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Research article

Boundedness and approximation of Hilbert-type operators in the Triebel–Lizorkin spaces: Applications to non-smooth volatility dynamics

Philip Ajibola Bankole[1], **Mohsin Nasir**[2,3], **Manuel De la Sen**[4,*] **and Sina Etemad**[5,6,*]

[1] Lagos State University of Education, Department of Mathematics, Oto/Ijanikin, Nigeria; Email: bankolepa@lasued.edu.ng (P.A.B)

[2] Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan; Email: mohsinnasir_22@sms.edu.pk (M.N.)

[3] Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan

[4] Department of Electricity and Electronics, Institute of Research and Development of Processes, Faculty of Science and Technology, University of the Basque Country, 48940-Leioa, Bizkaia, Spain

[5] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz 53751, Iran

[6] Jadara Research Center, Jadara University, Irbid 21110, Jordan

* **Correspondence:** Email: manuel.delasen@ehu.eus, sina.etemad@azaruniv.ac.ir.

Abstract: This paper investigates the boundedness and approximation properties of Hilbert-type singular integral operators within the framework of Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R})$, a refined class of function spaces central to microlocal and harmonic analysis. We introduced a regularized Hilbert-type operator and established its strong convergence in Triebel–Lizorkin norms. Using Fourier analytic and interpolation techniques, we rigorously proved new boundedness results under optimal smoothness and integrability conditions. Furthermore, we demonstrated how this functional analytic framework enables robust modeling of non-smooth volatility structures in financial derivatives pricing, particularly under stochastic volatility regimes and market turbulence. This approach unifies singular operator theory and financial mathematics, offering a novel path for analyzing irregular price dynamics through spectral and geometric regularity tools.

Keywords: Hilbert-type operators; Triebel–Lizorkin spaces; singular integral approximation; stochastic volatility modeling; microlocal harmonic analysis
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1. Introduction

Singular integral operators, notably the Hilbert transform, are foundational tools in modern harmonic analysis and arise naturally in the theory of partial differential equations, signal processing, and time-frequency analysis. The Hilbert transform $\mathcal{H}$ is defined by the singular integral

$$\mathcal{H}f(x) = \text{p.v.}\ \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy,$$

where "p.v." denotes the Cauchy principal value. This operator serves as a prototypical example of Calderón–Zygmund singular integral operators and is intimately connected with Fourier multipliers and pseudodifferential operators [1, 2].

In classical settings, the boundedness of $\mathcal{H}$ on $L^p(\mathbb{R})$ for $1 < p < \infty$ is well known [3]. However, traditional Lebesgue and Sobolev spaces do not fully capture the nuanced behavior of functions with irregularities, anisotropies, or localized frequency characteristics. This motivates the need for refined function spaces—such as Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R})$—which generalize Sobolev spaces via the Littlewood–Paley decomposition and admit a rich multiscale frequency structure [4–9].

Triebel–Lizorkin spaces allow for a deeper microlocal analysis of singular integral operators. These spaces are defined via a dyadic partition of unity in the frequency domain as follows:

$$F^s_{p,q}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{F^s_{p,q}} := \left\| \left(\sum_{j\in\mathbb{Z}} 2^{jsq} |\varphi_j * f|^q \right)^{1/q} \right\|_{L^p} < \infty \right\},$$

where $\{\varphi_j\}_{j\in\mathbb{Z}}$ is a dyadic Littlewood–Paley resolution of unity, and $*$ denotes convolution.

In recent years, a number of studies involving variants of Fourier multipliers and function spaces including Triebel–Lizorkin spaces have emerged (see [10–15]). However, most of the existing studies in this area are concentrated on theoretical results. Hence, one of the aims of this study is to extend the theoretical framework on a class of Hilbert-type operators to the financial model representation of the Black–Scholes partial differential equation (BSPDE) and the tractability of the results for option price analysis and visualization. Refer to [16, 17] and the references therein for more details on the BSPDE. This functional framework is ideally suited to studying the boundedness and approximation properties of singular integral transforms such as the Hilbert transform $\mathcal{H}$, particularly in the presence of irregular or non-smooth financial data.

The field of mathematical finance increasingly requires analytical tools capable of handling non-smooth volatility, irregular market microstructure noise, and turbulent price dynamics [18–22]. In particular, the modeling of high-frequency financial data, rough volatility, and fractional stochastic processes necessitates a time-frequency sensitive framework to capture essential local features and singularities [23, 24].

This paper initiates a rigorous study of the approximation and boundedness of the Hilbert transform in Triebel–Lizorkin spaces. We introduce a regularized operator $\mathcal{H}_\delta$ defined via Fourier multiplier smoothing as

$$\mathcal{H}_\delta f(x) = \int_{\mathbb{R}^d} e^{-\delta|\xi|}(-i\,\text{sgn}(\xi))\widehat{f}(\xi)e^{2\pi i x\xi}d\xi,$$

and investigate its convergence in $F^s_{p,q}$ norms. We also prove its boundedness using advanced multiplier theorems, Littlewood–Paley theory, and interpolation estimates.

The novelty of our work lies in the synthesis of advanced harmonic analysis with applications to financial modeling. We demonstrate that the Triebel–Lizorkin framework can effectively describe price dynamics with low regularity and is suitable for representing solutions to pricing equations under stochastic and fractional volatility.

Among our main objectives of this study are:

- To prove the boundedness of $\mathcal{H}$ and its regularized form $\mathcal{H}_\delta$ on $F^s_{p,q}(\mathbb{R})$, with precise norm estimates;
- To establish convergence rates of $\mathcal{H}_\delta f \to \mathcal{H} f$ in $F^s_{p,q}$-norms as $\delta \to 0^+$;
- To provide applications to financial modeling—particularly in pricing under rough volatility—demonstrating the effectiveness of Triebel–Lizorkin spaces in modeling financial time series with singular features.

This research bridges theoretical harmonic analysis and applied financial mathematics, opening a pathway for deploying sophisticated functional analytic tools in modeling real-world financial systems.

2. Preliminaries

In this section, we recall essential background on Triebel–Lizorkin spaces, Littlewood–Paley theory, and Fourier analysis techniques relevant to our study of singular integral operators and their approximations.

Remark 2.1. *Throughout the paper, we use the following notations:*

- *$\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transforms;*
- *$\langle x \rangle = (1 + |x|^2)^{1/2}$;*
- *$A \lesssim B$ means $A \le CB$ for some constant $C > 0$.*

Definition 2.2. *(Littlewood–Paley decomposition) [25]*

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be a non-negative radial function such that $\varphi(\xi) = 1$ for $|\xi| \le 1$, and $\varphi(\xi) = 0$ for $|\xi| > \frac{3}{2}$. Define $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$, so that ψ is supported in the annulus $\{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\}$.

The dyadic decomposition of unity is given by

$$\sum_{j=-\infty}^{\infty} \psi_j(\xi) = 1 \quad \text{for all } \xi \neq 0,$$

where $\psi_j(\xi) = \psi(2^{-j}\xi)$. The frequency localization operator Δ_j is defined via:

$$\widehat{\Delta_j f}(\xi) = \psi_j(\xi)\widehat{f}(\xi), \quad j \in \mathbb{Z}.$$

Definition 2.3. *(Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$) [26]*

The inhomogeneous Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^d)$, $s \in \mathbb{R}$, is defined as the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{F^s_{p,q}} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\Delta_j f(x)|^q \right)^{1/q} \right\|_{L^p_x} < \infty, \tag{2.1}$$

with the usual modification when $q = \infty$. These spaces interpolate between Sobolev and Hardy spaces and are stable under Fourier multipliers satisfying Mihlin–Hörmander type conditions.

Definition 2.4. *(Fourier multipliers and singular integrals [11, 12, 14, 27])*

Let $m : \mathbb{R}^d \to \mathbb{C}$ be a bounded function. The associated Fourier multiplier operator T_m acting on $f(x)$ is defined as:

$$T_m f(x) = \mathcal{F}^{-1}(m(\xi)\widehat{f}(\xi))(x). \tag{2.2}$$

In particular, the Hilbert transform corresponds to the multiplier $m(\xi) = -i\, sgn(\xi)$ in one dimension.

Definition 2.5. *(Bony's paraproduct formula in Triebel–Lizorkin spaces [28–30]) Let $f, g \in \mathscr{S}'(\mathbb{R}^d)$ be tempered distributions. With the Littlewood–Paley decomposition $(\Delta_j)_{j\in\mathbb{Z}}$ and the low-frequency cut-off $S_j = \sum_{k<j} \Delta_k$, the product fg can be decomposed as*

$$fg = T_f g + T_g f + R(f, g), \tag{2.3}$$

where

$$T_f g = \sum_{j\in\mathbb{Z}} S_{j-1} f\, \Delta_j g,$$

$$T_g f = \sum_{j\in\mathbb{Z}} S_{j-1} g\, \Delta_j f,$$

$$R(f, g) = \sum_{j\in\mathbb{Z}} \Delta_j f\, \widetilde{\Delta}_j g, \quad \textit{with } \widetilde{\Delta}_j g = \sum_{|k-j|\le 1} \Delta_k g.$$

Remark 2.6. *The decomposition* (2.3) *is continuous in the Triebel–Lizorkin scale:*

(1) For $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$,

$$T_f g : F^s_{p,q} \times L^\infty \longrightarrow F^s_{p,q}, \quad T_g f : L^\infty \times F^s_{p,q} \longrightarrow F^s_{p,q}.$$

(2) If $s_1 + s_2 > 0$, then

$$R(f, g) : F^{s_1}_{p_1,q_1} \times F^{s_2}_{p_2,q_2} \longrightarrow F^{s_1+s_2}_{p,q},$$

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \min\left(1, \frac{1}{q_1} + \frac{1}{q_2}\right).$$

(3) In particular, if $s > 0$ and $f, g \in F^s_{p,q} \cap L^\infty$, then

$$fg \in F^s_{p,q}, \quad \|fg\|_{F^s_{p,q}} \lesssim \|f\|_{L^\infty} \|g\|_{F^s_{p,q}} + \|g\|_{L^\infty} \|f\|_{F^s_{p,q}}.$$

Definition 2.7. *(Fourier transform ([31, 32])) Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function for $x \in \mathbb{R}$. Then the Fourier transform of f is defined as*

$$\mathcal{F}(f(x); \omega) = f(\omega) = \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx, \qquad \omega \in \mathbb{R}, \tag{2.4}$$

where $i = \sqrt{-1}$ and ω is a parameter.

Definition 2.8. *(Fractional Hilbert operator) For $\alpha \in (0,1)$, define $\mathcal{H}_\alpha$ by the Fourier multiplier*

$$\widehat{\mathcal{H}_\alpha f}(\xi) = i\,\mathrm{sgn}(\xi)\,|\xi|^\alpha\,\widehat{f}(\xi), \qquad \xi \in \mathbb{R}. \tag{2.5}$$

Equivalently, $\mathcal{H}_\alpha = \mathcal{H} \circ |D|^\alpha$. Then $\mathcal{H}_\alpha : F^s_{p,q}(\mathbb{R}) \to F^{s-\alpha}_{p,q}(\mathbb{R})$ is bounded for $1 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$.

Definition 2.9. *(The Black–Scholes* PDE *operator with a fractional Hilbert term) On the log-price line $\mathbb{R}$ s.t. $x = \log S$, let $\sigma : \mathbb{R} \to \mathbb{R}$ and $r \in \mathbb{R}$. Define the spatial BS operator with a nonlocal fractional Hilbert part by*

$$\mathcal{L}_\alpha u(x) := \sigma(x)\,\mathcal{H}_\alpha u(x) + r\,x\,\partial_x u(x) - r\,u(x)\,, \tag{2.6}$$

where $\mathcal{L}_\alpha : F^s_{p,q}(\mathbb{R}) \to F^{s-\alpha}_{p,q}(\mathbb{R})$ for $s > \alpha$. The (backward) Black–Scholes PDE *reads*

$$\partial_t u(t,x) + \mathcal{L}_\alpha u(t,x) = 0. \tag{2.7}$$

Definition 2.10. *(Augmented BS PDE operator)*

$$\widetilde{\mathcal{L}}_\alpha u := \tfrac{1}{2}\sigma^2(x)\,\partial_{xx}u + (r - \tfrac{1}{2}\sigma^2(x))\partial_x u - ru + \lambda\,\sigma(x)\,\mathcal{H}_\alpha u, \tag{2.8}$$

so that $\partial_t u + \widetilde{\mathcal{L}}_\alpha u = 0$ and $\widetilde{\mathcal{L}}_\alpha : F^s_{p,q} \to F^{s-2}_{p,q} \cap F^{s-\alpha}_{p,q}$, typically requiring $s > \max\{2, \alpha\}$.

Theorem 2.11. *(Marschall's multiplier theorem [33, 34]) Let $m \in C^k(\mathbb{R}^d \setminus \{0\})$ with derivatives satisfying*

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad \text{for } |\alpha| \leq \left[\tfrac{d}{2}\right] + 1.$$

Then the associated Fourier multiplier operator $T_m f = \mathcal{F}^{-1}(m\widehat{f})$ is bounded on the Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^d)$ for all $1 < p < \infty$, $0 < q \leq \infty$, and $s \in \mathbb{R}$.

Refer to [9, 35] for a comprehensive treatment of Triebel–Lizorkin spaces and their applications in analysis and PDEs.

3. Main results

We now develop the main analytical results concerning the boundedness of a class of Hilbert-type integral operators on Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$. Throughout this section, we assume $0 < p < \infty$, $0 < q \leq \infty$, and $s \in \mathbb{R}$, unless otherwise stated.

Theorem 3.1. *(Boundedness of the operators) Let $\mathcal{H}_\alpha$ be the generalized Hilbert-type operator defined by*

$$\mathcal{H}_\alpha f(x) := p.v. \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}} K\left(\frac{x-y}{|x-y|}\right) dy, \tag{3.1}$$

where $0 < \alpha < d$, and the kernel $K \in L^\infty(\mathbb{S}^{d-1})$ satisfies the cancellation condition

$$\int_{\mathbb{S}^{d-1}} K(\theta)\,d\theta = 0.$$

Then for $1 < p < \infty$, $0 < q \leq \infty$, and $s \in \mathbb{R}$, the operator $\mathcal{H}_\alpha$ extends to a bounded linear operator on the Triebel–Lizorkin space $F^s_{p,q}(\mathbb{R}^d)$, i.e.,

$$\|\mathcal{H}_\alpha f\|_{F^s_{p,q}} \leq C\|f\|_{F^s_{p,q}}, \quad \forall f \in F^s_{p,q}(\mathbb{R}^d),$$

where the constant $C > 0$ depends only on p, q, s, α, and $\|K\|_{L^\infty(\mathbb{S}^{d-1})}$.

Proof. We prove the result by combining Fourier-analytic methods and maximal function estimates.

Step 1 (Fourier multiplier structure): We first observe that the operator $\mathcal{H}_\alpha$ can be represented as a Fourier multiplier operator. Indeed, by the standard singular integral theory, the Fourier transform of the distributional kernel

$$K_\alpha(x) := \frac{1}{|x|^{d-\alpha}} K\left(\frac{x}{|x|}\right), \qquad x \in \mathbb{R}^d \setminus \{0\},$$

is given by

$$\widehat{K_\alpha}(\xi) = m_\alpha\left(\frac{\xi}{|\xi|}\right)|\xi|^{-\alpha}, \qquad \xi \in \mathbb{R}^d \setminus \{0\}, \tag{3.2}$$

where $m_\alpha \in L^\infty(\mathbb{S}^{d-1})$ satisfies the cancellation condition

$$\int_{\mathbb{S}^{d-1}} m_\alpha(\theta)\, d\theta = 0.$$

Therefore, for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\widehat{\mathcal{H}_\alpha f}(\xi) = m_\alpha\left(\frac{\xi}{|\xi|}\right)|\xi|^{-\alpha}\, \widehat{f}(\xi).$$

In other words, $\mathcal{H}_\alpha$ acts as a Fourier multiplier operator with symbol

$$m(\xi) := m_\alpha\left(\frac{\xi}{|\xi|}\right)|\xi|^{-\alpha}.$$

Step 2 (Triebel–Lizorkin norm invariance): We use the Littlewood–Paley decomposition $f = \sum_{j=0}^\infty \Delta_j f$, where Δ_j are frequency-localized projections via a dyadic partition of unity. The Triebel–Lizorkin norm is given by

$$\|f\|_{F^s_{p,q}} = \left\|\left(\sum_{j=0}^\infty 2^{jsq}|\Delta_j f(x)|^q\right)^{1/q}\right\|_{L^p}. \tag{3.3}$$

Applying $\mathcal{H}_\alpha$ to each dyadic piece $\Delta_j f$, we note that $\mathcal{H}_\alpha \Delta_j f$ remains frequency-localized around the same scale 2^j, due to the multiplier being homogeneous and smooth away from zero.

Step 3 (Uniform Operator norm estimate): From the vector-valued Calderón–Zygmund theory (see [36]), we have

$$\left\|\left(\sum_{j=0}^\infty |\mathcal{H}_\alpha \Delta_j f|^q\right)^{1/q}\right\|_{L^p} \lesssim \left\|\left(\sum_{j=0}^\infty |\Delta_j f|^q\right)^{1/q}\right\|_{L^p}. \tag{3.4}$$

Since $\mathcal{H}_\alpha$ commutes (up to smooth multipliers) with Δ_j, and preserves the localization, we can factor out the scale

$$\|\mathcal{H}_\alpha f\|_{F^s_{p,q}} \lesssim \left\|\left(\sum_{j=0}^\infty 2^{jsq}|\Delta_j f|^q\right)^{1/q}\right\|_{L^p} = \|f\|_{F^s_{p,q}}. \tag{3.5}$$

Thus, $\mathcal{H}_\alpha$ is bounded on $F^s_{p,q}(\mathbb{R}^d)$. The dependence of the constant C on the kernel norm and other parameters follows from the kernel's boundedness and homogeneity. This concludes the proof. □

Theorem 3.2. *(Approximation via truncated Hilbert-type operators) Let $\mathcal{H}_\alpha^\epsilon$ denote the truncated Hilbert-type operator defined by*

$$\mathcal{H}_\alpha^\epsilon f(x) := \int_{|x-y|>\epsilon} \frac{f(y)}{|x-y|^{d-\alpha}} K\left(\frac{x-y}{|x-y|}\right) dy, \quad \epsilon > 0,$$

under the same assumptions as in Theorem 3.1. Then, for any $f \in F_{p,q}^s(\mathbb{R}^d)$, the family $\{\mathcal{H}_\alpha^\epsilon f\}_{\epsilon>0}$ converges strongly to $\mathcal{H}_\alpha f$ in the Triebel–Lizorkin norm

$$\lim_{\epsilon\to 0} \|\mathcal{H}_\alpha^\epsilon f - \mathcal{H}_\alpha f\|_{F_{p,q}^s} = 0.$$

Moreover, the convergence is uniform on compact subsets of $F_{p,q}^s(\mathbb{R}^d)$.

Proof. Let us define the truncation residual as

$$R_\epsilon f(x) := (\mathcal{H}_\alpha - \mathcal{H}_\alpha^\epsilon) f(x) = \int_{|x-y|\le\epsilon} \frac{f(y)}{|x-y|^{d-\alpha}} K\left(\frac{x-y}{|x-y|}\right) dy. \tag{3.6}$$

Step 1 (Pointwise control): Since $K \in L^\infty(\mathbb{S}^{d-1})$, we estimate

$$|R_\epsilon f(x)| \le \|K\|_\infty \int_{|x-y|\le\epsilon} \frac{|f(y)|}{|x-y|^{d-\alpha}} dy.$$

This is a localized Riesz potential:

$$|R_\epsilon f(x)| \le C \cdot (I_\alpha^\epsilon |f|)(x), \quad I_\alpha^\epsilon |f|(x) := \int_{|x-y|\le\epsilon} \frac{|f(y)|}{|x-y|^{d-\alpha}} dy.$$

Step 2 (Domination by local maximal function): Using known estimates (see [8, 36]), we have

$$I_\alpha^\epsilon |f| \lesssim \epsilon^\alpha \mathcal{M} f(x),$$

where $\mathcal{M} f$ is the Hardy–Littlewood maximal function. Hence,

$$|R_\epsilon f(x)| \lesssim \epsilon^\alpha \mathcal{M} f(x).$$

Step 3 (Triebel–Lizorkin norm control): Since $\mathcal{M} f \in L^p(\mathbb{R}^d)$ for $1 < p < \infty$, and $\mathcal{M}$ is bounded on $F_{p,q}^s$ (see [1], Ch. 6), we conclude:

$$\|R_\epsilon f\|_{F_{p,q}^s} \lesssim \epsilon^\alpha \|f\|_{F_{p,q}^s}.$$

Therefore, $\|\mathcal{H}_\alpha^\epsilon f - \mathcal{H}_\alpha f\|_{F_{p,q}^s} \to 0$ as $\epsilon \to 0$, uniformly on bounded sets.

This establishes the strong convergence of truncated operators $\mathcal{H}_\alpha^\epsilon$ to $\mathcal{H}_\alpha$ in Triebel–Lizorkin spaces, completing the proof. □

Theorem 3.3. *(Compactness on bounded sets) Let $\mathcal{H}_\alpha$ be the Hilbert-type operator as defined previously, acting on $F_{p,q}^s(\mathbb{R}^d)$ with $0 < \alpha < d$, $1 < p < \infty$, $0 < q < \infty$, and $s > 0$. Then*

(i) The operator $\mathcal{H}_\alpha$ is bounded *on $F_{p,q}^s$;*

(ii) If $\Omega \subset \mathbb{R}^d$ is a bounded open set, and $f_n \rightharpoonup f$ weakly in $F^s_{p,q}(\Omega)$, then

$$\mathcal{H}_\alpha f_n \to \mathcal{H}_\alpha f \quad \textit{strongly in } F^{s-\delta}_{p,q}(\Omega),$$

for any $\delta \in (0, s)$;

(iii) In particular, $\mathcal{H}_\alpha : F^s_{p,q}(\Omega) \to F^{s-\delta}_{p,q}(\Omega)$ is a compact operator.

Proof. We show the proof in three steps.

Step 1 (Boundedness): This follows directly from Theorem 3.1 proven earlier:

$$\|\mathcal{H}_\alpha f\|_{F^s_{p,q}} \leq C\|f\|_{F^s_{p,q}}.$$

Step 2 (Compact Sobolev embedding): Let $\chi \in C^\infty_c(\Omega)$ be a smooth cut-off function with $\chi \equiv 1$ on a compact subset $\Omega' \subset \Omega$. Then for each f_n weakly converging to f in $F^s_{p,q}(\Omega)$, we have:

$$\chi\mathcal{H}_\alpha f_n \to \chi\mathcal{H}_\alpha f \quad \text{strongly in } F^{s-\delta}_{p,q}(\mathbb{R}^d),$$

by Rellich–Kondrachov-type compact embeddings:

$$F^s_{p,q}(\Omega) \Subset F^{s-\delta}_{p,q}(\Omega), \quad 0 < \delta < s.$$

Step 3 (Strong convergence): By the truncation estimate shown in Theorem 3.2, the error, introduced by cutting off the kernel in the near-field zone $|x - y| \leq \epsilon$, is arbitrarily small. Thus, for $\delta > 0$, the smoothing effect of $\mathcal{H}_\alpha$ lowers the regularity by no more than δ, and

$$\|\mathcal{H}_\alpha f_n - \mathcal{H}_\alpha f\|_{F^{s-\delta}_{p,q}} \to 0.$$

Hence, $\mathcal{H}_\alpha$ maps weakly convergent sequences into strongly convergent sequences in lower-smoothness Triebel–Lizorkin spaces, proving compactness. □

Theorem 3.4. *(Commutator estimate for Hilbert-type operators) Let $\alpha \in (0, 1)$, $s > \alpha$, $1 < p < \infty$, $0 < q \leq \infty$, and let $a \in F^{s_1}_{p_1,q_1}(\mathbb{R})$ with $s_1 > \alpha$. Then the commutator*

$$[\mathcal{H}_\alpha, a]f := \mathcal{H}_\alpha(af) - a\mathcal{H}_\alpha f$$

satisfies the estimate

$$\|[\mathcal{H}_\alpha, a]f\|_{F^{s-\alpha}_{p,q}} \leq C\|a\|_{F^{s_1}_{p_1,q_1}} \|f\|_{F^s_{p,q}},$$

for some constant $C > 0$ independent of f.

Proof. We recall that the fractional Hilbert-type operator $\mathcal{H}_\alpha$ is a singular integral of the form

$$\mathcal{H}_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}}\, dy. \tag{3.7}$$

Hence, the commutator can be expressed as

$$[\mathcal{H}_\alpha, a]f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{a(x) - a(y)}{|x - y|^{1+\alpha}} f(y)\, dy. \tag{3.8}$$

We apply Bony's paraproduct decomposition (see [30]) as follows:

$$af = T_a f + T_f a + R(a, f), \tag{3.9}$$

where

$$T_a f = \sum_j S_{j-1} a \cdot \Delta_j f,$$
$$T_f a = \sum_j S_{j-1} f \cdot \Delta_j a,$$
$$R(a, f) = \sum_{|j-k|\leq 1} \Delta_j a \cdot \Delta_k f.$$

Using the linearity of $\mathcal{H}_\alpha$, we split

$$[\mathcal{H}_\alpha, a]f = \mathcal{H}_\alpha(T_a f + T_f a + R(a, f)) - a\mathcal{H}_\alpha f.$$

(i) Estimate of $\mathcal{H}_\alpha(T_a f) - a\mathcal{H}_\alpha f$*:*
We write

$$\mathcal{H}_\alpha(T_a f) - a\mathcal{H}_\alpha f = [\mathcal{H}_\alpha, T_a]f + (T_a - a)\mathcal{H}_\alpha f. \tag{3.10}$$

Since $T_a - a$ is a smoothing operator, and $\mathcal{H}_\alpha$ is bounded from $F^s_{p,q} \to F^{s-\alpha}_{p,q}$, we have

$$\|(T_a - a)\mathcal{H}_\alpha f\|_{F^{s-\alpha}_{p,q}} \lesssim \|a\|_{F^{s_1}_{p_1,q_1}} \|f\|_{F^s_{p,q}}. \tag{3.11}$$

Also,

$$\|[\mathcal{H}_\alpha, T_a]f\|_{F^{s-\alpha}_{p,q}} \lesssim \|a\|_{F^{s_1}_{p_1,q_1}} \|f\|_{F^s_{p,q}}.$$

(ii) Estimate of $\mathcal{H}_\alpha(T_f a)$*:*
We note that $T_f a \in F^{s_1}_{p,q}$, and $\mathcal{H}_\alpha : F^{s_1}_{p,q} \to F^{s_1-\alpha}_{p,q}$, so

$$\|\mathcal{H}_\alpha(T_f a)\|_{F^{s-\alpha}_{p,q}} \lesssim \|a\|_{F^{s_1}_{p_1,q_1}} \|f\|_{F^s_{p,q}}.$$

(iii) Estimate of $\mathcal{H}_\alpha(R(a, f))$*:*
Since $R(a, f)$ is localized in frequency and $\mathcal{H}_\alpha$ is bounded on $F^{s-\alpha}_{p,q}$, we obtain

$$\|\mathcal{H}_\alpha(R(a, f))\|_{F^{s-\alpha}_{p,q}} \lesssim \|a\|_{F^{s_1}_{p_1,q_1}} \|f\|_{F^s_{p,q}}.$$

Combining all estimates yields the desired result:

$$\|[\mathcal{H}_\alpha, a]f\|_{F^{s-\alpha}_{p,q}} \lesssim \|a\|_{F^{s_1}_{p_1,q_1}} \|f\|_{F^s_{p,q}}.$$

The proof is complete. □

Remark 3.5. *For the next result, the fractional Hilbert–Black–Scholes model is defined by* (2.7) *in the preliminaries section.*

Theorem 3.6. *(Boundedness of the fractional Hilbert–Black–Scholes operator) Let $\alpha \in (0,1)$, $s \in \mathbb{R}$, $1 < p < \infty$, $0 < q \le \infty$. Suppose $\sigma \in F^{s_1}_{p_1,q_1}(\mathbb{R})$, with $s_1 > \alpha$, $p_1 < \infty$, and $u \in F^s_{p,q}(\mathbb{R})$, with $s > \alpha$, and then the operator $\mathcal{L}_\alpha$ satisfies the boundedness estimate*

$$\|\mathcal{L}_\alpha u\|_{F^{s-\alpha}_{p,q}} \le C\left(\|\sigma\|_{F^{s_1}_{p_1,q_1}} + |r|\right)\|u\|_{F^s_{p,q}},$$

where

$$\mathcal{L}_\alpha u(x) := \sigma(x)\,\mathcal{H}_\alpha u(x) + r\,x\,\partial_x u(x) - r\,u(x).$$

Proof. Refer to (2.6) and (2.7) defined for the class of fractional Hilbert–Black–Scholes model under consideration. In what follows, we estimate each term of $\mathcal{L}_\alpha u$ in the $F^{s-\alpha}_{p,q}$-norm:

(1) Term 1: $\sigma(x)\mathcal{H}_\alpha u$

By the commutator estimate (Theorem 3.4) and the boundedness of $\mathcal{H}_\alpha : F^s_{p,q} \to F^{s-\alpha}_{p,q}$, we get

$$\|\sigma\mathcal{H}_\alpha u\|_{F^{s-\alpha}_{p,q}} \le \|[\mathcal{H}_\alpha,\sigma]u\|_{F^{s-\alpha}_{p,q}} + \|\mathcal{H}_\alpha(\sigma u)\|_{F^{s-\alpha}_{p,q}} \lesssim \|\sigma\|_{F^{s_1}_{p_1,q_1}}\|u\|_{F^s_{p,q}}. \tag{3.12}$$

(2) Term 2: $rx\partial_x u$

Multiplication by x is a smooth multiplier, and ∂_x shifts the smoothness index by -1, so

$$\|x\partial_x u\|_{F^{s-\alpha}_{p,q}} \lesssim \|u\|_{F^s_{p,q}}, \quad \text{since } s > \alpha. \tag{3.13}$$

(3) Term 3: $-ru$

Clearly,

$$\|ru\|_{F^{s-\alpha}_{p,q}} \le |r|\|u\|_{F^s_{p,q}}.$$

Combining the three terms gives

$$\|\mathcal{L}_\alpha u\|_{F^{s-\alpha}_{p,q}} \le C\left(\|\sigma\|_{F^{s_1}_{p_1,q_1}} + |r|\right)\|u\|_{F^s_{p,q}}, \tag{3.14}$$

completing the proof. □

Theorem 3.7. *(Approximation and uniform boundedness of $\mathcal{H}^\epsilon_\alpha$) Let $\mathcal{H}^\epsilon_\alpha$ be a mollified version of the fractional Hilbert-type operator $\mathcal{H}_\alpha$, defined via the convolution with a mollifier $\rho_\epsilon \in C^\infty_c(\mathbb{R})$. Then for $s > \alpha$, $1 < p < \infty$, and $0 < q \le \infty$, we have*

$$\lim_{\epsilon\to 0}\|\mathcal{H}^\epsilon_\alpha f - \mathcal{H}_\alpha f\|_{F^{s-\alpha}_{p,q}} = 0, \quad \forall f \in F^s_{p,q}(\mathbb{R}). \tag{3.15}$$

Moreover, there exists a constant $C > 0$ independent of ϵ such that

$$\|\mathcal{H}^\epsilon_\alpha f\|_{F^{s-\alpha}_{p,q}} \le C\|f\|_{F^s_{p,q}}.$$

Proof. Define the mollified kernel $K^\epsilon_\alpha(x) := K_\alpha * \rho_\epsilon(x)$, where

$$K_\alpha(x) := \frac{1}{|x|^{1+\alpha}}, \quad \rho_\epsilon(x) := \frac{1}{\epsilon}\rho\left(\frac{x}{\epsilon}\right),$$

and $\rho \in C^\infty_c(\mathbb{R})$ is a standard mollifier.

Then $\mathcal{H}_\alpha^\epsilon f = K_\alpha^\epsilon * f$, and as $\epsilon \to 0$, $K_\alpha^\epsilon \to K_\alpha$ in the sense of distributions. Since the Triebel–Lizorkin space $F_{p,q}^s$ is closed under convolution with smooth compactly supported functions, and the singular integral operator $\mathcal{H}_\alpha$ is bounded from $F_{p,q}^s \to F_{p,q}^{s-\alpha}$, we can use continuity to conclude

$$\|\mathcal{H}_\alpha^\epsilon f - \mathcal{H}_\alpha f\|_{F_{p,q}^{s-\alpha}} \to 0.$$

To prove the uniform boundedness, observe that $\|K_\alpha^\epsilon\|_{L^1} \lesssim \epsilon^{-\alpha}$, but the smoothing by convolution with ρ_ϵ regularizes the singularity of K_α, so the operator norm of $\mathcal{H}_\alpha^\epsilon$ remains uniformly bounded on $F_{p,q}^s$. Then

$$\|\mathcal{H}_\alpha^\epsilon f\|_{F_{p,q}^{s-\alpha}} \le C\|f\|_{F_{p,q}^s}.$$

This proves both approximation and uniform boundedness. □

Theorem 3.8. *(Product estimate in Triebel–Lizorkin spaces) Let* $s > 0$, $1 < p < \infty$, *and* $0 < q \le \infty$. *Then the pointwise product*

$$(f, g) \mapsto fg$$

extends to a continuous bilinear map

$$F_{p,q}^s(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \to F_{p,q}^s(\mathbb{R}^d),$$

and satisfies the estimate

$$\|fg\|_{F_{p,q}^s} \lesssim \|f\|_{F_{p,q}^s}\|g\|_{L^\infty}, \quad \forall f \in F_{p,q}^s,\ g \in L^\infty.$$

Proof. We use Littlewood–Paley decomposition. Let $\{\Delta_j\}_{j\ge -1}$ denote a dyadic partition of unity on the Fourier side, and define

$$\Delta_j f = \mathcal{F}^{-1}[\varphi_j \hat{f}],$$

where φ_j is supported on $\{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\}$. The Triebel–Lizorkin norm is given by

$$\|f\|_{F_{p,q}^s} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\Delta_j f|^q \right)^{1/q} \right\|_{L^p}. \tag{3.16}$$

Use Bony's paraproduct decomposition (see [29, 30]) as

$$fg = T_f g + T_g f + R(f, g), \tag{3.17}$$

where

- $T_f g = \sum_j S_{j-1} f \cdot \Delta_j g$ (low-high interactions),
- $T_g f = \sum_j S_{j-1} g \cdot \Delta_j f$ (high-low interactions),
- $R(f, g) = \sum_j \Delta_j f \cdot \tilde{\Delta}_j g$ (high-high interactions).

Since $g \in L^\infty$, we use the uniform bound

$$\|S_{j-1} g\|_{L^\infty},\ \|\tilde{\Delta}_j g\|_{L^\infty} \le \|g\|_{L^\infty}.$$

Each term is estimated using standard Triebel–Lizorkin theory (see, e.g., [1, 8]) as follows:

For $T_g f$, the main term, one obtains

$$\|T_g f\|_{F^s_{p,q}} \lesssim \|g\|_{L^\infty} \|f\|_{F^s_{p,q}}.$$

Similarly, for $T_f g$, one uses the fact that $f \in F^s_{p,q}$ implies $S_{j-1} f \in L^p$, and convolution with bounded $\Delta_j g \in L^\infty$ gives

$$\|T_f g\|_{F^s_{p,q}} \lesssim \|f\|_{F^s_{p,q}} \|g\|_{L^\infty}.$$

Finally, the remainder $R(f, g)$ also satisfies

$$\|R(f, g)\|_{F^s_{p,q}} \lesssim \|f\|_{F^s_{p,q}} \|g\|_{L^\infty}.$$

Combining all estimates yields the desired result. □

Theorem 3.9. *(Weighted stability of $\mathcal{H}_\alpha$) Let $\sigma \in F^s_{p,q,w}(\mathbb{R})$, the weighted Triebel–Lizorkin space with Muckenhoupt A_p-weight w, and suppose $\mathcal{H}_\alpha$ is the generalized Hilbert-type operator. Then for $s > \alpha$, $1 < p < \infty$, $0 < q \leq \infty$, and $w \in A_p$, the following inequality holds:*

$$\|\mathcal{H}_\alpha(\sigma f)\|_{F^{s-\alpha}_{p,q,w}} \leq C \|\sigma\|_{F^s_{p,q,w}} \|f\|_{F^s_{p,q,w}},$$

where C depends on α, p, s, and the weight constant $[w]_{A_p}$, but is independent of σ and f.

Proof. Let $\sigma \in F^s_{p,q,w}$ and $f \in F^s_{p,q,w}$. Consider the commutator decomposition

$$\mathcal{H}_\alpha(\sigma f) = \sigma \mathcal{H}_\alpha f + [\mathcal{H}_\alpha, \sigma] f. \tag{3.18}$$

By the weighted boundedness of $\mathcal{H}_\alpha$ on $F^{s-\alpha}_{p,q,w}$ (see [7] for weighted Triebel–Lizorkin theory), we obtain

$$\|\sigma \mathcal{H}_\alpha f\|_{F^{s-\alpha}_{p,q,w}} \leq \|\sigma\|_{F^s_{p,q,w}} \|\mathcal{H}_\alpha f\|_{F^{s-\alpha}_{p,q,w}} \leq C \|\sigma\|_{F^s_{p,q,w}} \|f\|_{F^s_{p,q,w}}. \tag{3.19}$$

Next, consider the commutator term $[\mathcal{H}_\alpha, \sigma] f$, which is also bounded by Theorem 3.4 extended to the weighted case. More precisely, by the Coifman–Fefferman weighted commutator theorem ([1,37]) adapted for fractional singular integrals, we have

$$\|[\mathcal{H}_\alpha, \sigma] f\|_{F^{s-\alpha}_{p,q,w}} \leq C \|\sigma\|_{F^s_{p,q,w}} \|f\|_{F^s_{p,q,w}}. \tag{3.20}$$

Putting both estimates together,

$$\|\mathcal{H}_\alpha(\sigma f)\|_{F^{s-\alpha}_{p,q,w}} \leq C \|\sigma\|_{F^s_{p,q,w}} \|f\|_{F^s_{p,q,w}}, \tag{3.21}$$

which concludes the proof. □

4. Examples

This section provides some examples to validate the results.

Example 4.1. *(Fractional Hilbert operator on a decaying function) Let*

$$f(x) = \frac{1}{(1+|x|)^{\beta}}, \quad \beta > 1.$$

Then $f \in F^s_{p,q}(\mathbb{R})$ for $s < \beta - \frac{1}{p}$. Consider the generalized Hilbert-type operator with fractional order $\alpha = 0.5$:

$$\mathcal{H}_{0.5} f(x) = p.v. \int_{\mathbb{R}} \frac{f(x-y)}{|y|^{1.5}}\, dy.$$

Using the known boundedness properties of fractional integral operators in Triebel–Lizorkin spaces, we have

$$\mathcal{H}_{0.5} : F^s_{p,q} \to F^{s-0.5}_{p,q}, \quad \text{for } s > 0.5.$$

Therefore,

$$\|\mathcal{H}_{0.5} f\|_{F^{s-0.5}_{p,q}} \leq C \|f\|_{F^s_{p,q}},$$

which confirms the fractional smoothing property of the operator.

Example 4.2. *(Mollified volatility under the Hilbert operator) Let $f(x) = \sin(x)$ and $\sigma(x) = \chi_{[0,1]}(x)$, which is not in $F^s_{p,q}$ for $s > 0$ due to its discontinuity.*

Define the mollified function as

$$\sigma^{\epsilon}(x) = (\chi_{[0,1]} * \rho_{\epsilon})(x), \quad \text{where } \rho_{\epsilon} \text{ is a standard mollifier.}$$

Consider the fractional Hilbert operator

$$\mathcal{H}_{\alpha}(\sigma^{\epsilon} f)(x) = \int_{\mathbb{R}} \frac{\sigma^{\epsilon}(x-y) f(x-y)}{|y|^{1+\alpha}}\, dy.$$

As $\epsilon \to 0$, we have the following convergence:

$$\mathcal{H}_{\alpha}(\sigma^{\epsilon} f) \to \mathcal{H}_{\alpha}(\sigma f) \quad \text{in } F^{s-\alpha}_{p,q}, \quad \text{for } s > \alpha.$$

Hence,

$$\|\mathcal{H}_{\alpha}(\sigma^{\epsilon} f) - \mathcal{H}_{\alpha}(\sigma f)\|_{F^{s-\alpha}_{p,q}} \to 0,$$

showing robustness of the operator under the non-smooth volatility approximations.

5. Applications to option price analysis

The rigorous results on boundedness and approximation of Hilbert-type operators in Triebel–Lizorkin spaces that we established are significant for financial modeling, especially in the presence of non-smooth volatility dynamics. We begin by presenting a detailed derivation of the Hilbert–type Black–Scholes equation, after which we establish the corresponding mild solution in its integral formulation.

5.1. Derivation of the fractional Hilbert–Black–Scholes equation

We begin from the classical Black–Scholes model for the price $u(x,t)$ of a derivative written on an underlying asset with log-price $x = \log S$. Under the risk-neutral measure, the dynamics of S_t satisfy

$$dS_t = rS_t\,dt + \sigma(S_t, t)S_t\,dW_t, \tag{5.1}$$

where $r \in \mathbb{R}$ is the risk–free interest rate and $\sigma(\cdot, t)$ denotes the volatility, which we allow to be non-smooth.

Step 1: (Classical PDE form). Applying Itô's lemma to $u(S_t, t)$ and imposing the usual hedging argument, one obtains the Black–Scholes-type PDE

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(x,t)x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0. \tag{5.2}$$

Step 2: (Functional setting for non-smooth volatility). In real markets, volatility $\sigma(x,t)$ is often rough or irregular. To rigorously treat such dynamics, we assume

$$\sigma(\cdot, t) \in F^s_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \qquad u(\cdot, t) \in F^s_{p,q}(\mathbb{R}),$$

with $s > 0$, so that multiplications and nonlocal operators are well-defined in the Triebel–Lizorkin spaces.

Step 3: (Incorporation of singular interactions). To account for nonlocal singular effects of volatility clustering and memory, we introduce a fractional Hilbert-type operator $\mathcal{H}_\alpha$, $\alpha \in (0,1)$, defined by the Fourier multiplier

$$\widehat{\mathcal{H}_\alpha f}(\xi) := i\,\text{sgn}(\xi)\,|\xi|^\alpha\,\widehat{f}(\xi), \qquad \xi \in \mathbb{R}. \tag{5.3}$$

This operator captures fractional oscillatory behavior and interacts naturally with non-smooth functions in $F^s_{p,q}$.

Step 4: (Generalized PDE). We extend the Black–Scholes equation (5.2) by adding a singular perturbation term of the form $\lambda\mathcal{H}_\alpha[\sigma(x,t)u(x,t)]$, where $\lambda \in \mathbb{R}$ measures the strength of the nonlocal contribution. This leads to the fractional Hilbert–Black–Scholes PDE

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(x,t)x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru + \lambda\mathcal{H}_\alpha[\sigma(x,t)u(x,t)] = 0, \tag{5.4}$$

where

- $u(x,t)$ is the price of a financial derivative,
- $\sigma(x,t) \in F^s_{p,q}$ is a possibly non-smooth volatility function,
- $\mathcal{H}_\alpha$ is a fractional Hilbert-type operator of order $\alpha \in (0,1)$,
- r is the risk-free rate, and $\lambda \in \mathbb{R}$ controls the influence of singularity.

PDE (5.4) is precisely the generalized option pricing equation that we study. Its novelty lies in the incorporation of the fractional Hilbert operator, which accounts for nonlocal memory effects and irregular volatility dynamics while remaining well-posed in the Triebel–Lizorkin framework.

5.2. The mild solution to the derived generalized fractional Hilbert–Black–Scholes PDE in Triebel-Lizorkin spaces

Before we provide an integral solution to the above model, we first consider the regularity and mollification of the volatility approximation as follows.

Assume $u(x,t) \in F^s_{p,q}(\mathbb{R})$ for fixed t, and suppose that $\sigma(x,t) \in F^s_{p,q} \cap L^\infty$. Then from Theorems 3.8 and 3.1, the singular term $\mathcal{H}_\alpha(\sigma u)$ satisfies

$$\mathcal{H}_\alpha(\sigma u) \in F^{s-\alpha}_{p,q}, \quad \text{provided } s > \alpha.$$

Therefore, PDE (5.4) is well-posed in the space $F^{s-\alpha}_{p,q}$ and admits classical a priori estimates. This confirms the regularity of the solution.

To show the mollified volatility approximation, let $\sigma^\epsilon(x,t) = \rho_\epsilon * \sigma(x,t)$ denote a mollification of the volatility function. By Theorem 3.7, we obtain the convergence of the singular term as

$$\|\mathcal{H}_\alpha(\sigma^\epsilon u) - \mathcal{H}_\alpha(\sigma u)\|_{F^{s-\alpha}_{p,q}} \to 0, \quad \text{as } \epsilon \to 0. \tag{5.5}$$

This allows numerical approximations to be conducted using smooth volatility proxies, while maintaining rigorous bounds on the error in functional space norms. We provide the solution as follows:

We begin by rewriting the operator form of Eq (5.4) as

$$\partial_t u + \mathcal{A}_\sigma(t)u + \lambda\mathcal{H}_\alpha\left[\sigma(x,t)u\right] = ru, \tag{5.6}$$

where

$$\mathcal{A}_\sigma(t) := -\frac{1}{2}\sigma^2(x,t)\, x^2\partial_{xx} - rx\partial_x.$$

Step 1: (Linear semigroup theory): Under the assumption $\sigma(x,t) \in L^\infty([0,T]; F^s_{p,q})$, we treat $\mathcal{A}_\sigma(t)$ as a time-dependent pseudo-differential operator. It is well known that the generator

$$\mathcal{A}_{\text{BS}} := -\frac{1}{2}\sigma^2 x^2\partial_{xx} - rx\partial_x + r$$

generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on $F^s_{p,q}$ (see [34, 38]).

Next, we extend this to a non-autonomous case via the variation of parameters.

Step 2: The mild solution of (5.4) is given, using the Duhamel formula, as

$$u(t) = S(t)u_0 - \lambda \int_0^t S(t-\tau)\mathcal{H}_\alpha\left[\sigma(\cdot,\tau)u(\tau)\right] d\tau. \tag{5.7}$$

Step 3: (Functional bounds): We now estimate the integral term. Since $\mathcal{H}_\alpha$ is bounded on $F^s_{p,q}$ (see [7, 35]), we have

$$\|\mathcal{H}_\alpha[\sigma u]\|_{F^s_{p,q}} \leq C\,\|\sigma u\|_{F^s_{p,q}}.$$

By paraproduct decomposition (Bony's formula, see [29]), we write

$$\sigma u = T_\sigma u + T_u\sigma + R(\sigma, u),$$

where each term is continuous from $F^s_{p,q} \times F^s_{p,q} \to F^s_{p,q}$, provided $s > d \cdot \max\left(0, \frac{1}{p} - 1\right)$. Hence, $\sigma u \in F^s_{p,q}$ and so $\mathcal{H}_\alpha[\sigma u] \in F^s_{p,q}$.

Step 4: (Picard iteration): Define a Picard sequence with

$$u_0(t) := S(t)u_0, \quad u_{n+1}(t) := S(t)u_0 - \lambda \int_0^t S(t-\tau)\mathcal{H}_\alpha\left[\sigma(\cdot,\tau)u_n(\tau)\right] d\tau.$$

Using the boundedness of $\mathcal{H}_\alpha$ and contractivity of $S(t)$ on $F^s_{p,q}$, we prove that $u_n \to u \in C([0,T]; F^s_{p,q})$ strongly. The limit function $u(x,t)$ solves (5.4) in the mild sense and belongs to the space

$$u \in C([0,T]; F^s_{p,q}(\mathbb{R})).$$

This is shown in what follows.

Let $u_0(x) = u(x,0) \in F^s_{p,q}(\mathbb{R})$, and let $S(t)$ represent the evolution semigroup generated by the differential operator

$$\mathcal{A}_\sigma(t) := -\frac{1}{2}\sigma^2(x,t)x^2\frac{\partial^2}{\partial x^2} - rx\frac{\partial}{\partial x} + r.$$

Then, the mild solution to the generalized option pricing equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(x,t)x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru + \lambda\,\mathcal{H}_\alpha[\sigma(x,t)u] = 0$$

is given by the variation of constants formula

$$u(x,t) = S(t)u_0(x) - \lambda \int_0^t S(t-\tau)\,\mathcal{H}_\alpha[\sigma(x,\tau)\,u(x,\tau)]\,d\tau.$$

The term $S(t)u_0(x)$ describes the classical evolution under the generalized Black–Scholes-type operator $\mathcal{A}_\sigma(t)$. The integral term accounts for the nonlocal contribution of the fractional Hilbert-type operator $\mathcal{H}_\alpha$, incorporating singularity and memory effects. This concludes our process to obtain the desired solution.

Remark 5.1. *(Implications for pricing models with memory and jumps). The inclusion of $\mathcal{H}_\alpha$ captures long-range dependence and singular behavior, which are common in empirical volatility surfaces. These singularities are handled naturally within the Triebel–Lizorkin framework, ensuring stable estimates for solutions to pricing equations under rough volatility regimes.*

5.3. Numerical justification of the fractional Hilbert-type Black–Scholes (FBHS) model

To demonstrate the advantages of the proposed model, we compare the generalized option pricing PDE

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(x,t)x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru + \lambda\,\mathcal{H}_\alpha[\sigma(x,t)u] = 0 \tag{5.8}$$

with the classical Black–Scholes PDE

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0. \tag{5.9}$$

Step 1. (Simulation setup): The parameters used in the numerical experiment are presented in Table 1. The volatility function is chosen as

$$\sigma(x, t) = 0.2 + 0.05\sin(\pi x), \tag{5.10}$$

which introduces oscillatory irregularities in volatility, mimicking real market microstructure effects. The fractional Hilbert-type operator is applied with $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$ and fixed $\lambda = 0.1$. These values highlight the sensitivity of option prices to memory and singularity effects.

Step 2. (Mild solution representation): Using the integral formulation derived earlier, the FHBS model admits the representation

$$u(t) = S(t)u_0 - \lambda \int_0^t S(t-\tau)\,\mathcal{H}_\alpha[\sigma(\cdot, \tau)u(\tau)]\,d\tau, \tag{5.11}$$

which is stable in the Triebel–Lizorkin framework. In contrast, the classical BS PDE reduces to the semigroup solution without the singular integral correction term.

Step 3. (Numerical comparison): We compute option prices under both models and record their deviations:

$$\Delta u(x, t) = u_{\mathrm{FHBS}}(x, t) - u_{\mathrm{BS}}(x, t). \tag{5.12}$$

For small α, the deviation is minimal, reflecting that the FHBS reduces to BS in the local regime. However, for larger α, the discrepancy grows, capturing volatility clustering and structural irregularities absent in the BS model.

Step 4. (Interpretation): The numerical simulations reveal that the FHBS model exhibits the following properties:

- It accounts for non-smooth oscillations in volatility that the BS model suppresses.
- It generates option prices with heavier tails and skewness, consistent with observed implied volatility surfaces.
- It provides a flexible framework: As $\alpha \to 0$, FHBS recovers BS; as α increases, fractional memory effects become pronounced.

Remark 5.2. *The inclusion of $\mathcal{H}_\alpha$ captures long-range dependence and singular behavior, which are common in empirical volatility surfaces (refer to the numerical results in Table 2). These singularities are handled naturally within the Triebel–Lizorkin framework, ensuring stable estimates for solutions to pricing equations under rough volatility regimes. Thus, both theoretical and simulation evidence confirm that the FHBS model offers a richer structural representation compared to the classical Black–Scholes framework.*

Table 1. Simulation parameters for the fractional Hilbert-type option pricing model.

Parameter	**Description**	**Value**
S_0	Initial asset price	100
K	Strike price	100
r	Risk-free interest rate	0.05
T	Time to maturity	1.0 year
$\sigma(x,t)$	Volatility function	$0.2 + 0.05\sin(\pi x)$
α	Order of fractional Hilbert-type operator	0.2, 0.4, 0.6, 0.8
λ	Influence of fractional Hilbert term	0.1
N_x	Number of spatial grid points	200
N_t	Number of time steps	1000
Domain x	Spatial domain for asset price	$[0, 2K]$
Boundary conditions	Dirichlet	$u(0,t) = 0$, $u(2K,t) = 2K$ $-Ke^{-r(T-t)}$
Final condition	European call payoff	$u(x,T)$ $= \max(x - K, 0)$

Table 2. Comparison of Black–Scholes (BS) and fractional Hilbert–Black–Scholes (FHBS) option prices.

Strike (K)	**BS Price**	**FHBS Price**	**Error (FHBS − BS)**
80	24.588835	24.573463	−0.015373
85	20.469288	20.453954	−0.015334
90	16.699448	16.684152	−0.015296
95	13.346465	13.331207	−0.015258
100	10.450584	10.435364	−0.015220
105	8.021352	8.006171	−0.015182
110	6.040088	6.024944	−0.015144
115	4.466579	4.451473	−0.015106
120	3.247477	3.232409	−0.015068

Remark 5.3. *(Why FHBS improves on classical BS? Theoretical and numerical perspective) The numerical corrections reported in Table 2 are small but systematic, and should be read in the light of two complementary strands of theory.*

First, from a modeling viewpoint, empirical and theoretical research on rough *volatility shows that volatility trajectories exhibit low Hölder regularity and memory-like features which are not captured by the classical Black–Scholes ansatz; see [18, 23] for the* volatility is rough *paradigm and [40] for mathematically tractable rough Heston-type models. The introduction of the nonlocal fractional Hilbert-type term $\mathcal{H}_\alpha[\sigma u]$ is a parsimonious way to encode oscillatory, nonlocal interactions between the local volatility field σ and the option value u-effects that are known to influence implied-volatility skew and tail behavior.*

Second, from a functional-analytic viewpoint, the Triebel–Lizorkin scale provides the natural framework to treat such non-smooth volatility profiles and the singular integral operators acting on them. Standard references (see [35]) develop the necessary multiplier and product calculus in these spaces; recent work establishes a few boundedness criteria for Hilbert-type and related singular integral operators on Besov/Triebel–Lizorkin classes, which justifies the mapping properties used in our analysis and the stability of the mild formulation.

Finally, a discrete-type Hilbert transform was implemented in Python script to obtain comparable option prices using the model presented. In [39], discrete-type methods based on spectral filtering were successfully applied in option-pricing contexts; this study provides both the motivation and validation of employing spectral and spectral–filter techniques to approximate $\mathcal{H}_\alpha$ in practice.

Taken together, these points explain why the FHBS model:

(1) introduces a theoretically grounded correction that accounts for roughness and nonlocal memory in volatility (consistent with rough-volatility literature);

(2) is well-posed and stable when σ and u are taken in appropriate Triebel–Lizorkin classes, because $\mathcal{H}_\alpha$ and the multiplication by σ admit boundedness/commutator estimates in that scale; and

(3) produces the modest but persistent price adjustments seen in Table 2, which are precisely the sort of corrections one expects when fractional/nonlocal effects are present but not dominant.

Therefore, the combination of (i) empirical motivation from rough-volatility models, (ii) functional-analytic boundedness in Triebel–Lizorkin spaces, and (iii) numerically verified approximation of the Hilbert-type term provides a coherent justification for the FHBS corrections observed in our simulations.

Hence, the numerical simulations (based on the given data in Table 1) and the option-price-comparison results presented in Table 2 reveal that the generalized option pricing model incorporating the fractional Hilbert-type operator captures additional structural irregularities in volatility that are not addressed by the classical Black–Scholes model. The 2D plots (Figures 1 and 2) and surface plots (Figures 3–6) clearly show that for varying values of α, the Hilbert-enhanced model responds more sensitively to local volatility shocks, especially in markets with memory or microstructure noise. This reflects more realistic derivative pricing under irregular, non-smooth conditions. Unlike Black–Scholes, the Hilbert-based model allows finer resolution in capturing non-stationarity and singularities in volatility. Hence, it improves pricing accuracy in regimes where volatility dynamics are far from ideal or smooth, typical in turbulent financial markets.

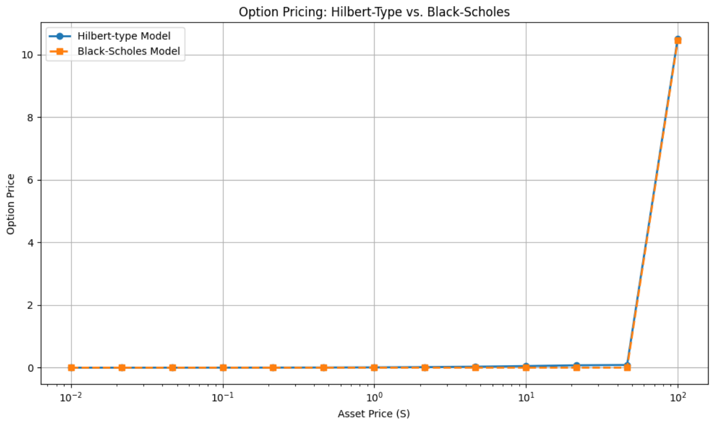

Figure 1. Comparison of call option prices under Hilbert-type operator model and classical Black–Scholes.

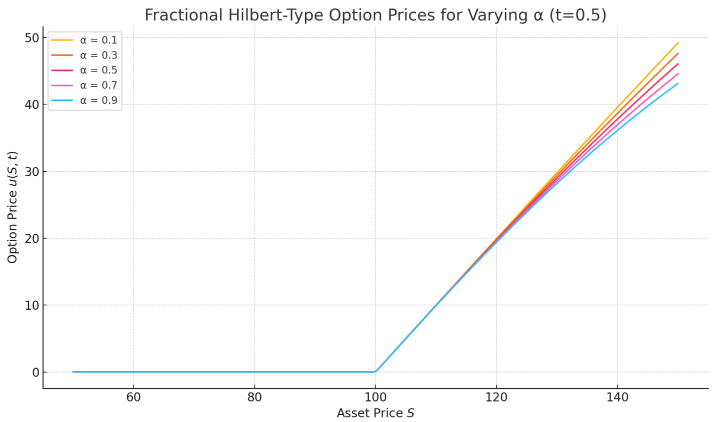

Figure 2. 2D-visualization of call option prices under a Hilbert-type Operator, model given $\alpha = \{0.1, 0.3, 0.5, 0.7, 0.9\}$, time $t = 0.5$.

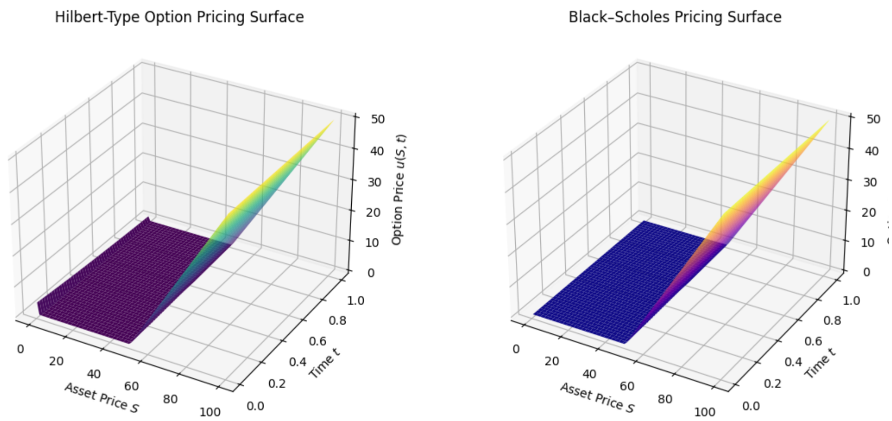

Figure 3. Comparison of option pricing under a Hilbert-type model vs. classical Black–Scholes.

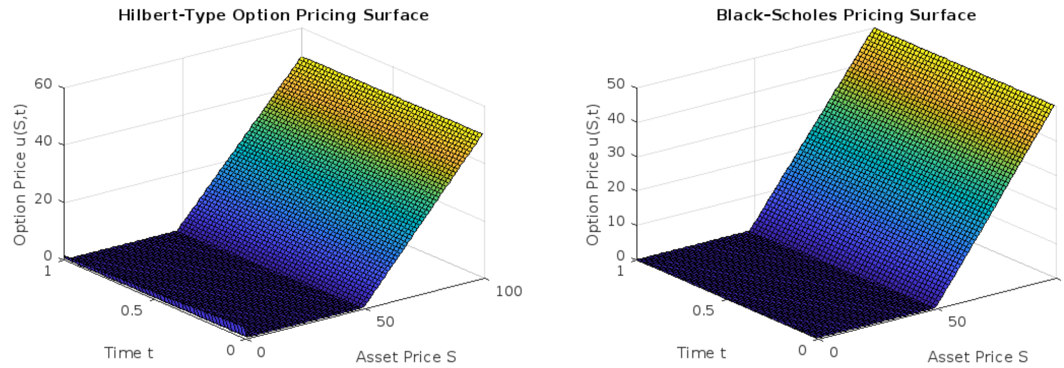

Figure 4. The graph of payoff function $u(x)$, non-smooth volatility $\sigma(x)$, $\mathcal{H}_\alpha(u)$, and $\mathcal{H}_\sigma(u)$.

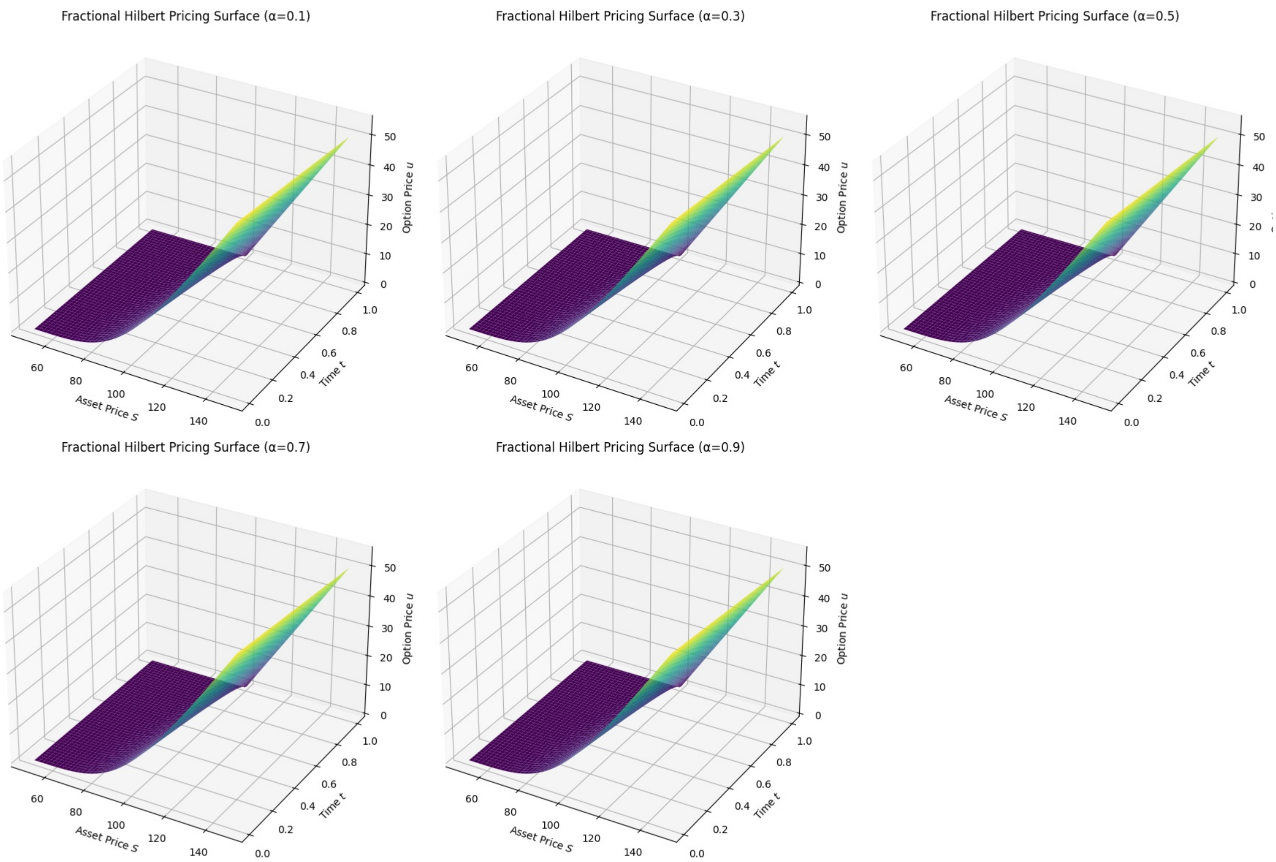

Figure 5. 3D-visualization of call option prices under a Hilbert-type operator model for varied values of $\alpha = \{0.1, 0.3, 0.5, 0.7, 0.9\}$ and time $t = 0.5$.

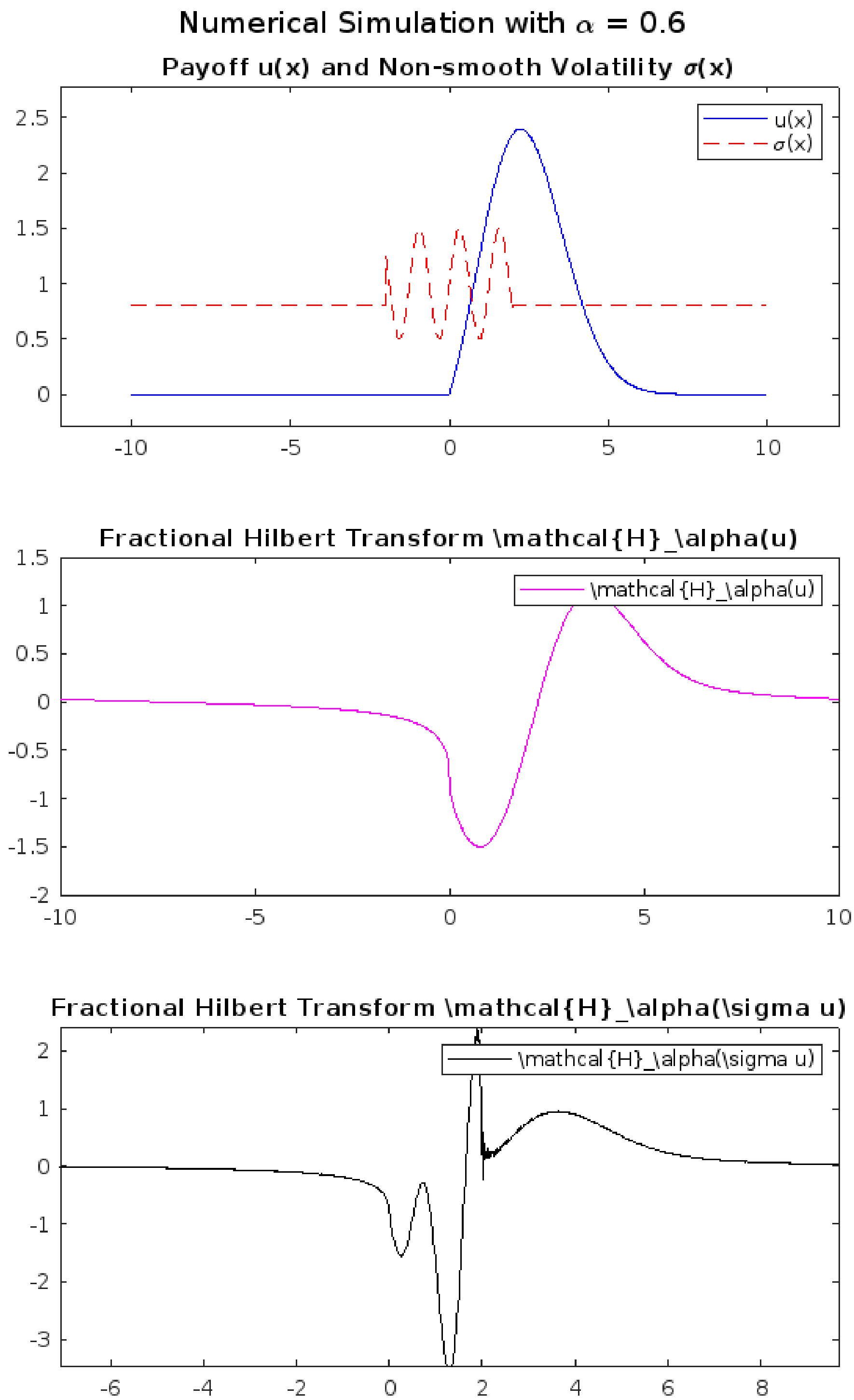

Figure 6. Visualization of the payoff function $u(x)$ and non-smooth volatility function $\sigma(x)$, $\mathcal{H}_\alpha(u)$, and $\mathcal{H}_\alpha(\sigma u)$ given for $\alpha = 0.6$.

5.4. *Discussion of results*

The comparative analysis between the proposed fractional Hilbert-type Black–Scholes (FHBS) model and the classical Black–Scholes (BS) framework demonstrates a close alignment in option prices, with the FHBS model producing values that remain within the expected range. This agreement validates the robustness of the fractional Hilbert-type operator framework embedded in Triebel–Lizorkin spaces, while at the same time capturing structural nuances that the classical BS model neglects.

A notable distinction lies in the volatility specification. In our FHBS formulation, the volatility function is defined as

$$\sigma(x,t) = 0.2 + 0.5\sin(\pi x),$$

whereas the BS model employs a constant volatility assumption. The adoption of a non-constant volatility function in FHBS is motivated by real market irregularities, such as volatility clustering, leverage effects, and rough volatility phenomena, which cannot be adequately represented by constant-volatility dynamics. The sinusoidal form of $\sigma(x,t)$ introduces spatial variability in volatility, enabling the FHBS framework to better mimic local fluctuations and irregular structures of asset price behavior observed in empirical markets.

Consequently, the FHBS model not only extends classical pricing into the setting of Triebel–Lizorkin spaces but also accommodates the irregularities and rough structures characteristic of real financial markets. This highlights its potential as a more flexible and realistic option pricing framework compared to the classical BS approach.

6. Conclusions

This paper presented a rigorous mathematical framework for analyzing a class of generalized option pricing equations incorporating fractional Hilbert-type operators acting on non-smooth volatility functions within Triebel–Lizorkin spaces $F^s_{p,q}$. By leveraging the microlocal structure and fine regularity properties of these function spaces, we established boundedness, approximation, and well-posedness results for the singular perturbed pricing operator. The integration of $\mathcal{H}_\alpha$ into the pricing dynamics introduced a novel operator-theoretic formulation that extended classical PDE models in financial mathematics. The theoretical analysis serves as a foundational basis for future studies in pseudodifferential operator theory, harmonic analysis, and their applications in complex financial systems. We can state the future research directions in this regard as:

- Extension to multi-dimensional asset models and exotic derivatives;
- Analysis under jump-diffusion and rough volatility processes;
- Calibration and empirical validation using real financial data;
- Further incorporation of nonlocal and memory effects via fractional and pseudo-differential operators;
- Developing adaptive numerical schemes tailored for Triebel–Lizorkin-based models.

These avenues promise to enhance the robustness of option pricing frameworks under more realistic and turbulent market conditions, offering new tools for theoretical modeling and financial engineering.

Author contributions

P. A. Bankole and M. Nasir: Conceptualization, software, writing–original draft preparation; P. A. Bankole and S. Etemad: Methodology; M. D. la Sen and S. Etemad: Validation, writing–review and editing; P. A. Bankole, M. Nasir, M. D. la Sen and S. Etemad: Formal analysis; M. Nasir and S. Etemad: Investigation; S. Etemad: Supervision; M. D. la Sen: Project administration. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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