Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(9): 21916–21928.
DOI: 10.3934/math.2025976
Received: 18 May 2025
Revised: 17 August 2025
Accepted: 09 September 2025
Published: 22 September 2025

Research article

Existence and uniqueness of solutions for fractional Volterra-Fredholm equations in Banach spaces of order $\eta \in (1, 2)$

Mdi Begum Jeelani[1,*], **Farva Hafeez**[2] **and Nouf AbdulRahman Alqahtani**[1]

[1] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh

[2] Department of Mathematics and Statistics, University of Lahore, Sargodha, Pakistan

* **Correspondence:** Email: mbshaikh@imamu.edu.sa.

Abstract: The primary objective of this paper is to investigate and establish existence and uniqueness results for solutions of nonlinear Volterra-Fredholm integro-differential equations (VFIDEs) of fractional order, specifically for $1 < \eta < 2$. By leveraging fixed-point theorems and contraction mapping principles within Banach spaces, we derive comprehensive results for both one-dimensional and two-dimensional nonlinear fractional-order equations. By presenting sufficient conditions, we ensure the existence and uniqueness of a fixed point associated with the operator form of the VFIDEs. Our analysis provides a rigorous framework for understanding the behavior of such equations, and the results obtained in this study enhance our knowledge of fractional integro-differential equations (FIDEs). To illustrate the practical application of these theoretical results, two examples are provided that demonstrate the uniqueness of solutions.

Keywords: Volterra-Fredholm integro-differential equations; fixed point theorem; Riemann-Liouville derivative; Caputo derivative
Mathematics Subject Classification: 45J05, 45M10, 46E15, 47G20

1. Introduction

Fractional integro-differential equations (FIDEs) represent a powerful mathematical framework for modeling complex systems with memory effects and non-local phenomena. By combining fractional calculus and integro-differential operators, FIDEs can capture the intricate dynamics of systems in various fields. FIDEs have numerous applications across various fields, including physics, biology, engineering, finance, and materials science. In physics, Kumar et al. [1] introduced a novel Morgan-Voyce collocation technique for solving FIDEs involving Caputo and Atangana-Baleanu derivatives, demonstrating its effectiveness in modeling complex physical systems such as viscoelastic materials, oscillatory processes with memory, and nonlinear optical phenomena. In the biological sciences, Abro et al. [2] presented a detailed analysis of plasma dilution using FIDEs, demonstrating their effectiveness in modeling complex biological processes. In the field of engineering, Raghavendran et al. [3] applied artificial neural networks to investigate the existence and controllability of solutions for impulsive fractional VFIDEs, highlighting the power of hybrid computational techniques. In the context of finance, Ali et al. [4] explored stochastic fractional differential equations in a financial context, presenting a dynamic competition model solved efficiently via the pseudospectral method.

Recent advances in the study of FIDEs have significantly expanded their theoretical foundations and practical applications. Several notable works in 2024 have contributed to this progress. For instance, Sadek [5] developed advanced fractional backward differentiation methods to solve two-term fractional differential Sylvester matrix equations, enhancing computational techniques in the field. Zhai et al. [6] investigated positive solutions for a novel system of Hadamard FIDEs on infinite intervals, providing new insights into existence theory for such systems. Moreover, Obukhovskii et al. [7] explored the topological properties of solution sets for semi-linear fractional differential inclusions with non-convex right-hand sides, enriching the understanding of solution behavior in fractional differential inclusions. Additionally, Alsallami et al. [8] investigated the dynamics of fractional q-integro-differential equations with infinite time delays, which offer new mathematical perspectives on time-dependent models. Furthermore, Raghavendran et al. [9] applied artificial neural networks to study the existence and controllability of impulsive fractional VFIDEs, demonstrating their application in solving real-world complex problems.

The benefits of FIDEs lie in their ability to accurately model complex systems that traditional differential equations cannot, enabling better understanding, prediction, and decision-making. Guran et al. [10] applied fixed point results to nonlinear fractional and integral differential equations. El Ghazouani et al. [11] studied nonlinear fuzzy fractional VFIDEs and established results concerning the existence, uniqueness, and Ulam-Hyers stability of solutions. Albugami et al. [12] focused on two-dimensional fractional nonlinear Fredholm integro-differential equations, presenting computational methods and proving the existence and uniqueness of their solutions. Integro-differential equations (IDEs) are evolution equations that combine differential and integral terms, describing complex systems where the rate of change of a function depends on its past values and integrals.

Bicer et al. [13] developed a numerical method using Boole polynomials for functional IDEs with hybrid delays. Miah et al. [14] proved existence and uniqueness for two-dimensional fractional-order nonlinear IDEs with delay. Adebisi et al. [15] applied the Chebyshev least square method to solve VFIDEs. Kumar et al. [16] developed product integration techniques tailored for solving FIDEs, offering improved accuracy and stability in handling nonlocal operators. Researchers such as Alvi et al. [17] and [18], Singh et al. [19] and [20], Ramakrishnan et al. [21], and Badshah et al. [22] have made significant contributions to the development of FIDEs, advancing our understanding of their theoretical foundations, numerical solution methods, and applications.

Researchers have continued to advance the field, exploring new applications and refining existing methods, such as Ali et al. [23], who explored the dynamics of nonlocal coupled systems of fractional q-IDEs with infinite delay, and Alaofi et al. [24], who provided a comprehensive analysis on the existence and uniqueness of solutions for fractional q-IDEs. Their work has paved the way for the application of FIDEs in diverse disciplines, enabling the modeling and analysis of complex systems that exhibit non-local behavior and memory effects. Alanzi et al. [25] focused on proving the existence and uniqueness of solutions to fractional integro-stochastic differential equations, utilizing methods from fixed-point theory and stochastic processes. Gunasekar et al. [26] explored fractional VFIDEs involving state-dependent delays and established key results concerning solution existence, uniqueness, and various stability properties, such as Ulam-Hyers stability. Bekri et al. [27] studied unique solutions for Caputo-type fractional BVPs using Banach contraction. This work extends their results to more general FIDEs with broader kernel and integral conditions.

Future research is expected to focus on the potential of FIDEs in emerging areas such as quantum computing, artificial intelligence, and climate modeling. Additionally, scientists will explore novel numerical schemes and machine learning-based approaches to solve these equations, and achieve more accurate and efficient simulations. As the field continues to evolve, new applications are anticipated in areas such as personalized medicine, sustainable energy, and advanced materials, driving innovation and discovery in the years to come. The increasing relevance of fractional calculus in modeling real-world phenomena has driven the need for more advanced methods to solve nonlinear VFIDEs of fractional order. In particular, for fractional orders $1 < \eta < 2$, the complexity of these equations presents significant challenges in terms of existence and uniqueness of solutions.

This paper seeks to address these challenges by providing a comprehensive framework for proving the existence and uniqueness of solutions for both one-dimensional and two-dimensional nonlinear VFIDEs. By applying fixed-point theorems and contraction mapping principles in Banach spaces, we derive new sufficient conditions that ensure the existence of a unique solution. Our results enhance the theoretical understanding of fractional-order integro-differential equations and pave the way for their application in diverse scientific fields. The motivation for this work lies in the growing need for robust mathematical tools to model complex systems that exhibit nonlocal behavior, memory effects, and hereditary properties–features that are often captured by fractional-order models. The methods and results presented here will contribute to the broader understanding of fractional dynamics and provide a foundation for future work in both applied and theoretical contexts.

Building on previous research, this study presents novel uniqueness results for a class of Caputo FIDEs (CFIDEs), specifically the Volterra-Fredholm type, which is characterized by its distinctive integral structure:

$$ {}_0^CD_t^\eta b(z,t) = \Lambda b(z,t) + \int_p^a Q(z',t,b(z',t))\,dz' + \int_p^q Q'(z',t,b(z',t))\,dz', \tag{1.1}$$

$$ {}_0^CD_t^\eta b(z,t) = \Lambda b(z,t) + \int_p^a \int_0^t Q(z',t,b(z',t))\,dz'\,ds + \int_p^q \int_0^t Q'(z',t,b(z',t))\,dz'\,ds, \tag{1.2}$$

with initial conditions

$$ b(z,0) = b_0(z), \quad b'(z,0) - k(b) = b_m(z), \tag{1.3}$$

where ${}_0^CD_t^\eta$ be the Caputo fractional derivative (CFD) of order $1 < \eta < 2$, $t \in [0,\mathcal{T}]$, $k : C_t([p,q] \times [0,\mathcal{T}]) \to C_t([p,q] \times [0,\mathcal{T}])$ be the continuous function, and Λ be the constant coefficient.

The organization of this paper is as follows: Section 2 outlines fundamental concepts and lemmas in fractional calculus. Section 3 establishes novel results on the existence and uniqueness of solutions for a specific type of CFIDEs. In Section 4, two examples are presented to concretely demonstrate the uniqueness of solutions. Finally, Section 5 provides a summary and concluding remarks.

2. Preliminaries

Fractional derivatives and integrals have multiple mathematical definitions, with various approaches. The RL and Caputo derivatives are among the most commonly used definitions in fractional calculus. We present key definitions that will be applied throughout this paper.

Definition 2.1. *[28] For $\eta > 0$, the fractional integral is given by*

$$ {}_p\mathrm{I}_t^{\eta} b(z,t) = \frac{1}{\Gamma(\eta)} \int_p^t (t-s)^{\eta-1} b(z,s)ds, \quad \eta \in \mathbf{R}^+, $$

where $\mathbf{R}^+$ denotes the set of positive real numbers.

Definition 2.2. *[28] For a function $l(z,t)$, the CFD of order η can be expressed as*

$$ {}_pD_t^{\eta} l(z,t) = \begin{cases} \frac{1}{\Gamma(w-\eta)} \int_p^t (t-s)^{w-\eta-1} \frac{\partial^w l(z,s)}{\partial s^w} ds, & w-1 < \eta < w, \\ \frac{\partial^w l(z,t)}{\partial t^w}, & \eta = w, \ \ w \in \mathbf{N}. \end{cases} $$

In this context, η denotes the order of the derivative, potentially real or complex.

We restrict our analysis to real and positive η, leading to the properties outlined below:
(1) ${}_pL_t^{\eta}\, {}_pL_t^{u} l(z) = L^{\eta+u} l(z), \ \eta, u < 0.$
(2) ${}_pL_t^{\eta} l^{\mu}(z) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\eta+1)} l^{\mu+\eta}(z).$
(3) ${}_pD_t^{\eta} l^{\mu}(z) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\eta+1)} l^{\mu-\eta}(z), \ \eta > 0, \ \mu > -1.$
(4) ${}_pL_t^{\eta}\ {}_pD_t^{\eta} l(z) = l(z) - \sum_{x=0}^{w-1} l^{(x)}(0^+) \frac{(z-p)^x}{x!}, \ t > 0.$

Definition 2.3. *[29] A mapping $T : P \to P$ on a normed space $(P, \|.\|_\infty)$ has a fixed point $y \in P$ if $Ty = y$.*

Definition 2.4. *[30] A mapping T on a normed space $(P, \|.\|_\infty)$ is contractive if there exists a constant $v \in (0,1)$, so that*

$$ \|Ty - Th\|_\infty \le v\|y-h\|_\infty, \quad \forall\, y, h \in P. $$

Lemma 2.1. *[31] Using Definitions 2.1 and 2.2, let $b \in C_t([p,q] \times [0,\mathcal{T}])$ have continuous nth-order partial derivatives with respect to t. Then, $b(z,t)$ is the solution of* (1.1)–(1.3) *if and only if b satisfies*

$$ \begin{aligned} b(z,t) &= b_0(z) + b_m(z) + k(b) + {}_0\mathrm{I}_t^{\eta} b(z,t) + \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} \Big[\int_p^a Q(z', s, b(z',s))dz' \\ &\quad + \int_p^q Q'(z', s, b(z',s))dz' \Big] ds, \quad b \in [p,q], \quad t \in [0,\mathcal{T}]. \end{aligned} $$

Lemma 2.2. *[31] By using Definitions 2.1 and 2.2, let $b \in C_t([p,q] \times [0,\mathcal{T}])$ be a function with continuous nth-order partial derivatives in t. Then, $b(t)$ is a solution of* (1.2)-(1.3) *if and only if b satisfies*

$$ \begin{aligned} b(z,t) &= b_0(z) + b_m(z) + k(b) + {}_0\mathrm{I}_t^{\eta} b(z,t) + \frac{1}{\Gamma(\eta)} \int_0^t \int_p^a \int_0^{\zeta} (t-\zeta)^{\eta-1} Q(z', s, b(z',s))\, ds\, dz'\, d\zeta \\ &\quad + \frac{1}{\Gamma(\eta)} \int_0^t \int_p^q \int_0^{\zeta} (t-\zeta)^{\eta-1} Q'(z', s, b(z',s))\, ds\, dz'\, d\zeta. \end{aligned} $$

3. Main results

We present existence and uniqueness results for Eqs (1.1) and (1.2) with the initial condition from Eq (1.3). Our analysis relies on the following assumptions:

(R_1) Assume there exist positive constants M_1 and M_2 such that the following holds for all $b_1, b_2 \in C_t([p,q] \times [0,\mathcal{T}])$:

$$\begin{aligned} |Q(z', s, b_1(z', s)) - Q(z', s, b_2(z', s))| &\leq M_1|b_1 - b_2|, \\ |Q'(z', s, b_1(z', s)) - Q'(z', s, b_2(z', s))| &\leq M_2 b_1 - b_2|. \end{aligned}$$

(R_2) Assume that k satisfies $|k(b)| \leq K$ for some $K > 0$ and for all $b \in C_t([p,q] \times [0,\mathcal{T}])$.
(R_3) There exists a constant $\tilde{K}$ such that $|k(b_1) - k(b_2)| \leq \tilde{K}|b_1 - b_2|$ for all $b_1, b_2 \in C_t([p,q] \times [0,\mathcal{T}])$.

Remark 3.1. *The Lipschitz assumptions* (R_1)*,* (R_2)*, and* (R_3) *are standard for applying the Banach fixed point theorem and are realistic for many smooth kernels in theory and applications. However, these conditions can be restrictive when dealing with non-smooth, discontinuous, or singular kernels commonly found in real-world models. Such limitations may reduce the direct applicability of the current theorem. Extending the results to handle non-Lipschitz kernels remains a topic for future research.*

Theorem 3.1. *Suppose that assumptions* (R_1)*,* (R_2)*, and* (R_3) *hold. If*

$$\left[\tilde{K} + \frac{\mathcal{T}^\eta (q-p)(M_1 + M_2)}{\Gamma(\eta + 1)}\right] < 1,$$

then the FIDEs (1.1)–(1.3) *possess a unique solution.*

Proof. According to Lemma 2.1, the function b is the solution of (1.1)–(1.3) if and only if b satisfies the following equation:

$$\begin{aligned} b(z,t) &= b_0(z) + b_m(z) + k(b) + {}_0\mathrm{I}_t^\eta b(z,t) + \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} \Big[\int_p^a Q(z', s, b(z', s))\, dz' \\ &\quad + \int_p^q Q'(z', s, b(z', s))\, dz' \Big] ds, \end{aligned}$$

by transformation, we convert the Cauchy problem (1.1)–(1.3) into a fixed-point problem and consider the operator $\mathcal{T} : C_t([p,q] \times [0,\mathcal{T}]) \to C_t([p,q] \times [0,\mathcal{T}])$ defined as

$$\begin{aligned} (\mathcal{T} b)(z,t) &= b_0(z) + b_m(z) + k(b) + {}_0\mathrm{I}_t^\eta b(z,t) + \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} \Big[\int_p^a Q(z', s, b(z', s))\, dz' \\ &\quad + \int_p^q Q'(z', s, b(z', s))\, dz' \Big] ds. \end{aligned}$$

Thus, if b is a fixed point of $\mathcal{T}$ by Definition 2.3, then b satisfies (1.1)–(1.3). To show that $\mathcal{T}$ has a fixed point $b \in C_t([p,q] \times [0,\mathcal{T}])$, consider $b_1, b_2 \in C_t([p,q] \times [0,\mathcal{T}])$ such that

$$b_1(z,t) = b_0(z) + b_m(z) + k(b_1) + {}_0\mathrm{I}_t^\eta b(z,t) + \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} \Big[\int_p^a Q(z', s, b_1(z', s))\, dz'$$

$$+\int_p^q Q'(z',s,b_1(z',s))\,dz'\Big]ds,$$

as well as

$$\begin{aligned} b_2(z,t) &= b_0(z)+b_m(z)+k(b_2)+{}_0\mathrm{I}_t^{\eta}b(z,t)+\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}\Big[\int_p^a Q(z',s,b_2(z',s))\,dz' \\ &\quad +\int_p^q Q'(z',s,b_2(z',s))\,dz'\Big]ds. \end{aligned}$$

Consequently, we get

$$\begin{aligned} &|(\mathcal{T}b_1)(z,t)-(\mathcal{T}b_2)(z,t)| \\ \le\ & \Big|b_0(z)+b_m(z)+k(b_1)+{}_0\mathrm{I}_t^{\eta}b(z,t)+\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} \\ &\times\Big[\int_p^a Q(z',s,b_1(z',s))\,dz'+\int_p^q Q'(z',s,b_1(z',s))\,dz'\Big]ds \\ &-b_0(z)-b_m(z)-k(b_2)-{}_0\mathrm{I}_t^{\eta}b(z,t)\times\Big[\int_p^a Q(z',s,b_2(z',s))\,dz'+\int_p^q Q'(z',s,b_2(z',s))\,dz'\Big]ds\Big| \\ \le\ & |k(b_1)-k(b_2)|+\Big|\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}\Big[\int_p^a [Q(z',s,b_1(z',s))\,dz' \\ &-Q(z',s,b_2(z',s))]\,dz'+\int_p^q [Q'(z',s,b_1(z',s))-Q'(z',s,b_2(z',s))]\,dz'\Big]ds\Big| \\ \le\ & \tilde{K}|b_1-b_2|+\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}\Big[M_1|b_1(z',s)-b_2(z',s)|\,dz'+\int_p^q M_2|b_1(z',s)-b_2(z',s)|\,dz'\Big]ds \\ \le\ & \Big[\tilde{K}+\frac{(M_1+M_2)}{\Gamma(\eta)}\frac{t^{\eta}}{\eta}(q-p)\Big]|b_1-b_2| \\ \le\ & \Big[\tilde{K}+\frac{(M_1+M_2)(q-p)\mathcal{T}^{\eta}}{\Gamma(\eta+1)}\Big]|b_1-b_2|, \end{aligned}$$

which indicates that

$$\|(\mathcal{T}b_1)(z,t)-(\mathcal{T}b_2)(z,t)\|_\infty \le \Big[\tilde{K}+\frac{(M_1+M_2)(q-p)\mathcal{T}^{\eta}}{\Gamma(\eta+1)}\Big]\|b_1-b_2\|_\infty,$$

since $\frac{(M_1+M_2)(q-p)\mathcal{T}^{\eta}}{\Gamma(\eta+1)}<1$, it follows that $\mathcal{T}$ is a contractive mapping (see Definition 2.4) which implies the existence of a unique fixed point. Hence, Eq (1.1) has a unique solution. □

Remark 3.2. *The assumptions (R_1), (R_2), and (R_3) impose Lipschitz continuity conditions on the nonlinear functions Q and Q', and k, ensuring boundedness and controlled behavior of the operator $\mathcal{T}$. These conditions are essential to establish that $\mathcal{T}$ is a contraction mapping on the space $C_t([p,q]\times[0,\mathcal{T}])$. Moreover, the constraint*

$$\frac{(M_1+M_2)(q-p)\mathcal{T}^{\eta}}{\Gamma(\eta+1)}<1,$$

reflects the requirement of a sufficiently small time interval $\mathcal{T}$ to guarantee the unique solvability of the FIDEs system. This approach aligns with standard techniques in fractional differential equations *where local existence and uniqueness are established via fixed-point theory. Moreover, when Q and Q' are globally Lipschitz in their third argument, M_1 and M_2 can be estimated by bounding their partial derivatives with respect to b, making the condition practical and verifiable.*

Theorem 3.2. *Suppose that assumptions* (R_1), (R_2), *and* (R_3) *hold. If*

$$\left[\tilde{K} + \frac{\mathcal{T}^{\eta+1}(q-p)(M_1+M_2)}{\Gamma(\eta+2)}\right] < 1,$$

then the FIDEs (1.2)-(1.3) *possess a unique solution.*

Proof. According to Lemma 2.2, the function b is a solution of (1.2)-(1.3) if and only if b satisfies the following equation given below:

$$\begin{aligned} b(z,t) &= b_0(z) + b_m(z) + k(b) + {}_0\mathrm{I}_t^{\eta} b(z,t) + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^a\int_0^{\zeta}(t-\zeta)^{\eta-1}Q(z',s,b(z',s))\,ds\,dz'\,d\zeta \\ &\quad + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^q\int_0^{\zeta}(t-\zeta)^{\eta-1}Q'(z',s,b(z',s))\,ds\,dz'\,d\zeta, \end{aligned}$$

by transformation, we convert the Cauchy problem (1.2)-(1.3) into a fixed-point problem and consider the operator $\mathcal{T} : C_t([p,q]\times[0,\mathcal{T}]) \to C_t([p,q]\times[0,\mathcal{T}])$ defined as follows:

$$\begin{aligned} (\mathcal{T}b)(z,t) &= b_0(z) + b_m(z) + k(b) + {}_0\mathrm{I}_t^{\eta} b(z,t) + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^a\int_0^{\zeta}(t-\zeta)^{\eta-1}Q(z',s,b(z',s))\,dsdz'\,d\zeta \\ &\quad + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^q\int_0^{\zeta}(t-\zeta)^{\eta-1}Q'(z',s,b(z',s))\,ds\,dz'\,d\zeta. \end{aligned}$$

Thus, if b is a fixed point of $\mathcal{T}$, then, by Definition 2.3, b satisfies (1.2)-(1.3). To show that $\mathcal{T}$ has a fixed point $b \in C_t([p,q]\times[0,\mathcal{T}])$, consider $b_1, b_2 \in C_t([p,q]\times[0,\mathcal{T}])$. So we obtain

$$\begin{aligned} b_1(z,t) &= b_0(z) + b_m(z) + k(b_1) + {}_0\mathrm{I}_t^{\eta} b(z,t) + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^a\int_0^{\zeta}(t-\zeta)^{\eta-1}Q(z',s,b_1(z',s))\,ds\,dz'\,d\zeta \\ &\quad + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^q\int_0^{\zeta}(t-\zeta)^{\eta-1}Q'(z',s,b_1(z',s))\,ds\,dz'\,d\zeta, \end{aligned}$$

as well as

$$\begin{aligned} b_2(z,t) &= b_0(z) + b_m(z) + k(b_2) + {}_0\mathrm{I}_t^{\eta} b(z,t) + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^a\int_0^{\zeta}(t-\zeta)^{\eta-1}Q(z',s,b_2(z',s))\,ds\,dz'\,d\zeta \\ &\quad + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^q\int_0^{\zeta}(t-\zeta)^{\eta-1}Q'(z',s,b_2(z',s))\,ds\,dz'\,d\zeta. \end{aligned}$$

Hence, we have

$$\begin{aligned} &|(\mathcal{T}b_1)(z,t) - (\mathcal{T}b_2)(z,t)| \\ \leq\ &\Big|b_0(z) + b_m(z) + k(b_1) + {}_0\mathrm{I}_t^{\eta} b(z,t) + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^a\int_0^{\zeta}(t-\zeta)^{\eta-1}Q(z',s,b_1(z',s))\,ds\,dz'\,d\zeta \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\eta)}\int_0^t\int_p^q\int_0^\zeta (t-\zeta)^{\eta-1}Q'(z',s,b_1(z',s))\,ds\,dz'\,d\zeta - b_0(z) - b_m(z) - k(b_2) - {}_0\mathrm{I}_t^\eta b(z,t)\\
&-\frac{1}{\Gamma(\eta)}\int_0^t\int_p^q\int_0^\zeta (t-\zeta)^{\eta-1}Q(z',s,b_2(z',s))\,ds\,dz'\,d\zeta\\
&-\frac{1}{\Gamma(\eta)}\int_0^t\int_p^q\int_0^\zeta (t-\zeta)^{\eta-1}Q'(z',s,b_2(z',s))\,ds\,dz'\,d\zeta\Bigg|\\
\leq\ & |k(b_1)-k(b_2)| + \Bigg|\frac{1}{\Gamma(\eta)}\int_0^t\int_p^a\int_0^\zeta (t-\zeta)^{\eta-1}[Q(z',s,b_1(z',s)) - Q(z',s,b_2(z',s))]\,ds\,dz'\,d\zeta\\
&+\frac{1}{\Gamma(\eta)}\int_0^t\int_p^q\int_0^\zeta (t-\zeta)^{\eta-1}[Q'(z',s,b_1(z',s)) - Q'(z',s,b_2(z',s))]\,ds\,dz'\,d\zeta\Bigg|\\
\leq\ & \tilde{K}|b_1-b_2| + \frac{1}{\Gamma(\eta)}\int_0^t\int_p^a\int_0^\zeta (t-\zeta)^{\eta-1}M_1|b_1(z',s)-b_2(z',s)|\,ds\,dz'\,d\zeta\\
&+\frac{1}{\Gamma(\eta)}\int_0^t\int_p^q\int_0^\zeta (t-\zeta)^{\eta-1}M_2|b_1(z',s)-b_2(z',s)|\,ds\,dz'\,d\zeta\\
\leq\ & \left[\tilde{K} + \frac{(M_1+M_2)}{\Gamma(\eta)}\frac{t^{\eta+1}}{\eta(\eta+1)}(q-p)\right]|b_1-b_2|\\
\leq\ & \left[\tilde{K} + \frac{(M_1+M_2)(q-p)\mathcal{T}^{\eta+1}}{\Gamma(\eta+2)}\right]|b_1-b_2|,
\end{aligned}
$$

this implies that

$$
\|(\mathcal{T}b_1)(z,t) - (\mathcal{T}b_2)(z,t)\|_\infty \leq \left[\tilde{K} + \frac{(M_1+M_2)(q-p)\mathcal{T}^{\eta+1}}{\Gamma(\eta+2)}\right]\|b_1-b_2\|_\infty,
$$

since $\frac{(M_1+M_2)(q-p)\mathcal{T}^{\eta+1}}{\Gamma(\eta+2)} < 1$, it follows that $\mathcal{T}$ is a contractive mapping by Definition 2.4, which implies the existence of a unique fixed point. Hence, (1.2) has a unique solution. □

4. Examples

Example 1. Consider the FIDE with $\eta = \frac{3}{2}$:

$$
{}^{C}D^{\frac{3}{2}}b(z,t) = \Lambda b(z,t) + \int_0^t\int_p^a Q(z',s,b(z',s))\,ds\,dz' + \int_0^t\int_p^q Q'(z',s,b(z',s))\,ds\,dz',
$$

subject to the initial conditions

$$
b(z,0) = b_0(z), \quad \frac{\partial b}{\partial z}(z,0) - k(b) = b_m(z).
$$

To ensure the uniqueness of the solution, we verify the condition from Theorem 3.1

$$
\left[\tilde{K} + \frac{\mathcal{T}^{\frac{3}{2}}(q-p)(M_1+M_2)}{\Gamma(\frac{5}{2})}\right] < 1,
$$

where $M_1 = \alpha,\ M_2 = \beta$, and $\tilde{K} = \gamma$ are positive constants. Choosing $\gamma = 0.2,\ \alpha = 0.1,\ \beta = 0.1,\ \mathcal{T} = 2$, and $q - p = 1$. Using $\Gamma(\frac{5}{2}) \approx 1.33$, the inequality becomes

$$0.2 + \frac{2^{\frac{3}{2}}(1)(0.1 + 0.1)}{1.33} < 1, \quad 0.2 + 0.425 = 0.625 < 1,$$

which satisfies the uniqueness condition. Thus, by Theorem 3.1, the equation admits a unique solution.

To capture realistic phenomena, the kernels are specified as

$$Q(z', s, b) = \sin(z')b(z', s), \quad Q'(z', s, b) = e^{-z'}b(z', s),$$

modeling oscillatory and exponentially decaying memory effects, respectively. The initial conditions correspond to the value and the first time derivative of b at $t = 0$. Since the fractional derivative order is $\frac{3}{2}$, two initial conditions are necessary. Using the Taylor expansion

$$b(z, h) \approx b_0(z) + h[k(b_0(z)) + b_m(z)],$$

we obtain the starting value for the numerical scheme at the first time step h. A fractional Adams-Bashforth-Moulton predictor-corrector method is implemented to solve the equation numerically on $t \in [0, 2]$ with step size $h = 0.02$. The integral terms are approximated via quadrature rules. The iterative scheme updates the function b as

$$b_n = b_0 + \frac{h^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\sum_{j=0}^{n} a_{n-j} f(t_j, b_j),$$

where

$$f(t, b) = \gamma b + \int_0^t \int_p^a \sin(z')b(z', s)\, ds\, dz' + \int_0^t \int_p^q e^{-z'} b(z', s)\, ds\, dz'.$$

The numerical results indicate that the solution behaves regularly, exhibiting memory effects and spatial interactions consistent with the kernel structure and fractional order. Due to the fractional dynamics, the solution evolves more slowly than its integer-order counterparts while remaining bounded and stable. These results validate the theoretical findings.

Hence, the problem is well-posed with a unique solution under the chosen parameters, and the simulations confirm the model's applicability to real-world fractional systems.

Example 2. Consider the FIDE with $\eta = \frac{5}{4}$:

$${}^{C}D^{\frac{5}{4}} b(z, t) = \Lambda b(z, t) + \int_0^t \int_p^a \int_0^{\zeta} Q(z', s, b(z', s))\, ds\, dz'\, d\zeta + \int_0^t \int_p^q \int_0^{\zeta} Q'(z', s, b(z', s))\, ds\, dz'\, d\zeta,$$

subject to the initial conditions

$$b(z, 0) = b_0(z), \quad \frac{\partial b}{\partial z}(z, 0) - k(b) = b_m(z).$$

To ensure the uniqueness of the solution, we verify the condition from Theorem 3.2

$$\left[\tilde{K} + \frac{\mathcal{T}^{\frac{5}{4}+1}(q - p)(M_1 + M_2)}{\Gamma(\frac{9}{4})}\right] < 1,$$

where $M_1 = \alpha$, $M_2 = \beta$ and $\tilde{K} = \gamma$ are positive constants. Choosing $\gamma = 0.1$, $\alpha = 0.13$, $\beta = 0.12$, $\mathcal{T} = 1.5$, and $q - p = 1$. Using $\Gamma(\frac{9}{4}) \approx 1.754$, the inequality becomes

$$0.1 + \frac{1.5^{\frac{9}{4}}(1)(0.13 + 0.12)}{1.754} < 1, \quad 0.1 + 0.289 = 0.389 < 1.$$

Thus, the condition is satisfied, and by Theorem 3.2, the equation admits a unique solution. To model realistic effects, we choose the kernels as

$$Q(z', s, b) = \cos(z')e^{-s}b(z', s), \quad Q'(z', s, b) = \frac{s}{1 + s^2}b(z', s).$$

The kernel Q represents damped spatial oscillations, while Q' reflects saturating memory growth behavior, often observed in viscoelastic systems. Since $\eta = \frac{5}{4} \in (1, 2)$, two initial conditions are required. The Taylor expansion provides the first time-step approximation

$$b(z, h) \approx b_0(z) + h[k(b_0(z)) + b_m(z)].$$

A fractional Adams-Bashforth-Moulton predictor-corrector scheme is used on $t \in [0, 1.5]$ with step size $h = 0.015$. The integrals are approximated via numerical quadrature. The scheme updates the solution using

$$b_n = b_0 + \frac{h^{\frac{5}{4}}}{\Gamma(\frac{9}{4})} \sum_{j=0}^{n} a_{n-j} f(t_j, b_j),$$

where

$$f(t, b) = \gamma b + \int_0^t \int_p^q \int_0^\zeta \cos(z')e^{-s}b(z', s)\, ds\, dz'\, d\zeta + \int_0^t \int_p^q \int_0^\zeta \frac{s}{1 + s^2}b(z', s)\, ds\, dz'\, d\zeta.$$

Numerical simulations reveal a slowly evolving solution with visible memory effects, especially due to the fractional order and nested integrals.

These observations validate the theoretical findings and show that the problem is well-posed under the chosen parameters. The model is suitable for capturing complex fractional dynamics in real-world applications.

5. Conclusions

This study's primary objective was to establish novel existence and uniqueness criteria for solutions to Caputo fractional VFIDEs, thereby advancing our understanding of these complex mathematical models. By leveraging the fixed point theorem in Banach spaces, contraction mapping principles, and exploring the intricacies of fractional calculus within the specified order of $1 < \eta < 2$, we derived significant results that shed new light on the behavior of such equations in one- and two-dimensional spaces. These findings not only contribute to the theoretical foundations of fractional differential equations but also pave the way for future research in this domain, enabling the development of innovative solutions to real-world problems. The methodologies employed in this paper demonstrate the potential for further exploration and application in solving complex problems involving FIDEs, with potential implications for fields such as physics, engineering, and economics. Furthermore, the results obtained in this study lay the groundwork for future investigations into the properties and applications of Caputo fractional VFIDEs, underscoring the value of continued research in this area.

Author contributions

Mdi Begum Jeelani: Conceptualization, Methodology, Formal analysis, Writing–original draft preparation, Supervision, Project administration; Farva Hafeez: Investigation, Validation, Data curation, Writing–original draft preparation, Writing–review and editing; Nouf AbdulRahman Alqahtani: Resources, Visualization, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2502).

Conflict of interest

The authors declare no conflicts of interest.

References

1. S. Kumar, A. Chaudhary, S. Malik, V. Singh, A new Morgan-Voyce collocation technique and its applications for solving fractional integro-differential equations, *Comput. Appl. Math.*, **44** (2025), 369. https://doi.org/10.1007/s40314-025-03325-5
2. K. A. Abro, A. Atangana, J. F. Gomez-Aguilar, A comparative analysis of plasma dilution based on fractional integro-differential equation: an application to biological science, *Int. J. Model. Simul.*, **43** (2023), 1–10. https://doi.org/10.1080/02286203.2021.2015818
3. P. Raghavendran, T. Gunasekar, S. Gochhait, Application of artificial neural networks for existence and controllability in impulsive fractional Volterra-Fredholm integro-differential equations, *Appl. Math. Sci. Eng.*, **32** (2024), 2436440. https://doi.org/10.1080/27690911.2024.2436440
4. I. Ali, S. U. Khan, A dynamic competition analysis of stochastic fractional differential equation arising in finance via pseudospectral method, *Mathematics*, **11** (2023), 1–16. https://doi.org/10.3390/math11061328
5. L. Sadek, The methods of fractional backward differentiation formulas for solving two-term fractional differential Sylvester matrix equations, *Appl. Set Valued Anal. Optim.*, **6** (2024), 137–155. https://doi.org/10.23952/asvao.6.2024.2.02
6. C. Zhai, L. Bai, Positive solutions for a new system of Hadamard fractional integro-differential equations on an infinite interval, *J. Nonlinear Funct. Anal.*, **2024** (2024), 27. https://doi.org/10.23952/jnfa.2024.27

7. V. Obukhovskii, G. Petrosyan, M. Soroka, J. C. Yao, On topological properties of solution sets of semilinear fractional differential inclusions with non-convex right-hand side, *J. Nonlinear Var. Anal.*, **8** (2024), 95–108. https://doi.org/10.23952/jnva.8.2024.1.05

8. S. A. M. Alsallami, K. R. Raslan, E. M. Khalil, S. Abdel-Khalek, A. A. E. Ibrahim, K. K. Ali, Exploring the dynamics of fractional q-integro-differential equations with infinite time delays: a study in mathematical analysis, *J. Math.*, **2024** (2024), 3381147. https://doi.org/10.1155/2024/3381147

9. P. Raghavendran, T. Gunasekar, S. Gochhait, Application of artificial neural networks for existence and controllability in impulsive fractional Volterra-Fredholm integro-differential equations, *Appl. Math. Sci. Eng.*, **32** (2024), 2436440. https://doi.org/10.1080/27690911.2024.2436440

10. L. Guran, Z. D. Mitrovic, G. S. M. Reddy, A. Belhenniche, S. Radenovic, Applications of a fixed point result for solving nonlinear fractional and integral differential equations, *Fractal Fract.*, **5** (2021), 211. https://doi.org/10.3390/fractalfract5040211

11. A. El Ghazouani, M. Elomari, S. Melliani, Existence, uniqueness, and UH-stability results for nonlinear fuzzy fractional Volterra-Fredholm integro-differential equations, *J. Nonlinear Complex Data Sci.*, **25** (2025), 457–477. https://doi.org/10.1515/jncds-2024-0019

12. A. Albugami, N. A. Alharbi, A. M. S. Mahdy, Computational methods, existence and uniqueness for solving 2-D fractional nonlinear Fredholm integro-differential equation, *Eur. J. Pure Appl. Math.*, **18** (2025), 5924. https://doi.org/10.29020/nybg.ejpam.v18i2.5924

13. K. E. Bicer, H. G. Dag, Numerical method based on boole polynomial for solution of general functional integro-differential equations with hybrid delays, *TWMS J. Appl. Eng. Math.*, **14** (2024), 1264–1277.

14. B. A. Miah, M. Sen, R. Murugan, D. Gupta, Existence and uniqueness result for two-dimensional fraction order nonlinear IDE with delay and its numerical investigation, *Int. J. Appl. Comput. Math.*, **11** (2025), 1–20. https://doi.org/10.1007/s40819-025-01886-x

15. A. F. Adebisi, K. A. Okunola, Solution of Volterra-Fredholm integro-differential equations using Chebyshev least square method, *Sci. World J.*, **20** (2025), 271–275. https://doi.org/10.4314/swj.v20i1.36

16. S. Kumar, P. Yadav, V. K. Singh, Product integration techniques for fractional integro-differential equations, *Math. Methods Appl. Sci.*, **48** (2025), 2833–2858. https://doi.org/10.1002/mma.10464

17. J. Alavi, H. Aminikhah, An efficient parametric finite difference and orthogonal spline approximation for solving the weakly singular nonlinear time-fractional partial integro-differential equation, *Comput. Appl. Math.*, **42** (2023), 350. https://doi.org/10.1007/s40314-023-02491-8

18. J. Alavi, H. Aminikhah, Orthogonal cubic spline basis and its applications to a partial integro-differential equation with a weakly singular kernel, *Comput. Appl. Math.*, **40** (2021), 55. https://doi.org/10.1007/s40314-021-01442-5

19. A. Singh, N. Srivastava, Y. Kumar, V. K. Singh, Computational approach for two-dimensional fractional integro-differential equations, *Int. J. Appl. Comput. Math.*, **10** (2024), 155. https://doi.org/10.1007/s40819-024-01785-7

20. A. Singh, N. Srivastava, S. Singh, V. K. Singh, Computational technique for multi-dimensional non-linear weakly singular fractional integro-differential equation, *Chin. J. Phys.*, **80** (2022), 305–333. https://doi.org/10.1016/j.cjph.2022.04.015

21. S. Ramakrishnan, A. K. Singh, Stochastic dynamics of a nonlinear vibration energy harvester subjected to a combined parametric and external random excitation: the distinct cases of Itô and Stratonovich stochastic integration, *Int. J. Non-Linear Mech.*, **162** (2024), 104700. https://doi.org/10.1016/j.ijnonlinmec.2024.104700

22. F. Badshah, K. U. Tariq, A. Bekir, R. N. Tufail, H. Ilyas, Lump, periodic, travelling, semi-analytical solutions and stability analysis for the Ito integro-differential equation arising in shallow water waves, *Chaos Solitons Fract.*, **182** (2024), 114783. https://doi.org/10.1016/j.chaos.2024.114783

23. K. K. Ali, K. R. Raslan, A. A. E. Ibrahim, M. S. Mohammad, Exploring the dynamics of nonlocal coupled systems of fractional q-integro-differential equations with infinite delay, *Heliyon*, **10** (2024), e33399. https://doi.org/10.1016/j.heliyon.2024.e33399

24. Z. M. Alaofi, K. R. Raslan, A. A. E. Ibrahim, K. K. Ali, Comprehensive analysis on the existence and uniqueness of solutions for fractional q-integro-differential equations, *J. Supercomput.*, **80** (2024), 23848–23866. https://doi.org/10.1007/s11227-024-06305-4

25. A. R. A. Alanzi, S. S. Alshqaq, R. Fakhfakh, A. B. Makhlouf, Existence and uniqueness results for a class of fractional integro-stochastic differential equations, *Fractal Fract.*, **9** (2025), 1–14. https://doi.org/10.3390/fractalfract9010042

26. T. Gunasekar, P. Raghavendran, K. S. Nisar, Existence, uniqueness, and stability results of fractional Volterra-Fredholm integro-differential equations with state-dependent delay, *Qual. Theory Dyn. Syst.*, **24** (2025), 54. https://doi.org/10.1007/s12346-024-01185-8

27. Z. Bekri, S. Aljohani, M. E. Samei, A. Akgul, A. Belhenniche, A. Aloqaily, et al., Unique solution analysis for generalized Caputo-type fractional BVP via Banach contraction, *Eur. J. Pure Appl. Math.*, **18** (2025), 5979. https://doi.org/10.29020/nybg.ejpam.v18i2.5979

28. A. A. Kilbas, H. M. Srivastava,, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

29. Q. H. Ansari, D. R. Sahu, *Fixed point theory and variational principles in metric spaces*, Cambridge University Press, 2023. https://doi.org/10.1017/9781009351430

30. X. W. Su, L. D. Liu, Existence of solution for boundary value problem of nonlinear fractional differential equation, *Appl. Math. J. Chin. Univ.*, **22** (2007), 291–298. https://doi.org/10.1007/s11766-007-0306-2

31. K. H. Hussain, A. A. Hamoud, N. M. Mohammed, Some new uniqueness results for fractional integro-differential equations, *Nonlinear Funct. Anal. Appl.*, **24** (2019), 827–836.

AIMS Press