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Research article

A unified concept of periodicity on any time scale and applications

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Abstract: We introduce a novel definition of periodicity on arbitrary time scales, dependent on a strictly increasing and differentiable function. This removes the commonly used and restrictive assumption of a periodic time scale to define periodic functions. Our new definition furthermore allows for a wider class of functions to be studied using the theory of periodic systems. After providing crucial properties of these periodic functions, such as the translation invariance of integrals of periodic functions, we apply the concept of this new periodicity to linear dynamic equations. We provide necessary and sufficient conditions for a linear dynamic equation to have such a periodic solution and discuss its uniqueness.

Keywords: periodicity; time scales; existence; uniqueness; global stability; linear dynamic equations
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1. Introduction

The study of dynamic equations on time scales, introduced by Stefan Hilger in 1988, has gained increasing attention due to its unifying characteristics of the continuous and discrete calculus as well as its potential in numerous applications [1, 2]. Dynamic equations on time scales allow the modeling of processes that change not necessarily continuously or discretely, but rather based on a general time domain, called a *time scale*. The mathematical interest in such equations stems mainly from their unifying characteristics of differential and difference equations. Instead of studying differential equations and difference equations separately, the theory of time scales allows the study of both equations within one unified framework. Given that dynamics of differential and difference equations may behave drastically different, the analysis of dynamic equations on time scales may provide insights into the relevance of the underlying time structure of such vastly different behaviors.

Furthermore, the benefit of dynamic equations on time scales in applications is twofold. On one hand, dynamic equations may describe complex systems and their changes more accurately. For example, it allows for the modeling of hibernating species, where differential equations are well equipped to describe changes during the species' active period, but discrete equations are needed to model the hibernation and the next active period [2]. On the other hand, dynamic equations on time scales may allow for a reduction in the complexity of a continuous model by incorporating some of the model's complexity in the underlying time structure. For example, instead of working with discontinuous or piecewise defined model parameters, one may be able to capture such discontinuities in the construction of the time scale and use the time scales analogue of continuous model parameters, simplifying the model structure on time scales.

Independent of the specific time structure, available methods to analyze dynamical systems reduce as the complexity of a system increases. That is, differential, difference, and dynamic equations with constant coefficients are in general much easier to investigate than nonautonomous systems for each category. However, for a special subclass of nonautonomous systems, so-called periodic systems, mathematical tools such as Floquet theory have been developed to aid its analysis. The interest in periodic dynamical systems dates back to its application in celestial mechanics and solid state physics [3]. The implementation of periodic coefficients in differential or difference equations remains a popular tool to represent oscillatory behavior such as seasonal changes in ecology [4].

Given the potential of time scales to reduce the model complexity on one hand and allow for more realistic modeling on the other hand, it is natural to extend the study of periodic systems to time scales. The difficulty is, however, that classical periodicity may not be well-defined on time scales. More precisely, the classical definition of periodicity, $f(t+\omega) = f(t)$ for all t in the domain of f, requires that $t+\omega$ is also in the domain. Thus, if f is defined on a time scale that is any nonempty closed subset of the real numbers, $t+\omega$ must not necessarily be in that time scale, rendering the classical definition of periodicity useless. To enable the study of periodic systems on time scales, many works assumed a periodic time scale [5–7]. Here, a periodic time scale refers to any time structure that guarantees that if t is in the time scale, then $t \pm \omega$ is also in the time scale. This is however rather restrictive and many popular time scales, such as the quantum time domain $q^{\mathbb{N}_0}$ for $q > 1$ used in physical applications [8–10], do not satisfy this condition. In the special case of quantum time scales, a periodicity definition was presented in [11]. There, the authors called a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ ω-periodic if $q^{\omega} f(q^{\omega} t) = f(t)$ and developed the corresponding Floquet theory on quantum calculus.

More recently, the authors in [12] revisited the definition of periodicity and provided a definition for the special case of isolated time scales. These are time domains consisting of time points that are a positive distance apart but are not necessarily equidistant. One may understand an isolated time scale as a generalized discrete time domain $\mathbb{T}_{\mathcal{I}} = \{\ldots, t_{-1}, t_0, t_1, \ldots\}$. A function $f : \mathbb{T}_{\mathcal{I}} \to \mathbb{R}$ was introduced as ω-periodic if

$$(t_{i+\omega+1} - t_{i+\omega}) f(t_{i+\omega}) = (t_{i+1} - t_i) f(t_i)$$

for all $t_i \in \mathbb{T}_{\mathcal{I}}$. Although nonstandard, this definition preserved several useful properties of periodic functions and collapses to the classical formulation $f(t+h\omega) = f(t)$ if $\mathbb{T}_{\mathcal{I}}$ consists of equidistant points with distance h. Motivated by this definition, the authors then extended the concept of periodicity to define ν-periodicity in the continuous case [13], before returning to time scales theory and introducing the concept of ν-periodicity on time scales, unifying the concepts for continuous and arbitrary discrete time domains and their generalizations.

Around the same time, a different definition of periodicity for a distinct class of time scales was introduced in [14]. In this approach, the ω-periodicity of a function depends on multivariate shift functions $\delta_\pm$ and a value ω that is part of the time scale. This definition differs from ours and does not encompass the concept of periodicity as established in quantum calculus. Also, a significant restriction arises in this case: the time scale itself must be periodic in its shifts. Despite its seemingly general framework, this requirement limits its applicability, as it is not always satisfied. For example, the time scale $\mathbb{T} = \{n^n \mid n \in \mathbb{N}\}$ does not satisfy this condition for any fixed $\omega \in \mathbb{T}$ [12, Appendix A.6]. In addition, it should be noted that the results presented in this paper have not been given yet for the periodicity concept in [14].

On the other hand, the definition of Δ-periodicity also presented in [14], for isolated time scales and considering the function $\nu = \sigma^\omega$, aligns closely with the definition of ω-periodicity provided in [12]. However, the notion of periodicity introduced in this paper is more general than that presented in [14], even when restricted to isolated time scales. This increased generality is due to the flexibility allowed in the choice of the function ν within our framework. In our definition of ω-periodicity, ν is required only to be an increasing and differentiable function, providing many possibilities in its selection. For instance, we can define $\nu(t) = \log(t)$, thereby allowing the exponential function to be ν-periodic. This level of generality is not achievable under Adivar's definition of Δ-periodicity or periodicity, as these require the shift operators to satisfy the following condition: If $t \in [t_0, \infty)_\mathbb{T}$, then $(t, t_0) \in D_+$ and $\delta_+(t, t_0) = t$, which is clearly incompatible with the logarithm function.

This is just one example, but many more general choices for the function ν can be accommodated within this framework, showing the generality of our definition and consequently of our results.

We emphasize that the key principle guiding our definition of periodicity is rooted in the geometric properties of periodic functions. Specifically, it is a well-established fact that the area under the graph of a traditional ω-periodic function remains constant over intervals of length ω. Our general definition preserves this fundamental property across any time scale, demonstrating that this approach provides the most natural and accurate interpretation of periodic functions in the context of time scales. Unlike definitions relying on additive properties, our framework captures the true essence of periodicity, ensuring consistency with its geometric foundation while extending its applicability.

2. Time scales preliminaries

In this section, we briefly summarize useful time scales fundamentals and refer the interested reader to the introductory books on time scales [1, 2]. A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. To capture the underlying time structure, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ defined by (put $\inf \emptyset = \sup \mathbb{T}$)

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\},$$

and the *graininess operator* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ defined by

$$\mu(t) := \sigma(t) - t$$

are introduced, see [2, Definition 1.1]. If $\sigma(t) > t$, then we say that t is right-scattered. If $\sigma(t) = t$, then we say that t is right-dense. Similarly, left-scattered and left-dense points are defined. For a function $f : \mathbb{T} \to \mathbb{R}$, we put $f^\sigma := f \circ \sigma$. We call f *rd-continuous* provided it is continuous at all right-dense points and its left-sided limits exist (as finite values) at all left-dense points, see [1, Definition 1.58]. The set of all *rd-continuous* functions is denoted by $C_{\text{rd}} = C_{\text{rd}}(\mathbb{T}, \mathbb{R})$.

Definition 2.1 (see [1, Definition 2.25]). *A function $p : \mathbb{T} \to \mathbb{R}$ is called (positively) regressive provided $1 + \mu p \neq 0$ ($1 + \mu p > 0$) on $\mathbb{T}$. The set of all rd-continuous and (positively) regressive functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ ($\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R})$).*

The set of rd-continuous and regressive functions, $\mathcal{R}$, is a crucial set in the theory of time scales. In fact, most theorems in this manuscript require functions to be rd-continuous and regressive. Note that if $\mathbb{T} = \mathbb{R}$, then all continuous functions are rd-continuous and regressive.

The following circle-plus addition turns $(\mathcal{R}, \oplus)$ into an Abelian group.

Definition 2.2 (see [2, p. 13]). *Define circle plus and circle minus for $f, g \in \mathcal{R}$ by*

$$f \oplus g = f + g + \mu f g, \quad f \ominus g = \frac{f - g}{1 + \mu g}.$$

Definition 2.3 (see [1, Definition 1.10]). *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$, where $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$ if $\mathbb{T}$ has a left-scattered maximum M and otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t.

If $\mathbb{T} = \mathbb{R}$, then the delta-derivative is equivalent to the classical derivative, and if $\mathbb{T} = \mathbb{Z}$, then the delta-derivative of a function $f : \mathbb{Z} \to \mathbb{R}$ collapses to the forward difference.

For differentiable $f, g : \mathbb{T} \to \mathbb{R}$, the product rule and the quotient rule (see [1]) read as

$$(fg)^\Delta = f^\Delta g^\sigma + f g^\Delta = f^\Delta g + f^\sigma g^\Delta \qquad \text{and} \qquad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - f g^\Delta}{g g^\sigma},$$

where the latter assumes $g, g^\sigma \neq 0$, and the "simple useful formula" says

$$f^\sigma = f + \mu f^\Delta.$$

In the rest of this paper, we also use the notation $f^{\Delta\sigma} = (f^\Delta)^\sigma$.

Integration on time scales is defined in terms of antiderivatives. F is an antiderivative of f if $F^\Delta = f$ holds. By [1, Theorem 1.74], every rd-continuous function possesses an antiderivative. For $s, t \in \mathbb{T}$, we then define

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s).$$

If $\mathbb{T} = \mathbb{R}$, then the delta integral coincides with the classical Riemann integral, and for isolated time scales $\mathbb{T} = \mathbb{T}_{\mathcal{I}}$,

$$\int_t^s f(\tau)\Delta\tau = \sum_{\tau \in [t,s) \cap \mathbb{T}_{\mathcal{I}}} \mu(\tau) f(\tau), \quad t, s \in \mathbb{T}_{\mathcal{I}}, \quad t < s.$$

Relevant for our analysis is the following chain rule. For that, we define for $i \in \mathbb{N}$,

$$\Omega^i(\mathbb{T}) := \{\nu \in C^i_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}), \text{ such that } \nu^\Delta > 0 \text{ and } \nu(\mathbb{T}) = \mathbb{T}\}, \tag{2.1}$$

where $C^i_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ is the set of functions $f : \mathbb{T} \to \mathbb{R}$, $f(\mathbb{T}) \subset \mathbb{R}$ that are i-times delta-differentiable with an rd-continuous ith delta-derivative.

Remark 2.4. For $\nu \in \Omega^i(\mathbb{T})$ for some $i \in \{1, 2, \ldots\}$, ν and σ commute, that is, $\nu \circ \sigma = \sigma \circ \nu$. The proof is provided in the Appendix. Note that this further implies $\mu\nu^\Delta = \mu^\nu$ because $\mu\nu^\Delta = \nu^\sigma - \nu = \sigma^\nu - \nu = \mu^\nu$.

Theorem 2.5 (see [1, Theorem 1.93])**.** *Consider $\nu \in \Omega^1(\mathbb{T})$. Let $w : \mathbb{T} \to \mathbb{R}$. If $\nu^\Delta(t)$ and $w^\Delta(\nu(t))$ exist for $t \in \mathbb{T}^\kappa$, then*

$$(w \circ \nu)^\Delta = (w^\Delta \circ \nu)\nu^\Delta. \tag{2.2}$$

Theorem 2.6 (see [1, Theorem 1.98])**.** *Let $\nu \in \Omega^1(\mathbb{T})$. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with an rd-continuous derivative, then for $t, s \in \mathbb{T}$,*

$$\int_t^s f(\tau)\nu^\Delta(\tau)\Delta\tau = \int_{\nu(t)}^{\nu(s)} (f \circ \nu^{-1})(\tau)\Delta\tau. \tag{2.3}$$

It follows from Theorem 2.6 by replacing f with $f^\nu := f \circ \nu$ that for $\nu \in \Omega^1(\mathbb{T})$ and $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, then for $t, s \in \mathbb{T}$,

$$\int_{\nu(t)}^{\nu(s)} f(\tau)\Delta\tau = \int_t^s \nu^\Delta(\tau) f(\nu(\tau))\Delta\tau. \tag{2.4}$$

The time scales exponential function can be defined as the unique solution to a linear dynamic equation.

Theorem 2.7 (see [1, Theorem 2.33])**.** *Let $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. Then the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1$$

possesses a unique solution, denoted by $e_p(\cdot, t_0)$.

If the time scale is isolated, then the time scales exponential function for $p \in \mathcal{R}$ can be expressed as

$$e_p(t, t_0) = \prod_{s \in [t_0, t) \cap \mathbb{T}} (1 + \mu(s)p(s)), \quad t > t_0.$$

If $\mathbb{T} = \mathbb{R}$, then the exponential dynamic equation is identical to the classical exponential function

$$e_p(t, t_0) = e^{\int_{t_0}^t p(s)\mathrm{d}s}.$$

Useful properties of the dynamic exponential function for $p, q \in \mathcal{R}$ and $t, s, r \in \mathbb{T}$ are summarized below (see [1, Theorems 2.36, 2.39, and 2.48]):

i) $e_0(t, s) = 1$ and $e_p(t, t) = 1$,	ii) $e_{p \oplus q}(t, s) = e_p(t, s)e_q(t, s)$,
iii) $e_{p \ominus q}(t, s) = \dfrac{e_p(t, s)}{e_q(t, s)}$,	iv) $e_{\ominus p}(t, s) = e_p(s, t) = \dfrac{1}{e_p(t, s)}$,
v) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,	vi) $e_p(t, \sigma(s)) = \dfrac{e_p(t, s)}{1 + \mu(s)p(s)}$,
vii) $e_p^\Delta(\cdot, s) = p e_p(\cdot, s)$,	viii) $e_p^\Delta(s, \cdot) = -p e_p^\sigma(s, \cdot)$,
ix) $p \in \mathcal{R}^+$ implies $e_p(t, s) > 0$,	x) $e_p(t, r)e_p(r, s) = e_p(t, s)$.

Theorem 2.8 (see [2, Theorems 2.74 and 2.77])**.** *If $p \in \mathcal{R}$, $f : \mathbb{T} \to \mathbb{R}$, $t_0 \in \mathbb{T}$, and $y_0 \in \mathbb{R}$, then the unique solution of $y^\Delta = p(t)y + f(t)$ with $y(t_0) = y_0$ is given by*

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)\Delta s. \tag{2.5}$$

Furthermore, the unique solution of $y^\Delta = -p(t)y^\sigma + f(t)$ with $y(t_0) = y_0$ is given by

$$y(t) = e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)f(s)\Delta s. \tag{2.6}$$

Proposition 2.9. *Let $\nu \in \Omega^1(\mathbb{T})$ and $f \in \mathcal{R}$. Then, for $h(t) := e_f(\nu(t), t)$,*

$$h^\Delta = ((\nu^\Delta f^\nu) \ominus f)h \tag{2.7}$$

and

$$e_f(\nu(t), t) = e_f(\nu(t_0), t_0)e_{\nu^\Delta f^\nu \ominus f}(t, t_0). \tag{2.8}$$

Proof. Let $t_0 \in \mathbb{T}$ be arbitrary but fixed. Then

$$h(t) = e_f(\nu(t), t_0)e_f(t_0, t) = e_f(\nu(t), t_0)e_{\ominus f}(t, t_0)$$

and hence

$$\begin{aligned} h^\Delta(t) &\overset{(2.2)}{=} f(\nu(t))e_f(\nu(t), t_0)\nu^\Delta(t)e_{\ominus f}(\sigma(t), t_0) + e_f(\nu(t), t_0)(\ominus f(t))e_{\ominus f}(t, t_0) \\ &= \left\{f(\nu(t))\nu^\Delta(t)e_{\ominus f}(\sigma(t), t) + (\ominus f(t))\right\} e_f(\nu(t), t_0)e_{\ominus f}(t, t_0) = \left(\nu^\Delta f^\nu \ominus f\right)(t)h(t), \end{aligned}$$

confirming the first claim. The first equality holds due to the product rule and the identity in (2.2) as indicated by the equation number above the equality sign*. To agree with the second claim, note that by (2.7), h solves a first-order dynamic equation of the form $y^\Delta = py$ with $p = \nu^\Delta f^\nu \ominus f$. Since $f \in \mathcal{R}$ and $\nu^\Delta f^\nu \in \mathcal{R}$ (because for $\nu \in \Omega^1(\mathbb{T})$, $\sigma \circ \nu = \nu \circ \sigma$ so that $\mu\nu^\Delta = \mu^\nu$, implying that $1+\mu\nu^\Delta f^\nu = (1+\mu f)^\nu$), and we have $p \in \mathcal{R}$, so that $h(t) = e_p(t, t_0)h(t_0)$, confirming the claim in (2.8). □

3. Periodicity on time scales

We define periodicity on an arbitrary time scale $\mathbb{T}$ via a strictly increasing function $\nu \in \Omega^1(\mathbb{T})$. Throughout the remainder of this manuscript, we assume that $\nu \in \Omega^1(\mathbb{T})$ with Ω^1 defined in (2.1) for $i = 1$.

Definition 3.1. *A function $f : \mathbb{T} \to \mathbb{R}$ is called ν-periodic provided*

$$\nu^\Delta f^\nu = f, \qquad \text{where} \quad f^\nu = f \circ \nu. \tag{3.1}$$

The set of rd-continuous functions that satisfy (3.1) *is denoted by $\mathcal{P}_\nu(\mathbb{T})$ (for short $\mathcal{P}_\nu$).*

*Henceforth, to assist the reader in following our calculations, we put corresponding identities that are applied in the calculations above the corresponding equality sign.

Example 3.2. 1) Let $\mathbb{T} = \mathbb{R}$. Let $\omega \in \mathbb{R}$. We define $\nu(t) = t + \omega$ for all $t \in \mathbb{R}$. Then ν is strictly increasing and $\nu(\mathbb{R}) = \mathbb{R}$. Moreover, $\nu'(t) = 1$ for all $t \in \mathbb{R}$. Hence an $f : \mathbb{R} \to \mathbb{R}$ is ν-periodic if and only if

$$f(t) \overset{(3.1)}{=} \nu^{\Delta}(t) f^{\nu}(t) = \nu'(t) f(\nu(t)) = f(\nu(t)) = f(t + \omega) \quad \text{for all} \quad t \in \mathbb{R},$$

i.e., if and only if f is ω-periodic in the classical sense.

2) Let $\mathbb{T} = \mathbb{Z}$. Let $\omega \in \mathbb{Z}$. We define $\nu(t) = t + \omega$ for all $t \in \mathbb{Z}$. Then ν is strictly increasing and $\nu(\mathbb{Z}) = \mathbb{Z}$. Moreover,

$$\nu^{\Delta}(t) = \Delta(\nu(t)) = \nu(t+1) - \nu(t) = t + 1 + \omega - (t + \omega) = 1 \quad \text{for all} \quad t \in \mathbb{Z}.$$

Hence an $f : \mathbb{Z} \to \mathbb{Z}$ is ν-periodic if and only if

$$f(t) \overset{(3.1)}{=} \nu^{\Delta}(t) f^{\nu}(t) = \Delta(\nu(t)) f(\nu(t)) = f(\nu(t)) = f(t + \omega) \quad \text{for all} \quad t \in \mathbb{Z},$$

i.e., if and only if f is ω-periodic in the classical sense.

3) Let $\mathbb{T}$ be a periodic time scale (see [15, Definition 1]), i.e., there exists $\omega \in \mathbb{T}$ such that $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$. We define $\nu(t) = t + \omega$ for all $t \in \mathbb{Z}$. Then ν is strictly increasing and $\nu(\mathbb{Z}) = \mathbb{Z}$, see [15, Theorem 2]. Moreover, $\nu^{\Delta}(t) = 1$ for all $t \in \mathbb{T}$, see [15, Theorem 2]. Hence an $f : \mathbb{T} \to \mathbb{T}$ is ν-periodic if and only if

$$f(t) \overset{(3.1)}{=} \nu^{\Delta}(t) f^{\nu}(t) = f(\nu(t)) = f(t + \omega) \quad \text{for all} \quad t \in \mathbb{T},$$

i.e., if and only if f is ω-periodic in the "classical" sense.

4) Let $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$ with $q > 1$. Let $\omega \in \mathbb{Z}$. We define $\nu(t) = q^{\omega} t$. Then ν is strictly increasing and $\nu(\mathbb{T}) = \mathbb{T}$. Moreover,

$$\nu^{\Delta}(t) = \frac{\nu(qt) - \nu(t)}{(q-1)t} = \frac{q^{\omega} q t - q^{\omega} t}{(q-1)t} = q^{\omega}$$

for $t \in \mathbb{T} \setminus \{0\}$, and this is also true for $t = 0$, considering the limit as $t \to 0$. Hence an $f : \mathbb{T} \to \mathbb{T}$ is ν-periodic if and only if

$$f(t) \overset{(3.1)}{=} \nu^{\Delta}(t) f^{\nu}(t) = q^{\omega} f(\nu(t)) = q^{\omega} f(q^{\omega} t) \quad \text{for all} \quad t \in \mathbb{T},$$

i.e., if and only if f is ω-periodic in the sense of [11, Definition 3.1].

5) Let $\mathbb{T} = \bigcup_{i \in \mathbb{Z}} \{t_i\}$ with $t_i < t_{i+1}$ for all $i \in \mathbb{Z}$. Let $\omega \in \mathbb{Z}$. We define $\nu(t_i) = t_{i+\omega}$ for all $i \in \mathbb{Z}$. Then ν is strictly increasing and $\nu(\mathbb{T}) = \mathbb{T}$. Moreover,

$$\nu^{\Delta}(t_i) = \frac{\nu(t_{i+1}) - \nu(t_i)}{t_{i+1} - t_i} = \frac{t_{i+\omega+1} - t_{i+\omega}}{t_{i+1} - t_i} \quad \text{for all} \quad i \in \mathbb{Z}.$$

Hence an $f : \mathbb{T} \to \mathbb{T}$ is ν-periodic if and only if

$$f(t_i) \overset{(3.1)}{=} \nu^{\Delta}(t_i) f^{\nu}(t_i) = \frac{t_{i+\omega+1} - t_{i+\omega}}{t_{i+1} - t_i} f(\nu(t_i)) = \frac{t_{i+\omega+1} - t_{i+\omega}}{t_{i+1} - t_i} f(t_{i+\omega}) \quad \text{for all} \quad i \in \mathbb{Z},$$

i.e., if and only if f is ω-periodic in the sense of [12, Definition 4.1].

6) Let $\mathbb{T} = \mathbb{R}$. Let $\nu : \mathbb{R} \to \mathbb{R}$ be differentiable and strictly increasing with $\nu(\mathbb{R}) = \mathbb{R}$. Then an $f : \mathbb{R} \to \mathbb{R}$ is ν-periodic if and only if

$$f(t) \overset{(3.1)}{=} \nu^{\Delta}(t) f^{\nu}(t) = \nu'(t) f(\nu(t)) \quad \text{for all} \quad t \in \mathbb{R},$$

i.e., if and only if f is ν-periodic in the sense of [13, Definition 1].

All of the situations discussed in Example 3.2 are known from the various given sources. However, we can apply our results to time scales that are not covered in the previous literature. One such time scale is discussed in Example A3 in the Appendix.

The following properties of ν-periodic functions can be easily verified using (3.1).

Lemma 3.3. *If $f \in \mathcal{P}_{\nu}$, then $(\mu f)^{\nu} = \mu f$.*

Proof. Let $f \in \mathcal{P}_{\nu}$. Then,

$$(\mu f)^{\nu} = \mu^{\nu} f^{\nu} = \mu \nu^{\Delta} f^{\nu} = \mu f,$$

where we used $\mu \nu^{\Delta} = \mu^{\nu}$ from Remark 2.4. □

Lemma 3.4. *If $f, g \in \mathcal{P}_{\nu}$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$, $f \oplus g$, and $\ominus g$ are also ν-periodic.*

Proof. Since

$$\nu^{\Delta}(\alpha f + \beta g)^{\nu} = \alpha \nu^{\Delta} f^{\nu} + \beta \nu^{\Delta} g^{\nu} = \alpha f + \beta g,$$

the first claim follows. Furthermore,

$$\begin{aligned} \nu^{\Delta}(f \oplus g)^{\nu} &= \nu^{\Delta}(f + g + \mu f g)^{\nu} = \nu^{\Delta} f^{\nu} + \nu^{\Delta} g^{\nu} + \nu^{\Delta} \mu \nu^{\Delta} f^{\nu} g^{\nu} \\ &= f + g + \mu f g = f \oplus g. \end{aligned}$$

Lastly, note that

$$\nu^{\Delta}(\ominus g)^{\nu} = \nu^{\Delta} \frac{-g^{\nu}}{1 + \mu^{\nu} g^{\nu}} = \frac{-g}{1 + \mu \nu^{\Delta} g^{\nu}} = \frac{-g}{1 + \mu g} = \ominus g,$$

which completes the proof. □

Lemma 3.5. *Consider $f : \mathbb{T} \to \mathbb{R}$ such that $f \in \mathcal{P}_{\nu}$. Then $f \in \mathcal{P}_{\nu \circ \nu}$.*

Proof. Since $\nu^{\Delta} f^{\nu} = f$, $\widetilde{\nu} := \nu \circ \nu$ implies

$$\widetilde{\nu}^{\Delta} f^{\widetilde{\nu}} = (\nu \circ \nu)^{\Delta} f^{\nu \circ \nu} \overset{(2.2)}{=} (\nu^{\Delta})^{\nu} \nu^{\Delta} (f^{\nu})^{\nu} = \nu^{\Delta} (\nu^{\Delta} f^{\nu})^{\nu} \overset{(3.1)}{=} \nu^{\Delta} f^{\nu} \overset{(3.1)}{=} f,$$

which completes the proof. □

Theorem 3.6. *Consider $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ and $\nu \in \Omega^1(\mathbb{T})$. Let*

$$F_{\nu}(t) := \int_{t}^{\nu(t)} f(\tau) \Delta \tau, \qquad t \in \mathbb{T}. \tag{3.2}$$

Then $F_{\nu}^{\Delta} = \nu^{\Delta} f^{\nu} - f$. Thus, if $f \in \mathcal{P}_{\nu}$, then F_{ν} is constant.

Proof. Define $H(t) := \int_{t_0}^{t} f(\tau)\Delta\tau$ for arbitrary but fixed $t_0 \in \mathbb{T}$. Hence

$$F_\nu(t) = H(\nu(t)) - H(t).$$

Then, the claim follows from (2.2). □

Theorem 3.7 (Conservation of areas)**.** *If $f \in \mathcal{P}_\nu$ and $t, s \in \mathbb{T}$, then*

$$\int_t^{\nu(t)} f(\tau)\Delta\tau = \int_s^{\nu(s)} f(\tau)\Delta\tau \quad \text{and} \quad \int_t^{s} f(\tau)\Delta\tau = \int_{\nu(t)}^{\nu(s)} f(\tau)\Delta\tau. \tag{3.3}$$

Proof. Let $f \in \mathcal{P}_\nu$. The first equality in (3.3) follows immediately from Theorem 3.6. The second equality is derived by splitting the integrals and applying the first identity. □

Remark 3.8. The two properties in (3.3) are key properties of periodic functions. These area-preserving properties link the definition of periodicity introduced in this work to previously formulated definitions for special cases of time scales (see [16] for quantum time scales, see [12] for isolated time scales, and see [13] for the continuous time scale). Thus, our Definition 3.1 can be understood as a unification of these concepts and its extension to arbitrary time scales.

Theorem 3.9. *Let $f \in \mathcal{P}_\nu \cap \mathcal{R}$. Then, for all $t, s \in \mathbb{T}$,*

$$e_f(\nu(t), t) = e_f(\nu(s), s) \quad \text{and} \quad e_f(\nu(t), \nu(s)) = e_f(t, s). \tag{3.4}$$

Proof. For $f \in \mathcal{P}_\nu \cap \mathcal{R}$, $\nu^\Delta f^\nu = f$ so that $\nu^\Delta f^\nu \ominus f \equiv 0$. Thus, by (2.8), $e_f(\nu(t), t) = e_f(\nu(s), s)$, and the first equality follows. The second equality is a consequence of the first equality and the semi-group property $e_f(a, b) = e_f(a, c)e_f(c, b)$ for $a, b, c \in \mathbb{T}$. □

4. Existence of periodic solutions to linear dynamic equations

In this section, we apply the definition of periodicity introduced in this manuscript to discuss the existence and uniqueness of ν-periodic solutions to linear dynamic equations. For the remainder of this manuscript, we let $\nu \in \Omega^2(\mathbb{T})$, defined in (2.1).

For $f \in \mathcal{R}$ and $\nu \in \Omega^2(\mathbb{T})$, we introduce $E_f : \mathbb{T} \to \mathbb{R}$ as

$$E_f(t) := \nu^\Delta(t) e_f(\nu(t), t). \tag{4.1}$$

Lemma 4.1. *For $f \in \mathcal{R}$, $f, f^\nu \neq 0$, the identity*

$$\nu^{\Delta\Delta} + \nu^{\Delta\sigma}\nu^\Delta f^\nu = \nu^\Delta f \tag{4.2}$$

is equivalent to each of the following statements:

i) E_f *is constant, where* E_f *is defined in* (4.1).

ii) $\nu^\Delta f^\nu \ominus f = -\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}}$.

iii) $\frac{(\ominus f)^\nu}{\ominus f} = \frac{f^\nu}{f} \cdot \frac{\nu^{\Delta\sigma}}{\nu^\Delta}$, *if* $\mu \neq 0$.

Proof. We note that $E_f = \nu^\Delta h$, where h is defined in Proposition 2.9. Then,

$$\begin{aligned} E_f^\Delta &= \nu^{\Delta\Delta} h + \nu^{\Delta\sigma} h^\Delta \overset{(2.7)}{=} \left\{ \nu^{\Delta\Delta} + \nu^{\Delta\sigma}(\nu^\Delta f^\nu \ominus f) \right\} h \\ &= (\nu^{\Delta\Delta} + \mu\nu^{\Delta\Delta} f + \nu^{\Delta\sigma}\nu^\Delta f^\nu - \nu^{\Delta\sigma} f) \frac{h}{1+\mu f} \\ &= \left(\nu^{\Delta\Delta} + \nu^{\Delta\sigma}\nu^\Delta f^\nu - \nu^\Delta f \right) \frac{h}{1+\mu f}, \end{aligned}$$

confirming the equivalence between (4.2) and i). Next, ii) is equivalent to

$$\nu^\Delta f^\nu = -\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} \oplus f = -\frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} + f - \mu f \frac{\nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} = \frac{f(\nu^{\Delta\sigma} - \mu\nu^{\Delta\Delta}) - \nu^{\Delta\Delta}}{\nu^{\Delta\sigma}} = \frac{f\nu^\Delta - \nu^{\Delta\Delta}}{\nu^{\Delta\sigma}},$$

and this is equivalent to (4.2). Lastly, iii) is equivalent to

$$\begin{aligned} 0 &= \frac{(\ominus f)^\nu}{\ominus f} - \frac{f^\nu}{f} \cdot \frac{\nu^{\Delta\sigma}}{\nu^\Delta} = \frac{f^\nu}{f} \left\{ \frac{1+\mu f}{1+\mu^\nu f^\nu} - \frac{\nu^{\Delta\sigma}}{\nu^\Delta} \right\} \\ &= \frac{f^\nu}{f} \cdot \frac{1}{(1+\mu^\nu f^\nu)\nu^\Delta} \cdot \left\{ (1+\mu f)\nu^\Delta - (1+\mu^\nu f^\nu)\nu^{\Delta\sigma} \right\} \\ &= \frac{f^\nu}{f} \cdot \frac{1}{(1+\mu^\nu f^\nu)\nu^\Delta} \cdot \left\{ -\mu\nu^{\Delta\Delta} + \mu f \nu^\Delta - \mu^\nu f^\nu \nu^{\Delta\sigma} \right\} \\ &= \frac{f^\nu}{f} \cdot \frac{\mu}{(1+\mu^\nu f^\nu)\nu^\Delta} \cdot \left\{ -\nu^{\Delta\Delta} + f\nu^\Delta - \nu^\Delta f^\nu \nu^{\Delta\sigma} \right\}, \end{aligned}$$

where we used $\mu\nu^\Delta = \nu^\sigma - \nu = \mu^\nu$ for the last equal sign, and for $\mu \neq 0$, this is equivalent to (4.2). □

In the special case when $\mathbb{T} = \mathbb{R}$, Lemma 4.1 is consistent with [13, Lemma 17]. Moreover, if $\mathbb{T}$ is an isolated time scale and $\nu = \sigma^\omega$, then Lemma 4.1 collapses to [12, Lemma 6.2 and Theorem 6.3], supporting once more the claim that Definition 3.1 is a unification of the concept of periodicity on arbitrary time scales.

By Lemma 4.1, $f, e_f(\cdot, t_0) \in \mathcal{P}_\nu$ for $f \neq 0$, if and only if $\nu^{\Delta\Delta} \equiv 0$, restricting the time scale. Thus, in contrast to the continuous time domain, for general time scale $\mathbb{T}$, $0 \neq f \in \mathcal{P}_\nu$ does not guarantee that its corresponding dynamic exponential function is also ν-periodic.

4.1. Homogeneous dynamic equations

We consider the homogeneous first-order linear dynamic equation

$$x^\Delta = a(t)x \tag{4.3}$$

for $a \in \mathcal{R}$. By (2.5), the solution is given by

$$x(t) = e_a(t, t_0)x(t_0), \qquad t_0, t \in \mathbb{T}. \tag{4.4}$$

By [1, Theorem 2.44], if $a \in \mathcal{R}$, then $e_a(t, t_0) \neq 0$ for all $t \in \mathbb{T}$. Thus, we may consider $x(t_0) \neq 0$ henceforth.

Theorem 4.2. *A nontrivial solution of* (4.3) *is ν-periodic if and only if a satisfies* (4.2) *and there exists $t_0 \in \mathbb{T}$ such that $E_a(t_0) = 1$.*

Proof. If x is a nontrivial solution of (4.3), then $x(t) = e_a(t, t_0)x(t_0) \neq 0$. Then $x \in \mathcal{P}_\nu$ iff

$$x(t) = \nu^\Delta(t)x(\nu(t)) = \nu^\Delta(t)e_a(\nu(t), t_0)x(t_0) = \nu^\Delta(t)e_a(\nu(t), t)e_a(t, t_0)x(t_0) = E_a(t)x(t),$$

i.e., $E_a(t) = 1$ for all $t \in \mathbb{T}$. By property i) in Lemma 4.1, this holds if and only if a satisfies (4.2) and $E_a(t_0) = 1$. □

Theorem 4.2 highlights that the classical statement of the existence of a periodic solution in the case of periodic coefficients does not necessarily hold on time scales but depends on the underlying time structure.

Example 4.3. If $a \in \mathcal{R}$ is constant, then $0 \neq x$ solving (4.3) can only be ν-periodic if $\nu^{\Delta\Delta} = 0$ whenever $\nu^{\Delta\sigma} = 1$ and $\frac{\nu^{\Delta\Delta}}{\nu^\Delta(1-\nu^{\Delta\sigma})} = a$ whenever $\nu^{\Delta\sigma} \neq 1$, posing restrictions on the function ν and the underlying time scale $\mathbb{T}$. More precisely, let $z := \nu^\Delta$. Then the required condition is equivalent to $z^\Delta = az(z^\sigma - 1)$ so that for $u := z^{-1}$, we obtain a linear dynamic equation $u^\Delta = -a + au^\sigma$. By (2.6), assuming also $-a \in \mathcal{R}$, the solution is

$$u(t) = e_{\ominus(-a)}(t, t_0)u(t_0) + \int_{t_0}^{t} (-a)e_{\ominus(-a)}(t, s)\Delta s = 1 + e_{\ominus(-a)}(t, t_0)(u(t_0) - 1).$$

Thus,

$$\nu^\Delta(t) = \frac{\nu^\Delta(t_0)e_{-a}(t, t_0)}{\nu^\Delta(t_0)e_{-a}(t, t_0) + 1 - \nu^\Delta(t_0)}$$

and since $\nu \in \Omega^1(\mathbb{T})$ requires $\nu^\Delta > 0$, we require

$$\operatorname{sgn}(e_{-a}(t, t_0)) = \operatorname{sgn}(\nu^\Delta(t_0)e_{-a}(t, t_0) + 1 - \nu^\Delta(t_0)).$$

Theorem 4.4. *Let $a \in \mathcal{R}$ satisfy* (4.2). *If there exists $t_0 \in \mathbb{T}$ such that*

$$\nu^\Delta(t_0)e_a(\nu(t_0), t_0) = 1,$$

then all solutions of (4.3) *are ν-periodic. In all other cases,* (4.3) *has no nontrivial ν-periodic solution.*

Proof. This follows directly from Theorem 4.2. □

If $\mathbb{T} = \mathbb{R}$, then Theorem 4.4 collapses to the statement in [13, Theorem 19].

Example 4.5. If $\mathbb{T} = \mathbb{T}_\mathcal{I}$, where $\mathbb{T}_\mathcal{I}$ is an isolated time scale and $\nu = \sigma^\omega$, then Theorem 4.4 coincides with [12, Theorem 6.4] after using [12, Lemma 6.2] to express the required condition in the form of (4.2). More precisely, consider now the isolated time scale $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$ and $\nu(t) = q^\omega t$. Then $\nu^\Delta(t) = q^\omega$ and Theorem 4.4 states that if a that is ν-periodic, which is equivalent to condition (4.2), and $q^\omega e_a(q^\omega t_0, t_0) = 1$ for some $t_0 \in q^{\mathbb{N}_0}$, then all solutions are ν-periodic. Otherwise, if no such $t_0 \in q^{\mathbb{N}_0}$ exists and $a \in \mathcal{P}_\nu$, then only the trivial solution $x \equiv 0$ is ν-periodic.

Example 4.6. Consider again the quantum time scale $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$ but with a different $\nu \in \Omega^1(\mathbb{T})$, namely, $\nu(t) = q^{2+\left\lfloor \frac{100i}{1+i} \right\rfloor} t$ for $t = q^i$. Then ν is strictly increasing and

$$\nu^{\Delta}(t) = q^2 \frac{q^{1+\left\lfloor \frac{100(i+1)}{2+i} \right\rfloor} - q^{\left\lfloor \frac{100i}{1+i} \right\rfloor}}{q-1}.$$

Let $a \in \mathcal{R} \cap \mathcal{P}_{\nu}$. Then, Theorem 4.4 implies that if there exists $t_0 = q^k \in \mathbb{T}$ such that

$$q^2 \frac{q^{1+\left\lfloor \frac{100(k+1)}{2+k} \right\rfloor} - q^{\left\lfloor \frac{100k}{1+k} \right\rfloor}}{q-1} \prod_{i=0}^{T} (1 + (q-1) q^i a(q^i)) = 1,$$

where

$$T = \rho(\nu(t_0)) = q^{1+\left\lfloor \frac{100i}{1+i} \right\rfloor} q^k,$$

then all solutions to $x^{\Delta} = a(t)x$ are ν-periodic.

We note that the same analysis can be conducted for the homogeneous dynamic equation

$$x^{\Delta} = a(t) x^{\sigma} \tag{4.5}$$

for $-a \in \mathcal{R}$. By (2.6), its solution is given by

$$x(t) = e_{\ominus(-a)}(t, t_0) x(t_0), \quad t_0 \in \mathbb{T}. \tag{4.6}$$

All previous results still uphold but need to be adjusted to $\ominus(-a)$ instead of a. We summarize them briefly below but omit their proofs.

Lemma 4.7. *A function* $0 \neq x \in \mathcal{P}_{\nu}$ *solves* (4.5) *if and only if a satisfies*

$$\nu^{\Delta\Delta} + \nu^{\Delta\sigma} \nu^{\Delta} (\ominus(-a))^{\nu} = \nu^{\Delta} (\ominus(-a)). \tag{4.7}$$

Lemma 4.8. *Let* $-a \in \mathcal{R}$ *satisfy* (4.7). *If there exists* $t_0 \in \mathbb{T}$ *such that* $E_{\ominus(-a)}(t_0) = 1$, *or equivalently,* $\nu^{\Delta}(t_0) x(\nu(t_0)) = x(t_0)$ *for* $x(t_0) \neq 0$, *then all solutions of* (4.5) *are* ν*-periodic. In all other cases,* (4.5) *has no nontrivial* ν*-periodic solution.*

Remark 4.9. Using the definition of the operation $\ominus$, (4.7) can equivalently be expressed as

$$\nu^{\Delta\Delta}(1 + \mu^{\nu} a^{\nu})(1 + \mu a) + \nu^{\Delta\sigma} \nu^{\Delta} a^{\nu} (1 + \mu a) = \nu^{\Delta} a (1 + \mu^{\nu} a^{\nu}).$$

Note that in this case, if $a \in \mathbb{R}$ is constant, then we have

$$\nu^{\Delta\Delta}(1 + \mu^{\nu} a)(1 + \mu a) + (\nu^{\Delta})^{\sigma} (\nu^{\Delta}) a (1 + \mu a) = \nu^{\Delta} a (1 + \mu^{\nu} a),$$

which certainly differs from the condition in Remark 4.3. Thus, given a time scale $\mathbb{T}$, one may find $\nu \in \Omega^1(\mathbb{T})$ so that $x^{\Delta} = ax$ has a ν-periodic solution for constant a, but the same time scale may not yield a suitable $\widetilde{\nu} \in \Omega^1(\mathbb{T})$ so that $x^{\Delta} = ax^{\sigma}$ has a $\widetilde{\nu}$-periodic solution for the same constant a. This highlights once more the importance of the underlying time scale $\mathbb{T}$ and the periodicity function ν.

4.2. Nonhomogeneous dynamic equations

In this subsection, we address the existence and uniqueness of ν-periodic solutions to a nonhomogeneous first-order linear dynamic equation of the form

$$x^{\Delta} = a(t)x + b(t), \quad t \in \mathbb{T}, \tag{4.8}$$

where $a \in \mathcal{R}$ and $b \in C_{\mathrm{rd}}$. By (2.5), the solution to (4.8) is given by

$$x(t) = e_a(t, t_0)x(t_0) + \int_{t_0}^{t} e_a(t, \sigma(s))b(s)\Delta s, \quad t_0 \in \mathbb{T}. \tag{4.9}$$

Throughout this section, we let $\nu \in \Omega^2(\mathbb{T})$ and recall that $\nu \circ \sigma = \sigma \circ \nu$.

Theorem 4.10. *If $a \in \mathcal{R}$ satisfies* (4.2)*, $b \in C_{\mathrm{rd}}$ satisfies*

$$\nu^{\Delta}\nu^{\Delta\sigma}b^{\nu} = b \quad \text{on} \quad \mathbb{T}, \tag{4.10}$$

and there exists $t_0 \in \mathbb{T}$ such that $E_a(t_0) \neq 1$, where E_a is defined in (4.1)*, then* (4.8) *has a unique nontrivial ν-periodic solution given by*

$$\overline{x}(t) = \lambda \int_{t}^{\nu(t)} e_a(t, \sigma(s))b(s)\Delta s, \tag{4.11}$$

where

$$\lambda := \frac{E_a(t_0)}{1 - E_a(t_0)} \in \mathbb{R}. \tag{4.12}$$

Proof. The solution x of (4.8) is given by (4.9). Clearly, if $x \equiv 0$, then it is ν-periodic. Let $x \neq 0$. Then

$$\begin{aligned}
\nu^{\Delta}(t)x(\nu(t)) &= \nu^{\Delta}(t)\left\{e_a(\nu(t), t_0)x(t_0) + \int_{t_0}^{\nu(t)} e_a(\nu(t), \sigma(s))b(s)\Delta s\right\} \\
&= \nu^{\Delta}(t)e_a(\nu(t), t)\left\{x(t) + \int_{t}^{\nu(t)} e_a(t, \sigma(s))b(s)\Delta s\right\} \\
&= E_a(t)\left\{x(t) + \int_{t}^{\nu(t)} e_a(t, \sigma(s))b(s)\Delta s\right\} \qquad (4.13)\\
&= E_a(t_0)\left\{x(t) + \int_{t}^{\nu(t)} e_a(t, \sigma(s))b(s)\Delta s\right\}, \qquad (4.14)
\end{aligned}$$

where, in the last equality, we used Lemma 4.1 i) and the assumption that a satisfies (4.2). Thus, if $\overline{x}$ is a ν-periodic solution, then

$$\overline{x}(t) = \frac{E_a(t_0)}{1 - E_a(t_0)} \int_{t}^{\nu(t)} e_a(t, \sigma(s))b(s)\Delta s = \lambda \int_{t}^{\nu(t)} e_a(t, \sigma(s))b(s)\Delta s,$$

and hence $\overline{x}$ is given by (4.11). On the other hand, to confirm that $\overline{x}$ given in (4.11) is indeed a solution and ν-periodic, we first note that for $t_0 \in \mathbb{T}$,

$$\overline{x}(t) = \lambda e_a(t, t_0)\left\{\int_{t_0}^{\nu(t)} e_a(t_0, \sigma(s))b(s)\Delta s - \int_{t_0}^{t} e_a(t_0, \sigma(s))b(s)\Delta s\right\}$$

and thus

$$\begin{aligned}
\overline{x}^{\Delta}(t) &= \lambda a(t)e_a(t,t_0)\int_t^{\nu(t)} e_a(t_0,\sigma(s))b(s)\Delta s \\
&\quad + \lambda e_a(\sigma(t),t_0)\left\{\nu^{\Delta}(t)e_a(t_0,\sigma(\nu(t)))b(\nu(t)) - e_a(t_0,\sigma(t))b(t)\right\} \\
&= a(t)\overline{x}(t) + \lambda\left\{\nu^{\Delta}(t)e_a(\sigma(t),\sigma(\nu(t)))b(\nu(t)) - b(t)\right\} \\
&= a(t)\overline{x}(t) + b(t) + \frac{\lambda}{e_a(\nu(\sigma(t)),\sigma(t))}\nu^{\Delta}(t)b(\nu(t)) - b(t)(1+\lambda) \\
&= a(t)\overline{x}(t) + b(t) + \frac{\lambda\nu^{\Delta}(t)}{E_a(\sigma(t))}\nu^{\Delta\sigma}(t)b(\nu(t)) - b(t)(1+\lambda) \\
&\overset{\text{Lem 4.1 } i)}{=} a(t)\overline{x}(t) + b(t) + \frac{\lambda\nu^{\Delta}(t)}{E_a(\sigma(t_0))}\nu^{\Delta\sigma}(t)b(\nu(t)) - b(t)(1+\lambda) \\
&\overset{(4.10)}{=} a(t)\overline{x}(t) + b(t) + \frac{\lambda b(t)}{E_a(\sigma(t_0))} - b(t)(1+\lambda) \\
&= a(t)\overline{x}(t) + b(t) + \left\{\lambda\frac{1-E_a(\sigma(t_0))}{E_a(\sigma(t_0))} - 1\right\}b(t) \\
&\overset{(4.12)}{=} a(t)\overline{x}(t) + b(t),
\end{aligned}$$

so that $\overline{x}$ solves (4.8). Lastly, we show that $\nu^{\Delta}(t)\overline{x}(\nu(t)) = \overline{x}(t)$. We have

$$\begin{aligned}
\nu^{\Delta}(t)\overline{x}(\nu(t)) &\overset{(2.4)}{=} \nu^{\Delta}(t)\lambda\int_t^{\nu(t)} \nu^{\Delta}(s)e_a(\nu(t),\sigma(\nu(s)))b(\nu(s))\Delta s \\
&= \lambda E_a(t_0)\int_t^{\nu(t)} e_a(t,\sigma(s))\frac{\nu^{\Delta}(s)b(\nu(s))}{e_a(\nu(\sigma(s)),\sigma(s))}\Delta s \\
&\overset{(4.10)}{=} \lambda\int_t^{\nu(t)} e_a(t,\sigma(s))\frac{E_a(t_0)}{e_a(\nu(\sigma(s)),\sigma(s))}\frac{b(s)}{\nu^{\Delta}(\sigma(s))}\Delta s \\
&= \lambda\int_t^{\nu(t)} e_a(t,\sigma(s))\frac{E_a(t_0)}{E_a(\sigma(s))}b(s)\Delta s \\
&= \lambda\int_t^{\nu(t)} e_a(t,\sigma(s))b(s)\Delta s = \overline{x}(t).
\end{aligned}$$

The proof is complete. □

Remark 4.11. If $\mathbb{T} = \mathbb{T}_{\mathcal{I}}$ for an arbitrary but isolated time scale $\mathbb{T}_{\mathcal{I}}$ and $\nu(t) = \sigma^{\omega}(t)$, then Theorem 4.10 is consistent with [12, Theorem 7.5]. If $\mathbb{T} = \mathbb{R}$, then Theorem 4.10 collapses to [13, Theorem 27].

Theorem 4.12. *Let $a \in \mathcal{R}$ satisfy* (4.2) *and let $b \in C_{\mathrm{rd}}$ satisfy* (4.10)*. If x solves* (4.8) *with*

$$\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$$

for some $t_0 \in \mathbb{T}$, then x is ν-periodic.

Proof. Define

$$g(t) := \nu^{\Delta}(t)x(\nu(t)) - x(t).$$

Then

$$\begin{aligned}
g^{\Delta}(t) &= \nu^{\Delta\Delta}(t)x(\nu(t)) + \nu^{\Delta}(\sigma(t))x^{\Delta}(\nu(t))\nu^{\Delta}(t) - x^{\Delta}(t) \\
&\overset{(4.8)}{=} \nu^{\Delta\Delta}(t)x(\nu(t)) + \nu^{\Delta}(t)\nu^{\Delta}(\sigma(t))\left[a(\nu(t))x(\nu(t)) + b(\nu(t))\right] - \left[a(t)x(t) + b(t)\right] \\
&\overset{(4.10)}{=} \nu^{\Delta\Delta}(t)x(\nu(t)) + \nu^{\Delta}(t)\nu^{\Delta}(\sigma(t))a(\nu(t))x(\nu(t)) - a(t)x(t) \\
&\overset{(4.2)}{=} \nu^{\Delta}(t)a(t)x(\nu(t)) - a(t)x(t) = a(t)g(t).
\end{aligned}$$

Thus, $g(t) = e_a(t, t_0)g(t_0)$ for $t_0 \in \mathbb{T}$. Since $g(t_0) = 0$, $g(t) \equiv 0$, which completes the proof. □

Theorem 4.13. *Let $a \in \mathcal{R}$ and $b \in C_{\rm rd}$ such that a satisfies* (4.2) *and b satisfies* (4.10). *Assume there exists $t_0 \in \mathbb{T}$ such that $E_a(t_0) = 1$. If*

$$\int_{t_0}^{\nu(t_0)} e_a(t_0, \sigma(s))b(s)\Delta s = 0, \tag{4.15}$$

then all solutions of (4.8) *are ν-periodic. Otherwise, no nontrivial solution of* (4.8) *is ν-periodic.*

Proof. Let x be a solution of (4.8). By using $E_a(t_0) = 1$ in (4.13), we get

$$\nu^{\Delta}(t)x(\nu(t)) = x(t) + \int_{t}^{\nu(t)} e_a(t, \sigma(s))b(s)\Delta s.$$

Hence, by (4.15), $\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$ so that the claim follows by Theorem 4.12. □

The above results for (4.8) can also be extended to the second form of a linear dynamic equation, namely,

$$x^{\Delta} = a(t)x^{\sigma} + b(t), \quad t \in \mathbb{T}, \tag{4.16}$$

where $-a \in \mathcal{R}$ and $b \in C_{\rm rd}$. By (2.6), the solution to (4.16) is given by

$$x(t) = e_{\ominus(-a)}(t, t_0)x(t_0) + \int_{t_0}^{t} e_{\ominus(-a)}(t, s)b(s)\mathrm{d}s, \quad t_0 \in \mathbb{T}. \tag{4.17}$$

Note that the same results obtained for (4.8) apply but for a replaced by $\ominus(-a)$. We already argued that the analogue of condition (4.2) for the homogeneous case in (4.5) is (4.7). Since the condition on b, (4.10), is independent of a, the condition remains the same. We state below the corresponding theorems for (4.16) below but omit their proofs.

Theorem 4.14. *If $-a \in \mathcal{R}$ satisfies* (4.7)*, $b \in C_{\rm rd}$ satisfies* (4.10)*, and there exists $t_0 \in \mathbb{T}$ such that $E_{\ominus(-a)}(t_0) \neq 1$, then* (4.16) *has a unique ν-periodic solution given by*

$$\overline{x} = \Lambda \int_{t}^{\nu(t)} e_{\ominus(-a)}(t, s)b(s)\Delta s, \quad \Lambda = \frac{E_{\ominus(-a)}(t_0)}{1 - E_{\ominus(-a)}(t_0)}. \tag{4.18}$$

Other results obtained for (4.8) can now be rephrased for (4.16) accordingly.

Theorem 4.15. *Let $-a \in \mathcal{R}$ satisfy* (4.7) *and let $b \in C_{\rm rd}$ satisfy* (4.10). *If x solves* (4.16) *with $\nu^{\Delta}(t_0)x(\nu(t_0)) = x(t_0)$, for some $t_0 \in \mathbb{T}$, then x is ν-periodic.*

Theorem 4.16. *Let $-a \in \mathcal{R}$ satisfy* (4.7) *and let $b \in C_{\mathrm{rd}}$ satisfy* (4.10)*. Assume there exists $t_0 \in \mathbb{T}$ such that $E_{\ominus(-a)}(t_0) = 1$. If*

$$\int_{t_0}^{\nu(t_0)} e_{\ominus(-a)}(t_0, s)b(s)\Delta s = 0,$$

then all solutions of (4.16) *are ν-periodic. Otherwise, no nontrivial solution of* (4.16) *is ν-periodic.*

5. Conclusions

In this work, we introduced a novel definition of periodicity with respect to a strictly increasing and delta-differentiable function ν. We say a function f is ν-periodic provided it satisfies the functional equation (3.1). This definition finally allows the concept of periodicity on arbitrary time scales without the restrictive assumption of a periodic time scale. Our introduced definition is consistent with a recently introduced concept of generalized periodicity in the real numbers [13] and collapses, for the special case of $\nu = \sigma^{\omega}$, to a definition of periodicity recently introduced for isolated time scales [12]. Thus, our definition of ν-periodic functions unifies these earlier results that considered specific time domains and, in some cases, even specific expressions of ν.

Furthermore, if $\nu(t) = t + \omega$, the definition coincides with the classical definition of ω-periodicity, that is, $f(t+\omega) = f(t)$ on periodic time scales that include the classical discrete and continuous spaces. Our definition of ν-periodicity guarantees two key properties, namely the invariance of integrals of periodic functions. That is, the area underneath a periodic function over the length of its period remains constant and the integration bounds of a periodic function can always be shifted by the period without impacting its value. These crucial aspects hold also in our generalized definition of periodicity, see Theorem 3.7.

We applied our concept of periodicity to scalar linear dynamic equations and provided sufficient and necessary conditions for the existence of ν-periodic solutions. We first focused on the existence and uniqueness of ν-periodic solutions to linear homogeneous and then nonhomogeneous dynamic equations. For each of these two classes, we considered both types of dynamic equation with and without a σ-operator on the right-hand side.

For each of the four dynamic equations, we provided necessary and also sufficient conditions for the existence of a ν-periodic solution, providing new insights into solutions to nonautonomous dynamic equations. In the special case when $\nu(t) = t + \omega$ and the time scale is periodic, including isolated time scales discussed in [12], known results are recovered from our presented theorems. In contrast to these classical results that require the model coefficient to be either constant or itself periodic, our generalization of periodicity can aid the analysis of nonautonomous and (in the classical sense) nonperiodic dynamic equations, by providing a classification of solutions that satisfy the identity (3.1) and the translation invariant properties in (3.3). Hence, the introduction of ν-periodic functions can be thought of as a tool of reducing system complexity of nonautonomous dynamic equations.

Author contributions

All authors contributed equally to the formulation, analysis, and draft writing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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Appendix

Lemma A1. *Consider a strictly increasing function $\nu : \mathbb{T} \to \mathbb{T}$ with $\nu(\mathbb{T}) = \mathbb{T}$. Then the following holds: $t \in \mathbb{T}$ is right-scattered (right-dense) if and only if $\nu(t) \in \mathbb{T}$ is right-scattered (right-dense).*

Proof. We first show that for fixed $t \in \mathbb{T}$ that is i) right-dense (or ii) right-scattered), $\nu(t) \in \mathbb{T}$ is i) right-dense (or ii) right-scattered). Case i) Let $t \in \mathbb{T}$ be right-dense. Then there exists a sequence $\{t_n\}_n \subset \mathbb{T}$ such that $t_n > t_{n+1}$ and $\lim_{t\to\infty} t_n = t$. Since ν is strictly increasing and rd-continuous, $\nu_n := \nu(t_n) \in \mathbb{T}$ such that $\nu_{n+1} < \nu_n$ and $\lim_{\to\infty} \nu_n = \nu(\lim_{n\to\infty} t_n) = \nu(t)$ so that $\nu(t)$ is also right-dense. Case ii) To show that if $t \in \mathbb{T}$ is right-scattered, then so is $\nu(t)$, suppose instead that $\nu(t)$ is right-dense. Then, there exists a sequence $\{\nu_n\}_n$ with $\nu_{n+1} < \nu_n$ and $\lim_{n\to\infty} \nu_n = \nu(t)$. Since $\nu(\mathbb{T}) = \mathbb{T}$, there exists $\{s_n\}_n$ with $s_n \in \mathbb{T}$ such that $\nu_n = \nu(s_n)$. Since ν is strictly increasing, $s_{n+1} < s_n$ and by the rd-continuity of ν, $\lim_{n\to\infty} s_n = t$, violating the assumption that t is right-scattered.

It is left to show the reverse statement, that is, if $\nu(t) \in \mathbb{T}$ is right-scattered (right-dense), then $t \in \mathbb{T}$ is right-scattered (right-dense). That is, we show that if $\nu(t) \in \mathbb{T}$ is i) right-dense (or ii) right-scattered), then $t \in \mathbb{T}$ that is i) right-dense (or ii) right-scattered). Case i) Let $\nu(t) \in \mathbb{T}$ be right-dense so that $\nu(t) = \sigma(\nu(t))$. To show that t is also right-dense, we proceed in the same fashion. That is, due to $\nu(t)$ being right-dense, there exists a sequence $\{\nu_n\}_n \subset \mathbb{T}$ such that $\nu_{n+1} < \nu_n$ and $\lim_{n\to\infty} \nu_n = \nu(t)$. Since $\nu(\mathbb{T}) = \mathbb{T}$, there exist $\{s_n\}_n \subset \mathbb{T}$ such that $\nu(s_n) = \nu_n$. By the properties of ν, $s_n > s_{n+1}$ and $\lim_{n\to\infty} s_n = t$, implying that t is right-dense. Case ii) Let $\nu(t) \in \mathbb{T}$ be right-scattered so that $\nu(t) < \sigma(\nu(t))$. To show that t is also right-scattered, we proceed by contradiction. Let t be right-dense. Then there exists a decreasing sequence $\{t_n\}_{n=1}^{\infty} \subset \mathbb{T}$ with $t_n > t$ for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} t_n = t$. Since $\nu(t_n) \in \mathbb{T}$ and ν is strictly increasing, we have $\nu(t_n) > \nu(t_{n+1})$. Furthermore, by the continuity of ν, $\lim_{n\to\infty} \nu(t_n) = \nu(t)$, so that $\sigma(\nu(t)) = \nu(t)$, resulting in a contradiction that $\nu(t)$ is right-scattered. □

Theorem A2. *Consider a strictly increasing function $\nu : \mathbb{T} \to \mathbb{T}$ with $\nu(\mathbb{T}) = \mathbb{T}$. Then, $\nu \circ \sigma = \sigma \circ \nu$.*

Proof. First note that if t is right-dense, then, by Lemma A1, $\nu(t)$ is also right-dense and we have $\sigma(\nu(t)) = \nu(t) = \nu(\sigma(t))$, confirming the desired claim. Thus, if the desired equality was false for some $T \in \mathbb{T}$, then T would have to be right-scattered, and, by Lemma A1, also $\nu(T)$. First, suppose that there exists $T \in \mathbb{T}$ such that

$$(\sigma \circ \nu)(T) > (\nu \circ \sigma)(T). \tag{A.1}$$

Then,

$$\sigma(\nu(T)) \overset{(A.1)}{>} \nu(\sigma(T)) > \nu(T),$$

where the second inequality holds since T is right-scattered and ν is strictly increasing. Thus, there exists $s = \nu(\sigma(T)) \in \mathbb{T}$ such that $\tau < s < \sigma(\tau)$ for $\tau = \nu(T) \in \mathbb{T}$, contradicting the definition of σ. Now, consider the case that there exists $T \in \mathbb{T}$ such that

$$(\sigma \circ \nu)(T) < (\nu \circ \sigma)(T). \tag{A.2}$$

Then, since T is right-scattered and ν is strictly increasing,

$$\nu(T) < \sigma(\nu(T)) \overset{(A.1)}{<} \nu(\sigma(T)). \tag{A.3}$$

Thus, there exists $s = \nu(\sigma(T)) \in \mathbb{T}$ such that $\nu(T) < s < \nu(\sigma(T))$. Since $\nu(\mathbb{T}) = \mathbb{T}$, there exists τ such that $s = \nu(\tau)$. Furthermore, since ν is strictly increasing, $T < \tau < \sigma(T)$, again resulting in a contradiction of the definition of σ. This completes the proof. □

Example A3. Let us consider the time scale

$$\mathbb{T} = \bigcup_{i \in \mathbb{Z}} \left([2i, 2i+1] \cup \{2i - \tau_i\} \right),$$

where $0 < \tau_i < 1$ for all $i \in \mathbb{Z}$. Note that if τ_i are pairwise distinct, then $\mathbb{T}$ is neither a periodic time scale nor an isolated time scale, and hence the previous definitions of periodicity (see Example 3.2) do not apply to $\mathbb{T}$. An illustration of $\mathbb{T}$ is shown in Figure A1. For the purpose of this illustration, we used

$$\tau_i = \frac{i^2 + 5}{2i^2 + 6} \quad \text{for} \quad i \in \mathbb{Z}.$$

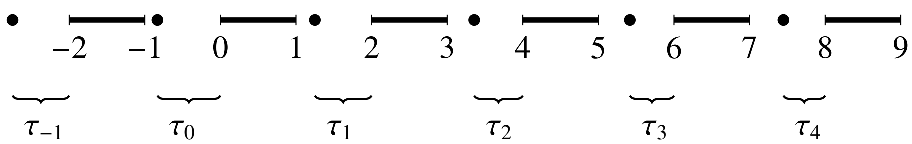

Figure A1. Illustration of $\mathbb{T}$.

For this time scale $\mathbb{T}$, the forward jump operator is

$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in [2i, 2i+1) \quad \text{for} \quad i \in \mathbb{Z}, \\ t + 1 - \tau_i & \text{if} \quad t = 2i - 1 \quad \text{for} \quad i \in \mathbb{Z}, \\ t + \tau_i & \text{if} \quad t = 2i - \tau_i \quad \text{for} \quad i \in \mathbb{Z}. \end{cases}$$

Now we define

$$\nu(t) = \begin{cases} t + 2 & \text{if} \quad t \in [2i, 2i+1] \quad \text{for} \quad i \in \mathbb{Z}, \\ t + 2 - \Delta\tau_i & \text{if} \quad t = 2i - \tau_i \quad \text{for} \quad i \in \mathbb{Z}. \end{cases}$$

Then ν is strictly increasing and $\nu(\mathbb{T}) = \mathbb{T}$. Indeed, we now calculate ν^Δ. Let $i \in \mathbb{Z}$. Then

$$\nu^\Delta(t) = 1 \quad \text{for} \quad t \in [2i, 2i-1),$$

$$\begin{aligned}\nu^\Delta(2i-1) &= \frac{\nu(\sigma(2i-1)) - \nu(2i-1)}{\sigma(2i-1) - (2i-1)} = \frac{\nu(2i-\tau_i) - \nu(2i-1)}{(2i-\tau_i) - (2i-1)} \\ &= \frac{(2i+2-\tau_{i+1}) - (2i+1)}{1-\tau_i} = \frac{1-\tau_{i+1}}{1-\tau_i},\end{aligned}$$

and

$$\begin{aligned}\nu^\Delta(2i-\tau_i) &= \frac{\nu(\sigma(2i-\tau_i)) - \nu(2i-\tau_i)}{\sigma(2i-\tau_i) - (2i-\tau_i)} = \frac{\nu(2i) - \nu(2i-\tau_i)}{2i-(2i-\tau_i)} \\ &= \frac{(2i+2) - (2i+2-\tau_{i+1})}{\tau_i} = \frac{\tau_{i+1}}{\tau_i}.\end{aligned}$$

In summary,

$$\nu^\Delta(t) = \begin{cases} 1 & \text{if} \quad t \in [2i, 2i+1) \quad \text{for} \quad i \in \mathbb{Z}, \\ \frac{1-\tau_{i+1}}{1-\tau_i} & \text{if} \quad t = 2i-1 \quad \text{for} \quad i \in \mathbb{Z}, \\ \frac{\tau_{i+1}}{\tau_i} & \text{if} \quad t = 2i-\tau_i \quad \text{for} \quad i \in \mathbb{Z}. \end{cases}$$

Now let $f : \mathbb{T} \to \mathbb{R}$. Let $i \in \mathbb{Z}$. Then

$$\nu^\Delta(t) f(\nu(t)) = \begin{cases} f(t+2) & \text{if} \quad t \in [2i, 2i+1) \quad \text{for} \quad i \in \mathbb{Z}, \\ \frac{1-\tau_{i+1}}{1-\tau_i} f(2i+1) & \text{if} \quad t = 2i-1 \quad \text{for} \quad i \in \mathbb{Z}, \\ \frac{\tau_{i+1}}{\tau_i} f(2i+2-\tau_{i+1}) & \text{if} \quad t = 2i-\tau_i \quad \text{for} \quad i \in \mathbb{Z}. \end{cases}$$

So f is ν-periodic if and only if it is "normal periodic" on $[2i, 2i+1)$ for all $i \in \mathbb{Z}$ and

$$f(2i+2-\tau_{i+1}) = \frac{\tau_i}{\tau_{i+1}} f(2i-\tau_i) \quad \text{and} \quad f(2i+1) = \frac{1-\tau_i}{1-\tau_{i+1}} f(2i-1) \quad \text{for all} \quad i \in \mathbb{Z}. \tag{A.4}$$

It is easy to solve the recursions in (A.4) and to arrive at

$$f(2i-1) = \frac{1-\tau_0}{1-\tau_i} f(-1) \quad \text{and} \quad f(2i-\tau_i) = \frac{\tau_0}{\tau_i} f(-\tau_0) \quad \text{for all} \quad i \in \mathbb{Z}. \tag{A.5}$$

By choosing

$$f(-1) = f(-\tau_0) = 0,$$

we could have a ν-periodic f as depicted in Figure A2.

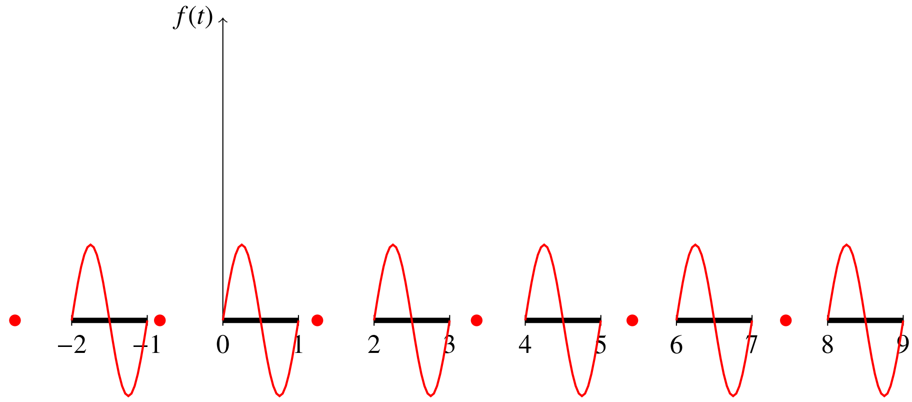

Figure A2. Example of a ν-periodic function on $\mathbb{T}$.

By choosing $f(-1) = \frac{1}{1-\tau_0}$ and $f(-\tau_0) = \frac{1}{\tau_0}$, (A.5) turns into

$$f(2i-1) = \frac{1}{1-\tau_i} \quad \text{and} \quad f(2i-\tau_i) = \frac{1}{\tau_i} \quad \text{for all} \quad i \in \mathbb{Z}, \tag{A.6}$$

and we could have a ν-periodic f as depicted in Figure A3.

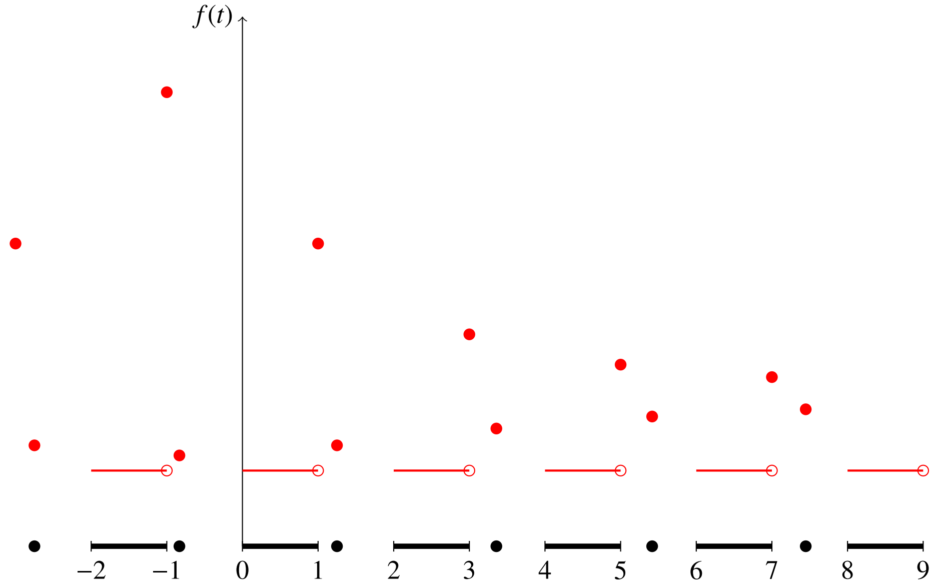

Figure A3. Example of a ν-periodic function on $\mathbb{T}$.