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Research article

Approximation properties of a moment-based modification of Bernstein operators

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Abstract: In this paper, a moment-based modification of the classical Bernstein operators is introduced. The proposed operators incorporate both a domain transformation and an adjustment by the second central moment to improve the approximation properties. We investigate their convergence behavior using tools such as the Lipschitz class and Peetre's $\kappa-$functional. Quantitative estimates are established based on the classical and second-order modulus of continuity. Furthermore, a Voronovskaja-type theorem is provided to analyze the asymptotic behavior. Theoretical results are supported by numerical examples and graphical illustrations that demonstrate the effectiveness of the operators.

Keywords: Bernstein operators; rate of convergence; modulus of continuity; Voronovskaja-type theorem
Mathematics Subject Classification: 41A25, 41A36

1. Introduction

The theory of positive linear operators is fundamental to approximation theory, providing powerful tools for investigating how functions can be approximated by simpler forms, such as polynomials. Among these, the Bernstein operators, first introduced by Bernstein in 1912 [1], occupy a central position. Their construction not only provided a constructive proof of the Weierstrass approximation theorem but also laid the foundation for a vast field of research into operator-based approximation. In addition to their theoretical importance in approximation theory, Bernstein-type operators and their modifications have found applications in several practical contexts. For example, they are widely used in computer-aided geometric design (CAGD) and computer graphics, where the classical Bernstein basis underlies Bézier curves and surfaces. Their shape-preserving properties make them suitable for image processing and curve fitting. Moreover, they often arise in probability theory and statistical modeling, since they are connected with binomial distributions and moment estimations. They have also been applied in the analysis of differential equations, signal processing, and control theory, where positive linear operators provide effective approximation procedures.

Bernstein operators are defined for $m \in \mathbb{N}$ as

$$B_m(f;\xi) = \sum_{\nu=0}^{m} \binom{m}{\nu} \xi^{\nu}(1-\xi)^{m-\nu} f\left(\frac{\nu}{m}\right), \tag{1.1}$$

which forms a sequence of positive linear operators converging uniformly to f on $[0,1]$. Due to their properties, such as positivity, linearity and shape-preserving properties, researchers have widely studied and extended Bernstein operators in many directions. Various generalizations of Bernstein operators have been proposed over the decades to address different approximation challenges, including those on unbounded intervals and in weighted function spaces. Notable examples include the Bernstein–Stancu [2], Bernstein–Kantorovich [3], Bernstein–Durrmeyer [4], Bernstein–Schurer [5], Bernstein–Chlodowsky [6], and q-Bernstein operators [7]. Each variant introduces new parameters or structural modifications to improve convergence properties or extend the domain of applicability.

One such modification is given by Çilo [8] as a natural linear transformation of the classical Bernstein definition on the interval $[0,1]$, as follows:

$$C_m(f;\xi) = \frac{1}{2^m} \sum_{\nu=0}^{m} \binom{m}{\nu} (1+\xi)^{\nu}(1-\xi)^{m-\nu} f\left(\frac{2\nu}{m}-1\right), \tag{1.2}$$

where $\xi \in [-1,1]$ and $f \in C[-1,1]$. C_m operators modify the classical Bernstein operators by adapting them to the symmetric interval $[-1,1]$ using a linear transformation of the argument. This adaptation preserves positivity and linearity while ensuring convergence on $[-1,1]$.

Usta presented another modification that was obtained by using the second central moment of the Bernstein operators, as follows [9]:

$$B_m^{*}(f;\xi) = \frac{1}{m} \sum_{\nu=0}^{m} \binom{m}{\nu} (\nu - m\xi)^2 \xi^{\nu-1} (1-\xi)^{m-\nu-1} f\left(\frac{\nu}{m}\right), \tag{1.3}$$

where $\xi \in (0,1)$. The operators B_m^{*} given in (1.3) apply a second central moment structure to the classical setting by incorporating the term $(\nu - m\xi)^2$ into the basis. A number of studies have been carried out following this study, with the focus being on new modification of Bernstein operators. For instance, in [10], the Kantorovich variant of the operators B_m^{*} is introduced; in [11], a parametric generalization of B_m^{*} is proposed; in [12], a new modification of Baskakov operators is defined using B_m^{*}; in [13], a beta-type modification of B_m^{*} is obtained; in [14], a new generalization of the Szász–Mirakjan operators based on B_m^{*} is presented; in [15], the q-analogue of B_m^{*} is studied; and in [16], a sequence of bi-variate α-modified Bernstein operators is constructed via B_m^{*}.

Inspired by the domain adaptation in (1.2) and the moment-based modification in (1.3), we introduce a new sequence of operators that combines the symmetric interval approach with the moment-based modification as

$$C_m^{*}(f;\xi) = \frac{1}{m2^m} \sum_{\nu=0}^{m} \binom{m}{\nu} (2\nu - m - m\xi)^2 (1+\xi)^{\nu-1} (1-\xi)^{m-\nu-1} f\left(\frac{2\nu}{m}-1\right), \tag{1.4}$$

where $m \in \mathbb{N}$, $\xi \in (-1, 1)$, and $f \in C[-1, 1]$. More precisely, the second central moment-like term $(2\nu - m - m\xi)^2$ is incorporated into the definition of C_m^*, along with the Bernstein basis and transformation mapping $\nu \to \frac{2\nu}{m} - 1$, to obtain a sequence of modified positive linear operators acting on $(-1, 1)$.

The proposed modification has several advantages over the classical and previously modified Bernstein-type operators. By combining a symmetric domain transformation with a moment-based adjustment, the new operators achieve convergence rates that are at least as good as those established previously, as demonstrated by the modulus of continuity and Lipschitz class. Furthermore, their construction preserves positivity and linearity, ensuring that fundamental approximation properties remain valid. However, these improvements also come with certain limitations. As the operators are defined on the open interval $(-1, 1)$, continuity at the endpoints cannot be guaranteed, which restricts their applicability compared to classical Bernstein operators, which are defined on closed intervals. Additionally, the moment-based modification increases the algebraic complexity of the operators, resulting in more intricate expressions and potentially higher computational requirements in practical applications. Despite these drawbacks, the enhanced approximation behavior and theoretical contributions make the operators C_m^* a valuable addition to the family of Bernstein-type modifications.

Remark 1. *The definition* (1.4) *is given for* $\xi \in (-1, 1)$. *It can be argued that the definition is also valid for* $\xi \in \{-1, 1\}$*: The only singular terms in the sum are these for* $\nu = 0$ *or* $\nu = m$. *In both cases we multiply by* $(1 + \xi)^2$ *or* $(1 - \xi)^2$, *respectively, removing the singularity. This extended definition would always give* $C_m^*(f; -1) = C_m^*(f; 1) = 0$, *which might destroy the continuity of the function* $\xi \to C_m^*(f; \xi)$*; thus, the interval must remain open, namely* $(-1, 1)$.

Remark 2. *The operators* C_m^* *are linear and positive. Indeed, for* $\xi \in (-1, 1)$, *all coefficients in the representation of* $C_m^*(f; \xi)$ *are nonnegative. Hence, if* $f \geq 0$ *on* $[-1, 1]$, *then* $C_m^*(f; \xi) \geq 0$.

The objective of this study is to examine the approximation properties of the operators C_m^*. To this end, some necessary lemmas are presented, and the approximation properties of these operators are then analyzed using the classical and second-order modulus of continuity. Furthermore, a global error estimation in the Lipschitz class is given, and the Voronovskaja-type theorem is obtained. Some numerical examples and graphical illustrations are also provided.

2. Approximation properties of the operators C_m^*

In order to obtain approximation properties, we give some lemmas.

Lemma 1. *Let* (C_m^*) *be the sequence of positive linear operators defined by (1.4). For all* $m \in \mathbb{N}$ *and* $\xi \in (-1, 1)$, *we have*

$$C_m^*(1; \xi) = 1, \tag{2.1}$$

$$C_m^*(t; \xi) = \xi - \frac{1}{m} 2\xi, \tag{2.2}$$

$$C_m^*\left(t^2; \xi\right) = \xi^2 + \frac{1}{m^2}(-7m\xi^2 + 3m + 6\xi^2 - 2), \tag{2.3}$$

$$C_m^*\left(t^3; \xi\right) = \xi^3 + \frac{1}{m^3}\left(-15m^2\xi^3 + 9m^2\xi + 38m\xi^3 - 26m\xi - 24\xi^3 + 16\xi\right), \tag{2.4}$$

$$C_m^*\left(t^4;\xi\right)=\xi^4+\frac{1}{m^4}\left(-26m^3\xi^4+18m^3\xi^2+131m^2\xi^4-122m^2\xi^2\right.$$
$$\left.+15m^2-226m\xi^4+224m\xi^2-30m+120\xi^4-120\xi^2+16\right). \tag{2.5}$$

Proof. In the B_m operators, the nodes used inside f are $\frac{\nu}{m}$ for $\nu=0,1,2,...,m$ in the interval $[0,1]$; whereas in the C_m operators, they are $\frac{2\nu}{m}-1$ for $\nu=0,1,2,...,m$ in the interval $[-1,1]$. In light of this explanation, for $k\in\mathbb{N}$, the recurrence relation between the C_m and B_m operators has been obtained as follows:

$$\begin{aligned}C_m\left(t^k;\xi\right)&=\sum_{\nu=0}^{m}\binom{m}{\nu}\left(\frac{1+\xi}{2}\right)^{\nu}\left(\frac{1-\xi}{2}\right)^{m-\nu}\left(\frac{2\nu}{m}-1\right)^{k}\\&=\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,2^{k-j}B_m\left(t^{k-j};\frac{1+\xi}{2}\right).\end{aligned}$$

The following equations can be derived from this relationship:

$$\begin{aligned}C_m\left(1;\xi\right)&=B_m\left(1;\frac{1+\xi}{2}\right)=1,\\C_m\left(t;\xi\right)&=2B_m\left(t;\frac{1+\xi}{2}\right)-1=2\left(\frac{1+\xi}{2}\right)-1=\xi,\\C_m\left(t^2;\xi\right)&=4B_m\left(t^2;\frac{1+\xi}{2}\right)-4B_n\left(t;\frac{1+\xi}{2}\right)+1\\&=4\left[\left(\frac{1+\xi}{2}\right)^2+\frac{\left(\frac{1+\xi}{2}\right)\left(1-\frac{1+\xi}{2}\right)}{m}\right]-4\left(\frac{1+\xi}{2}\right)+1\\&=\xi^2+\frac{1}{m}\left(1-\xi^2\right).\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}C_m(t^3;\xi)&=\xi^3+\left(\frac{3m-2}{m^2}\right)\xi(1-\xi^2),\\C_m(t^4;\xi)&=\xi^4+\frac{1}{m^3}\left(6m^2\xi^2-6m^2\xi^4+11m\xi^4-14m\xi^2+3m-6\xi^4+8\xi^2-2\right),\\C_m(t^5;\xi)&=\xi^5+\frac{1}{m^4}\,\xi\left(-10m^3\xi^4+10m^3\xi^2+35m^2\xi^4-50m^2\xi^2+15m^2-50m\xi^4\right.\\&\quad\left.+80m\xi^2-30m+24\xi^4-40\xi^2+16\right),\\C_m(t^6;\xi)&=\xi^6+\frac{1}{m^5}\left(-15m^4\xi^6+15m^4\xi^4+85m^3\xi^6-130m^3\xi^4+45m^3\xi^2-225m^2\xi^6+405m^2\xi^4\right.\\&\quad\left.-195m^2\xi^2+15m^2+274m\xi^6-530m\xi^4+286m\xi^2-30m-120\xi^6+240\xi^4-136\xi^2+16\right).\end{aligned}$$

An important point to note is that the values of $C_m(t^k;\xi)$ can be obtained either by direct computation, as given in [8] for $k\in\{0,1,2,3,4\}$, or by using the recurrence relation between the operators C_m and B_m, as given above.

Now, from the fact that

$$C_m^*\left(t^k;\xi\right) = \frac{m}{1-\xi^2}\left[C_m\left(t^{k+2};\xi\right) - 2\xi C_m\left(t^{k+1};\xi\right) + \xi^2 C_m\left(t^k;\xi\right)\right],$$

we can obtain the following equalities:

For $f(t) = 1$, we have

$$\begin{aligned} C_m^*(1;\xi) &= \frac{1}{m2^m}\sum_{\nu=0}^{m}\binom{m}{\nu}(2\nu - m - m\xi)^2(1+\xi)^{\nu-1}(1-\xi)^{m-\nu-1} \\ &= \frac{m}{1-\xi^2}\left[C_m\left(t^2;\xi\right) - 2\xi C_m(t;\xi) + \xi^2 C_m(1;\xi)\right] \\ &= \frac{m}{1-\xi^2}\left[\xi^2 + \frac{(1-\xi)(1+\xi)}{m} - 2\xi^2 + \xi^2\right] \\ &= 1. \end{aligned}$$

For $f(t) = t$, we have

$$\begin{aligned} C_m^*(t;\xi) &= \frac{1}{m2^m}\sum_{\nu=0}^{m}\binom{m}{\nu}(2\nu - m - m\xi)^2(1+\xi)^{\nu-1}(1-\xi)^{m-\nu-1}\left(\frac{2\nu}{m} - 1\right) \\ &= \frac{m}{1-\xi^2}\left[C_m\left(t^3;\xi\right) - 2\xi C_m\left(t^2;\xi\right) + \xi^2 C_m(t;\xi)\right] \\ &= \frac{m}{1-\xi^2}\left[\xi^3 + \frac{3m-2}{m^2}\xi(1-\xi)(1+\xi) - 2\xi\left(\xi^2 + \frac{(1-\xi)(1+\xi)}{m}\right) + \xi^3\right] \\ &= \xi - \frac{1}{m}2\xi. \end{aligned}$$

Similarly, taking into account the values of the operators C_m at test functions, (2.3)–(2.5) can be obtained through straightforward calculations. □

Lemma 2. *For all $m \in \mathbb{N}$ and $\xi \in (-1, 1)$, we have*

$$C_m^*(t-\xi;\xi) = -\frac{1}{m}2\xi,$$

and

$$C_m^*\left((t-\xi)^2;\xi\right) = \frac{1}{m^2}(-3\xi^2 m + 3m + 6\xi^2 - 2).$$

Proof. The desired results can be obtained from the linearity of the operators C_m^* and the above-given lemma. □

Lemma 3. *For all $m \in \mathbb{N}$ and $\xi \in (-1, 1)$, we have the following limits:*

$$\begin{aligned} \lim_{m\to\infty} mC_m^*(t-\xi;\xi) &= -2\xi, \\ \lim_{m\to\infty} mC_m^*\left((t-\xi)^2;\xi\right) &= -3\xi^2 + 3, \\ \lim_{m\to\infty} m^2 C_m^*\left((t-\xi)^4;\xi\right) &= 15\xi^4 - 30\xi^2 + 15. \end{aligned}$$

Proof. From Lemma 1 and the linearity of the operators C_m^*, we can obtain the above limits. □

Theorem 1. *For every $f \in C[-1,1]$, $C_m^*(f;\xi)$ converges uniformly to f on each compact subset of $(-1,1)$ as $m \to \infty$.*

Proof. To prove the theorem, it is helpful to consider the results of Lemma 1. As can clearly be seen, $C_m^*(1;\xi)$, $C_m^*(t;\xi)$, and $C_m^*\left(t^2;\xi\right)$ converge uniformly to 1, ξ, and ξ^2, respectively, in the limit case on $(-1,1)$. Therefore, by the well-known Korovkin theorem [17], the proof is completed. □

Remark 3. *Although the classical Korovkin theorem is formulated for continuous functions on closed intervals, it can be suitably extended to open intervals by considering local uniform convergence. Since the operators C_m^* are from $C[-1,1]$ to $C(-1,1)$, we apply a localized version of the Korovkin-type theorem. Specifically, we verify the convergence of the operators on the test functions $\{1,t,t^2\}$ and establish that $\lim_{m\to\infty} C_m^*(f;\xi) = f(\xi)$ holds uniformly on each compact subset of $(-1,1)$ for all $f \in C[-1,1]$. This validates the approximation properties of the sequence (C_m^*) within the framework of the Korovkin-type theorem adapted to open intervals.*

The degree of local approximation for the operators C_m^* will now be determined by the classical modulus of continuity, as defined in [18], for a function f in $C[-1,1]$:

$$\omega_1(f;\delta) = \sup_{\substack{|t-\xi|\le\delta \\ t,\xi\in[-1,1]}} |f(t)-f(\xi)|.$$

Recall that the modulus of continuity has the following property:

$$\omega_1(f;\alpha) = \omega_1\left(f;\frac{\alpha}{\beta}\beta\right) \le \left(1+\frac{\alpha}{\beta}\right)\omega_1(f;\beta) \text{ for } \alpha \ge 0,\ \beta > 0.$$

Theorem 2. *For each $f \in C[-1,1]$ and $\xi \in (-1,1)$, we have*

$$|C_m^*(f;\xi) - f(\xi)| \le 2\omega_1(f;\delta_m(\xi)),$$

where

$$\delta_m(\xi) = \sqrt{\frac{-3\xi^2 m + 3m + 6\xi^2 - 2}{m^2}}.$$

Proof. Applying the property that $C_m^*(1;\xi) = 1$, together with the linearity of the sequence (C_m^*) and the property of $\omega_1(f;\delta)$, has led to the following result:

$$\begin{aligned}
\left|C_m^*(f;\xi) - f(\xi)\right| &= \left|\frac{1}{m2^m}\sum_{\nu=0}^{m}\binom{m}{\nu}(2\nu-m-m\xi)^2(1+\xi)^{\nu-1}(1-\xi)^{m-\nu-1} f\left(\frac{2\nu}{m}-1\right) - f(\xi)\right| \\
&\le \frac{1}{m2^m}\sum_{\nu=0}^{m}\binom{m}{\nu}(2\nu-m-m\xi)^2(1+\xi)^{\nu-1}(1-\xi)^{m-\nu-1}\left|f\left(\frac{2\nu}{m}-1\right) - f(\xi)\right| \\
&\le \frac{1}{m2^m}\sum_{\nu=0}^{m}\binom{m}{\nu}(2\nu-m-m\xi)^2(1+\xi)^{\nu-1}(1-\xi)^{m-\nu-1}\,\omega_1\left(f,\left|\frac{2\nu}{m}-1-\xi\right|\right) \\
&\le \frac{1}{m2^m}\sum_{\nu=0}^{m}\binom{m}{\nu}(2\nu-m-m\xi)^2(1+\xi)^{\nu-1}(1-\xi)^{m-\nu-1}\left(1+\frac{1}{\delta^2}\frac{(2\nu-m-m\xi)^2}{m^2}\right)\omega_1(f;\delta)
\end{aligned}$$

$$= \left(1 + \frac{1}{\delta^2} \frac{-3\xi^2 m + 3m + 6\xi^2 - 2}{m^2}\right) \omega_1 (f; \delta).$$

The desired result is obtained by choosing

$$\delta_m(\xi) = \sqrt{\frac{-3\xi^2 m + 3m + 6\xi^2 - 2}{m^2}},$$

as demonstrated above. □

The focus of the next part of this study is to determine the rate of convergence of the sequence of operators (C_m^*) in a Lipschitz class defined as follows:

$$Lip_M(\Theta) = \left\{ f : |f(y) - f(\xi)| \le M |y - \xi|^\Theta, \xi, y \in (-1, 1) \right\},$$

where M is a positive constant and $0 < \Theta \le 1$.

Theorem 3. *Let $f \in Lip_M(\Theta)$ and $\Theta \in (0, 1]$. Then for $\xi \in (-1, 1)$ and $m > 2$, the following estimate holds:*

$$\left| C_m^*(f; \xi) - f(\xi) \right| \le M \left(\frac{3m - 2}{m^2} \right)^{\frac{\Theta}{2}}.$$

Proof. Let $f \in Lip_M(\Theta)$, $\xi \in (-1, 1)$, and $0 < \Theta \le 1$; then we have

$$\begin{aligned} \left| C_m^*(f; \xi) - f(\xi) \right| &\le \frac{1}{m2^m} \sum_{\nu=0}^{m} \binom{m}{\nu} (2\nu - m - m\xi)^2 (1 + \xi)^{\nu-1} (1 - \xi)^{m-\nu-1} \left| f\left(\frac{2\nu}{m} - 1 \right) - f(\xi) \right| \\ &\le \frac{M}{m2^m} \sum_{\nu=0}^{m} \binom{m}{\nu} (2\nu - m - m\xi)^2 (1 + \xi)^{\nu-1} (1 - \xi)^{m-\nu-1} \left| \frac{2\nu}{m} - 1 - \xi \right|^\Theta. \end{aligned}$$

Applying the Hölder's inequality by taking $p = \frac{2}{\Theta}$ and $q = \frac{2}{2 - \Theta}$, we have

$$\begin{aligned} \left| C_m^*(f; \xi) - f(\xi) \right| &\le M \left\{ \frac{1}{m2^m} \sum_{\nu=0}^{m} \binom{m}{\nu} (2\nu - m - m\xi)^2 (1 + \xi)^{\nu-1} (1 - \xi)^{m-\nu-1} \left(\frac{2\nu}{m} - 1 - \xi \right)^2 \right\}^{\frac{\Theta}{2}} \\ &\le M \left(\sqrt{\frac{-3\xi^2 m + 3m + 6\xi^2 - 2}{m^2}} \right)^\Theta. \end{aligned}$$

Since

$$\sqrt{\frac{-3\xi^2 m + 3m + 6\xi^2 - 2}{m^2}} \le \sqrt{\frac{3m - 2}{m^2}},$$

the proof is completed. □

In what follows, a quantitative estimate of the operators C_m^* is to be considered by means of the second-order modulus of continuity of f, as defined in [18]. To establish this estimate, we first review the definitions of the function spaces and concepts that will be employed.

The Peetre's κ-functional is defined by

$$\kappa(f;\delta) := \inf_{g\in C^2[-1,1]} \left\{\|f-g\|_{C[-1,1]} + \delta\|g''\|_{C[-1,1]}\right\}, \tag{2.6}$$

where

$$C^2[-1,1] = \{f \in C[-1,1] : f'' \in C[-1,1]\},$$

with the norm

$$\|f\|_{C^2[-1,1]} = \|f\|_{C[-1,1]} + \|f'\|_{C[-1,1]} + \|f''\|_{C[-1,1]}, \tag{2.7}$$

and $C[-1,1] = \{f : [-1,1] \to \mathbb{R} : f \text{ is continuous}\}$, with the sup-norm.

The second-order modulus of continuity of $f \in C[-1,1]$ is to be considered as well:

$$\omega_2(f;\delta) = \sup_{0<h<\delta}\ \sup_{\xi,\xi+2h\in(-1,1)} |f(\xi+2h) - 2f(\xi+h) + f(\xi)|, \tag{2.8}$$

where $\delta > 0$. The following relationship exists between the second-order modulus of continuity and the Peetre's $\kappa-$ functional:

$$\kappa(f;\delta) \le B\omega_2\left(f;\sqrt{\delta}\right), \tag{2.9}$$

where B is a positive constant [19].

Lemma 4. *For $\xi \in (-1,1)$ and $f \in C[-1,1]$, we have*

$$|C_m^*(f;\xi) \le \|f\|_{C[-1,1]}.$$

Proof. For the operators C_m^*, we have

$$\begin{aligned}
\left|C_m^*(f;\xi)\right| &= \left|\frac{1}{m2^m}\sum_{\nu=0}^{m}\binom{m}{\nu}(2\nu-m-m\xi)^2(1+\xi)^{\nu-1}(1-\xi)^{m-\nu-1} f\left(\frac{2\nu}{m}-1\right)\right| \\
&\le \frac{1}{m2^m}\sum_{\nu=0}^{m}\binom{m}{\nu}(2\nu-m-m\xi)^2(1+\xi)^{\nu-1}(1-\xi)^{m-\nu-1}\left|f\left(\frac{2\nu}{m}-1\right)\right| \\
&\le \|f\|_{C[-1,1]} C_m^*(1;\xi) \\
&= \|f\|_{C[-1,1]}.
\end{aligned}$$

□

Theorem 4. *Let $f \in C[-1,1]$ and $\xi \in (-1,1)$. Then for any $m \in \mathbb{N}$ there exists a positive constant B such that*

$$\left|C_m^*(f;\xi) - f(\xi)\right| \le B\omega_2(f;\eta_m(\xi)) + 2\omega_1(f;\rho_m(\xi))$$

where

$$\eta_m(\xi) = \sqrt{\frac{-3\xi^2 m + 3m + 10\xi^2 - 2}{2m^2}}$$

and $\rho_m(\xi) = \left|\frac{-2\xi}{m}\right|$.

Proof. Let us define the operators $C_m^{**} : C[-1,1] \to C(-1,1)$ by

$$C_m^{**}(g;\xi) = C_m^{*}(g;\xi) - g\left(\frac{(m-2)\xi}{m}\right) + g(\xi). \tag{2.10}$$

From Lemma 1, we have

$$C_m^{**}(1;\xi) = 1, \tag{2.11}$$

$$\begin{aligned} C_m^{**}(t-\xi;\xi) &= C_m^{*}(t-\xi;\xi) - \left(\frac{(m-2)\xi}{m} - \xi\right) + \xi - \xi \\ &= \frac{-2\xi}{m} - \left(\frac{(m-2)\xi}{m} - \xi\right) + \xi - \xi \\ &= 0. \end{aligned} \tag{2.12}$$

On the other hand, from the Taylor expansion of $g \in C^2[-1,1]$, we can write

$$g(t) = g(\xi) + (t-\xi)g'(\xi) + \int_\xi^t (t-s)g''(s)ds,\ t \in (-1,1).$$

By applying the operators C_m^{**} to the above equality, we obtain

$$\begin{aligned} C_m^{**}(g;\xi) &= C_m^{**}\left(g(\xi) + (t-\xi)g'(\xi) + \int_\xi^t (t-s)g''(s)ds;\xi\right) \\ &= g(\xi) + C_m^{**}\left((t-\xi)g'(\xi);\xi\right) + C_m^{**}\left(\int_\xi^t (t-s)g''(s)ds;\xi\right). \end{aligned}$$

Thus, we have

$$C_m^{**}(g;\xi) - g(\xi) = g'(\xi)C_m^{**}(t-\xi;\xi) + C_m^{**}\left(\int_\xi^t (t-s)g''(s)ds;\xi\right).$$

With the help of (2.10) and (2.12), we may write

$$\begin{aligned} C_m^{**}(g;\xi) - g(\xi) &= C_m^{**}\left(\int_\xi^t (t-s)g''(s)ds;\xi\right) \\ &= C_m^{*}\left(\int_\xi^t (t-s)g''(s)ds;\xi\right) - \int_\xi^{\frac{(m-2)\xi}{m}} \left(\frac{(m-2)\xi}{m} - s\right)g''(s)ds \\ &+ \int_\xi^\xi (\xi-s)\,g''(s)ds. \end{aligned} \tag{2.13}$$

Additionally, we can write

$$\begin{aligned} \left|\int_\xi^t (t-s)g''(s)ds\right| &\le \int_\xi^t |t-s|\,|g''(s)|\,ds \le \|g''\|_{C[-1,1]}\left|\int_\xi^t |t-s|\,ds\right| \\ &\le \frac{1}{2}(t-\xi)^2\|g''\|_{C[-1,1]} \end{aligned} \tag{2.14}$$

and

$$\left|\int_{\xi}^{\frac{(m-2)\xi}{m}}\left(\frac{(m-2)\xi}{m}-s\right)g''(s)ds\right| \leq \|g''\|_{C[-1,1]}\int_{\xi}^{\frac{(m-2)\xi}{m}}\left(\frac{(m-2)\xi}{m}-s\right)ds$$
$$= \|g''\|_{C[-1,1]}\frac{2\xi^2}{m^2}. \tag{2.15}$$

By using (2.14) and (2.15) in (2.13), we obtain

$$\begin{aligned}\left|C_m^{**}(g;\xi)-g(\xi)\right| &\leq \frac{1}{2}\|g''\|_{C[-1,1]}C_m^*\left((t-\xi)^2;\xi\right)+\|g''\|_{C[-1,1]}\frac{2\xi^2}{m^2}\\ &= \|g''\|_{C[-1,1]}\left(\frac{1}{2}C_m^*\left((t-\xi)^2;\xi\right)+\frac{2\xi^2}{m^2}\right)\\ &= \|g''\|_{C[-1,1]}\eta_m^2(\xi),\end{aligned}$$

where

$$\eta_m(\xi) = \sqrt{\frac{1}{2}C_m^*\left((t-\xi)^2;\xi\right)+\frac{2\xi^2}{m^2}} = \sqrt{\frac{-3\xi^2m+3m+10\xi^2-2}{2m^2}}.$$

Moreover, from Lemma 4, we have

$$\begin{aligned}\left|C_m^{**}(g;\xi)\right| &= \left|C_m^*(g;\xi)-g\left(\frac{(m-2)\xi}{m}\right)+g(\xi)\right|\\ &\leq \left|C_m^*(g;\xi)\right|+\left|g\left(\frac{(m-2)\xi}{m}\right)\right|+|g(\xi)|\\ &\leq 3\|g\|_{C[-1,1]}.\end{aligned}$$

Now, using the definitions of the operators C_m^{**} and the modulus of continuity, as well as some basic calculations, we have

$$\begin{aligned}\left|C_m^*(f;\xi)-f(\xi)\right| &= \left|C_m^{**}(f;\xi)-f(\xi)+f\left(\frac{(m-2)\xi}{m}\right)-f(\xi)+g(\xi)-g(\xi)+C_m^{**}(g;\xi)-C_m^{**}(g;\xi)\right|\\ &\leq \left|C_m^{**}(f-g;\xi)-(f-g)(\xi)\right|+\left|C_m^{**}(g;\xi)-g(\xi)\right|+\left|f\left(\frac{(m-2)\xi}{m}\right)-f(\xi)\right|\\ &\leq 4\left(\|f-g\|_{C[-1,1]}+\|g''\|_{C[-1,1]}\eta_m^2(\xi)\right)+2\omega_1\left(f;\rho_m(\xi)\right),\end{aligned} \tag{2.16}$$

where

$$\rho_m(\xi) = \left|\frac{(m-2)\xi}{m}-\xi\right| = \left|\frac{-2\xi}{m}\right|.$$

Hence, by taking the infimum over all $g \in C^2[-1,1]$ in (2.16), we have

$$\left|C_m^*(f;\xi)-f(\xi)\right| \leq 4\kappa(f;\eta_m^2(\xi))+2\omega_1(f;\rho_m(\xi)).$$

Finally, by using the inequality (2.9), we obtain

$$\left|C_m^*(f;\xi)-f(\xi)\right| \leq B\omega_2(f;\eta_m(\xi))+2\omega_1(f;\rho_m(\xi)),$$

which completes the proof. □

Now, we give the Voronovskaja-type theorem for the operators C_m^*.

Theorem 5. *Let* $f \in C^2[-1,1]$. *Then the following equality holds:*

$$\lim_{m\to\infty} m\left(C_m^*(f;\xi) - f(\xi)\right) = -2\xi f'(\xi) + \frac{-3\xi^2+3}{2} f''(\xi).$$

Proof. From the Taylor expansion of f, we may write

$$f(t) = f(\xi) + f'(\xi)(t-\xi) + \frac{1}{2} f''(\xi)(t-\xi)^2 + \epsilon(t,\xi)(t-\xi)^2,$$

where $\epsilon(t,\xi) = \frac{f''(\sigma)-f''(\xi)}{2} \to 0$ as $t \to \xi$. Here, σ is between ξ and t. By applying the operators C_m^* to both sides on the above equation, we have

$$C_m^*(f;\xi) - f(\xi) = f'(\xi) C_m^*(t-\xi;\xi) + \frac{1}{2} f''(\xi) C_m^*\left((t-\xi)^2;\xi\right) + C_m^*\left(\epsilon(t,\xi)(t-\xi)^2;\xi\right).$$

So, we have

$$\begin{aligned}\lim_{m\to\infty} m\left[C_m^*(f;\xi) - f(\xi)\right] =& f'(\xi) \lim_{m\to\infty} mC_m^*(t-\xi;\xi) + \frac{1}{2} f''(\xi) \lim_{m\to\infty} mC_m^*\left((t-\xi)^2;\xi\right) \\ &+ \lim_{m\to\infty} mC_m^*\left(\epsilon(t,\xi)(t-\xi)^2;\xi\right).\end{aligned}$$

On the other hand, via Cauchy–Schwarz inequality, we write

$$\lim_{m\to\infty} mC_m^*\left(\epsilon(t,\xi)(t-\xi)^2;\xi\right) \le \sqrt{\lim_{m\to\infty} C_m^*\left(\epsilon^2(t,\xi);\xi\right)} \sqrt{\lim_{m\to\infty} m^2 C_m^*\left((t-\xi)^4;\xi\right)}.$$

Since $\lim_{m\to\infty} C_m^*\left(\epsilon^2(t,\xi);\xi\right) = 0$ and by Lemma 3,

$$\lim_{m\to\infty} m^2 C_m^*\left((t-\xi)^4;\xi\right) = 15\xi^4 - 30\xi^2 + 15,$$

so

$$\lim_{m\to\infty} mC_m^*\left(\epsilon(t,\xi)(t-\xi)^2;\xi\right) = 0.$$

Finally, with the help of Lemma 3, we obtain

$$\lim_{m\to\infty} m\left(C_m^*(f;\xi) - f(\xi)\right) = -2\xi f'(\xi) + \frac{-3\xi^2+3}{2} f''(\xi).$$

□

3. Numerical examples

In the subsequent section, the convergence of the operators will be illustrated through the utilization of numerical examples. The following graphical comparisons will be presented: those between the operators and the functions.

Example 1. *For $m = 50$, $m = 100$, and $m = 150$, the convergence of $C_m^*(f;\xi)$ to $f(\xi) = \frac{1}{1+\xi^2}$ is shown in Figure 1 for $\xi \in [-0.8, 0.8]$.*

Example 2. *For $m = 20$, $m = 30$, and $m = 50$, the convergence of $C_m^*(f;\xi)$ to $f(\xi) = cos\xi + \xi^3$ is shown in Figure 2 for $\xi \in [-0.8, 0.8]$.*

Example 3. *Let us define the function $f(\xi) = \frac{1}{1+\xi^2}$. For $-1 < \xi < 1$, the error estimate of the function f found by the first modulus of continuity for the operators C_m^* is given in Table 1. As can be seen from the values in Table 1, the error bound of the function decreases as the m values increase.*

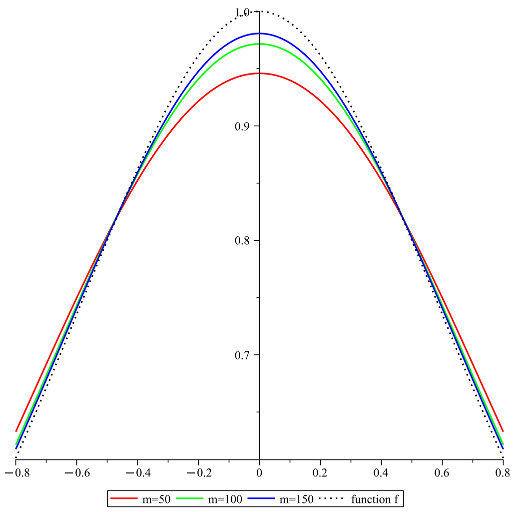

Figure 1. The convergence of $C_m^*(f;\xi)$ to $f(\xi) = \frac{1}{1+\xi^2}$ for $m = 50$, $m = 100$, and $m = 150$.

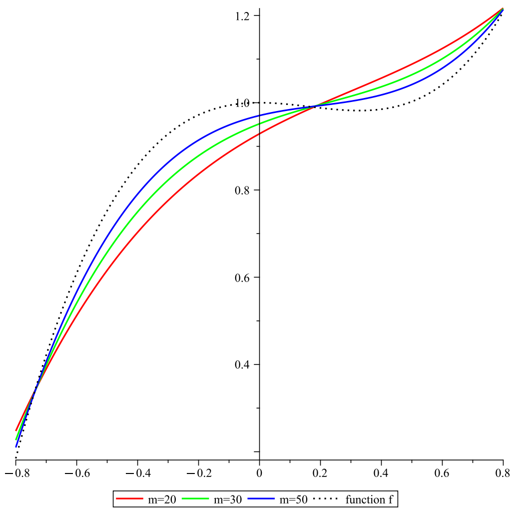

Figure 2. The convergence of $C_m^*(f;\xi)$ to $f(\xi) = cos\xi + \xi^3$ for $m = 20$, $m = 30$, and $m = 50$.

Table 1. Error estimation of the function $f(\xi) = \frac{1}{1+\xi^2}$ for the operators $C_m^*(f;\xi)$.

m	Error estimate by the operators $C_m^*(f;\xi)$
10^1	0.6098216645
10^2	0.1995498353
10^3	0.0611718537
10^4	0.0190373507
10^5	0.0059856275
10^6	0.0018892479
10^7	0.0005970714
10^8	0.0001887743
10^9	0.0000596920
10^{10}	0.0000188759
10^{11}	0.0000059690
10^{12}	0.0000018875
10^{13}	0.0000005969
10^{14}	0.0000001887
10^{15}	0.0000000596

4. Conclusions

This study introduced a moment-based modification of Bernstein operators on the symmetric interval, establishing their fundamental approximation properties. The analysis is supported by quantitative estimates and a Voronovskaja-type theorem, providing a theoretical foundation. Numerical examples and graphical illustrations demonstrated the efficiency of the proposed operators. Potential future research directions include exploring multivariate extensions, studying weighted function spaces, and investigating applications in fields such as signal processing and numerical analysis. These directions may further enhance the applicability and usefulness of the modified operators.

Use of Generative-AI tools declaration

The author declares that she has not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no competing financial interest.

References

1. S. N. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités, *Commun. Kharkov Math. Soc.*, **13** (1912), 1-2.

2. D. D. Stancu, Asupra unei generalizări a polinoamelor lui Bernstein, *Stud. Univ. Babes-Bolyai Ser. Math.-Phys.*, **14** (1969), 31–45.

3. L. V. Kantorovich, Sur certains développements suivant les polynômes de la forme de S. Bernstein I, II, *C. R. Acad. Sci. U.R.S.S.*, **20** (1930), 563–568, 595–600.

4. J. L. Durrmeyer, *Une formule d'inversion de la transformée de Laplace: application à la théorie des moments*, Thése de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.

5. F. Schurer, Linear positive operators in approximation theory, *Mathematical Institute of the Technological University Delft, Report*, 1962.

6. I. Chlodowsky, Sur le développement des fonctions définies dans un intervalle infini en séries de polynomes de M. S. Bernstein, *Compositio Math.*, **4** (1937), 380–393.

7. G. M. Phillips, Bernstein polynomials based on the q-integers, *Ann. Numer. Math.*, **4** (1997), 511–518.

8. A. Çilo, *In* $[-1, 1]$ *ranges Bernstein polynomials approach properties and approach speed*, MS. Thesis, Harran University, 2012.

9. F. Usta, On new modification of Bernstein operators: theory and applications, *Iran. J. Sci. Technol. Trans. Sci.*, **44** (2020), 1119–1124. https://doi.org/10.1007/s40995-020-00919-y

10. A. Senapati, A. Kumar, T. Som, Convergence analysis of modified Bernstein–Kantorovich type operators, *Rend. Circ. Mat. Palermo, II. Ser.*, **72** (2023), 3749–3764. https://doi.org/10.1007/s12215-022-00860-6

11. M. Sofyalıoğlu, K. Kanat, B. Çekim, Parametric generalization of the modified Bernstein operators, *Filomat*, **36** (2022), 1699–1709. https://doi.org/10.2298/FIL2205699S

12. W. T. Cheng, S. A. Mohiuddine, Construction of a new modification of Baskakov operators on $(0, \infty)$, *Filomat*, **37** (2023), 139–154. https://doi.org/10.2298/FIL2301139C

13. Q. B. Cai, M. Sofyalıoğlu, K. Kanat, B. Çekim, Some approximation results for the new modification of Bernstein-Beta operators, *AIMS Math.*, **7** (2022), 1831–1844. https://doi.org/10.3934/math.2022105

14. N. I. Mahmudov, M. Kara, Generalization of Szász–Mirakjan operators and their approximation properties, *J. Anal.*, **33** (2025), 1687–1710. https://doi.org/10.1007/s41478-025-00890-0

15. Y. S. Wu, W. T. Cheng, F. L. Chen, Y. H. Zhou, Approximation theorem for new modification of q-Bernstein operators on $(0, 1)$, *J. Funct. Spaces*, **4** (2021), 6694032. https://doi.org/10.1155/2021/6694032

16. N. Rao, M. Shahzad, N. K. Jha, Study of two dimensional α-modified Bernstein bi-variate operators, *Filomat*, **39** (2025), 1509–1522. https://doi.org/10.2298/FIL2505509R

17. P. P. Korovkin, On convergence of linear positive operators in the space of continuous functions (Russian), *Dokl. Akad. Nauk SSSR (NS)*, **90** (1953), 961–964.

18. Z. Ditzian, V. Totik, *Moduli of smoothness*, 1 Ed., Springer-Verlag, 1987. https://doi.org/10.1007/978-1-4612-4778-4

19. R. A. DeVore, G. G. Lorentz, *Constructive approximation*, Vol. 303, Springer, 1993.