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Research article

Some refinements of the Cauchy-Schwarz inequality via orthogonal projections

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Abstract: In this paper, we present several refinements of the operator Cauchy-Schwarz inequality for positive operators. Our main result strengthens the classical form of this inequality and serves as a foundation for deriving a series of new inequalities that both generalize and improve upon existing results in the literature. Furthermore, we investigate substantial improvements to the Cauchy–Schwarz inequality by employing orthogonal projections, leading to sharper bounds in various settings. Additionally, we obtain a new perspective on the Cauchy–Schwarz inequality by showing that both the inner product and the product of norms can be characterized as extremal values of projection-dependent expressions. Several related inequalities are also established, many of which recover or extend recent contributions by other authors.

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1. Introduction

In mathematics, inequalities have played a significant role in various fields for many years. A pivotal moment in the study of inequalities occurred with the publication of the book "*Inequalities*" by G. H. Hardy, J. Littlewood, and J. Polya in 1934 [1]. This influential work not only shaped the discipline but also provided valuable insights, techniques, and applications, establishing inequalities as a well-defined area of study. Another crucial contribution was made in 1961 when Edwin F. Beckenbach and R. Bellman released their important book on this topic [2]. This work further advanced the field of inequalities, emphasizing its relevance and offering new perspectives for research.

These significant texts have had a profound effect on the study of inequalities, laying a strong foundation for future inquiries. For additional information, readers can consult the references provided. Inspired by the extensive history of inequalities and their practical applications, this paper seeks to enhance the classical Cauchy–Schwarz inequality. Our aim is to deepen our understanding of this inequality and explore its implications.

Before addressing our primary topic, it is beneficial to review some well-known and extensively studied inequalities in inner product spaces, which may be either real or complex. For our purposes, we will consider $\mathcal{H}$ as a complex Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. Among the fundamental inequalities in inner product spaces is the Cauchy–Schwarz inequality (CSI), which is widely recognized and utilized. It can be stated as follows:

$$|\langle u, v\rangle| \leq \|u\|\|v\|, \tag{1.1}$$

for all $u, v \in \mathcal{H}$. Equality in (1.1) holds if and only if a complex number $\mu \in \mathbb{C}$ exists such that $u = \mu v$.

In Buzano's paper [3], a revised version of the Cauchy–Schwarz inequality was presented, known as the Buzano inequality (BuI):

$$|\langle x, z\rangle\langle z, y\rangle| \leq \frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\|y\|\right)\|z\|^2, \tag{1.2}$$

for any $x, y, z \in \mathcal{H}$. This inequality serves as a notable extension of the Cauchy–Schwarz inequality and has significant applications across various areas of mathematics.

Another important inequality in the literature was established by A. Selberg (see, for example, [4, p. 394]). For the vectors $x, z_1, \ldots, z_n$ in $\mathcal{H}$, where $z_i \neq 0$ for all $i \in \{1, \ldots, n\}$, we can apply Selberg's inequality (SI), which states that:

$$\sum_{i=1}^{n} \frac{|\langle x, z_i\rangle|^2}{\sum_{j=1}^{n} \left|\langle z_i, z_j\rangle\right|} \leq \|x\|^2 . \tag{1.3}$$

Selberg's inequality plays a significant role in harmonic analysis and mathematical physics, with numerous applications. It has been extensively studied, as evidenced by notable works such as [5–7]. Moreover, when the vectors z_i are orthonormal for all $i \in \{1, \ldots, n\}$, the inequality (1.3) simplifies to the well-known Bessel's inequality [4].

Refining the Cauchy–Schwarz inequality has both theoretical and practical importance. From a theoretical viewpoint, sharper inequalities provide deeper insight into the geometry of Hilbert spaces and operator behavior. On the practical side, such refinements play an important role in numerous fields including numerical linear algebra, quantum information theory, and spectral theory. For instance, in quantum mechanics, the CSI is often used in the derivation of uncertainty relations, and its refinement may lead to tighter bounds. In numerical analysis, improved versions of the CSI contribute to better error estimates in iterative methods and stability analyses.

Despite the large number of generalizations of the CSI, many of them do not take the structure of positive operators or orthogonal projections into account, which naturally arise in applications. This motivates the need for new results that incorporate such structures while still refining the classical inequality.

For further insights into the Cauchy–Schwarz inequality (CSI) and its refinements, we recommend reviewing several classical and modern contributions. Bombieri's work [8] provides a fundamental large sieve inequality that highlights the strength of the CSI in number theory. Dragomir and collaborators [9–11] present various refinements and applications of the CSI and Bessel-type inequalities in inner product spaces. More recent developments and extensions, such as those connected with Selberg- and Heinz–Kato–Furuta-type inequalities, can be found in the works of Fujii and co-authors [12, 13], while Steele's monograph [14] gives a comprehensive exposition of the CSI and its role as a central tool in mathematical inequalities.

Throughout this manuscript, we denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ as $\mathcal{B}(\mathcal{H})$. The identity operator is represented by I. For any operator $T \in \mathcal{B}(\mathcal{H})$, we indicate its adjoint by T^*.

An operator T is considered positive, written as $T \geq 0$, if it satisfies the condition $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Furthermore, for the operators T and S, the notation $T \geq S$ signifies that $T - S$ is a positive operator.

Assuming T is positive, we can apply the operator Cauchy–Schwarz inequality as follows:

$$|\langle Tx, y\rangle| \leq \langle Tx, x\rangle^{\frac{1}{2}} \langle Ty, y\rangle^{\frac{1}{2}}, \tag{1.4}$$

for any vectors $x, y \in \mathcal{H}$.

In this paper, we aim to refine the inequality (1.4). Our main result enhances this inequality, leading to several new inequalities that generalize and improve upon the existing results in the literature. Additionally, we investigate significant refinements of the Cauchy–Schwarz inequality using orthogonal projections and prove several related inequalities that recover and extend the recent findings by various authors.

2. Some preliminaries

In this section, we will discuss fundamental concepts, definitions, and motivations that are crucial for understanding the main results of this paper. First, we note that the inequality (1.4) can be expressed using the square root of the positive operator $T \in \mathcal{B}(\mathcal{H})$ as follows:

$$|\langle Tx, y\rangle| \leq \|T^{\frac{1}{2}}x\| \|T^{\frac{1}{2}}y\|.$$

Additionally, (1.4) leads to the following useful result:

$$\|Tx\|^2 \leq \|T\| \langle Tx, x\rangle,$$

which is valid for any positive operator T and any $x \in \mathcal{H}$.

In [15, Proposition 1], a refined version of (1.4) is presented. Specifically, if T is a positive operator in $\mathcal{B}(\mathcal{H})$ and $\alpha \in [0, 1]$, then the following holds:

$$\begin{aligned} |\langle Tx, y\rangle|^2 &\leq (1-\alpha)\langle Tx, x\rangle^{\frac{1}{2}} \langle Ty, y\rangle^{\frac{1}{2}} |\langle Tx, y\rangle| + \alpha \langle Tx, x\rangle \langle Ty, y\rangle \\ &\leq \langle Tx, x\rangle \langle Ty, y\rangle, \end{aligned} \tag{2.1}$$

for any $x, y \in \mathcal{H}$.

It is straightforward to observe that the parameters in the inequality, which do not depend on the operator T or the vectors x and y, satisfy the condition

$$(1-\alpha)+\alpha = 1.$$

Motivated by this property and the proof of (2.1), we will consider the following subset of functions defined on the interval $[0,1]$:

$$\mathcal{S} = \{f : [0,1] \to [0,1],\ f(\alpha) + f(1-\alpha) = 1 \text{ for any } \alpha \in [0,1]\}. \tag{2.2}$$

For instance, it is easy to show that

$$f_1(\alpha) = \alpha, \quad f_2(\alpha) = \frac{1}{2}\left(\frac{1}{2}+\alpha\right), \quad \text{and} \quad f_3(\alpha) = \sin^2\left(\frac{\pi}{2}\alpha\right)$$

belong to $\mathcal{S}$.

We will now present a characterization of the set $\mathcal{S}$ defined above.

Proposition 2.1. *$f \in \mathcal{S}$ if and only if an antysimmetric functions with two arguments, i.e. $\Phi(x,z) = -\Phi(z,x)$, exists such that*

$$f(\alpha) = \frac{1}{2} + \Phi(\alpha, 1-\alpha),$$

where

$$-\frac{1}{2} \le \Phi(\alpha, 1-\alpha) \le \frac{1}{2},$$

for all $\alpha \in [0,1]$.

Proof. In [16], Polyanin and Manzhirov studied and solved the following functional equation:

$$f(\alpha) + f(a-\alpha) = g(\alpha), \tag{2.3}$$

where the function g satisfies the condition $g(\alpha) = g(a-\alpha)$, with $\alpha, a \in \mathbb{R}$. The general solution of (2.3) is given by:

$$f(\alpha) = \frac{1}{2}g(\alpha) + \Phi(\alpha, a-\alpha),$$

where any antisymmetric function with two arguments Φ. In particular, if we take $g(\alpha) = 1$ for all $\alpha \in [0,1]$ and set $a = 1$, then the general solution of the functional equation presented in (2.2) is given by

$$f(\alpha) = \frac{1}{2} + \Phi(\alpha, 1-\alpha).$$

To obtain all functions in $\mathcal{S}$, it is necessary to impose conditions on the antisymmetric function Φ defined above, ensuring that the range of f is contained within $[0,1]$. Specifically, in addition to being antisymmetric, Φ must satisfy the following condition:

$$-\frac{1}{2} \le \Phi(\alpha, 1-\alpha) \le \frac{1}{2},$$

for all $\alpha \in [0,1]$. □

Notice that the functions f_1 and f_3 mentioned earlier both belong to $\mathcal{S}$ and are related in the following way:

$$f_3(\alpha) = \sin^2\left(\frac{\pi}{2} f_1(\alpha)\right).$$

In the following result, we will demonstrate, without providing a proof since it is nearly immediate, that this holds in general for any function f_1 that belongs to $\mathcal{S}$.

Lemma 2.1. *Let $f \in \mathcal{S}$. Then, $g(\alpha) = \cos^2\left(\frac{\pi}{2} f(\alpha)\right)$ and $h(\alpha) = \sin^2\left(\frac{\pi}{2} f(\alpha)\right)$ also belong to $\mathcal{S}$.*

For the upcoming discussion, it is important to recall the Selberg operator defined as follows.

Definition 2.1. *Given a subset $\mathcal{Z} = \{z_i : i = 1, \dots, n\}$ of nonzero vectors in the Hilbert space $\mathcal{H}$, the Selberg operator $S_{\mathcal{Z}}$ is defined by*

$$S_{\mathcal{Z}} = \sum_{i=1}^{n} \frac{z_i \otimes z_i}{\sum_{j=1}^{n} |\langle z_i, z_j \rangle|} \in \mathcal{B}(\mathcal{H}),$$

where $x \otimes y$ denotes a rank-one operator defined by $x \otimes y(z) = \langle z, y \rangle x$, with x, y, and z being vectors in the Hilbert space $\mathcal{H}$.

Using the Selberg operator, we can reformulate the (SI) as follows:

$$0 \le \langle S_{\mathcal{Z}} x, x \rangle = \sum_{i=1}^{n} \frac{|\langle x, z_i \rangle|^2}{\sum_{j=1}^{n} |\langle z_i, z_j \rangle|} \le \langle x, x \rangle,$$

for any $x \in \mathcal{H}$. Thus, the (SI) implies that every Selberg operator is a positive contraction; i.e., $0 \le S_{\mathcal{Z}} \le I$.

We conclude this preliminary section by noting that within the space of bounded linear operators $\mathcal{B}(\mathcal{H})$, there is a special group called Hilbert-Schmidt operators, denoted as $\mathcal{B}_2(\mathcal{H})$. This group forms its own Hilbert space, with an inner product defined by

$$\langle X, Y \rangle_2 = \sum_{i=1}^{\infty} \langle X e_i, Y e_i \rangle = \operatorname{tr}(Y^* X),$$

for any operators X, Y in $\mathcal{B}_2(\mathcal{H})$. Here, $\{e_i\}_{i=1}^{\infty}$ is any orthonormal basis of $\mathcal{H}$, and $\operatorname{tr}(\cdot)$ is the trace function.

When looking at non-zero operators $X, Y \in \mathcal{B}_2(\mathcal{H})$, the angle between them, $\alpha_{X,Y} \in [0, \pi]$, is calculated using the formula:

$$\cos(\alpha_{X,Y}) = \frac{\operatorname{Re}\langle X, Y \rangle_2}{\|X\|_2 \|Y\|_2},$$

where the Hilbert-Schmidt norm $\|X\|_2$ is defined as $\|X\|_2^2 = \langle X, X \rangle_2$.

3. Main results

In this section, we present our main results. The primary finding of this manuscript is a generalization of (2.1), which naturally refines (1.4).

Theorem 3.1. *Assume that T is a positive operator in $\mathcal{B}(\mathcal{H})$, $f \in \mathcal{S}$, and that $r > 0$. Then, for any $x, y \in \mathcal{H}$, we have:*

$$\begin{aligned} |\langle Tx, y\rangle|^r &\leq f(1-\alpha)\|T^{\frac{1}{2}}x\|^{\frac{r}{2}}\|T^{\frac{1}{2}}y\|^{\frac{r}{2}}|\langle Tx, y\rangle|^{\frac{r}{2}} + f(\alpha)\|T^{\frac{1}{2}}x\|^r\|T^{\frac{1}{2}}y\|^r \\ &\leq \|T^{\frac{1}{2}}x\|^r\|T^{\frac{1}{2}}y\|^r. \end{aligned}$$

Proof. Let T be a positive operator in $\mathcal{B}(\mathcal{H})$, and let $\alpha \in [0,1]$. Since $f(\alpha) + f(1-\alpha) = 1$ for any $f \in \mathcal{S}$, it follows that for any $x, y \in \mathcal{H}$, we have

$$\begin{aligned} |\langle Tx, y\rangle|^r &= [f(\alpha) + f(1-\alpha)]|\langle Tx, y\rangle|^r \\ &= f(\alpha)|\langle Tx, y\rangle|^r + f(1-\alpha)|\langle Tx, y\rangle|^r. \end{aligned}$$

Applying inequality (1.4) and using the fact that $T^{\frac{1}{2}}$ exists due to the positivity of T, we obtain that

$$|\langle Tx, y\rangle|^s \leq \langle Tx, x\rangle^{\frac{s}{2}}\langle Ty, y\rangle^{\frac{s}{2}} = \|T^{\frac{1}{2}}x\|^s\|T^{\frac{1}{2}}y\|^s$$

for any $s > 0$. Combining the previous inequalities for $s = r$ and $s = \frac{r}{2}$, we conclude that

$$\begin{aligned} |\langle Tx, y\rangle|^r &= f(\alpha)|\langle Tx, y\rangle|^r + f(1-\alpha)|\langle Tx, y\rangle|^r \\ &= f(\alpha)|\langle Tx, y\rangle|^{\frac{r}{2}}|\langle Tx, y\rangle|^{\frac{r}{2}} + f(1-\alpha)|\langle Tx, y\rangle|^r \\ &\leq f(1-\alpha)\|T^{\frac{1}{2}}x\|^{\frac{r}{2}}\|T^{\frac{1}{2}}y\|^{\frac{r}{2}}|\langle Tx, y\rangle|^{\frac{r}{2}} + f(\alpha)\|T^{\frac{1}{2}}x\|^r\|T^{\frac{1}{2}}y\|^r \\ &\leq f(1-\alpha)\langle Tx, x\rangle^{\frac{r}{2}}\langle Ty, y\rangle^{\frac{r}{2}} + f(\alpha)\langle Tx, x\rangle^{\frac{r}{2}}\langle Ty, y\rangle^{\frac{r}{2}} \\ &= [f(\alpha) + f(1-\alpha)]\langle Tx, x\rangle^{\frac{r}{2}}\langle Ty, y\rangle^{\frac{r}{2}} \\ &= \langle Tx, x\rangle^{\frac{r}{2}}\langle Ty, y\rangle^{\frac{r}{2}} = \|T^{\frac{1}{2}}x\|^r\|T^{\frac{1}{2}}y\|^r. \end{aligned}$$

This finishes the proof. □

Remark 3.1. *It is immediate that by considering $f(\alpha) = \alpha$ with $\alpha \in [0,1]$ and $r = 2$, we obtain* (2.1).

Next, as an application of our findings, we will demonstrate that several inequalities, both recently established by the authors of this manuscript and by others, can be derived by considering different positive operators T, functions f, and values of r. The first application of Theorem 3.1, obtained by considering an orthogonal projection, can be stated in the following result.

Theorem 3.2. *Let P be an orthogonal projection on $\mathcal{H}$ and $f \in \mathcal{S}$. Then, for any $x, y \in \mathcal{H}$, $\alpha \in [0,1]$ and $r > 0$, we have*

$$\begin{aligned} |\langle Px, y\rangle - \langle x, y\rangle|^r &\leq f(1-\alpha)\|P^\perp x\|^{\frac{r}{2}}\|P^\perp y\|^{\frac{r}{2}}\left|\langle P^\perp x, y\rangle\right|^{\frac{r}{2}} + f(\alpha)\|P^\perp x\|^r\|P^\perp y\|^r \\ &\leq \|P^\perp x\|^r\|P^\perp y\|^r \\ &\leq (\|x\|\|y\| - \|Px\|\|Py\|)^r \\ &\leq (\|x\|\|y\|)^r, \end{aligned}$$

where $P^\perp = I - P$.

Proof. As is well-known, if P is an orthogonal projection, then $P^{\perp}$ is as well. Then, by Theorem 3.1, for any $x, y \in \mathcal{H}$ and $f \in \mathcal{S}$, we have

$$\begin{aligned}|\langle Px, y\rangle - \langle x, y\rangle|^r &= |\langle P^{\perp}x, y\rangle|^r \\ &\leq f(1-\alpha)\|P^{\perp}x\|^{\frac{r}{2}}\|P^{\perp}y\|^{\frac{r}{2}}\left|\langle P^{\perp}x, y\rangle\right|^{\frac{r}{2}} + f(\alpha)\|P^{\perp}x\|^r\|P^{\perp}y\|^r \\ &\leq \|P^{\perp}x\|^r\|P^{\perp}y\|^r.\end{aligned}$$

Taking into account that $\|P^{\perp}z\|^2 = \|z\|^2 - \|Pz\|^2$ for any $z \in \mathcal{H}$ and using the inequality $(a^2-b^2)(c^2-d^2) \leq (ac-bd)^2$ for all $a, b, c, d \geq 0$, we obtain

$$\begin{aligned}\|P^{\perp}x\|^r\|P^{\perp}y\|^r &= \left[\left(\|x\|^2 - \|Px\|^2\right)\left(\|y\|^2 - \|Py\|^2\right)\right]^{\frac{r}{2}} \\ &\leq [\|x\|\|y\| - \|Px\|\|Py\|]^r \\ &\leq (\|x\|\|y\|)^r.\end{aligned}$$

Hence, the proof is complete. □

Example 3.1. *In this example, we illustrate in a simple case, that the bound obtained in Theorem 3.2 is indeed sharper than the well-known estimate*

$$|\langle Px, y\rangle - \langle x, y\rangle|^r \leq (\|x\|\|y\| - \|Px\|\|Py\|)^r.$$

Let $\mathcal{H} = \mathbb{R}^2$ with the standard inner product and let P be the orthogonal projection onto span$\{e_1\}$*, i.e.,*

$$P = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \qquad P^{\perp} = I - P = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Consider

$$x = \begin{bmatrix}1\\ 1\end{bmatrix}, \qquad y = \begin{bmatrix}1\\ 2\end{bmatrix}.$$

Then

$$\langle x, y\rangle = 3, \qquad \langle Px, y\rangle = 1,$$

so

$$\left|\langle Px, y\rangle - \langle x, y\rangle\right|^2 = \left|1-3\right|^2 = 4.$$

Moreover,

$$\|P^{\perp}x\| = 1, \qquad \|P^{\perp}y\| = 2, \qquad \textit{and} \qquad \langle P^{\perp}x, y\rangle = 2.$$

Choose $f \in \mathcal{S}$ with $f(t) = t$, take $\alpha = \frac{1}{2}$ and $r = 2$. Applying Theorem 3.2 yields

$$\left|\langle Px, y\rangle - \langle x, y\rangle\right|^2 \leq f(1-\alpha)\,\|P^{\perp}x\|\,\|P^{\perp}y\|\left|\langle P^{\perp}x, y\rangle\right| + f(\alpha)\,\|P^{\perp}x\|^2\,\|P^{\perp}y\|^2.$$

With the data above, the right-hand side is equal to 4. Hence, in this case, the refined bound from Theorem 3.2 is exact:

$$\left|\langle Px, y\rangle - \langle x, y\rangle\right|^2 = 4.$$

On the other hand, the classical estimate gives

$$(\|x\|\,\|y\| - \|Px\|\,\|Py\|)^2 = (\sqrt{2}\,\sqrt{5} - 1)^2 = (\sqrt{10} - 1)^2 \approx 4.6776,$$

so we obtain the strict chain

$$\left|\langle Px, y\rangle - \langle x, y\rangle\right|^2 = 4 < (\sqrt{10} - 1)^2 = (\|x\|\,\|y\| - \|Px\|\,\|Py\|)^2.$$

Remark 3.2. *It is noteworthy that the inequality derived in Theorem 3.2 serves to generalize the result presented in Theorem 3.1 of [17].*

Two of the authors of this manuscript derived the following inequalities in [17], employing a proof based on different techniques: Suppose that P is an orthogonal projection. We then have:

$$|\langle Px, y\rangle - \langle x, y\rangle|^2 \leq \|y\|^2(\|x\|^2 - \|Px\|^2),$$

for every $x, y, z \in \mathcal{H}$. *In particular, if* $y \neq 0$ *and* $z \neq 0$, *we obtain*

$$\frac{\|z\|^2}{\|y\|^2}\left|\frac{\langle x, z\rangle\langle z, y\rangle}{\|z\|^2} - \langle x, y\rangle\right|^2 \leq \|x\|^2\,\|z\|^2 - |\langle x, z\rangle|^2. \tag{3.1}$$

The inequality (3.1) *is a generalization of the well-known Ostrowski result in a inner product space.*

As a consequence of Theorem 3.2, we obtain the following refinements.

Corollary 3.1. *Let* P *be an orthogonal projection on* $\mathcal{H}$ *and* $f \in \mathcal{S}$. *Then, for any* $x, y \in \mathcal{H}$ *and* $\alpha \in [0, 1]$, *we have*

$$\begin{aligned} |\langle Px, y\rangle - \langle x, y\rangle|^2 &\leq f(1-\alpha)\|P^{\perp}x\|\|P^{\perp}y\|\left|\langle P^{\perp}x, y\rangle\right| + f(\alpha)\|P^{\perp}x\|^2\|P^{\perp}y\|^2 \\ &\leq \|P^{\perp}x\|^2\|P^{\perp}y\|^2 \\ &\leq \|y\|^2(\|x\|^2 - \|Px\|^2), \end{aligned} \tag{3.2}$$

where $P^{\perp} = I - P$.

In particular, if $y \neq 0$ *and* $z \neq 0$, *we derive*

$$\begin{aligned} \left|\frac{\langle x, z\rangle\langle z, y\rangle}{\|z\|^2} - \langle x, y\rangle\right|^2 &\leq f(1-\alpha)\|P_0^{\perp}x\|\|P_0^{\perp}y\|\left|\langle P_0^{\perp}x, y\rangle\right| + f(\alpha)\|P_0^{\perp}x\|^2\|P_0^{\perp}y\|^2 \\ &\leq (\|x\|^2 - \|P_0x\|^2)(\|y\|^2 - \|P_0y\|^2) \\ &\leq \frac{\|y\|^2}{\|z\|^2}\left(\|x\|^2\|z\|^2 - |\langle x, z\rangle|^2\right), \end{aligned} \tag{3.3}$$

with $P_0 = z_0 \otimes z_0$ *is the one-dimensional orthogonal projection , where* $z_0 = \frac{z}{\|z\|}$.

Proof. The inequality (3.2) follows directly from Theorem 3.2 by setting $r = 2$ and applying the fundamental properties of orthogonal projections P and $P^{\perp}$. To establish the inequality (3.3), we examine (3.2) under the specific condition $P = P_0$. In this and similar situations, the following holds:

$$\begin{aligned} \left|\frac{\langle x, z\rangle\langle z, y\rangle}{\|z\|^2} - \langle x, y\rangle\right|^2 &= |\langle P_{z_0}x, y\rangle - \langle x, y\rangle|^2 \\ &\leq f(1-\alpha)\|P_0^{\perp}x\|\|P_0^{\perp}y\|\left|\langle P_0^{\perp}x, y\rangle\right| + f(\alpha)\|P_0^{\perp}x\|^2\|P_0^{\perp}y\|^2 \\ &\leq (\|x\|^2 - \|P_0x\|^2)(\|y\|^2 - \|P_0y\|^2) \\ &\leq \|y\|^2(\|x\|^2 - \|P_0x\|^2) \\ &= \|y\|^2\left(\|x\|^2 - \frac{1}{\|z\|^2}|\langle x, z\rangle|^2\right) \\ &= \frac{\|y\|^2}{\|z\|^2}\left(\|x\|^2\|z\|^2 - |\langle x, z\rangle|^2\right). \end{aligned}$$

This achieves the proof. □

Remark 3.3. *We note that the inequality* (3.3) *represents a sharper variant of the classical Cauchy-Schwarz inequality.*

Now we are in a position to derive a refinement of Ostrowski's inequality in inner product spaces, utilizing functions that belong to the set $\mathcal{S}$.

Corollary 3.2. *Let $f \in \mathcal{S}$. Then, for any $x, y, z \in \mathcal{H}$ such that $z \neq 0$, $\langle x, z\rangle = 1$, $\langle y, z\rangle = 0$, and $\alpha \in [0, 1]$, we have*

$$\begin{aligned}
&\frac{\|y\|^2}{\|z\|^2} \\
&\leq f(\alpha)\frac{\|y\|}{\|z\|}\sqrt{\|x\|^2\|z\|^2 - 1}|\langle x, y\rangle| + f(\alpha)\frac{\|y\|^2}{\|z\|^2}(\|x\|^2\|z\|^2 - 1) + \frac{\|y\|^2}{\|z\|^2} - |\langle x, y\rangle|^2 \\
&\leq \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2.
\end{aligned}$$

We are now in a position to derive, by considering the identity operator in Theorem (3.1), the following refinements and inequalities associated with Buzano's inequality (1.2).

Corollary 3.3. *Let $x, y \in \mathcal{H}$, and $f \in \mathcal{S}$. Then for any $\alpha \in [0, 1]$ and $r \geq 1$, we have*

$$\begin{aligned}
|\langle x, y\rangle|^r &\leq f(1-\alpha)\|x\|^{\frac{r}{2}}\|y\|^{\frac{r}{2}}|\langle x, y\rangle|^{\frac{r}{2}} + f(\alpha)\|x\|^r\|y\|^r \\
&\leq \|x\|^r\|y\|^r. \qquad (3.4)
\end{aligned}$$

Proof. The inequality (3.4) is derived from Theorem 3.1 by selecting $T = I$.

□

Starting from the BuI and applying a similar argument to the one used in the proof of Theorem 3.1, we can derive the following generalization of that inequality.

Proposition 3.1. *Let $x, y, z \in \mathcal{H}$ with $\|z\| = 1$, and $f \in \mathcal{S}$. Then for any $\alpha \in [0, 1]$ and $r \geq 1$, we have*

$$\begin{aligned}
|\langle x, z\rangle\langle z, y\rangle|^r &\leq \frac{1}{2}f(\alpha)|\langle x, y\rangle|^r + \frac{1}{2}(1 + f(1-\alpha))\|x\|^r\|y\|^r \\
&\leq \|x\|^r\|y\|^r.
\end{aligned}$$

Proof. Considering the convex function $g(t) = t^r$ for $t \in [0, \infty)$ and applying Buzano's inequality, we obtain:

$$\begin{aligned}
|\langle x, z\rangle\langle z, y\rangle|^r &\leq \frac{1}{2}|\langle x, y\rangle|^r + \frac{1}{2}\|x\|^r\|y\|^r \\
&= \frac{1}{2}[f(\alpha) + f(1-\alpha)]|\langle x, y\rangle|^r + \frac{1}{2}\|x\|^r\|y\|^r \\
&= \frac{1}{2}f(\alpha)|\langle x, y\rangle|^r + \frac{1}{2}[1 + f(1-\alpha)]\|x\|^r\|y\|^r.
\end{aligned}$$

Finally, the last inequality is obtained by applying the Cauchy-Schwarz inequality (CSI) along with the identity satisfied by the function f. □

Corollary 3.4. *Let $x, y, z \in \mathcal{H}$ with $\|z\| = 1$. Then for any $\theta \geq 0$ and $r \geq 1$, we have*

$$\begin{aligned} |\langle x, z\rangle\langle z, y\rangle|^r &\leq \frac{1}{4\theta+4}|\langle x, y\rangle|^r + \frac{4\theta+3}{4\theta+4}\|x\|^r\|y\|^r \\ &\leq \|x\|^r\|y\|^r. \end{aligned} \tag{3.5}$$

Proof. The proof follows directly by setting $f(\alpha) = \alpha$ with $\alpha = \frac{1}{2(\theta+1)}$ for $\theta \geq 0$, and then applying Proposition 3.1. □

Remark 3.4. *The inequality* (3.5) *for the case $r = 2$ was recently obtained by Guesba and Garayev in [18]. Furthermore, for any $r \geq 1$, they established the following result:*

$$|\langle x, z\rangle\langle z, y\rangle|^r \leq \frac{1}{2\theta+2}|\langle x, y\rangle|^r + \frac{2\theta+1}{2\theta+2}\|x\|^r\|y\|^r.$$

This inequality can be derived from Proposition 3.1 by setting $f(\alpha) = \alpha$ and choosing $\alpha = \frac{1}{\theta+1}$.

Our next main objective of this paper is to offer significant refinements of the Cauchy-Schwarz inequality through the use of orthogonal projections. To achieve our goals, we first recall the result from [17]: Let P be an orthogonal projection, and let $x, y \in \mathcal{H}$; in this case

$$|\langle x, y\rangle| \leq \|x\|\|y\| + |\langle Px, y\rangle| - \|Px\|\|Py\| \leq \|x\|\|y\|.$$

We will now present a refinement of this inequality.

Theorem 3.3. *Let P be an orthogonal projection on $\mathcal{H}$ and $f \in \mathcal{S}$. Then, for any $x, y \in \mathcal{H}$ and $\alpha \in [0, 1]$, we have*

$$\begin{aligned} &|\langle x, y\rangle| \\ &\leq f(1-\alpha)\|P^\perp x\|^{\frac{1}{2}}\|P^\perp y\|^{\frac{1}{2}} \left|\langle P^\perp x, y\rangle\right|^{\frac{1}{2}} + f(\alpha)\|P^\perp x\|\|P^\perp y\| + |\langle Px, y\rangle| \\ &\leq \sqrt{\|x\|^2 - \|Px\|^2}\sqrt{\|y\|^2 - \|Py\|^2} + |\langle Px, y\rangle| \\ &\leq \|x\|\|y\| - \|Px\|\|Py\| + |\langle Px, y\rangle| \\ &\leq \|x\|\|y\|. \end{aligned}$$

Proof. Decomposing $x = Px + P^\perp x$ for any $x \in \mathcal{H}$ and applying the triangle inequality, we obtain that For any $x, y \in \mathcal{H}$, we have

$$|\langle x, y\rangle| = \left|\langle Px + P^\perp x, y\rangle\right| \leq \left|\langle P^\perp x, y\rangle\right| + |\langle Px, y\rangle|.$$

Applying Theorem 3.2 to $\left|\langle P^\perp x, y\rangle\right|$, it follows that:

$$\begin{aligned} |\langle x, y\rangle| &\leq \left|\langle P^\perp x, y\rangle\right| + |\langle Px, y\rangle| \\ &\leq f(1-\alpha)\|P^\perp x\|^{\frac{1}{2}}\|P^\perp y\|^{\frac{1}{2}} \left|\langle P^\perp x, y\rangle\right|^{\frac{1}{2}} \\ &\quad + f(\alpha)\|P^\perp x\|\|P^\perp y\| + |\langle Px, y\rangle| \\ &\leq \|P^\perp x\|\|P^\perp y\| + |\langle Px, y\rangle|. \end{aligned}$$

Since $\|P^{\perp}x\| = \sqrt{\|x\|^2 - \|Px\|^2}$ holds for any $x \in \mathcal{H}$, and using the elementary inequality

$$(\alpha^2 - \beta^2)(\gamma^2 - \delta^2) \leq (\alpha\gamma - \beta\delta)^2$$

which is valid for any real numbers α, β, γ, and δ, we deduce that:

$$\begin{aligned}
|\langle x, y\rangle| &\leq \sqrt{\|x\|^2 - \|Px\|^2}\sqrt{\|y\|^2 - \|Py\|^2} + |\langle Px, y\rangle| \\
&\leq \|x\|\|y\| - \|Px\|\|Py\| + |\langle Px, y\rangle| \\
&\leq \|x\|\|y\|.
\end{aligned}$$

Hence, the proof is complete. □

Motivated by the preceding proof, given an orthogonal P, we define the following functions:

$$\Phi_P : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, \qquad \Phi_P(x, y) := \|P^{\perp}x\|\|P^{\perp}y\| + |\langle Px, y\rangle|,$$

and

$$\Psi_P : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, \qquad \Psi_P(x, y) := \|P^{\perp}x\|\|P^{\perp}y\| + \|Px\|\|Py\|.$$

Such functions are symmetric and comparable due to the idempotence property of every orthogonal projection and the Cauchy-Schwarz inequality. More precisely, for any for any $x, y \in \mathcal{H}$, the following holds:

$$\Phi_P(x, y) \leq \Psi_P(x, y).$$

Additionally, by applying Proposition 3.3 and noting that

$$\|P^{\perp}x\|\|P^{\perp}y\| \leq \|x\|\|y\| - \|Px\|\|Py\|,$$

we conclude that both Φ_P and Ψ_P provide an enhancement of the Cauchy-Schwarz inequality. In particular,

$$|\langle x, y\rangle| \leq \Phi_P(x, y) \leq \Psi_P(x, y) \leq \|x\|\|y\|. \tag{3.6}$$

Remark 3.5. *Note that the following inequality holds for any $x, y \in \mathcal{H}$ and for any orthogonal projection P:*

$$\|Px\|\|Py\| - |\langle Px, y\rangle| \leq \|x\|\|y\| - |\langle x, y\rangle|, \tag{3.7}$$

because, by (3.6)*, we have*

$$0 \leq \|Px\|\|Py\| - |\langle Px, y\rangle| = \Psi_P(x, y) - \Phi_P(x, y) \leq \|x\|\|y\| - |\langle x, y\rangle|.$$

In conclusion, we obtain the following refinement of the classical Cauchy-Schwarz inequality:

$$|\langle x, y\rangle| \leq |\langle x, y\rangle| + \|Px\|\|Py\| - |\langle Px, y\rangle| \leq \|x\|\|y\|.$$

From now on, for convenience, we will use $\mathcal{P}(\mathcal{H})$ to denote the set of all orthogonal projections defined on the Hilbert space $\mathcal{H}$.

Theorem 3.4. *For any $x, y \in \mathcal{H}$, it holds that*

$$|\langle x, y\rangle| = \inf_{P\in\mathcal{P}(\mathcal{H})} \Phi_P(x, y) = \inf_{P\in\mathcal{P}(\mathcal{H})} \Psi_P(x, y),$$

and

$$\sup_{P\in\mathcal{P}(\mathcal{H})} \Phi_P(x, y) = \sup_{P\in\mathcal{P}(\mathcal{H})} \Psi_P(x, y) = \|x\|\|y\|.$$

Proof. If $x = 0$ or $y = 0$, the equalities hold trivially. Therefore, we can assume, without loss of generality, that $x \neq 0$. We consider $P_0 = x_0 \otimes x_0 \in \mathcal{P}(\mathcal{H})$ where $x_0 = \frac{x}{\|x\|}$. We then have

$$\Psi_{P_0}(x, y) = \|P_0x\|\|P_0y\| + \|P_0^{\perp}x\|\|P_0^{\perp}y\| = |\langle x, y\rangle|.$$

From (3.6), we conclude that

$$|\langle x, y\rangle| \le \inf_{P\in\mathcal{P}(\mathcal{H})} \Phi_P(x, y) \le \inf_{P\in\mathcal{P}(\mathcal{H})} \Psi_P(x, y) \le \Psi_{P_0}(x, y) = |\langle x, y\rangle|.$$

Thus, the identities involving the infimum are proven. Now, consider $P_1 = 0$, the null projection. In this case, we have:

$$\Phi_{P_1}(x, y) = \|P_1^{\perp}x\|\|P_1^{\perp}y\| + |\langle P_1x, y\rangle| = \|x\|\|y\|.$$

Again using the inequality (3.6), we obtain:

$$\|x\|\|y\| = \Phi_{P_1}(x, y) \le \sup_{P\in\mathcal{P}(\mathcal{H})} \Phi_P(x, y) \le \sup_{P\in\mathcal{P}(\mathcal{H})} \Psi_P(x, y) \le \|x\|\|y\|.$$

Thus, we can conclude the proof. □

The following result is obtained as a consequence of Proposition 3.3.

Corollary 3.5. *For any $f \in \mathcal{S}$ and $x, y \in \mathcal{H}$, it holds that*

$$\begin{aligned} |\langle x, y\rangle| &= \inf_{P\in\mathcal{P}(\mathcal{H})} \left\{\left|\langle P^{\perp}x, y\rangle\right| + |\langle Px, y\rangle|\right\} \\ &= \inf_{P\in\mathcal{P}(\mathcal{H})} \Lambda_{f,P}(x, y), \end{aligned}$$

where

$$\begin{aligned} \Lambda_{f,P}(x, y) = {}& f(1-\alpha)\|P^{\perp}x\|^{\frac{1}{2}}\|P^{\perp}y\|^{\frac{1}{2}} \left|\langle P^{\perp}x, y\rangle\right|^{\frac{1}{2}} \\ &+ f(\alpha)\|P^{\perp}x\|\|P^{\perp}y\| + |\langle Px, y\rangle| . \end{aligned}$$

In addition, we have

$$\begin{aligned} \|x\|\|y\| &= \sup_{P\in\mathcal{P}(\mathcal{H})} \left\{ \sqrt{\|x\|^2 - \|Px\|^2}\sqrt{\|y\|^2 - \|Py\|^2} + |\langle Px, y\rangle|\right\} \\ &= \sup_{P\in\mathcal{P}(\mathcal{H})} \left\{\|x\|\|y\| - \|Px\|\|Py\| + |\langle Px, y\rangle|\right\} . \end{aligned}$$

Proof. By the proof leading to Proposition 3.3 and, in particular, the following inequalities:

$$\begin{aligned}|\langle x, y\rangle| &\leq \left|\langle P^{\perp}x, y\rangle\right| + |\langle Px, y\rangle| \\ &\leq f(1-\alpha)\|P^{\perp}x\|^{\frac{1}{2}}\|P^{\perp}y\|^{\frac{1}{2}}\left|\langle P^{\perp}x, y\rangle\right|^{\frac{1}{2}} \\ &\quad + f(\alpha)\|P^{\perp}x\|\|P^{\perp}y\| + |\langle Px, y\rangle| \\ &\leq \|P^{\perp}x\|\|P^{\perp}y\| + |\langle Px, y\rangle| = \Phi_P(x, y)\end{aligned}$$

and

$$\begin{aligned}\Phi_P(x, y) &\leq \sqrt{\|x\|^2 - \|Px\|^2}\sqrt{\|y\|^2 - \|Py\|^2} + |\langle Px, y\rangle| \\ &\leq \|x\|\|y\| - \|Px\|\|Py\| + |\langle Px, y\rangle| \leq \|x\|\|y\|,\end{aligned}$$

which are valid for any $x, y \in \mathcal{H}$, $f \in \mathcal{S}$, and $P \in \mathcal{P}(\mathcal{H})$. So, the proof of this corollary follows directly from the previous inequalities and Theorem 3.4. □

We now present a new proof and refinement of the well-known fact that every orthogonal projection satisfies the classical Buzano inequality ([19]). The central idea of this result builds upon the proof of that inequality, proposed by Fuji and Kubo in [20], along with the inequalities derived in (3.6).

Theorem 3.5. *Let $P, Q \in \mathcal{P}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then*

$$\begin{aligned}|\langle Px, y\rangle| &\leq \frac{1}{2}\left(\Phi_Q((2P-I)x, y) + |\langle x, y\rangle|\right) \\ &\leq \frac{1}{2}\left(\Psi_Q((2P-I)x, y) + |\langle x, y\rangle|\right) \\ &\leq \frac{1}{2}\left(\|(2P-I)x\|\|y\| + |\langle x, y\rangle|\right) \\ &\leq \frac{1}{2}\left(\|x\|\|y\| + |\langle x, y\rangle|\right).\end{aligned}$$

Proof. We immediately find that for all $x, y \in \mathcal{H}$ and $P \in \mathcal{P}(\mathcal{H})$, the following holds:

$$\begin{aligned}2\,|\langle Px, y\rangle| &= |\langle 2Px, y\rangle| = |\langle (2P-I)x, y\rangle + \langle x, y\rangle| \\ &\leq |\langle (2P-I)x, y\rangle| + |\langle x, y\rangle|.\end{aligned}$$

As a consequence of the inequality (3.6), we have

$$|\langle (2P-I)x, y\rangle| \leq \Phi_Q((2P-I)x, y) \leq \Psi_Q((2P-I)x, y),$$

for any $Q \in \mathcal{P}(\mathcal{H})$ and $x, y \in \mathcal{H}$. By combining the preceding inequalities and utilizing the most recent inequality given in (3.6), we conclude that

$$\begin{aligned}2\,|\langle Px, y\rangle| &\leq |\langle (2P-I)x, y\rangle| + |\langle x, y\rangle| \\ &\leq \Phi_Q((2P-I)x, y) + |\langle x, y\rangle| \\ &\leq \Psi_Q((2P-I)x, y) + |\langle x, y\rangle| \\ &\leq \|(2P-I)x\|\|y\| + |\langle x, y\rangle|.\end{aligned}$$

The proof concludes by noting that $\|2P - I\| \leq 1$. □

Example 3.2. *Let $\mathcal{H} = \mathbb{R}^2$ with the standard inner product, and let*

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Let Q denote the orthogonal projection onto $\operatorname{span}\{(1,1)\}$. *A direct computation yields the following bounds for $|\langle Px, y\rangle|$:*

$$|\langle Px, y\rangle| = 1 < \frac{1}{2}\Big(\Phi_Q((2P-I)x, y) + |\langle x, y\rangle|\Big) = 2 < \frac{1}{2}\Big(\|x\|\,\|y\| + |\langle x, y\rangle|\Big) \approx 3.081.$$

This illustrates that the inequalities in Theorem 3.5 are strict, and that by choosing appropriate orthogonal projections Q, one can obtain bounds that are sharper than the classical Buzano estimate, providing greater flexibility in controlling the inner products.

Building upon the proof technique used in the previous result, we are now equipped to derive the following statement, which serves as a refinement of Proposition 3.1 in [21].

Corollary 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C} \setminus \{0\}$, with $\|\alpha T - I\| \leq 1$, and let $P \in \mathcal{P}(\mathcal{H})$. For any $x, y \in \mathcal{H}$, the following holds:*

$$\begin{aligned} |\langle Tx, y\rangle| &\leq \frac{1}{|\alpha|}\left(\Phi_P((\alpha T - I)x, y) + |\langle x, y\rangle|\right) \\ &\leq \frac{1}{|\alpha|}\left(\Psi_P((\alpha T - I)x, y) + |\langle x, y\rangle|\right) \\ &\leq \frac{1}{|\alpha|}\left(\|(\alpha T - I)x\|\|y\| + |\langle x, y\rangle|\right) \\ &\leq \frac{1}{|\alpha|}\left(\|x\|\|y\| + |\langle x, y\rangle|\right). \end{aligned}$$

Our final objective in this paper is to present various applications of our main result in broader contexts than those previously discussed, where we considered the identity operator or orthogonal projections. Specifically, we will focus on the Selberg operator and inequalities related to the angle between the operators in the $\mathcal{B}_2(\mathcal{H})$ ideal.

We begin with a result that emerges from the positivity of the Selberg operator associated with a set $\mathcal{Z} \subseteq \mathcal{H}$, in conjunction with Theorem 3.1. Specifically, we state the following result.

Proposition 3.2. *Given a subset $\mathcal{Z} = \{z_i : i = 1, \ldots, n\}$ of nonzero vectors in the Hilbert space $\mathcal{H}$, $f \in \mathcal{S}$, and $r \geq 1$, we have:*

$$\begin{aligned} |\langle S_{\mathcal{Z}}x, y\rangle|^r &\leq f(1-\alpha)\left\|S_{\mathcal{Z}}^{\frac{1}{2}}x\right\|^{\frac{r}{2}}\left\|S_{\mathcal{Z}}^{\frac{1}{2}}y\right\|^{\frac{r}{2}}|\langle S_{\mathcal{Z}}x, y\rangle|^{\frac{r}{2}} + f(\alpha)\left\|S_{\mathcal{Z}}^{\frac{1}{2}}x\right\|^{r}\left\|S_{\mathcal{Z}}^{\frac{1}{2}}y\right\|^{r} \\ &\leq \left\|S_{\mathcal{Z}}^{\frac{1}{2}}x\right\|^{r}\left\|S_{\mathcal{Z}}^{\frac{1}{2}}y\right\|^{r}. \end{aligned}$$

The next result deals with the operators in $\mathcal{B}_2(\mathcal{H})$.

Theorem 3.6. *Let $X, Y \in \mathcal{B}_2(\mathcal{H})$ be non-zero operators, and $f \in \mathcal{S}$. For any $\alpha \in [0,1]$, we have*

$$\begin{aligned}\cos^2(\alpha_{X,Y}) &\leq f(\alpha)\sqrt{\cos(\alpha_{|X^*|,|Y^*|})}\sqrt{\cos(\alpha_{|X|,|Y|})}\frac{|\langle X, Y\rangle_2|}{\|X\|_2\|Y\|_2} \\ &\quad + f(1-\alpha)\cos(\alpha_{|X^*|,|Y^*|})\cos(\alpha_{|X|,|Y|}) \\ &\leq \cos(\alpha_{|X^*|,|Y^*|})\cos(\alpha_{|X|,|Y|}).\end{aligned}$$

Proof. Let us note that for any X, Y in the Hilbert-Schmidt class and non-zero, and for any $\alpha \in [0,1]$, we have:

$$\begin{aligned}\cos^2(\alpha_{X,Y}) &= \left(\frac{\mathrm{Re}\langle X, Y\rangle_2}{\|X\|_2\|Y\|_2}\right)^2 \\ &\leq \frac{|\langle X, Y\rangle_2|^2}{\|X\|_2^2\|Y\|_2^2} = \frac{[f(\alpha)+f(1-\alpha)]|\langle X, Y\rangle_2|^2}{\|X\|_2^2\|Y\|_2^2} \\ &= \frac{f(\alpha)|\langle X, Y\rangle_2|^2}{\|X\|_2^2\|Y\|_2^2} + \frac{f(1-\alpha)|\langle X, Y\rangle_2|^2}{\|X\|_2^2\|Y\|_2^2}.\end{aligned}$$

By Theorem 2.6 in [22], we know that

$$|\langle X, Y\rangle_2|^2 \leq |\langle |X^*|, |Y^*|\rangle_2||\langle |X|, |Y|\rangle_2|. \tag{3.8}$$

Combining the two previous inequalities, we obtain:

$$\begin{aligned}\cos^2(\alpha_{X,Y}) &\leq \frac{f(\alpha)\sqrt{|\langle |X^*|, |Y^*|\rangle_2|}\sqrt{|\langle |X|, |Y|\rangle_2|}|\langle X, Y\rangle_2|}{\|X\|_2^2\|Y\|_2^2} \\ &\quad + \frac{f(1-\alpha)|\langle |X^*|, |Y^*|\rangle_2||\langle |X|, |Y|\rangle_2|}{\|X\|_2^2\|Y\|_2^2} \\ &= f(\alpha)\sqrt{\frac{|\langle |X^*|, |Y^*|\rangle_2|}{\|X\|_2\|Y\|_2}}\sqrt{\frac{|\langle |X|, |Y|\rangle_2|}{\|X\|_2\|Y\|_2}}\frac{|\langle X, Y\rangle_2|}{\|X\|_2\|Y\|_2} \\ &\quad + f(1-\alpha)\frac{|\langle |X^*|, |Y^*|\rangle_2|}{\|X\|_2\|Y\|_2}\frac{|\langle |X|, |Y|\rangle_2|}{\|X\|_2^2\|Y\|_2^2}.\end{aligned}$$

Using the fact that if Z, W are positive operators, then $\langle Z, W\rangle_2 \geq 0$, it follows that

$$\begin{aligned}\cos^2(\alpha_{X,Y}) &\leq f(\alpha)\sqrt{\cos(\alpha_{|X^*|,|Y^*|})}\sqrt{\cos(\alpha_{|X|,|Y|})}\frac{|\langle X, Y\rangle_2|}{\|X\|_2\|Y\|_2} \\ &\quad + f(1-\alpha)\cos(\alpha_{|X^*|,|Y^*|})\cos(\alpha_{|X|,|Y|}).\end{aligned}$$

Finally, note that by (3.8), we have:

$$\begin{aligned}\frac{|\langle X, Y\rangle_2|}{\|X\|_2\|Y\|_2} &\leq \sqrt{\frac{|\langle |X^*|, |Y^*|\rangle_2|}{\|X\|_2^2\|Y\|_2^2}}\sqrt{\frac{|\langle |X|, |Y|\rangle_2|}{\|X\|_2^2\|Y\|_2^2}} \\ &= \sqrt{\cos(\alpha_{|X^*|,|Y^*|})}\sqrt{\cos(\alpha_{|X|,|Y|})}.\end{aligned}$$

Therefore, we have shown that

$$\begin{aligned}\cos^2(\alpha_{X,Y}) &\le f(\alpha)\sqrt{\cos(\alpha_{|X^*|,|Y^*|})}\sqrt{\cos(\alpha_{|X|,|Y|})}\frac{|\langle X,Y\rangle_2|}{\|X\|_2\|Y\|_2}\\ &\quad + f(1-\alpha)\cos(\alpha_{|X^*|,|Y^*|})\cos(\alpha_{|X|,|Y|})\\ &\le \cos(\alpha_{|X^*|,|Y^*|})\cos(\alpha_{|X|,|Y|}).\end{aligned}$$

□

Remark 3.6. *Let us note that if we consider $\lambda \ge 0$, $\alpha = \frac{1}{1+\lambda} \in [0,1]$, and $f(\alpha) = \alpha$ in Theorem 3.6, we recover, as a particular case, the inequality (2.7) established by Zamani in [22].*

4. Conclusions

In this paper, we have presented several refinements and generalizations of the operator Cauchy–Schwarz inequality, particularly in the context of positive operators. By introducing new inequalities involving orthogonal projections and, more generally, positive operators, we have not only extended known results but also provided a unified framework that recovers various inequalities previously established in the literature.

These contributions open up new directions for research in several areas. In particular, the refined inequalities developed here may offer valuable tools in harmonic analysis, operator theory, and related branches of functional analysis. Moreover, since inequalities of the Cauchy–Schwarz type often play a crucial role in studying convergence in iterative methods, error bounds, and variational estimates, our results may find applications in numerical linear algebra and quantum information theory.

Natural open problems arise from this work. For instance, it would be interesting to determine whether the techniques employed here can be adapted to nonpositive or unbounded operators, or extended to more general settings such as Hilbert C^*-modules. Another promising direction is to study whether analogous inequalities hold under different operator means or to explore possible connections with matrix trace inequalities.

We hope that the refinements and methodologies presented in this manuscript inspire further developments and applications in the theory of operator inequalities.

Author contributions

S.A., C.C., and K.F. contributed to the conceptualization, visualization, resources, writing—review and editing, formal analysis, project administration, validation, and investigation. S.A. secured funding.

All authors declare that they have contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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