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Research article

Local max-norm regularity estimates for gradient solutions of variational inequalities involving a class of doubly degenerate parabolic operators arising from quanto option valuation analysis

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Abstract: This paper investigates a class of variational inequality problems governed by double-phase degenerate parabolic operators on cylindrical domains within bounded open sets, which arises from sensitivity analysis of quanto option valuation. The main result establishes max-norm regularity estimates for gradient solutions of the variational inequality.

Keywords: initial-boundary value problem for variational inequality; gradient solution; boundedness in the sup-norm; max-norm regularity estimate
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1. Introduction and application of variational inequalities in finance

Variational inequalities have demonstrated significant utility in the field of option pricing theory. Consider a Chinese investor purchasing an American-style call option in European markets with a strike price of K euros. The underlying European stock price follows the stochastic process:

$$\mathrm{d}S_1(t) = \mu_1 S_1(t)\mathrm{d}t + \sigma_1 S_1(t)\mathrm{d}W_1(t).$$

Furthermore, the Euro-to-Chinese yuan (CNY) exchange rate, denoted by $S_2(t)$, is governed by the dynamics:

$$\mathrm{d}S_2(t) = \mu_2 S_2(t)\mathrm{d}t + \sigma_2 S_2(t)\mathrm{d}W_2(t).$$

Upon exercising the option, the investor may choose to either: 1) Convert CNY to euros at the prevailing exchange rate $S_2(t)$, or 2) utilize a guaranteed exchange rate S_2^0 for the conversion. Consequently, the payoff of the American quanto option $V(t, S_1, S_2)$ at maturity T is given by:

$$f(S_1, S_2) = \max(S_2^0, S_2) \cdot \max\{S_1 - K, 0\}.$$

According to the literature [1–3], the value of the American-style quanto option at time t, denoted as $V(t, x_1, x_2)$, satisfies the following partial differential equation:

$$\begin{cases} \max\{L_0 V, f(e^{x_1}, e^{x_2}) - V\} = 0 \text{ in } \Omega \times (0, T), \\ V(x_1, x_2, T) = f(e^{x_1}, e^{x_2}) \text{ in } \Omega, \\ V(x_1, x_2, t) = 0 \text{ in } \partial\Omega \times (0, T), \end{cases} \tag{1}$$

where

$$L_0 V = D_t V + \frac{1}{2}\sigma_1^2 D^2_{x_1 x_1} V + \frac{1}{2}\sigma_2^2 D^2_{x_2 x_2} V + (r - \frac{1}{2}\sigma_1^2) D_{x_1} V + (r - \frac{1}{2}\sigma_2^2) D_{x_2} V - rV. \tag{2}$$

Here, $x_1 = \ln S_1$ and $x_2 = \ln S_2$. Clearly, in this case, Ω is a bounded open region on R^2_+. σ_1 represents the volatility of the European stock underlying the American option, σ_2 denotes the volatility of the EUR/CNY exchange rate, and r is the market's risk-free interest rate. The boundary condition $V(x_1, x_2, t) = 0 \text{ in } \partial\Omega \times (0, T)$ is designed to prevent excessive losses for both the investor and the option issuer due to excessively high or low prices of the underlying asset. The structure of the variational inequality (1) gives rise to the research problem investigated in this paper, which also serves as the key motivation for our study.

This paper investigates a class of variational inequality problems arising from credit risk bond valuation, formulated within the framework of a double-phase degenerate parabolic operator:

$$\begin{cases} \max\{L\omega, \omega_0 - \omega\} = 0 \text{ in } \Omega_T, \\ \omega(\,\cdot\,, 0) = \omega_0 \text{ in } \Omega, \\ \omega = 0 \text{ in } \partial\Omega \times (0, T). \end{cases} \tag{3}$$

Here, $L\omega$ represents a double-phase degenerate parabolic operator,

$$L\omega = \partial_t \omega - \sum_{i=1}^{N} [D_i(|D_i\omega|^{p-2} D_i\omega) + D_i(|D_i\omega|^{q-2} D_i\omega)] + r\omega^h \tag{4}$$

where p and q are constants that satisfy conditions $p \geq 2$ and $q \geq 2$. The power-type nonlinear term $r\omega^h$ fulfills conditions $h > 1$ and $r \geq 0$. Ω is a bounded connected open domain on R_N with smooth boundary $\partial\Omega$, and (x, t) denotes a point on the cylindrical domain $\Omega_T = \Omega \times (0, T)$ (with Ω being the base of the cylinder). The term $D_i\omega = D_{x_i}\omega$ corresponds to the first-order partial derivative of ω with respect to x_i , $D^2_{i,j} = D^2_{x_i x_j}$. Throughout this work, we assume the initial condition ω_0 satisfies:

$$\omega_0 \in W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^h(\Omega). \tag{5}$$

The notation $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$.

In recent years, the existence of solutions to variational inequalities of the form (1) has been extensively and actively studied [4, 5], which also serves as the foundation for the regularity analysis in this paper. The literature [6, 7] investigated a class of variational inequality problems (obstacle problems) formulated within an elliptic operator framework, where the associated differential operators are of p-Laplacian type. Specifically, reference [6] analyzed the global integrability of gradient solutions, while [7] established local L^p estimates for such solutions. Reference [8] examined the local L^p estimates for gradient solutions of variational inequalities under a parabolic Kirchhoff-type operator framework. By employing the Caffarelli–Kohn–Nirenberg inequality, Hölder's inequality, and Young's inequality, the authors derived an L^p-norm-based upper bound for the L^p norm of gradient solutions.

Due to the inequality constraints in variational inequalities, scholars have found it challenging to derive suitable integral inequalities for gradient solutions, which has hindered the development of gradient estimation in variational inequalities. Consequently, research on gradient estimation of solutions has predominantly focused on parabolic initial-boundary value problems [9–11]. Reference [9] investigated the influence of the smoothness of initial values and source terms on the global regularity of $p(x,t)$-Laplacian type parabolic equations. Reference [10] analyzed nonlinear Calderón–Zygmund type estimates for the spatial gradient of mixed local and nonlocal parabolic problems with measure data, presenting several gradient estimates concerning the L^1 norm. Meanwhile, Reference [11] obtained global gradient estimates for positive solutions of nonlinear parabolic equations under the Bakry–Émery Ricci curvature condition. In recent years, some scholars have explored the regularity of solutions to initial-boundary value problems governed by doubly degenerate parabolic operators [12, 13]. The structure of such operators is inherently complex, often leading to energy functions for gradient solutions with different orders when constructing energy inequalities. This fact has posed significant challenges to the study of gradient solution regularity.

This paper investigates the local max-norm regularity estimates for gradient solutions of variational inequalities governed by a class of doubly degenerate parabolic operators arising from quanto option valuation analysis. In Section 2, we present several useful lemmas that facilitate the proof of our main results, followed by a statement of the key theorems of this work. Section 3 establishes an energy inequality over Ω_T and its extension to local cylindrical domains. Our energy inequality incorporates the energy functional of gradient solutions induced by the doubly degenerate parabolic operator. Finally, by combining this energy inequality with auxiliary lemmas derived from existing literature, we obtain the max-norm regularity estimates for the spatial gradient.

2. Preliminaries and main results

The following lemma on rapid geometric convergence assists us in analyzing the boundedness and estimation of the infinity norm of the gradient solution [14].

Lemma 2.1. *Let the sequence $\{Z_n, n = 0, 1, 2, \cdots\}$ satisfy $Z_{n+1} \leq C b^n Z_n^{1+l}$ for non-negative constants C, b, and l. If $Z_0 \leq C^{-1/l} b^{-1/l^2}$ holds, then $Z_n \to 0$ as $n \to \infty$.*

Lemma 2.2. *Suppose a sequence $\{Z_n, n = 0, 1, 2, \cdots\}$ satisfies $Z_{n+1} \leq C b^n Z_n^{1+l}$ with non-negative constants C, b, and l. Then $l \in (0,\ 1)$ implies $Z_0 \leq C^{-1/l} b^{-1/l^2}$.*

The following Caffarelli-Kohn-Nirenberg inequality plays a crucial role in the proof, and readers may refer to [14] for further details.

Lemma 2.3. *(Caffarelli-Kohn-Nirenberg inequality) For any $\psi \in L^p(\Omega_T) \cap L^q(\Omega)$, there exists a non-negative constant C_{C-K-N}, which depends only on N and p, such that*

$$\int\int_{\Omega_T} |\psi|^{p\frac{(N+q)}{N}} \mathrm{d}x\mathrm{d}t \leq C_{C-K-N} \left(\int\int_{\Omega_T} |\nabla\psi|^p \mathrm{d}x\mathrm{d}t \right) \left(\operatorname*{esssup}_{t\in(0,T)} \int_{\Omega} |\psi|^q \mathrm{d}x \right)^{\frac{p}{N}}.$$

It is worth noting that in Lemma 2.3, the parameters p and q are only required to satisfy $p \geq 1$ and $q \geq 1$.

Regarding generalized solutions for (3), see [8] (details omitted here). Following established results, we list key a priori estimates for ω and $D_i\omega$. Observing that $\omega \geq \omega_0 \geq 0$ in Ω_T, we integrate the inequality $\omega L\omega \leq 0$ over $\Omega \times (0,t)$ to obtain

$$\begin{aligned}&\int_{\Omega\times(0,t)} \partial_\tau\omega \cdot \omega \mathrm{d}x\mathrm{d}\tau + \sum_{i=1}^{N} \int_{\Omega\times(0,t)} |D_i\omega|^p \mathrm{d}x\mathrm{d}\tau + \sum_{i=1}^{N} \int_{\Omega\times(0,t)} |D_i\omega|^q \mathrm{d}x\mathrm{d}\tau + r\int_{\Omega\times(0,t)} \omega^{h+1}\mathrm{d}x\mathrm{d}\tau \\ &\leq \sum_{i=1}^{N} \int_{\partial\Omega\times(0,t)} |D_i\pi|^{p-2}D_i\pi \cdot \pi\cos(x_i,\vec{\upsilon})\mathrm{d}S\,\mathrm{d}\tau + \sum_{i=1}^{N} \int_{\partial\Omega\times(0,t)} |D_i\pi|^{q-2}D_i\pi \cdot \pi\cos(x_i,\vec{\upsilon})\mathrm{d}S\,\mathrm{d}\tau.\end{aligned}$$

By the boundary condition $\omega = 0$ in $\partial\Omega \times (0,T)$ and $\int_{\Omega\times(0,t)} \partial_\tau\omega \cdot \omega \mathrm{d}x\mathrm{d}\tau = \frac{1}{2}\int_\Omega \omega^2 \mathrm{d}x - \frac{1}{2}\int_\Omega \omega_0^2 \mathrm{d}x$, we have

$$\frac{1}{2}\int_\Omega \omega^2\mathrm{d}x + \sum_{i=1}^{N} \int_{\Omega\times(0,t)} |D_i\omega|^p \mathrm{d}x\mathrm{d}\tau + \sum_{i=1}^{N} \int_{\Omega\times(0,t)} |D_i\omega|^q \mathrm{d}x\mathrm{d}\tau + r\int_{\Omega\times(0,t)} \omega^{h+1}\mathrm{d}x\mathrm{d}\tau \leq \frac{1}{2}\int_\Omega \omega_0^2\mathrm{d}x,$$

which, together with (5), implies that

$$\omega \in L^\infty(0,T;L^2(\Omega)) \cap L^{h+1}(\Omega_T), D_i\omega \in L^p(\Omega_T) \cap L^q(\Omega_T), i = 1,2,\cdots,N. \tag{6}$$

Note that the function $(\omega - \|\omega_0\|_{L^\infty(\Omega)})_+$ is non-negative, where $\|\omega_0\|_{L^\infty(\Omega)}$ denotes the supremum norm of ω_0 over the domain Ω. Therefore, integrating both sides of $(\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ Lu \leq 0$ over $\Omega \times (0,t)$, we obtain

$$\begin{aligned}&\int_{\Omega\times(0,t)} \partial_\tau\omega \cdot (\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ \mathrm{d}x\mathrm{d}\tau + \sum_{i=1}^{N} \int_{\Omega\times(0,t)} |D_i\omega|^{p-2}D_i\omega \cdot D_i(\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ \mathrm{d}x\mathrm{d}\tau \\ &+ \sum_{i=1}^{N} \int_{\Omega\times(0,t)} |D_i\omega|^{q-2}D_i\omega \cdot D_i(\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ \mathrm{d}x\mathrm{d}\tau + r\int_{\Omega\times(0,t)} \omega^h \cdot (\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ \mathrm{d}x\mathrm{d}\tau \\ &\leq \sum_{i=1}^{N} \int_{\partial\Omega\times(0,t)} |D_i\pi|^{p-2}D_i\pi \cdot (\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ \cos(x_i,\vec{\upsilon})\mathrm{d}S\,\mathrm{d}\tau \\ &\quad + \sum_{i=1}^{N} \int_{\partial\Omega\times(0,t)} |D_i\pi|^{q-2}D_i\pi \cdot (\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ \cos(x_i,\vec{\upsilon})\mathrm{d}S\,\mathrm{d}\tau.\end{aligned} \tag{7}$$

We begin by analyzing the right-hand side of inequality (7). From the boundary conditions of the variational inequality (3), it is straightforward to observe that

$$(\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ = 0 \text{ in } \partial\Omega \times (0,T)$$

which implies

$$\sum_{i=1}^{N} \int_{\partial\Omega\times(0,t)} |D_i\pi|^{p-2}D_i\pi \cdot (\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ \cos(x_i,\vec{\upsilon})\mathrm{d}S\,\mathrm{d}\tau = 0$$

and

$$\sum_{i=1}^{N} \int_{\partial\Omega\times(0,t)} |D_i\pi|^{q-2}D_i\pi \cdot (\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ \cos(x_i,\vec{\upsilon})\mathrm{d}S\,\mathrm{d}\tau = 0.$$

Next, we analyze the second and third terms on the left-hand side of (7). When $\omega \le \|\omega_0\|_{L^\infty(\Omega)}$ holds, we have $(\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ = 0$; when $\omega > \|\omega_0\|_{L^\infty(\Omega)}$ is satisfied, it follows that $(\omega - \|\omega_0\|_{L^\infty(\Omega)})_+ = \omega - \|\omega_0\|_{L^\infty(\Omega)}$. Consequently,

$$\sum_{i=1}^{N}\int_{\Omega\times(0,t)}|D_i\omega|^{p-2}D_i\omega\cdot D_i(\omega-\|\omega_0\|_{L^\infty(\Omega)})_+\mathrm{d}x\mathrm{d}\tau \ge \sum_{i=1}^{N}\int_{\Omega\times(0,t)}|D_i\omega|^{p}\mathrm{d}x\mathrm{d}\tau \ge 0$$

and

$$\sum_{i=1}^{N}\int_{\Omega\times(0,t)}|D_i\omega|^{q-2}D_i\omega\cdot D_i(\omega-\|\omega_0\|_{L^\infty(\Omega)})_+\mathrm{d}x\mathrm{d}\tau \ge \sum_{i=1}^{N}\int_{\Omega\times(0,t)}|D_i\omega|^{q}\mathrm{d}x\mathrm{d}\tau \ge 0.$$

Note that

$$r\int_{\Omega\times(0,t)}\omega^h\cdot(\omega-\|\omega_0\|_{L^\infty(\Omega)})_+\mathrm{d}x\mathrm{d}\tau \ge 0.$$

Combining the above results, we obtain

$$\int_{\Omega\times(0,t)}\partial_\tau\omega\cdot(\omega-\|\omega_0\|_{L^\infty(\Omega)})_+\mathrm{d}x\mathrm{d}\tau \le 0,$$

which implies that

$$\int_{\Omega}(\omega-\|\omega_0\|_{L^\infty(\Omega)})_+^2\mathrm{d}x \le \int_{\Omega}(\omega_0-\|\omega_0\|_{L^\infty(\Omega)})_+^2\mathrm{d}x = 0.$$

This further yields

$$\omega \le \|\omega_0\|_{L^\infty(\Omega)} \text{ in } \Omega_T. \tag{8}$$

For clarity in presenting our main results, we first introduce some necessary notation. Let us define

$$\mathrm{O}(\rho,\sigma) = \mathrm{O}(\rho,\sigma|(x_0,t_0)) = \Theta_\rho\times\Xi_\sigma = \{x\,|\,|x-x_0|<\rho\}\times(t_0-\sigma,t_0).$$

Our main results can then be stated as follows:

Theorem 2.4. *For any given $\mathrm{O}(\rho,\sigma)\subset\Omega_T$, there exists a non-negative constant C depending only on p, q, N, and $\|\omega_0\|_{L^\infty(\Omega)}$, such that the solution ω of variational inequality (3) satisfies*

$$\|D_i\omega\|_{L^\infty(O(\rho,\sigma))} \le C,\ i = 1,2,\cdots,N.$$

Theorem 2.5. *Let $\alpha\in(0,\ 1)$ be fixed, and let ω be a solution to the variational inequality (3). Then, there exists a constant $C = C(h,p,q,\|\omega_0\|_{L^\infty(\Omega)},|\Omega|)\ge 0$ such that*

$$\|D_i\omega\|_{L^\infty(\mathrm{O}_{\sigma\rho,\sigma\sigma})} \le C\left(\|D_i\omega\|_{L^p(\mathrm{O}_{\rho,\sigma})}^{\frac{\alpha(1+\alpha)}{2(p-1)}} + \|D_i\omega\|_{L^q(\mathrm{O}_{\rho,\sigma})}^{\frac{\alpha(1+\alpha)}{2(q-1)}}\right), i = 1,2,\cdots,N.$$

When either p or q is greater than 2, the constant α satisfies $0<\alpha<\min\{\frac{2}{p-2},\frac{2}{q-2}\}$. In the special case when $p = q = 2$, α can be any positive real number.

3. Energy inequality for the gradient solution

In this section, we establish an energy inequality concerning $D_i\omega$, which will later be employed to derive a sharper result compared to (6). Let $\theta_i = |D_i\omega|$, and define

$$\varphi_i = (\theta_i - \lambda)_+\psi^2\eta^2, \tag{9}$$

where λ is a non-negative constant to be determined, $\psi \in C_0^\infty(\Omega)$, $\eta \in C_0^\infty((0,T))$, and they further satisfy

$$0 \le \psi \le 1 \,\text{in}\, \Omega, 0 \le \eta \le 1 \,\text{in}\, (0,T). \tag{10}$$

Under these conditions, we obtain the following result.

Theorem 3.1. *Suppose that ω is a solution to the variational inequality (3), and let λ be a nonnegative constant. Then,*

$$\begin{aligned}&\sup_{t\in(0,T)} \int_\Omega \sum_{i=1}^N (\theta_i-\lambda)^2\psi^2\eta^2 \mathrm{d}x + \int_{\Omega_T} \sum_{i=1}^N \left(\sum_{j=1}^N (\theta_j^{p-2}+\theta_j^{q-2})|D_{i,j}^2\omega|^2 I_{\theta_i\ge\lambda}\right)\psi^2\eta^2 \mathrm{d}x\mathrm{d}t\\ &\le 2\int_{\Omega_T} \sum_{i=1}^N (\theta_i-\lambda)^2\psi^2\eta|\partial_t\eta|\mathrm{d}x\mathrm{d}t + 2rX_1 + 2(p-1)^2X_{2,p} + 2(q-1)^2X_{2,q},\end{aligned}$$

where

$$X_1 = \int_{\Omega_T} \sum_{i=1}^N |D_i(\omega^h)|(\theta_i-\lambda)_+\psi^2\eta^2 \mathrm{d}x\mathrm{d}t,$$

$$X_{2,p} = \int_{\Omega_T} \sum_{i=1}^N \left(\sum_{j=1}^N \theta_j^{p-2}(\theta_i-\lambda)_+^2|D_j\psi|^2\right)\psi\eta^2 \mathrm{d}x\mathrm{d}t, X_{2,q} = \int_{\Omega_T} \sum_{i=1}^N \left(\sum_{j=1}^N \theta_j^{q-2}(\theta_i-\lambda)_+^2|D_j\psi|^2\right)\psi\eta^2 \mathrm{d}x\mathrm{d}t.$$

Proof. From Eq (3), we deduce that $\omega = \omega_0$ holds when $L\omega < 0$. Under the assumptions on ω_0, we immediately obtain

$$D_i\omega \in L^\infty(\{(x,t)|\omega=\omega_0\}), i = 1,2,\cdots,N. \tag{11}$$

We now analyze the behavior of $L\omega = 0$ on Ω_T. By multiplying equation $D_iL\omega = 0$ through by φ_i and integrating over the domain Ω_t, we obtain:

$$\sum_{i=1}^N \int_{\Omega_t} \partial_t D_i\omega\varphi_i \,\mathrm{d}x\mathrm{d}t + \int_{\Omega_t} \sum_{i=1}^N D_i \sum_{j=1}^N D_j[(\theta_j^{p-2}+\theta_j^{q-2})D_j\omega]\varphi_i \,\mathrm{d}x\mathrm{d}t + r\sum_{i=1}^N \int_{\Omega_t} \mathrm{div}(\omega^h)\varphi_i \,\mathrm{d}x\mathrm{d}t = 0. \tag{12}$$

First consider $\int_{\Omega_t} \partial_t D_i\omega\varphi_i \,\mathrm{d}x\mathrm{d}t$ with λ non-negative. Note that when $\theta_i \le \lambda$, we have $\int_{\Omega_t} \partial_t D_i\omega\varphi_i \mathrm{d}x\mathrm{d}t = 0$. Integration by parts then gives:

$$\begin{aligned}&\int_{\Omega_t} \partial_t D_i\omega\varphi_i \mathrm{d}x\mathrm{d}t = \int_{\Omega_t} \partial_t\theta_i(\theta_i-\lambda)_+\psi^2\eta^2 \,\mathrm{d}x\mathrm{d}t\\ &= \frac{1}{2}\int_{\Omega_t} (\theta_i(\cdot,t)-\lambda)^2\psi^2\eta(\cdot,t)^2 \,\mathrm{d}x - \int_{\Omega_t} (\theta_i-\lambda)^2\psi^2\eta\partial_t\eta \,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{13}$$

Let us now analyze the second term on the left-hand side of (12). By applying integration by parts, we readily obtain

$$\begin{aligned}&\int_{\Omega_t} \sum_{i=1}^N \left[D_i \sum_{j=1}^N D_j[(\theta_j^{p-2}+\theta_j^{q-2})D_j\omega]\right]\varphi_i \,\mathrm{d}x\mathrm{d}t\\ &= \int_{\partial\Omega} \left(\sum_{i=1}^N \sum_{j=1}^N D_i[(\theta_j^{p-2}+\theta_j^{q-2})D_j\omega]\right)\varphi_i \cos(\vec{\nu}, x_j)\mathrm{d}S - \int_{\Omega_t} \left(\sum_{i=1}^N \sum_{j=1}^N D_i[(\theta_j^{p-2}+\theta_j^{q-2})D_j\omega]\right) D_j\varphi_i \,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $\vec{\nu}$ denotes the outward normal vector to $\partial\Omega$. Noting that $\omega = 0$ in $\partial\Omega \times (0,T)$, $\psi \in C_0^\infty(\Omega)$ and $\eta \in C_0^\infty((0,T))$, we consequently have

$$\int_{\Omega_t} \sum_{i=1}^{N} \left[D_i \sum_{j=1}^{N} D_j[(\theta_j^{p-2} + \theta_j^{q-2}) D_j\omega] \right] \varphi_i \,\mathrm{d}x\mathrm{d}t = -\int_{\Omega_t} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} D_i[(\theta_j^{p-2} + \theta_j^{q-2}) D_j\omega] \right) D_j\varphi_i \,\mathrm{d}x\mathrm{d}t.$$

By expanding the differential forms of $D_i[(\theta_j^{p-2} + \theta_j^{q-2})D_j\omega]$ and $D_j\varphi_i$, we obtain

$$-\int_{\Omega_t} D_i \left(\sum_{j=1}^{N} D_j[(\theta_j^{p-2} + \theta_j^{q-2}) D_j\omega] \right) \varphi_i \,\mathrm{d}x\mathrm{d}t = (p-1)X_3+(q-1)X_4+2(p-1)X_5+2(q-1)X_6, \quad (14)$$

where

$$X_3 = \int_{\Omega_t} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \theta_j^{p-2} |D_{i,j}^2\omega|^2 I_{\theta_i \geq \lambda} \right) \psi^2\eta^2 \mathrm{d}x\mathrm{d}t, X_4 = \int_{\Omega_t} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \theta_j^{q-2} |D_{i,j}^2\omega|^2 I_{\theta_i \geq \lambda} \right) \psi^2\eta^2 \mathrm{d}x\mathrm{d}t,$$

$$X_5 = \int_{\Omega_t} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \theta_j^{p-2} D_{i,j}^2\omega (\theta_i - \lambda)_+ \right) \psi\eta^2 D_j\psi \mathrm{d}x\mathrm{d}t, X_6 = \int_{\Omega_t} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \theta_j^{q-2} D_{i,j}^2\omega (\theta_i - \lambda)_+ \right) \psi\eta^2 D_j\psi \mathrm{d}x\mathrm{d}t.$$

Consequently, we have successfully transformed $\int_{\Omega_t} \partial_t D_i\omega \cdot \varphi_i \,\mathrm{d}x\mathrm{d}t$ using (13). Substituting both (13) and (14) into (12) yields

$$\begin{aligned} &\tfrac{1}{2} \int_{\Omega_t} \sum_{i=1}^{N} (\theta_i - \lambda)^2 \psi^2\eta^2 \mathrm{d}x + \int_{\Omega_t} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} (\theta_j^{p-2} + \theta_j^{q-2}) |D_{i,j}^2\omega|^2 I_{\theta_i \geq \lambda} \right) \psi^2\eta^2 \mathrm{d}x\mathrm{d}t \\ &\leq \int_{\Omega_t} \sum_{i=1}^{N} (\theta_i - \lambda)^2 \psi^2 \eta |\partial_t\eta| \mathrm{d}x\mathrm{d}t + rX_1 + 2(p-1)\,|X_5| + 2(q-1)\,|X_6|. \end{aligned} \quad (15)$$

We now estimate the upper bound of the third term X_5 on the right-hand side of (15) to combine it with the second term on the left-hand side. Applying Hölder's inequality with parameter $(\frac{1}{2}, \frac{1}{2})$ yields

$$X_5 \leq \frac{1}{4(p-1)} \int_{\Omega_t} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \theta_j^{p-2} |D_{i,j}^2\omega|^2 I_{\theta_i \geq \lambda} \right) \psi^2\eta^2 \mathrm{d}x\mathrm{d}t + (p-1)X_{2,p}, \quad (16)$$

and

$$X_6 \leq \frac{1}{4(q-1)} \int_{\Omega_t} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \theta_j^{q-2} |D_{i,j}^2\omega|^2 I_{\theta_i \geq \lambda} \right) \psi^2\eta^2 \mathrm{d}x\mathrm{d}t + (q-1)X_{2,q}. \quad (17)$$

Combining Eqs (16) and (17), we finally obtain the result stated in Theorem 3.1. □

We now proceed to refine Theorem 3.1 to establish a result valid in local cylindrical domains. First, we introduce some notation for local cylindrical regions in Ω_T. Let

$$\rho_n = \frac{2}{3}\rho + \frac{1}{3}\frac{1}{3^n}\rho, \;\; \sigma_n = \frac{2}{3}\sigma + \frac{1}{3}\frac{1}{3^n}\sigma, \;\; \mathrm{O}_n = \mathrm{O}(\rho_n, \sigma_n) = \Theta_{\rho_n} \times \Xi_{\sigma_n}. \quad (18)$$

Clearly, $O_0 = O(\rho,\sigma)$, $O_\infty = O(\frac{2}{3}\rho, \frac{2}{3}\sigma)$. To further improve the estimates for η and ψ, we require that ψ_n satisfies not only condition (10) regarding ψ but also the additional constraint:

$$\psi_n = 0 \,\text{in}\, \partial\Theta_{\rho_n},\ \psi_n(x) = 1 \,\text{in}\, \Theta_{\rho_{n+1}},\ |D_i\psi_n| \le \frac{3^n}{\rho}. \tag{19}$$

Under these conditions, we call ψ_n a cutoff factor on $\Theta_{\rho_{n+1}}$. Similarly, we assume η_n serves as a cutoff factor on $\Xi_{\theta_{n+1}}$, which, besides satisfying condition (10) concerning η, must also vanish at $t_0 - \sigma_n$ (i.e., η_n= 0 there) and satisfy

$$\eta_n(t) = 1 \,\text{in}\, \Xi_{\sigma_{n+1}},\ |D_i\eta_n| \le \frac{3^n}{\sigma}. \tag{20}$$

We conclude by defining $\lambda_n = \lambda - \frac{1}{3^n}\lambda$. Observe that the second term on the left-hand side of Theorem 3.1's conclusion is precisely X_3+X_4. From $p \ge 2$ and $q \ge 2$, it follows that

$$X_3 = \frac{4}{p^2}\int_{\Omega_t}\sum_{i=1}^{N}\sum_{j=1}^{N}|D_i(\theta_j-\lambda)^{\frac{1}{2}p}|^2\psi^2\eta^2\mathrm{d}x\mathrm{d}t, \tag{21}$$

$$X_4 = \frac{4}{p^2}\int_{\Omega_t}\sum_{i=1}^{N}\sum_{j=1}^{N}|D_i(\theta_j-\lambda)^{\frac{1}{2}q}|^2\psi^2\eta^2\mathrm{d}x\mathrm{d}t. \tag{22}$$

By selecting $\lambda = \lambda_{n+1}$ in Theorem 3.1 and combining (19)–(22), we obtain the following result.

Theorem 3.2. *Suppose that ω is a solution to the variational inequality (3), and let $\theta_i = |D_i\omega|$. Then for every $n = 1, 2, 3, \cdots$, it holds that*

$$\begin{aligned}
&\sup_{t\in(0,T)}\int_{\Theta_n}\sum_{i=1}^{N}(\theta_i-\lambda_{n+1})^2\psi^2\eta^2\mathrm{d}x + \tfrac{4}{p^2}\int_{O_n}\sum_{i=1}^{N}\sum_{j=1}^{N}|D_i(\theta_j-\lambda_{n+1})^{\frac{1}{2}p}|^2\psi^2\eta^2\mathrm{d}x\mathrm{d}t\\
&+\tfrac{4}{q^2}\int_{O_n}\sum_{i=1}^{N}\sum_{j=1}^{N}|D_i(\theta_j-\lambda_{n+1})^{\frac{1}{2}q}|^2\psi^2\eta^2\mathrm{d}x\mathrm{d}t\\
&\le 2\int_{O_n}\sum_{i=1}^{N}(\theta_i-\lambda_{n+1})^2\psi^2\eta|\partial_t\eta|\mathrm{d}x\mathrm{d}t + 2r\int_{O_n}\sum_{i=1}^{N}|D_i(\omega^h)|(\theta_i-\lambda_{n+1})_+\psi^2\eta^2\mathrm{d}x\mathrm{d}t\\
&\quad+2(p-1)^2\int_{O_n}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\theta_j^{p-2}(\theta_i-\lambda_{n+1})_+^2|D_j\psi|^2\right)\psi\eta^2\mathrm{d}x\mathrm{d}t\\
&\quad+2(q-1)^2\int_{O_n}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\theta_j^{q-2}(\theta_i-\lambda_{n+1})_+^2|D_j\psi|^2\right)\psi\eta^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{23}$$

4. Proof of main theorems

The objective of this section is to complete the proofs of Theorems 2.4 and 2.5, which establish results concerning the norm estimates of the spatial gradient $\nabla\omega$ of solutions in the vicinity of the point (x_0, t_0). We first analyze the existence of the supremum norm of $\nabla\omega$. For clarity of presentation, we define

$$\Phi_n = \sum_{i=1}^{N}\int_{O_n}(\theta_i-\lambda_n)_+^p\mathrm{d}x\mathrm{d}t. \tag{24}$$

Note that when $\theta_i > \lambda_{n+1}$, we have $(\theta_i - \lambda_{n+1})_+ > \frac{\lambda}{3^{n+1}}$ in O_n, which consequently implies

$$\sum_{i=1}^{N} \int_{\mathrm{O}_n} I_{\theta_i \geq \lambda_{n+1}} \mathrm{d}x\mathrm{d}t \leq \frac{3^{(n+1)p}}{\lambda^p} \sum_{i=1}^{N} \int_{\mathrm{O}_n} (\theta_i - \lambda_n)_+^p \mathrm{d}x\mathrm{d}t = \lambda^{-p} 3^{(n+1)p} \Phi_n. \tag{25}$$

Moreover, we require an additional result concerning Φ_n: for any $i = 1, 2, \cdots, N$ and $n = 1, 2, 3, \cdots$, the relation

$$\sum_{i=1}^{N} \int_{\mathrm{O}_n} \theta_i{}^p I_{\theta_i \geq \lambda_{n+1}} \mathrm{d}x\mathrm{d}t \leq 3^{(n+1)p} \sum_{i=1}^{N} \int_{\mathrm{O}_n} (\theta_i - \lambda_n)_+^p \mathrm{d}x\mathrm{d}t \tag{26}$$

holds. Indeed, under the condition $\theta_i < \lambda_n$, $(\theta_i - \lambda_n)_+^p = 0$ is satisfied. On the other hand, when $\theta_i \geq \lambda_n$, $(\theta_i - \lambda_n)_+^p = (\theta_i - \lambda_n)^p$ holds. Combined with $\lambda_{n+1} \geq \lambda_n$, this results in

$$\int_{\mathrm{O}_n} (\theta_i - \lambda_n)_+^p \mathrm{d}x\mathrm{d}t = \int_{\mathrm{O}_n} (\theta_i - \lambda_n)_+^p I_{\theta_i \geq \lambda_{n+1}} \mathrm{d}x\mathrm{d}t. \tag{27}$$

Observe that $\lambda_{n+1} \geq \lambda_n$ and $\lambda_n = \lambda_{n+1} \frac{3^{n+1}-3}{3^{n+1}-1}$, and hence

$$\int_{\mathrm{O}_n} (\theta_i - \lambda_n)_+^p \mathrm{d}x\mathrm{d}t \geq \int_{\mathrm{O}_n} \theta_i{}^p \left(1 - \frac{3^{n+1}-3}{3^{n+1}-1}\right)_+^p I_{\theta_i \geq \lambda_{n+1}} \mathrm{d}x\mathrm{d}t \geq \frac{1}{3^{(n+1)p}} \int_{\mathrm{O}_n} \theta_i{}^p I_{\theta_i \geq \lambda_{n+1}} \mathrm{d}x\mathrm{d}t \tag{28}$$

holds. By merging (27) and (28), (26) becomes evident.

Proof of Theorem 2.4. We establish the recursive relations in Lemma 2.1 and Lemma 2.2, with the convention that $p > q$. Observing that $\mathrm{O}_n \supset \mathrm{O}_{n+1}$ and $\psi_n \eta_n = 1$ in O_{n+1}, we can readily obtain

$$\begin{aligned} l\Phi_{n+1} &\leq \sum_{i=1}^{N} \int_{\mathrm{O}_n} |(\theta_i - \lambda_{n+1})_+^{p/2} \psi_n \eta_n|^2 \mathrm{d}x\mathrm{d}t \\ &\leq \left(\sum_{i=1}^{N} \int_{\mathrm{O}_n} |(\theta_i - \lambda_{n+1})_+^{p/2} \psi_n \eta_n|^{2(1+\alpha)} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{1+\alpha}} \left(\sum_{i=1}^{N} \int_{\mathrm{O}_n} I_{\theta_i \geq \lambda_{n+1}} \mathrm{d}x\mathrm{d}t \right)^{\frac{\alpha}{1+\alpha}}, \end{aligned} \tag{29}$$

by applying Hölder's inequality. Here, α is non-negative, and its exact value will be determined later. Substituting (25) into the second term on the right-hand side of the above expression, we easily obtain

$$\begin{aligned} l\Phi_{n+1} &\leq \sum_{i=1}^{N} \int_{\mathrm{O}_n} |(\theta_i - \lambda_{n+1})_+^{p/2} \psi_n \eta_n|^2 \mathrm{d}x\mathrm{d}t \\ &\leq \left(\sum_{i=1}^{N} \int_{\mathrm{O}_n} |(\theta_i - \lambda_{n+1})_+^{p/2} \psi_n \eta_n|^{2(1+\alpha)} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{1+\alpha}} 3^{(n+1)\frac{\alpha}{1+\alpha} p} \Phi_n{}^{\frac{\alpha}{1+\alpha}}. \end{aligned} \tag{30}$$

Note that in the above expression, we have adopted the convention that $\lambda \geq 1$. By comparing the conditions in Lemma 2.1 and Lemma 2.2, we analyze the upper bound estimate of $\sum_{i=1}^{N} \int_{\mathrm{O}_n} |(\theta_i - \lambda_{n+1})_+^{p/2} \zeta_n|^{2(1+\alpha)} \mathrm{d}x\mathrm{d}t$. Consequently, applying the Caffarelli–Kohn–Nirenberg inequality

(Lemma 2.3), we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\mathrm{O}_n}|(\theta_i-\lambda_{n+1})_+^{p/2}\psi_n\eta_n|^{2(1+\alpha)}\mathrm{d}x\mathrm{d}t\\&\le C_{C-K-N}\left(\sum_{i=1}^{N}\int_{\mathrm{O}_n}|D_i(\theta_i-\lambda_{n+1})_+^{p/2}\psi_n\eta_n|^2\mathrm{d}x\mathrm{d}t\right)\left(ess\sup_{t\in\Xi_n}\sum_{i=1}^{N}\int_{\Theta_n}(\theta_i-\lambda_{n+1})_+^2\psi_n\eta_n\mathrm{d}x\right)^{\frac{1}{2}\alpha p}.\end{aligned}\tag{31}$$

By reapplying the two terms on the right-hand side of estimate (23) to (31), we can easily deduce from formula (24) that

$$ess\sup_{t\in\Xi_n}\sum_{i=1}^{N}\int_{\Theta_n}(\theta_i-\lambda_{n+1})_+^2\psi_n\eta_n\mathrm{d}x\le X_8,\tag{32}$$

$$\sum_{i=1}^{N}\int_{\mathrm{O}_n}|D_i(\theta_i-\lambda_n)_+^{\frac{1}{2}p}|^2\psi_n\eta_n\mathrm{d}x\mathrm{d}t\le\frac{p^2}{4}X_8,\tag{33}$$

$$\sum_{i=1}^{N}\int_{\mathrm{O}_n}|D_i(\theta_i-\lambda_n)_+^{\frac{1}{2}q}|^2\psi_n\eta_n\mathrm{d}x\mathrm{d}t\le\frac{q^2}{4}X_8.\tag{34}$$

From (7), we obtain $\omega\le\|\omega_0\|_{L^\infty(\Omega)}$ in Ω_T. Thus, estimating the second term in (23), we express X_5 as

$$X_8=(\frac{3^n}{\sigma}+2rq\|\omega_0\|_{L^\infty(\Omega)}^{q-1})\sum_{i=1}^{N}\int_{\mathrm{O}_n}{\theta_i}^2I_{\theta_i\ge\lambda_{n+1}}\mathrm{d}x\mathrm{d}t+2(p-1)^2\frac{9^n}{\rho^2}X_9+2(q-1)^2\frac{9^n}{\rho^2}X_{10},$$

where

$$X_9=\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathrm{O}_n}\theta_i^2\theta_j^{p-2}I_{\theta_i\ge\lambda_{n+1}}\mathrm{d}x\mathrm{d}t,\ X_{10}=\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathrm{O}_n}\theta_i^2\theta_j^{q-2}I_{\theta_i\ge\lambda_{n+1}}\mathrm{d}x\mathrm{d}t.$$

Using (32)–(34) to estimate $\sum_{i=1}^{N}\int_{\mathrm{O}_n}|D_i(\theta_i-\lambda_{n+1})_+^{p/2}\zeta_n|^2\mathrm{d}x\mathrm{d}t$ in (31), we obtain

$$\sum_{i=1}^{N}\int_{\mathrm{O}_n}|(\theta_i-\lambda_{n+1})_+^{p/2}\zeta_n|^{2(1+\alpha)}\mathrm{d}x\mathrm{d}t\le C_{C-K-N}\frac{p^2}{4}{X_8}^{1+\frac{1}{2}\alpha p}.\tag{35}$$

Proceeding further, we simplify X_8 by estimating its upper bound using Φ_n. By appropriately selecting parameters $\frac{2}{p}$ and $\frac{p-2}{p}$, and applying the weighted Hölder inequality, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathrm{O}_n}\theta_i^p\theta_j^{p-2}I_{\theta_i\ge\lambda_{n+1}}\mathrm{d}x\mathrm{d}t\\&\le\left(N\sum_{i=1}^{N}\int_{\mathrm{O}_n}\theta_i^pI_{\theta_i\ge\lambda_{n+1}}\mathrm{d}x\mathrm{d}t\right)^{\frac{2}{p}}\left(N\sum_{j=1}^{N}\int_{\mathrm{O}_n}\theta_j^pI_{\theta_i\ge\lambda_{n+1}}\mathrm{d}x\mathrm{d}t\right)^{\frac{p-2}{p}}=N\sum_{i=1}^{N}\int_{\mathrm{O}_n}\theta_i^pI_{\theta_i\ge\lambda_{n+1}}\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{36}$$

Substituting (26) into (36) provides

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathrm{O}_n}\theta_i^2\theta_j^{p-2}I_{\theta_i\ge\lambda_{n+1}}\mathrm{d}x\mathrm{d}t\le N3^{(n+1)p}\Phi_n.\tag{37}$$

Observe that given our standing assumption $p \geq q$, an application of Hölder's inequality combined with the monotonicity property of L^p-norms in p yields

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\Omega_T}\theta_i^2\theta_j^{q-2}I_{\theta_i\geq\lambda_{n+1}}\mathrm{d}x\mathrm{d}t \leq N\sum_{i=1}^{N}\int_{\mathrm{O}_n}\theta_i^q I_{\theta_i\geq\lambda_{n+1}}\mathrm{d}x\mathrm{d}t \leq N\sum_{i=1}^{N}\int_{\mathrm{O}_n}\theta_i^p I_{\theta_i\geq\lambda_{n+1}}\mathrm{d}x\mathrm{d}t. \tag{38}$$

By substituting this result into the term X_8 in (35), we can construct an upper bound estimate for $\sum\limits_{i=1}^{N}\int_{Q_n}|(\theta_i-\lambda_{n+1})_+^{p/2}\zeta_n|^{2(1+\alpha)}\mathrm{d}x\mathrm{d}t$ formulated in terms of Φ_n, which is precisely

$$\begin{aligned}&\sum_{i=1}^{N}\int_{Q_n}|(\theta_i-\lambda_{n+1})_+^{p/2}\zeta_n|^{2(1+\alpha)}\mathrm{d}x\mathrm{d}t\\ &\leq \mathrm{C}(p,q,r,h,C_{C-K-N})(3^{np}\Phi_n+9^n\Phi_n)^{1+\frac{1}{2}\alpha p} \leq \mathrm{C}(p,q,r,h,C_{C-K-N})3^{n(1+\frac{1}{2}\alpha p)p}{\Phi_n}^{1+\frac{1}{2}\alpha p}.\end{aligned} \tag{39}$$

The above expression fully satisfies the conditions of Lemma 2.1 and Lemma 2.2. Therefore, by combining (30) and (38), we obtain

$$\Phi_{n+1} \leq C(p,q,r,h,C_{C-K-N})3^{n(1+\frac{1}{2}\alpha p)p}{\Phi_n}^{1+\frac{\alpha p}{2(1+\alpha)}}.$$

Under the additional constraint $0<\alpha\leq\frac{2}{p-2}$, we get $\frac{\alpha p}{2(1+\alpha)}\in(0,\ 1)$. By Lemma 2.1, when

$$\Phi_0 \leq \mathrm{C}(p,q,r,h,C_{C-K-N})^{-\frac{2(1+\alpha)}{\alpha}}3^{-\frac{2(2+\alpha p)(1+\alpha)^2}{\alpha^2 p}}, \tag{40}$$

we have

$$\Phi_n \to 0 \text{ as } n\to\infty.$$

Furthermore, (40) holds whenever λ is sufficiently large. Observe that $\mathrm{O}_\infty = \mathrm{O}(\frac{2}{3}\rho,\frac{2}{3}\sigma)$, which completes the proof of Theorem 2.4 for the case of $\mathrm{O}(\frac{2}{3}\rho,\frac{2}{3}\sigma)$.

When $p<q$, by restricting $0<\alpha\leq\frac{2}{q-2}$ and repeating the argument in (29)–(40), we obtain

$$\Phi_0 \leq \mathrm{C}(p,q,r,h,C_{C-K-N})^{-\frac{2(1+\alpha)}{\alpha}}3^{-\frac{4(1+\alpha)^2}{\alpha^2 q}}. \tag{41}$$

Consequently, Theorem 2.4 remains valid on $\mathrm{O}(\frac{2}{3}\rho,\frac{2}{3}\sigma)$ under the condition $p<q$. □

Proof of Theorem 2.5. Observe that Theorem 2.4 establishes the validity of $\nabla\omega\in L^\infty(\mathrm{O}(\rho,\sigma))$ for arbitrary $\mathrm{O}(\rho,\sigma)\subset\Omega_T$, from which both

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathrm{O}_n}\theta_i^2\theta_j^{p-2}I_{\theta_i\geq\lambda_{n+1}}\mathrm{d}x\mathrm{d}t \leq \left(\sup_{(x,t)\in\mathrm{O}_n}|\nabla\omega|\right)^{p-2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathrm{O}_n}\theta_i^2 I_{\theta_i\geq\lambda_{n+1}}\mathrm{d}x\mathrm{d}t \tag{42}$$

and

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathrm{O}_n}\theta_i^2\theta_j^{q-2}I_{\theta_i\geq\lambda_{n+1}}\mathrm{d}x\mathrm{d}t \leq \left(\sup_{(x,t)\in\mathrm{O}_n}|\nabla\omega|\right)^{q-2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathrm{O}_n}\theta_i^2 I_{\theta_i\geq\lambda_{n+1}}\mathrm{d}x\mathrm{d}t \tag{43}$$

follow. Assuming $p\geq q$ and substituting (42) and (43) into (35) yields

$$\sum_{i=1}^{N}\int_{\mathrm{O}_n}|(\theta_i-\lambda_{n+1})_+^{p/2}\zeta_n|^{2(1+\alpha)}\mathrm{d}x\mathrm{d}t \leq C_{C-K-N}\frac{p^2}{4}{C_1}^{1+\frac{1}{2}\alpha p}\left(\sum_{i=1}^{N}\int_{\mathrm{O}_n}{\theta_i}^2 I_{\theta_i\geq\lambda_{n+1}}\mathrm{d}x\mathrm{d}t\right)^{1+\frac{1}{2}\alpha p}, \tag{44}$$

where

$$C_1 = \frac{3^n}{\sigma} + 2rq\|\omega_0\|_{L^\infty(\Omega)}^{q-1} + 2(p-1)^2\frac{9^n}{\rho^2}\left(\sup_{(x,t)\in O_n}|\nabla\omega|\right)^{p-2} + 2(q-1)^2\frac{9^n}{\rho^2}\left(\sup_{(x,t)\in O_n}|\nabla\omega|\right)^{q-2}. \tag{45}$$

Using the monotonicity of the L^p-norm with respect to p, we obtain

$$\sum_{i=1}^{N}\int_{O_n}|(\theta_i-\lambda_{n+1})_+^{p/2}\zeta_n|^{2(1+\alpha)}\mathrm{d}x\mathrm{d}t \le C_{C-K-N}\frac{p^2}{4}{C_1}^{1+\frac{1}{2}\alpha p}3^{n(1+\frac{1}{2}\alpha p)p}{\Phi_n}^{1+\frac{1}{2}\alpha p}. \tag{46}$$

Substituting the above results into (30), we obtain

$$\Phi_{n+1} \le \left(C_{C-K-N}\frac{p^2}{4}{C_1}^{1+\frac{1}{2}\alpha p}3^{n(1+\frac{1}{2}\alpha p)p}{\Phi_n}^{1+\frac{1}{2}\alpha p}\right)^{\frac{1}{1+\alpha}}3^{(n+1)\frac{\alpha}{1+\alpha}p}{\Phi_n}^{\frac{\alpha}{1+\alpha}},$$

which simplifies to

$$\Phi_{n+1} \le 3^{\frac{\alpha}{1+\alpha}p}C_{C-K-N}^{\frac{1}{1+\alpha}}\left(\frac{p^2}{4}\right)^{\frac{1}{1+\alpha}}{C_1}^{\frac{1+\frac{3}{2}\alpha p}{1+\alpha}}3^{n\frac{(1+\frac{1}{2}\alpha p)p}{1+\alpha}}{\Phi_n}^{1+\frac{\alpha p}{2(1+\alpha)}}. \tag{47}$$

Furthermore, by Theorem 2.4, we have $\nabla\omega \in L^\infty(O(\rho,\sigma))$. Consequently, when λ is sufficiently large, $\Phi_n \to 0$ as $n \to \infty$. Applying Lemma 2.2 then yields

$$\Phi_0 \le \frac{1}{9}C_{C-K-N}^{-\frac{2}{\alpha p}}\left(p^2/4\right)^{-\frac{2}{\alpha p}}{C_1}^{-\frac{2+3\alpha p}{\alpha p}}3^{-\frac{2(1+\alpha)(2+\alpha p)}{\alpha^2 p}}. \tag{48}$$

Note that $\lambda_0 = 0$ in Φ_0, which completes the proof of Theorem 2.5. For the case when $p < q$, by interchanging the roles of p and q in the proof of (44)–(48), we obtain

$$\Phi_0 \le \frac{1}{9}C_{C-K-N}^{-\frac{2}{\alpha q}}\left(q^2/4\right)^{-\frac{2}{\alpha q}}{C_1}^{-\frac{2+3\alpha q}{\alpha q}}3^{-\frac{2(1+\alpha)(2+\alpha q)}{\alpha^2 q}}. \tag{49}$$

This establishes the conclusion of Theorem 2.5 for the $p < q$ case as well. □

5. Conclusions and discussions

This paper establishes a variational inequality initial-boundary value problem through mathematical modeling of quanto option valuation, expressed as

$$\begin{cases} \max\{L\omega, \omega_0-\omega\} = 0 \,\text{in}\, \Omega_T, \\ \omega(\,\cdot\,,0) = \omega_0 \,\text{in}\, \Omega, \\ \omega = 0 \,\text{in}\, \partial\Omega\times(0,T). \end{cases}$$

We analyze the local max-norm regularity estimates for its gradient solutions, as detailed in Theorems 2.3 and 2.4, where $L\omega$ represents a double-phase degenerate parabolic operator with

$$L\omega=\partial_t\omega - \sum_{i=1}^{N}[D_i(|D_i\omega|^{p-2}D_i\omega) + D_i(|D_i\omega|^{q-2}D_i\omega)] + r\omega^h.$$

First, by employing the test function $\varphi_i = (\theta_i - \lambda)_+ \times \psi^2\eta^2$, we establish the energy inequality for the gradient solution and its local version over the cylindrical domain O_n, as detailed in Theorem 3.2. Subsequently, we derive an upper bound estimate for Φ_{n+1}. Utilizing Theorem 3.2 along with the Hölder's inequality and Caffarelli-Kohn-Nirenberg inequality, we construct an upper bound for Φ_{n+1} composed of Φ_n, which takes the form of

$$\Phi_{n+1} \leq C(p, q, r, h, C_{C-K-N})3^{n(1+\frac{1}{2}\alpha p)p}{\Phi_n}^{1+\frac{\alpha p}{2(1+\alpha)}}.$$

This leads to the boundedness of the maximum norm for the gradient solution of variational inequality (1). Finally, we incorporate this maximum norm boundedness into the upper bound estimation of Φ_{n+1}, thereby obtaining a precise upper bound for the maximum norm of the gradient solution.

The following lemma and energy inequality have been widely employed in the literature to derive recursive estimates for the solution's energy functional.

Lemma 5.1. *Let* $1 < p \leq q < \infty$, $\theta \in (0, 1)$, *and* $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2}$ *hold. Then*

$$\|\nabla\omega\|_{L^p} \leq c(p, q)\|\omega\|^{\theta}_{L^q} \cdot \|\nabla\omega\|^{1-\theta}_{L^2}.$$

Due to the restrictive parameter conditions in this lemma, the obtained results are suboptimal. We therefore employ Hölder's inequality to refine the energy inequality (see Theorem 3.2), thereby establishing recursive estimates for the energy functional of gradient solutions. These estimates satisfy the applicability conditions of both Lemmas 2.1 and 2.2.

Author contributions

Wenwen Jiang: Conceptualization, Methodology, Formal analysis, Writing-original draft, Writing-review & editing; Jia Li: Writing-review & editing, Validation, Supervision, Project administration. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were employed in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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