sum_{k=1}^{\infty} C_k^{-1} \left(-a_i \exp\left[-\frac{4C_kT}{a_i^2}\right] + l_i \exp\left[-\frac{4C_kT}{l_i^2}\right] \right)}{\sum_{k=1}^{\infty} \left(\frac{1}{C_k} + \frac{8T}{b_i^2}\right) \exp\left[-\frac{4C_kT}{b_i^2}\right] - \sum_{k=1}^{\infty} \left(\frac{1}{C_k} + \frac{8T}{a_i^2}\right) \exp\left[-\frac{4C_kT}{a_i^2}\right]} \right)^{p_i-1}$$

$$\times \left(1 - \frac{\left|\frac{T^{\frac{-1}{2}}}{2}\right| \sum_{k=1}^{\infty} C_k^{-1} \left(-a_i \exp\left[-\frac{4C_kT}{a_i^2}\right] + l_i \exp\left[-\frac{4C_kT}{l_i^2}\right] \right)}{\sum_{k=1}^{\infty} \left(\frac{1}{C_k} + \frac{8T}{b_i^2}\right) \exp\left[-\frac{4C_kT}{b_i^2}\right] - \sum_{k=1}^{\infty} \left(\frac{1}{C_k} + \frac{8T}{a_i^2}\right) \exp\left[-\frac{4C_kT}{a_i^2}\right]} \right)^{\tilde{n}-p_i}$$

$$\times \left(\frac{\left|\frac{T^{-\frac{1}{2}}}{2}\right| \sum_{k=1}^{\infty} \left(\frac{1}{C_k} + \frac{8T}{l_i^2}\right) \exp\left[-\frac{4C_kT}{l_i^2}\right]}{\sum_{k=1}^{\infty} \left(\frac{1}{C_k} + \frac{2T}{b_i^2}\right) \exp\left[-\frac{C_kT}{b_i^2}\right] - \sum_{k=1}^{\infty} \left(\frac{1}{C_k} + \frac{2T}{a_i^2}\right) \exp\left[-\frac{C_kT}{a_i^2}\right]} \right)$$

$$= \prod_{i=1}^{n} \frac{\binom{\tilde{n}}{p_i}\left|\frac{T^{\frac{-1}{2}}}{2}\right|^{p_i}}{Q_i^{\tilde{n}}} \left(\sum_{k=1}^{\infty} C_k^{-1}\left(-a_i \exp\left[-4C_k T a_i^{-2}\right] + l_i \exp\left[-4C_k T l_i^{-2}\right]\right)\right)^{p_i - 1}$$

$$\times \left(Q_i - \left|\frac{T^{\frac{-1}{2}}}{2}\right| \sum_{k=1}^{\infty} C_k^{-1}\left(-a_i \exp\left[-4C_k T a_i^{-2}\right] + l_i \exp\left[-4C_k T l_i^{-2}\right]\right)\right)^{\tilde{n} - p_i} \tag{2.22}$$

$$\times \left(\sum_{k=1}^{\infty}\left(C_k^{-1} + 8 l_i^{-2} T\right)\exp\left[-4C_k T l_i^{-2}\right]\right),$$

where $Q_i = \sum_{k=1}^{\infty}\left(\frac{1}{C_k} + \frac{2T}{b_i^2}\right)\exp\left[-\frac{C_k T}{b_i^2}\right] - \sum_{k=1}^{\infty}\left(\frac{1}{C_k} + \frac{2T}{a_i^2}\right)\exp\left[-\frac{C_k T}{a_i^2}\right]$.

In addition, the joint distribution function of the random vector $\left\lfloor L_{1(p_1:\tilde{n})}, L_{2(p_2:\tilde{n})}, \ldots, L_{n(p_n:\tilde{n})}\right\rfloor$ is

$$\tilde{U}_{L_1(T), L_2(T), \ldots, L_n(T)(p_1:\tilde{n}, p_2:\tilde{n}, \ldots, p_n:\tilde{n})}(l_1, l_2, \ldots, l_n; t) = \prod_{i=1}^{n}\sum_{\tilde{i}=p_i}^{\tilde{n}}\binom{\tilde{n}}{\tilde{i}}\left(U_{L_i(T)}(l_i)\right)^{\tilde{i}}\left(1 - U_{L_i(T)}(l_i)\right)^{\tilde{n} - \tilde{i}}$$

$$= \prod_{i=1}^{n}\sum_{\tilde{i}=p_i}^{\tilde{n}}\binom{\tilde{n}}{\tilde{i}}\left(\frac{\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty} C_k^{-1}\left(-a_i \exp\left[-\frac{4C_k T}{a_i^2}\right] + l_i \exp\left[-\frac{4C_k T}{l_i^2}\right]\right)}{\sum_{k=1}^{\infty}\left(\frac{1}{C_k} + \frac{8T}{b_i^2}\right)\exp\left[-\frac{4C_k T}{b_i^2}\right] - \sum_{k=1}^{\infty}\left(\frac{1}{C_k} + \frac{8T}{a_i^2}\right)\exp\left[-\frac{4C_k T}{a_i^2}\right]}\right)^{\tilde{i}}$$

$$\times \left(1 - \frac{\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty} C_k^{-1}\left(-a_i \exp\left[-\frac{4C_k T}{a_i^2}\right] + l_i \exp\left[-\frac{4C_k T}{l_i^2}\right]\right)}{\sum_{k=1}^{\infty}\left(\frac{1}{C_k} + \frac{8T}{b_i^2}\right)\exp\left[-\frac{4C_k T}{b_i^2}\right] - \sum_{k=1}^{\infty}\left(\frac{1}{C_k} + \frac{8T}{a_i^2}\right)\exp\left[-\frac{4C_k T}{a_i^2}\right]}\right)^{\tilde{n} - \tilde{i}}$$

$$= \prod_{i=1}^{n}\sum_{\tilde{i}=p_i}^{\tilde{n}} \frac{\binom{\tilde{n}}{\tilde{i}}\left|\frac{T^{\frac{-1}{2}}}{2}\right|^{\tilde{i}}}{Q_i^{\tilde{n}}}\left(\sum_{k=1}^{\infty} C_k^{-1}\left(-a_i \exp\left[-4C_k T a_i^{-2}\right] + l_i \exp\left[-4C_k T l_i^{-2}\right]\right)\right)^{\tilde{i}}$$

$$\times \left(Q_i - \left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty} C_k^{-1}\left(-a_i \exp\left[-4C_k T a_i^{-2}\right] + l_i \exp\left[-4C_k T l_i^{-2}\right]\right)\right)^{\tilde{n} - \tilde{i}}. \tag{2.23}$$

Also, the q_i^{th} moment of the p_i^{th} order statistic, $L_{i(p_i:\tilde{n})}$, $i = 1, 2, \ldots, n$, is

$$E\left(L_{1(k:\tilde{n})}^{q_1} L_{2(k:\tilde{n})}^{q_2} \ldots L_{n(k:\tilde{n})}^{q_n}\right) = \prod_{i=1}^{n}\left(q_i \sum_{\tilde{j}=\tilde{n}-k+1}^{\tilde{n}} (-1)^{\tilde{j}=\tilde{n}-k+1}\binom{\tilde{j}-1}{\tilde{n}-k}\binom{\tilde{n}}{\tilde{j}}\int_{a_i}^{b_i} l_i^{q_i - 1}\left(1 - U_{L_i(T)}(l_i)\right)^{\tilde{j}} dl_i\right). \tag{2.24}$$

We consider $\aleph_{i\tilde{j}} = \int_{a_i}^{b_i} l_i^{q_i - 1}\left(1 - U_{L_i(T)}(l_i)\right)^{\tilde{j}} dl_i$, $Z_{ik} = \frac{-8a_i}{(\pi - 2k\pi)^2}$, and $G_k = 4C_k T$. This leads to

$$\aleph_{i\tilde{j}} = \int_{a_i}^{b_i} l_i^{q_i - 1}\left(1 - \left|\frac{T^{-\frac{1}{2}}}{2}\right| \sum_{k=1}^{\infty}\left(Z_{ik}\exp\left[-G_k a_i^{-2}\right] + 8l_i \exp\left[-G_k l_i^{-2}\right]\right)\right)^{\tilde{j}} dl_i .$$

From the binomial theorem, we have $\left(1 - U_{L_i(T)}(l_i)\right)^{\tilde{j}} = \sum_{\tilde{j}=0}^{p_i}\binom{p_i}{\tilde{j}}(-1)^{\tilde{j}}\left(U_{L_i(T)}(l_i)\right)^{\tilde{j}}$. Consequently,

$$\aleph_{i\tilde{j}} = \sum_{\tilde{j}=0}^{p_i}\binom{p_i}{\tilde{j}}(-1)^{\tilde{j}}\int_{a_i}^{b_i}\left(U_{L_i(T)}(l_i)\right)^{\tilde{j}} l_i^{q_i-1} dl_i = \sum_{\tilde{j}=0}^{p_i}\binom{p_i}{\tilde{j}}(-1)^{\tilde{j}}\left(\left|\frac{T^{-\frac{1}{2}}}{2}\right|\right)^{\tilde{j}}\int_{a_i}^{b_i} l_i^{q_i-1}\left(\sum_{k=1}^{\infty}\left(Z_{ik}\exp\left[-G_k a_i^{-2}\right] + 8l_i\exp\left[-G_k l_i^{-2}\right]\right)\right)^{\tilde{j}} dl_i .$$

When $\tilde{j} = 0,1$, we obtain, respectively,

$$\aleph_{0\tilde{j}} = \binom{p_i}{0}\int_{a_i}^{b_i} l_i^{q_i-1} dl_i = \frac{b_i^{q_i} - a_i^{q_i}}{q_i} \quad \text{and} \quad \aleph_{1\tilde{j}} = \binom{p_i}{1}(-1)\left|\frac{T^{-\frac{1}{2}}}{2}\right|\int_{a_i}^{b_i}\sum_{k=1}^{\infty}\left(Z_{ik}\exp\left[-G_k a_i^{-2}\right] + 8l_i\exp\left[-G_k l_i^{-2}\right]\right) l_i^{q_i-1} dl_i,$$

where the series $\sum_{k=1}^{\infty}\left(Z_{ik}\exp\left[-G_k a_i^{-2}\right] + 8l_i\exp\left[-G_k l_i^{-2}\right]\right)$ is convergent. This leads to

$$\aleph_{1\tilde{j}} = \binom{p_i}{1}(-1)\left|\frac{T^{-\frac{1}{2}}}{2}\right|\sum_{k=1}^{\infty}\int_{a_i}^{b_i}\left(Z_{ik}\exp\left[-G_k a_i^{-2}\right] + 8l_i\exp\left[-G_k l_i^{-2}\right]\right) l_i^{q_i-1} dl_i$$

$$= \binom{p_i}{1}(-1)\left|\frac{T^{-\frac{1}{2}}}{2}\right|\sum_{k=1}^{\infty}\frac{\exp\left[-\frac{G_k}{a_i^2}\right]\left(-4xp\left[-\frac{G_k}{a_i^2}\right]q_i\left(a_i^{1+q_i}Ei\left(\frac{3+q_i}{2},\frac{G_k}{a_i^2}\right) - b_i^{1+q_i}Ei\left(\frac{3+q_i}{2},\frac{G_k}{b_i^2}\right)\right) + \left(b_i^{q_i} - a_i^{q_i}\right)Z_{ik}\right)}{q_i},$$

where Ei denotes to the exponential integral function. At $\tilde{j} = 2$, we have

$$\left(\sum_{k=1}^{\infty}\left(Z_{ik}\exp\left[-G_k a_i^{-2}\right] + 8l_i\exp\left[-G_k l_i^{-2}\right]\right)\right)^2 = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(Z_{ik}\exp\left[-G_k a_i^{-2}\right] + 8l_i\exp\left[-G_k l_i^{-2}\right]\right)\left(Z_{i\tilde{n}}\exp\left[-G_{\tilde{n}} a_i^{-2}\right] + 8l_i\exp\left[-G_{\tilde{n}} l_i^{-2}\right]\right)$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(Z_{ik}Z_{i\tilde{n}}\exp\left[-(G_k + G_{\tilde{n}})a_i^{-2}\right] + 8l_i Z_{ik}\exp\left[-G_k a_i^{-2} - G_{\tilde{n}} l_i^{-2}\right] + 8l_i Z_{i\tilde{n}}\exp\left[-G_k l_i^{-2} - G_{\tilde{n}} a_i^{-2}\right] + 64 l_i^2\exp\left[-G_k l_i^{-2} - G_{\tilde{n}} l_i^{-2}\right]\right).$$

Consequently,

$$\aleph_{2\tilde{j}} = \binom{p_i}{2}\left(\left|\frac{T^{-\frac{1}{2}}}{2}\right|\right)^2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_{a_i}^{b_i}\left(Z_{ik}Z_{i\tilde{n}}\exp\left[-(G_k + G_{\tilde{n}})a_i^{-2}\right] + 8l_i Z_{ik}\exp\left[-G_k a_i^{-2} - G_{\tilde{n}} l_i^{-2}\right] + 8l_i Z_{i\tilde{n}}\exp\left[-G_k l_i^{-2} - G_{\tilde{n}} a_i^{-2}\right]\right.$$
$$\left. + 64 l_i^2\exp\left[-\left(G_k + G_{\tilde{n}}\right)l_i^{-2}\right]\right) l_i^{q_i-1} dl_i$$

$$= \binom{p_i}{2}\left(\left|\frac{T^{-\frac{1}{2}}}{2}\right|\right)^2 \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} Z_{ik} Z_{i\tilde{n}} \int_{a_i}^{b_i} \exp\left[-(G_k + G_{\tilde{n}})a_i^{-2}\right] l_i^{q_i-1} dl_i + 8Z_{ik}\int_{a_i}^{b_i} \exp\left[-G_k a_i^{-2} - G_{\tilde{n}} l_i^{-2}\right] l_i^{q_i} dl_i$$
$$+ 8Z_{i\tilde{n}}\int_{a_i}^{b_i} \exp\left[-G_k l_i^{-2} - G_{\tilde{n}} a_i^{-2}\right] l_i^{q_i} dl_i + 64\int_{a_i}^{b_i} \exp\left[-(G_k + G_{\tilde{n}}) l_i^{-2}\right] l_i^{q_i+1} dl_i,$$

where

$$\int_{a_i}^{b_i} \exp\left[-(G_k + G_{\tilde{n}})a_i^{-2}\right] l_i^{q_i-1} dl_i = \exp\left[-(G_k + G_{\tilde{n}})a_i^{-2}\right]\frac{b_i^{q_i} - a_i^{q_i}}{q_i},$$

$$\int_{a_i}^{b_i} l_i^{q_i} \exp\left[-(G_k a_i^{-2} + G_{\tilde{n}} l_i^{-2})\right] dl_i = \frac{1}{2}\exp\left[-G_k a_i^{-2}\right]\left(-a_i^{q_i+1} Ei\left(\frac{3+q_i}{2}, G_{\tilde{n}} a_i^{-2}\right) + b_i^{q_i+1} Ei\left(\frac{3+q_i}{2}, G_{\tilde{n}} b_i^{-2}\right)\right), \text{ and}$$

$$\int_{a_i}^{b_i} \exp\left[-(G_k + G_{\tilde{n}}) l_i^{-2}\right] l_i^{q_i+1} dl_i = \frac{1}{2}\left(G_k + G_{\tilde{n}}\right)$$
$$\times\left(-a_i^{q_i}\Gamma\left(-\left(1+\frac{q_i}{2}\right),(G_k + G_{\tilde{n}})a_i^{-2}\right)\left(\frac{a_i^2}{G_k + G_{\tilde{n}}}\right)^{-\frac{Q_i}{2}} + b_i^{q_i}\Gamma\left(-\left(1+\frac{q_i}{2}\right),(G_k + G_{\tilde{n}})b_i^{-2}\right)\left(\frac{b_i^2}{G_k + G_{\tilde{n}}}\right)^{-\frac{Q_i}{2}}\right).$$

Similarly, one can obtain the value of $\aleph_{i\tilde{j}}$ for any value of $\tilde{j}$, and then $E\left(L_{1(k:\tilde{n})}^{q_1} L_{2(k:\tilde{n})}^{q_2} \dots L_{n(k:\tilde{n})}^{q_n}\right)$ will be obtained from (2.24).

2.1.4. Modeling stock price disparities using Lorenz, Gini, and Bonferroni curves

The study of the variance of stock price spreads is a central issue in the quantitative analysis of financial markets, especially when using statistical models based on the multivariate distribution of stock price spreads over a limited range (double truncated multivariate distribution). Among the most well-known tools in this field is the Lorenz curve, a graphical means of measuring the unequal distribution of stock price spreads across a set of stocks. The random variables $L_i, i = 1,2,\dots,n$, are arranged in ascending order, and the cumulative relative distribution of values is calculated against the cumulative percentage of observations. The Lorenz curve is based on the existence of a joint cumulative distribution function (2.6) for the vector of spreads $[L_1, L_2, \dots, L_n]$ and represents the relationship between the cumulative percentage of assets and the cumulative percentage of the change in value. We can obtain the Lorenz curve for the independent random variables $L_i, i = 1,2,\dots,n$, from the following equation:

$$\tilde{L}(u_{L_1(T),L_2(T),\dots,L_n(T)}(l_1,l_2,\dots,l_n;t)) = \prod_{i=1}^{n}\left(\frac{\int_{a_i}^{l_i} l_i u_{L_i(T)}(l_i;t) dl_i}{\int_{a_i}^{b_i} l_i u_{L_i(T)}(l_i;t) dl_i}\right) = \prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{\infty}\left[-4T\left(\Gamma\left[0,4C_k T a_i^{-2}\right] - \Gamma\left[0,4C_k T l_i^{-2}\right]\right) + 4C_k \hat{A}_i\right]}{\sum_{k=1}^{\infty}\left[-4T\left(\Gamma\left[0,4C_k T a_i^{-2}\right] - \Gamma\left[0,4C_k T b_i^{-2}\right]\right) + 4C_k \hat{C}_i\right]}\right), \tag{2.25}$$

where

$$\hat{A}_i = 2\left[-2a_i^2 \exp\left[-4C_kTa_i^{-2}\right] + 2l_i^2 \exp\left[-4C_kTl_i^{-2}\right] + 8C_kT\left(\Gamma\left[0,4C_kTa_i^{-2}\right] - \Gamma\left[0,4C_kTl_i^{-2}\right]\right)\right],$$

$$\hat{C}_i = 2\left[-2a_i^2 \exp\left[-4C_kTa_i^{-2}\right] + 2b_i^2 \exp\left[-4C_kTb_i^{-2}\right] + 8C_kT\left(\Gamma\left[0,\exp\left[-4C_kTa_i^{-2}\right]\right] - \Gamma\left[0,\exp\left[-4C_kTb_i^{-2}\right]\right]\right)\right].$$

Accordingly, the Gini index is calculated as a quantitative measure of inequality, given from the formula $\tilde{G}(u_{L_1(T),\dots,L_n(T)}(l_1,\dots,l_n;t))$

$$= \prod_{i=1}^{n}\left(1 - \frac{2}{Z_i}\sum_{k=1}^{\infty}\left[-4T(b_i - a_i)\Gamma\left[0,4C_kTa_i^{-2}\right] + 4T(W_i + \chi_i) + \frac{\Lambda_i + 4\tilde{\Phi}_i + S_i - \left(128kC_k^2\pi^{-2}T(W_i + \chi_i)\right)}{8C_k}\right]\right), \tag{2.26}$$

where

$$W_i = 4\sqrt{\pi C_kT}\left(Erf\left[-2b_i\sqrt{C_kT}\right] - Erf\left[-2a_i\sqrt{C_kT}\right]\right),$$

$$\chi_i = b_i\left(\Gamma\left[0,4b_i^{-2}C_kT\right] - 2e^{-4b_i^{-2}C_kT}\right) - a_i\left(\Gamma\left[0,4a_i^{-2}C_kT\right] - 2e^{-4a_i^{-2}C_kT}\right),$$

$$\tilde{\Phi}_i = \frac{1}{6}\left(\exp\left[-\frac{(1+4k^{2)}\pi^2T}{2a_i^2}\right].\left[2a_i\exp\left[2k\pi^2Ta_i^{-2}\right](a_i^2 - 8C_k) - \sqrt{2}\exp\left[-\frac{(1+4k^{2)}\pi^2T}{2a_i^2}\right](-8C_kT)^{\frac{3}{2}}Erf\left[-2\sqrt{C_kT}a_i^{-1}\right]\right]\right.$$

$$\left.-\exp\left[-\frac{(1+4k^{2)}\pi^2T}{2b_i^2}\right].\left[2b_i\exp\left[2k\pi^2Tb_i^{-2}\right](b_i^2 - 8C_k) - \sqrt{2e}xp\left[-\frac{(1+4k^{2)}\pi^2T}{2b_i^2}\right](-8C_kT)^{\frac{3}{2}}Erf\left[-2\sqrt{C_kT}b_i^{-1}\right]\right]\right),$$

$$Z_i = \sum_{k=1}^{\infty}\left[-4T\left(\Gamma\left[0,4C_kTa_i^{-2}\right] - \Gamma\left[0,4C_kTb_i^{-2}\right]\right) + \frac{\hat{C}_i}{8C_k}\right],$$

$$S_i = 16C_kT(b_i - a_i)\Gamma\left[0,4C_kTa_i^2\right],$$

$$\Lambda_i = -4a_i^2(b_i - a_i)\exp\left[-8C_kT\right].$$

It indicates the degree of concentration of price differences; the closer it is to one, the more concentrated the changes are in a small number of stocks. On the other hand, the Bonferroni curve is a useful alternative in cases where the Lorenz curve is insensitive to the minimum variance. The Bonferroni curve is one such measure that has the benefit of being graphically displayed in the unit square and can also be related to the Lorenz curve and Gini index, as demonstrated in [35–37], where the Bonferroni curve of the random vector $\left[L_1, L_2,\dots, L_n\right]$ is given from (2.25) and (2.26) by

$$\tilde{B}(u_{L_1(T),\dots,L_n(T)}(l_1,\dots,l_n;t)) = \frac{\tilde{L}(u_{L_1(T),\dots,L_n(T)}(l_1,\dots,l_n;t))}{\tilde{G}(u_{L_1(T),\dots,L_n(T)}(l_1,\dots,l_n;t))}. \tag{2.27}$$

It represents the relative average of the categories with the least price changes, making it sensitive to price changes in the underperforming stock segment, which gives it particular importance in analyzing defensive or low-risk financial portfolios. When modeling price movements using a multivariate distribution (such as the multiple normal distribution or copula), these three measures allow us to understand the common variance between different price differences and reflect the nature of the common distribution and the asymmetry of risk associated with stocks. This becomes increasingly important in volatile market environments, where the three indicators can be used to assess risk distribution and improve diversification decisions in investment portfolios. According to [38], these metrics can also be used in life testing and reliability.

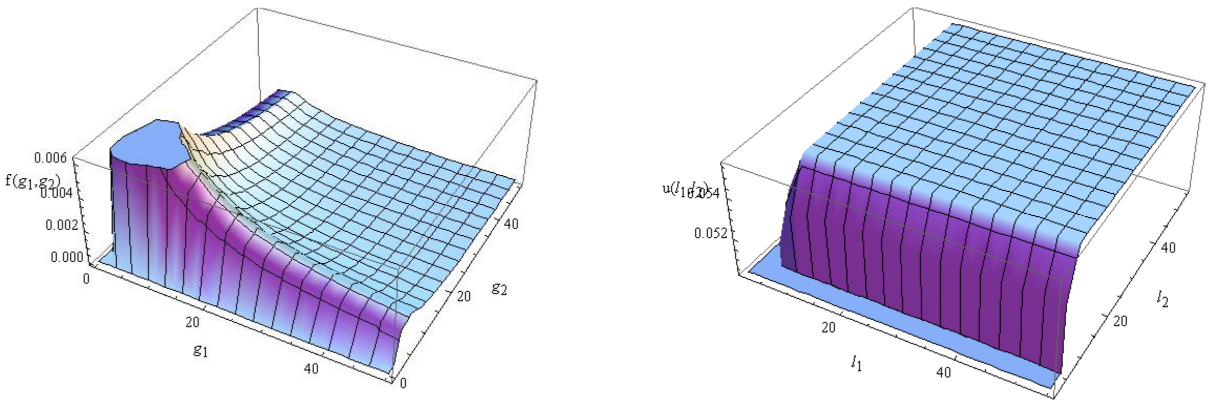
3. Bivariate distribution and its truncated version

The bivariate distribution of the difference between stock prices holds great importance in economics, especially in the areas of financial market analysis and fair pricing of derivative financial instruments. Thus, from (2.2), we can obtain the joint bivariate distribution of a Wiener range with random variables $G_1(T)$ and $G_2(T)$ given by

$$f_{G_1(T),G_2(T)}(g_1,g_2;t)=\prod_{i=1}^{2}\frac{4\sqrt{T}}{g_i^2}\sum_{k=1}^{\infty}\exp\left[-\frac{4C_kT}{g_i^2}\right],\ 0\le g_i\le\infty, i=1,2\,, \tag{3.1}$$

As appears in Figure 1(a). Also, Figure 1(b) shows the joint double truncated version of the random variables $L_1(T)$ and $L_2(T)$, which from (2.5) are given by

$$u_{L_1(T),L_2(T)}(l_1,l_2;t)=\prod_{i=1}^{2}\frac{\left|\frac{T^{-\frac{1}{2}}}{2}\right|\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{8T}{l_i^2}\right)\exp\left[-\frac{4C_kT}{l_i^2}\right]}{\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{2T}{b_i^2}\right)\exp\left[-\frac{C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{2T}{a_i^2}\right)\exp\left[-\frac{C_kT}{a_i^2}\right]},\ a_i\le l_i\le b_i, i=1,2\,. \tag{3.2}$$



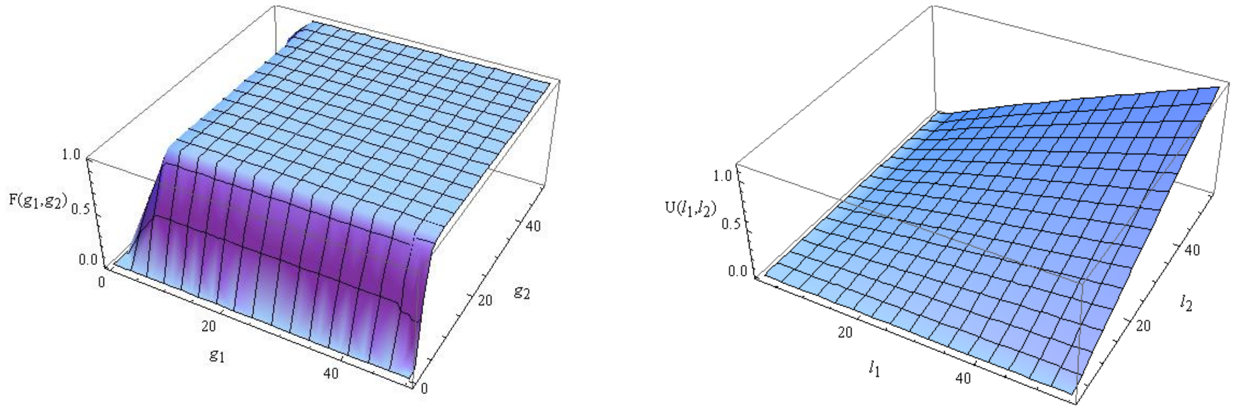
(a) $f_{G_1(T),G_2(T)}(g_1,g_2;t)$ (b) $u_{L_1(T),L_2(T)}(l_1,l_2;t)$

Figure 1. (a) The joint probability density function (3.1); and (b) the joint double truncated PDF (3.2).

For multivariate distributions of stock price spreads, the cumulative distribution function is a required tool for the investigation of the joint performance of returns. The cumulative distribution function can be used to describe the cumulative probabilities of common events, such as simultaneous decreases or increases in the prices of several financial assets. Financial dependence has been modeled, as in [39], which showed that the use of the cumulative distribution functions (especially in the context of non-normal distributions) allowed them to capture the phenomenon of dependence at extremes that cannot be captured by linear correlation coefficients. The cumulative distribution functions are applied to approximate conditional risk indexes such as the conditional value-at-risk suggested in [40], which approximates the probability of big losses in a single asset simultaneously with a reference asset. The cumulative distribution function thus provides a consistent probabilistic framework for approximating joint risk, derivative product valuation on multiple assets, and guiding optimal diversification policy in investment portfolios. Thus, from (2.3), the joint cumulative distribution function of $G_1(T)$ and $G_2(T)$ is given by

$$F_{G_1(T),G_2(T)}(g_1, g_2;t) = \prod_{i=1}^{2}\sum_{k=1}^{\infty}\left(\frac{g_i^2+8C_kT}{C_k g_i^2}\right)\exp\left[-\frac{8C_kT}{g_i^2}\right], \qquad 0 \le g_i \le \infty, i=1,2\,, \tag{3.3}$$

see Figure 2(a).



(a) $F_{G_1(T),G_2(T)}(g_1, g_2;t)$ (b) $U_{L_1(T),L_2(T)}(l_1, l_2;t)$

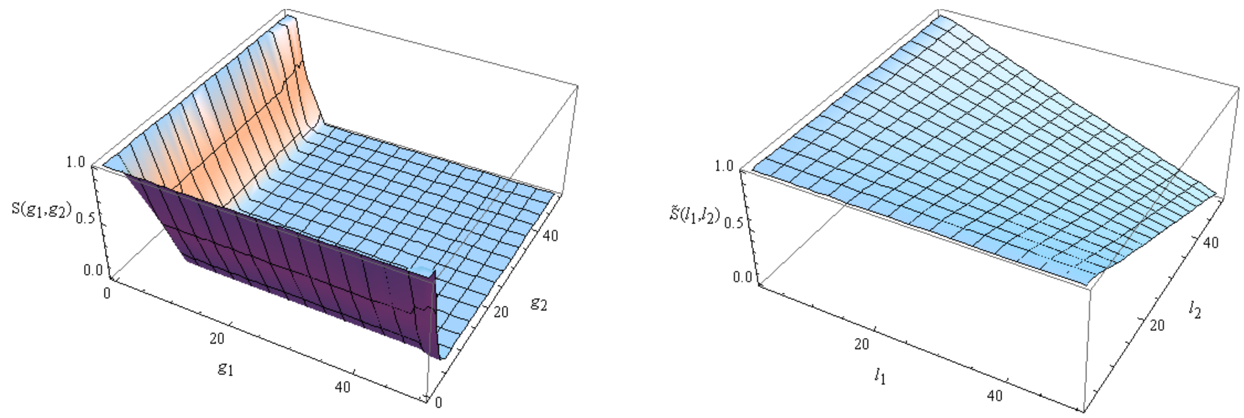
Figure 2. (a) The joint cumulative distribution function (3.3); and (b) the joint double truncated cumulative distribution function (3.4).

When stock price differences are limited to a certain range, as is the case in many realistic models that assume lower and upper bounds on price movements (due to price limits, volatility constraints, or bounded returns models), the practical and theoretical importance of the cumulative distribution function of the multivariate distribution increases. In this case, unlimited hypothetical distributions such as the normal distribution cannot be relied upon alone, making the cumulative distribution an essential tool for measuring joint probabilities within specific ranges. According to Genest and Favre [41], the use of cumulative distribution functions with distributions with finite support allows for a better description of the distribution of dependence between assets, especially when copula functions that separate margins and dependence structures within bounded ranges are used. Cherubini et al. [42] and Han and Zheng [43] also pointed out that this type of distribution is used in the evaluation of range-dependent derivatives, where the joint return is only calculated within a certain range of values, making integration over the cumulative distribution function essential for accurate pricing. In addition, predicting conditional risks and scenario probabilities within a given range requires precise tools, and the multivariate cumulative distribution function is the optimal tool for this. Thus, from (2.6) and for a bounded range $a_i < l_i < b_i, i = 1,2$, the joint double truncated cumulative distribution function is given by

$$U_{L_1(T),L_2(T)}(l_1,l_2;t)=\prod_{i=1}^{2}\left[\frac{\left|\frac{T^{\frac{-1}{2}}}{2}\right|\sum_{k=1}^{\infty}C_k^{-1}\left(-a_i\exp\left[-\frac{4C_kT}{a_i^2}\right]+l_i\exp\left[-\frac{4C_kT}{l_i^2}\right]\right)}{\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{8T}{b_i^2}\right)\exp\left[-\frac{4C_kT}{b_i^2}\right]-\sum_{k=1}^{\infty}\left(\frac{1}{C_k}+\frac{8T}{a_i^2}\right)\exp\left[-\frac{4C_kT}{a_i^2}\right]}\right],\ a_i<l_i<b_i, i=1,2\ , \quad (3.4)$$

see Figure 2(b).

As in (2.4) and (2.7), Figures 3(a),(b) show the joint survival function of the bivariate distribution of a Wiener range $S_{G_1(T),G_2(T)}(g_1,g_2;t)$ and its truncated version $\tilde{S}_{L_1(T),L_2(T)}(l_1,L_2;t)$, respectively. In financial models that assume bounded returns, the multivariate survival function gains advanced analytical importance, particularly in evaluating the probabilities associated with the right tail of a return distribution. Unlike the cumulative distribution function, which measures the probability that all variables are less than or equal to certain values, the survival function provides a computational framework for estimating the likelihood of returns exceeding certain bounds, which is crucial in risk management and extreme probability contexts. According to [8], survival functions become more accurate and appropriate when used to analyze scenarios involving "conditional exceedances", especially in portfolios containing correlated assets and affected by simultaneous movements in the upper tail. Similarly, Jaworski et al. [44] demonstrated that copula functions combined with survival functions allow tail dependence to be characterized efficiently, especially when the data are in a bounded range. In practice, in such applications as valuing barrier options or defining the probability of returns exceeding in the multi-asset portfolio, one has to employ multivariate survival functions in order to calibrate models to the actual behavior of market motion.



(a) $S_{G_1(T),G_2(T)}(g_1,g_2;t)$ (b) $\tilde{S}_{L_1(T),L_2(T)}(l_1,l_2;t)$

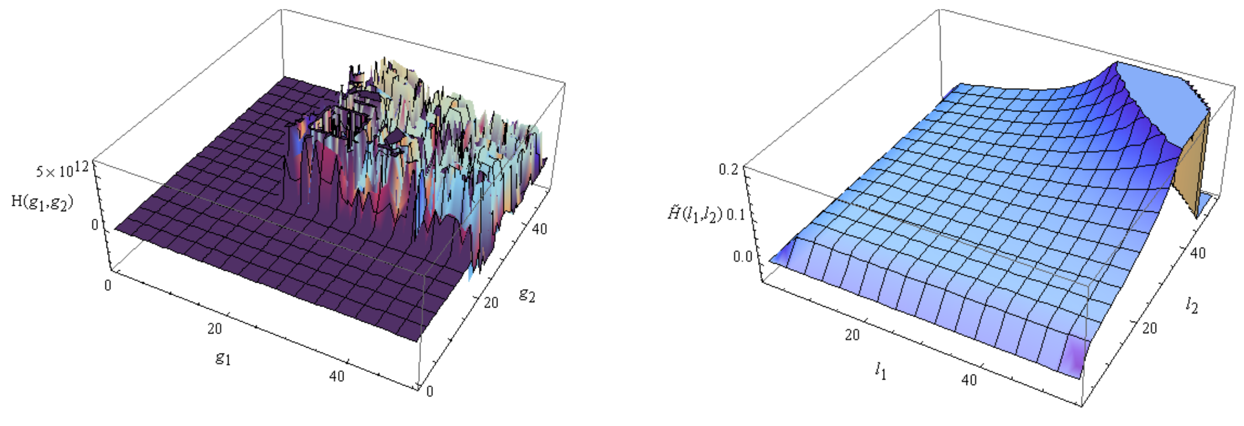
Figure 3. (a) The joint survival function $S_{G_1(T),G_2(T)}(g_1,g_2;t)$; and (b) the joint double truncated survival function $\tilde{S}_{L_1(T),L_2(T)}(l_1,l_2;t)$.

3.1. Some reliability properties

The hazard rate function is used in multivariate distributions to analyze the behavior of conditional probabilities of sudden or extreme changes in stock price spreads. It is particularly useful in predicting future risk conditional on past events, whether or not those spreads are range-bound. In the absence of a range-bound spread, the hazard rate function is used to capture the possibility of large and long-lasting market movements. It is useful for assessing the right or left tail of a distribution, supplementing measure instruments such as expected shortfall or conditional tail expectation. Therefore, the hazard rate function is used to assess systematic risk between correlated assets with unbounded return behavior and shows excellent efficiency for extreme scenario modeling and high dependencies. This function is given using (3.1) and (3.3) from the relation

$$H_{G_1(T),G_2(T)}(g_1,g_2;t)=\frac{f_{G_1(T),G_2(T)}(g_1,g_2;t)}{1-F_{G_1(T),G_2(T)}(g_1,g_2;t)} \tag{3.5}$$

as in Figure 4(a). When stock price spreads are restricted to a certain range, the hazard rate function helps in analyzing "edge risk" as it is used to estimate the probability of exceeding critical levels within a limited range. As Klein and Moeschberger [45] pointed out, the conditional hazard ratio provides accurate information about the timing of exceedances within the available range, especially in instruments that rely on predetermined scenarios such as range options or structured finance instruments. By using copula-based hazard models, as in [46], the dependence between variables can be separated from the spread margins, making estimation more flexible and realistic, both in the context of limited and unlimited price movements. Therefore, in the case of the truncated version of the bivariate Wiener range distribution, the joint hazard rate function is given, using (3.2) and (3.4), by $\tilde{H}_{L_1(T),L_2(T)}(l_1,l_2;t)=\dfrac{u_{L_1(T),L_2(T)}(l_1,l_2;t)}{1-U_{L_1(T),L_2(T)}(l_1,l_2;t)}$, see Figure 4(b).



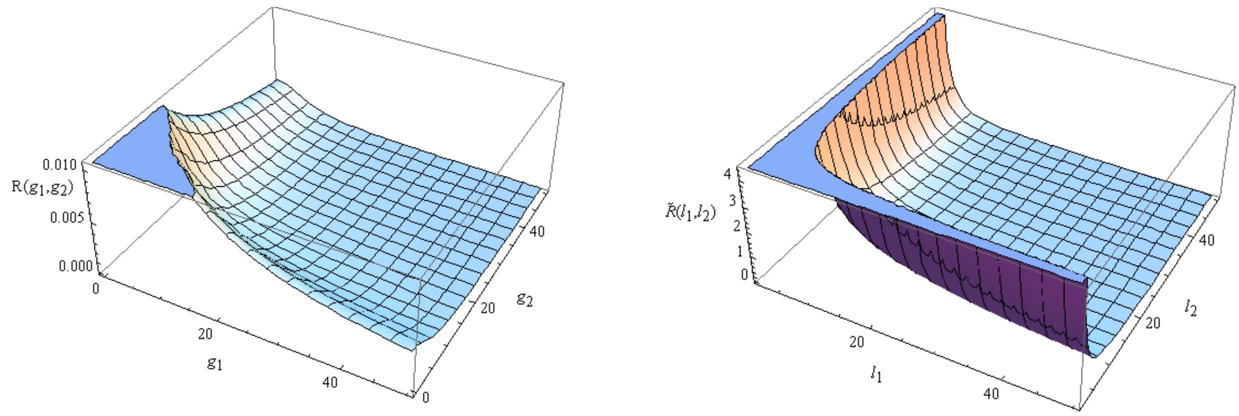
(a) $H_{G_1(T),G_2(T)}(g_1,g_2;t)$ (b) $\tilde{H}_{L_1(T),L_2(T)}(l_1,l_2;t)$

Figure 4. (a) The joint hazard rate function $H_{G_1(T),G_2(T)}(g_1,g_2;t)$; and (b) the joint double truncated hazard rate function $\tilde{H}_{L_1(T),L_2(T)}(l_1,l_2;t)$.

The reversed hazard rate function is a powerful analytical tool to examine multivariate probability distributions, particularly for stock price spreads, since it can measure the conditional probabilities of a price shift as it approaches a certain point from the left (i.e., before that value). Its use is in predicting the risk of premature price reversal or the likelihood of an abrupt flip before crossing a certain threshold. In the case of unlimited ranges of stock prices, the reversed hazard rate function is useful in tackling the probability of evading or avoiding sudden falls through examining the probability of remaining beneath a certain threshold despite the infinite range. Using this function allows us to model the regressive behavior of data and provides additional information to the standard hazard ratio, especially for heavy tail data, e.g., financial markets. Using (3.1) and (3.3), we get the reversed hazard rate function:

$$R_{G_1(T),G_2(T)}(g_1,g_2;t)=\frac{f_{G_1(T),G_2(T)}(g_1,g_2;t)}{F_{G_1(T),G_2(T)}(g_1,g_2;t)}, \tag{3.6}$$

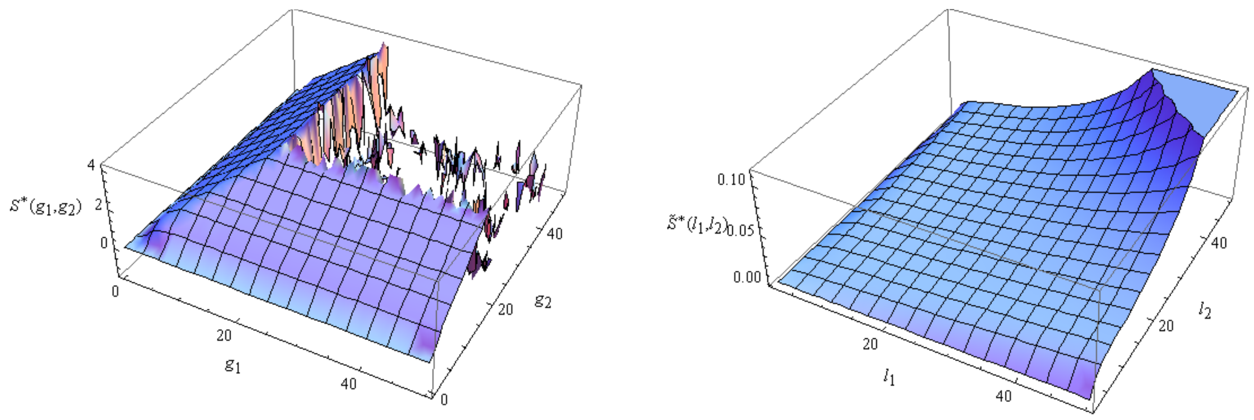
see Figure 5(a). In the case of range-bounded stock price spreads, the reversed hazard rate function provides an accurate measure of the probability of the price stabilizing below the upper bound of the range and is used to evaluate hedging strategies and control assets with expected upper bounds. This function is also crucial in determining the relation of variables as they approach the boundary, especially with models using copulas as joint distributions. In addition, it is also used in the examination of financial instruments such as capped options and reverse barrier options, whose outcome is subject to remaining under a specified level. Thus, this function is given from (3.2) and (3.4) by $\tilde{R}_{L_1(T),L_2(T)}(l_1,l_2;t)=\frac{u_{L_1(T),L_2(T)}(l_1,l_2;t)}{U_{L_1(T),L_2(T)}(l_1,l_2;t)}$, see Figure 5(b).



(a) $R_{G_1(T),G_2(T)}(g_1,g_2;t)$ (b) $\tilde{R}_{L_1(T),L_2(T)}(l_1,l_2;t)$

Figure 5. (a) The joint reversed hazard rate function $R_{G_1(T),G_2(T)}(g_1,g_2;t)$; and (b) the joint double truncated reversed hazard rate function $\tilde{R}_{L_1(T),L_2(T)}(l_1,l_2;t)$.

The second failure rate function is an advanced analytical tool in time-to-event statistical analysis and reliability theory. It is used to model the development of the failure rate or the risk itself over time or variables. For the multivariate distribution of stock price spreads, this function is necessary because it can estimate the probabilistic momentum of risk, i.e., the acceleration (or deceleration) of the probability of simultaneous changes in two or more stocks. When stock price spreads are not bounded by a specific range, the second failure rate function, given by $S^*_{G_1(T),G_2(T),\dots,G_n(T)}(g_1,g_2,\dots,g_n;t)$ $=\prod_{i=1}^{n}\log\left\{\frac{S_{G_i(T)}(g_i;t)}{S_{G_i(T)}(g_i+1;t)}\right\}$, is used to capture the behavior of increasing or decreasing risk at the tails of the distribution, as shown in Figure 6(a). This is particularly true during large and unexpected market movements, which are common in heavy-tailed models. This function enables us to distinguish between distributions with increasing or decreasing risk bias and helps us better assess the likelihood of simultaneous crashes or rare, high-impact events in financial markets. On the other hand, in the case of range-bound spreads, the second failure rate, given by the relation $\tilde{S}^*_{L_1(T),L_2(T),...,L_n(T)}(l_1,l_2,...,l_n;t)$ $=\prod_{i=1}^{n}\log\left\{\frac{\tilde{S}_{L_i(T)}(l_i;t)}{\tilde{S}_{L_i(T)}(l_i+1;t)}\right\}$, provides precise information about how risk changes near the boundary of the range, as in Figure 6(b), such as when the price approaches a known upper or lower bound. This is particularly useful in the pricing of derivatives with constraints, e.g., capped options or range accrual notes. The use of the second failure rate in multivariate distributions with bounded support therefore enables us to model the boundary behavior of the data and their co-dependence, especially when combined with copula models or conditional distributions.



(a) $S^*_{G_1(T),G_2(T),...,G_n(T)}(g_1,g_2,...,g_n;t)$ (b) $\tilde{S}^*_{L_1(T),L_2(T),...,L_n(T)}(l_1,l_2,...,l_n;t)$

Figure 6. (a) The joint second rate of failure function $S^*_{G_1(T),G_2(T),...,G_n(T)}(g_1,g_2,...,g_n;t)$; and (b) the joint double truncated second rate of failure function $\tilde{S}^*_{L_1(T),L_2(T),...,L_n(T)}(l_1,l_2,...,l_n;t)$.

4. Simulation and numerical validation with financial market data

To implement the proposed model based on the Wiener process range distribution in a continuous format, real financial data for Apple (AAPL) and Microsoft (MSFT) stocks during July 2025 were used. A real dataset of daily high–low price ranges was obtained from open financial data sources (e.g., NASDAQ via the Alpha Vantage API, https://www.alphavantage.co/). Moreover, the annual fluctuations in the prices of these stocks are clearly illustrated in Figure 7, which demonstrates the variability of their high–low ranges over the full year. Each trading day's range was determined using the difference between the high and low registered prices, as described in Table 1. Based on this data, the average daily range for AAPL was approximately \$3.4 with a standard deviation of approximately \$1.3, while the average daily range for MSFT was approximately \$6.7 with a standard deviation of \$3.8. An exceptional spike was observed on July 31, when the range exceeded \$23 due to a wide price fluctuation. Descriptive statistics showed that the range distributions were moderately skewed and flattened for AAPL, compared to being positively skewed and relatively high skewed for MSFT, indicating a higher concentration of large fluctuations in the latter.

Since the underlying price process evolves continuously and the theoretical Wiener range distribution is derived in continuous time, the nature of these data is directly consistent with the model's assumptions. Prior to calibration, the series were cleaned of missing values and any corporate events such as splits or dividends were treated in order to maintain consistency in the data. Calibration was performed by providing the volatility parameter σ for every stock. For AAPL, an estimate of $\hat{\sigma} \approx 0.018$ proved to be a good fit in simulating the empirical range distribution, while for MSFT, an estimate of $\hat{\sigma} \approx 0.032$ was consistent with the amplitude of the daily ranges. The estimation was performed either by maximizing the likelihood function or by minimizing the Kolmogorov–Smirnov distance between the empirical distribution and the joint CDF (2.3).

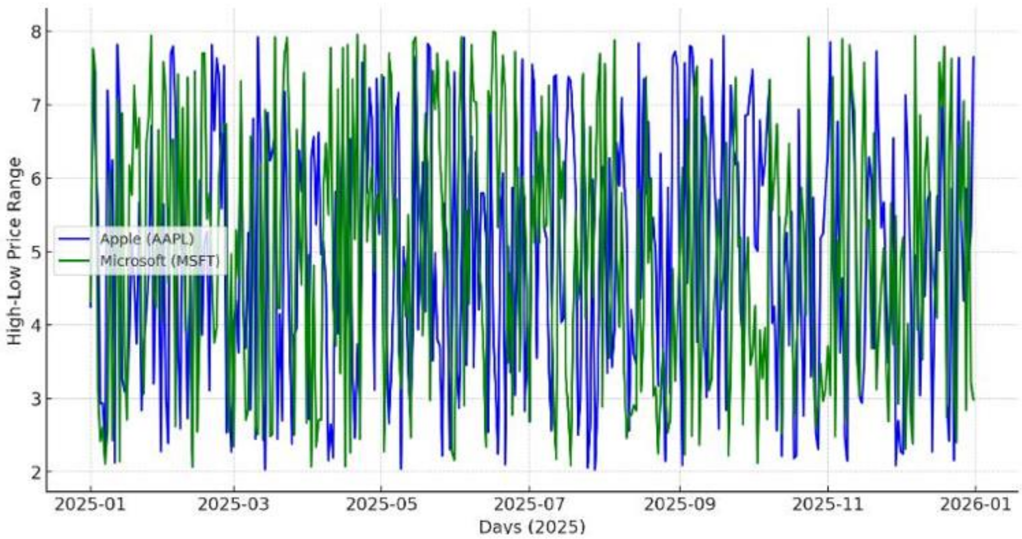


Figure 7. Fluctuations in the prices of AAPL and MSFT stocks during the year 2025 when the price range is between 2 and 8.

Table 1. Daily trading data for AAPL and MSFT with high–low price range differences.

Date	AAPL High	Low	Range	MSFT High	Low	Range	Average Daily Range	Daily Std Dev	AAPL Mean Price	MSFT Mean Price	AAPL Volatility Coefficient	MSFT Volatility Coefficient	AAPL Estimated Volatility	MSFT Estimated Volatility
7/1/2025	210.19	206.14	4.05	498.05	490.98	7.07	5.56	2.13546	208.165	494.515	0.019456	0.014297	0.010259	0.004318
7/2/2025	213.34	208.14	5.2	493.5	488.7	4.8	5	0.28284	210.74	491.1	0.024675	0.009774	0.001342	0.000576
7/3/2025	214.65	211.81	2.84	500.13	493.44	6.69	4.765	2.72236	213.23	496.785	0.013319	0.013467	0.012767	0.00548
7/7/2025	216.23	208.8	7.43	498.75	495.23	3.52	5.475	2.76478	212.515	496.99	0.034962	0.007083	0.01301	0.005563
7/8/2025	211.43	208.45	2.98	498.2	494.11	4.09	3.535	0.78488	209.94	496.155	0.014195	0.008243	0.003739	0.001582
7/9/2025	211.33	207.22	4.11	506.78	499.74	7.04	5.575	2.07182	209.275	503.26	0.019639	0.013989	0.0099	0.004117
7/10/2025	213.48	210.03	3.45	504.44	497.75	6.69	5.07	2.29102	211.755	501.095	0.016292	0.013351	0.010819	0.004572
7/11/2025	212.13	209.86	2.27	505.03	497.8	7.23	4.75	3.50725	210.995	501.415	0.010759	0.014419	0.016622	0.006995
7/14/2025	210.91	207.54	3.37	503.97	501.03	2.94	3.155	0.30405	209.225	502.5	0.016107	0.005851	0.001453	0.000605
7/15/2025	211.89	208.92	2.97	508.3	502.79	5.51	4.24	1.79605	210.405	505.545	0.014116	0.010899	0.008536	0.003553
7/16/2025	212.4	208.64	3.76	506.72	501.89	4.83	4.295	0.75660	210.52	504.305	0.017861	0.009578	0.003594	0.0015
7/17/2025	211.8	209.59	2.21	513.37	505.62	7.75	4.98	3.91737	210.695	509.495	0.010489	0.015211	0.018593	0.007689
7/18/2025	211.79	209.71	2.08	514.64	507.43	7.21	4.645	3.62745	210.75	511.035	0.00987	0.014109	0.017212	0.007098
7/21/2025	215.78	211.63	4.15	512.09	505.55	6.54	5.345	1.68998	213.705	508.82	0.019419	0.012853	0.007908	0.003321
7/22/2025	214.95	212.23	2.72	511.2	505.27	5.93	4.325	2.26981	213.59	508.235	0.012735	0.011668	0.010627	0.004466
7/23/2025	215.15	212.41	2.74	506.79	500.7	6.09	4.415	2.36880	213.78	503.745	0.012817	0.012089	0.011081	0.004702
7/24/2025	215.69	213.53	2.16	513.67	507.3	6.37	4.265	2.97692	214.61	510.485	0.010065	0.012478	0.013871	0.005832
7/25/2025	215.24	213.4	1.84	518.29	510.36	7.93	4.885	4.30628	214.32	514.325	0.008585	0.015418	0.020093	0.008373
7/28/2025	214.85	213.06	1.79	515	510.12	4.88	3.335	2.18496	213.955	512.56	0.008366	0.009521	0.010212	0.004263
7/29/2025	214.81	210.82	3.99	517.62	511.56	6.06	5.025	1.46371	212.815	514.59	0.018749	0.011776	0.006878	0.002844
7/30/2025	212.39	207.72	4.67	515.95	509.44	6.51	5.59	1.30107	210.055	512.695	0.022232	0.012698	0.006194	0.002538
7/31/2025	209.84	207.16	2.68	555.45	531.9	23.55	13.115	14.7573	208.5	543.675	0.012854	0.043316	0.070779	0.027144

The results of the goodness-of-fit tests, presented in Table 2, showed that the proposed model provided a good fit with the actual data. For AAPL, the estimated volatility coefficient was $\hat{\sigma} \approx 0.018$, and the Kolmogorov–Smirnov statistic (KS = 0.073, $p = 0.42$) and Anderson–Darling statistic (AD = 0.29, $p > 0.1$) were not significant, indicating a strong fit of the model. For MSFT, the estimated volatility coefficient was $\hat{\sigma} \approx 0.032$, with values of (KS = 0.089, $p = 0.21$) and (AD = 0.51, $p \approx 0.07$) demonstrating the model's ability to represent daily ranges, with greater sensitivity to the large fluctuations that characterize this stock. Probabilistic transformation (PIT) tests also confirmed the regularity of the time series after calibration and their closeness to the expected theoretical regularity.

Table 2. Results of the goodness-of-fit tests.

Stock	$\hat{\sigma}$	KS statistic	KS p-value	AD statistic	AD p-value	Notes
AAPL	0.018	0.073	0.42	0.29	> 0.10	Strong model fit
MSFT	0.032	0.089	0.21	0.51	≈ 0.07	Higher sensitivity to fluctuations

These results show that the model can capture daily price behavior in continuous-time markets. It works efficiently for stable shares like AAPL and is flexible enough to capture wider swings in MSFT. This provides a clear advantage over traditional models that rely on independence or non-intermittency assumptions.

5. Conclusions

A comprehensive mathematical framework was developed to model financial instruments sensitive to the range between maximum and minimum stock prices over a given interval. The model is based on the multivariate distribution of the range of independent Wiener processes, which capture stock price variations. We derived the general distribution of these variations and their truncated version from two perspectives, highlighting their effectiveness under random fluctuations. The main statistical properties of this distribution, such as reliability functions, statistical moments, stress-strength parameters, and order statistics, were analyzed, with a focus on the bivariate distribution. The truncated version of the bivariate distribution demonstrated higher accuracy, especially in highly volatile markets, enhancing its importance in risk management and derivatives pricing.

This work makes an important theoretical and practical contribution to analyzing stock price movements by their range. It also provides a foundation for developing analytical tools to improve risk management and financial forecasting. Based on these findings, future research may test the model under high-frequency trading episodes. Such episodes are marked by rapid volatility and structural complexity. Another direction is to extend the model to non-independent Wiener processes using copula functions. This extension preserves more realistic cross-sectional dependencies among asset returns. Moreover, future research may extend the model by adding jump processes or fractional Brownian motion to better capture long memory and sudden price shocks.

Author contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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