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Research article

Nonlinear Ω-Caputo fractional differential equations with infinite-point boundary conditions

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Abstract: This paper investigates the existence and uniqueness of solutions for a class of nonlinear Ω-Caputo fractional differential equations (CFDEs) supplemented with infinite-point boundary conditions. By constructing an appropriate operator framework and employing fixed-point (fp) theorems, including the Banach, the Schaefer, and the Schauder–Tychonoff fp theorems, we establish the existence and uniqueness criteria for the proposed boundary value problem (BVP). The analysis is conducted within suitable Banach spaces, taking into account the properties of the Ω-Caputo fractional derivative and the nonlocal nature of the boundary conditions. To substantiate the theoretical findings, a concrete example is presented to illustrate the applicability and effectiveness of the main results.

Keywords: Caputo fractional derivative; boundary value problems; existence of solutions; infinite-point boundary conditions; fixed-point theorems
Mathematics Subject Classification: 34K10, 34K37

1. Introduction

Fractional calculus (FC) has emerged as a powerful mathematical tool in recent decades [1–3], primarily due to its ability to accurately describe dynamic systems that exhibit memory effects and hereditary behavior. Unlike classical calculus, which is restricted to derivatives and integrals of integer order, FC generalizes these operators to arbitrary (noninteger) orders, allowing for a more nuanced and flexible modeling approach [4–6]. This generalization proves particularly valuable in capturing the intrinsic properties of complex phenomena that cannot be adequately explained using traditional integer-order models [7–9]. As a result, fractional-order models have found widespread applications across numerous scientific and engineering disciplines, including viscoelastic material analysis, anomalous diffusion processes, biological systems, control theory, and signal processing [10–12]. The growing interest in this area has led to the development of various definitions of fractional derivatives and integrals, each with its own theoretical advantages and practical implications [13–15].

FC, especially through the Ω-Caputo fractional derivative, has become a key tool in modeling and analyzing nonlinear dynamic systems that exhibit memory effects and complex hereditary behavior. By generalizing the classical fractional operators with the Ω-Caputo derivative, this approach provides enhanced flexibility and greater descriptive power [16–18]. Such generalizations allow researchers to capture intricate phenomena more accurately, leading to significant theoretical advancements and expanding potential applications in diverse disciplines, including physics, engineering, and biology.

In [19], the authors studied the existence of solutions of Caputo fractional differential equations (CFDEs) with m-point boundary conditions for $1 < q \leq 2$

$$-{}^{C}D^{q}u(t) = f(t, u(t), {}^{C}D^{q-1}u(t)), \ \ 0 < t < 1,$$

$$u(0) = 0, \ \ u(1) = \textstyle\sum_{i=1}^{m-2} \delta_i u(\mu_i),$$

where $0 < \delta_i, \mu_i < 1$ with $\sum_{i=1}^{m-2} \delta_i \mu_i < 1$.

In [20], the authors considered the following infinite-point CFDEs for $2 < \alpha \leq 3$

$${}^{C}D^{\alpha}_{0^+}u(t) + f(t, u(t), u'(t)) = 0, \ \ 0 < t < 1,$$

$$u(0) = u''(0) = 0, \ \ u'(1) = \textstyle\sum_{i=1}^{\infty} \eta_i u(\xi_i),$$

where $\eta_i \geq 0$ and $0 < \xi_1 < \xi_2 < ... < \xi_i < ... < 1$.

In [21], the authors investigated the existence of positive solutions of the following nonlinear fractional differential equation (FDE) with the Riemann–Liouville's fractional derivative for $n - 1 < \alpha \leq n$

$$D^{\alpha}_{0^+}u(t) + \lambda f(t, u(t)) = 0, \ \ 0 < t < 1,$$

$$u(0) = u'(0) = ... = u^{(n-2)}(0) = 0, \ \ u^{(i)}(1) = \textstyle\sum_{i=0}^{m-2} \eta_i u'(\xi_i),$$

where $\eta_i \geq 0$ and $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$.

In [22], the authors prove the positive solutions of the FDE with the nonlinear Riemann–Liouville's fractional derivative for $n - 1 < \alpha \leq n$

$$D^{\alpha}_{0^+}u(t) + \lambda h(t) f(t, u(t)) = 0, \ \ 0 < t < 1,$$

$$u(0) = u'(0) = ... = u^{(n-2)}(0) = 0, \ \ D^{\beta_0}_{0^+}u(1) = \sum_{i=1}^{m} \int_0^1 D^{\beta_i}_{0^+}u(t)dH_i(t),$$

where $\beta_0 \geq 1$ and $0 \leq \beta_1 < \beta_2 < ... < \beta_m \leq \beta_0 < \alpha - 1$.

In [23], the researchers studied the p-Laplacian FDE system with the Riemann–Liouville fractional derivative for infinite points

$$D^{\alpha}_{0^+}(\phi_{p_1}(D^{\gamma}_{0^+}u))(t) + \lambda^{\frac{1}{q_1-1}} f(t, u(t), D^{\mu_1}_{0^+}u(t), ..., D^{\mu_{n-2}}_{0^+}u(t), v(t)) = 0, \ \ 0 < t < 1,$$

$$D^{\beta}_{0^+}(\phi_{p_2}(D^{\delta}_{0^+}v))(t) + \lambda^{\frac{1}{q_2-1}} g(t, u(t), D^{\overline{\mu_1}}_{0^+}u(t), ..., D^{\overline{\mu_{n-2}}}_{0^+}u(t)) = 0, \ \ 0 < t < 1,$$

$$u^{(j)}(0) = 0, \ \ j = 0, 1, 2, ..., n-2, \ \ \ v^{(j)}(0) = 0, \ \ j = 0, 1, 2, ..., m-2,$$

$$D^{\gamma}_{0^+}u(0) = 0, \ \ D^{r_1}_{0^+}u(1) = \sum_{j=1}^{\infty} \nu_j D^{r_2}_{0^+}u(\xi_j), \ \ D^{\delta}_{0^+}v(0) = 0, \ \ D^{\overline{r_1}}_{0^+}v(1) = \sum_{j=1}^{\infty} \overline{\nu_j} D^{\overline{r_2}}_{0^+}u(\overline{\xi_j}),$$

where $\frac{1}{2} < \alpha, \beta \leq 1$, $n-1 < \gamma \leq n$, $m-1 < \delta \leq m$, $r_1, r_2 \in [2, n-2]$, $\overline{r_1}$, $\overline{r_2} \in [2, m-2]$, $i-1 < \mu_i \leq i$, $k-1 < \overline{\mu_i} \leq k$, $\nu_j, \overline{\nu_j} \geq 0$, $0 < \xi_1 < ... < \xi_j < ... < 1$, and $0 < \overline{\xi_1} < ... < \overline{\xi_j} < ... < 1$.

Inspired by the above mentioned articles, the following Ω-Caputo infinite-point FDE is studied in this work:

$$\begin{aligned} &{}^{C}D^{\alpha,\Omega}_{0^+}u(t) + \lambda h(t) f(t, u(t), {}^{C}D^{\beta_1,\Omega}_{0^+}u(t), ..., {}^{C}D^{\beta_{n-2},\Omega}_{0^+}u(t)) = 0, \ \ t \in (0,1), \\ &u(0) = u'(0) = u''(0) = ... = u^{(n-2)}(0) = 0, \\ &{}^{C}D^{\gamma_1,\Omega}_{0^+}u(1) = \sum_{j=1}^{\infty} \mu_j {}^{C}D^{\gamma_2,\Omega}_{0^+}u(\xi_j), \end{aligned} \tag{1.1}$$

where $n-1 < \alpha \leq n$, $n \geq 4$, $\gamma_1, \gamma_2 \in [1, n-3]$, $\gamma_2 \leq \gamma_1$, $i-1 < \beta_i \leq i$, $i = 1, 2, ..., n-2$, $\mu_j \geq 0$, $0 < \xi_1 < \xi_2 < ... < \xi_j < ... < 1$, $\mathcal{A} = \frac{\Gamma(n)}{\Gamma(n-\gamma_1)}(\Omega(1) - \Omega(0))^{n-\gamma_1-1} - \frac{\Gamma(n)}{\Gamma(n-\gamma_2)} \sum_{j=1}^{\infty} \mu_j (\Omega(\xi_j) - \Omega(0))^{n-\gamma_2-1} \neq 0$ $f \in C([0,1] \times \mathbb{R}^{n-1}, \mathbb{R})$, $f(t, u_1, u_2, ..., u_{n-1})$ is a continuous function, ${}^{C}D^{\alpha,\Omega}_{0^+}$, ${}^{C}D^{\beta_i,\Omega}_{0^+}$ $\{i = 1, 2, ..., n-2\}$, ${}^{C}D^{\gamma_1,\Omega}_{0^+}$, ${}^{C}D^{\gamma_2,\Omega}_{0^+}$ are generalized Ω-Caputo derivatives, $h \in C([0,1], \mathbb{R})$, and λ is a real number.

Equation (1.1) can be used to model physical and engineering systems with long-term memory and nonlocal effects. The Ω-Caputo fractional derivatives capture hereditary behavior, meaning that the present state of the system depends on its entire past history. Such characteristics are typical in viscoelastic materials, where the current stress depends on all previous strains, and in anomalous diffusion through heterogeneous media, where particle movement is influenced by spatial and temporal memory. The infinite-point boundary conditions correspond to situations in which the state at a given instant depends on an infinite number of past moments or spatial points, as occurs in distributed delay systems in control theory and in certain signal processing problems. Similar fractional modeling approaches are discussed in detail by Li and Zeng [24].

In the five papers above, where the equation structures were presented, various classes of fractional boundary value problems (FBVPs) were investigated. The works in [19–21] involve the standard Caputo or Riemann–Liouville fractional derivatives, while [22, 23] also address singular nonlinearities or positivity constraints. The boundary conditions considered in [19, 21, 22] are m-point or multi-point conditions, and [20] treats values at an infinite number of points, but none of these studies deals with the Ω-Caputo fractional derivative. Moreover, the analytical techniques used in these papers, such as Schauder's theorem, the Banach contraction principle, the mixed monotone operator method, and fixed-point (fp) index theory, are applied to classical fractional derivatives. In contrast, our work introduces an operator framework in Banach spaces tailored to the Ω-Caputo fractional derivative, combined with infinite-point nonlocal boundary conditions, and establishes the existence and uniqueness criteria using the Banach, the Schaefer, and the Schauder–Tychonoff fp theorems. To the best of our knowledge, this setting has not been investigated before in the literature.

2. Preliminaries

This section introduces the fundamental tools related to Ω-fractional calculus, such as definitions, notational conventions, and auxiliary results. These will serve as a foundation for the rest of the work. Specifically, we define m as $[\alpha]+1$, where $[\alpha]$ indicates the greatest integer less than or equal to α, and $\Gamma(\alpha)$ denotes the well-known gamma function.

Definition 2.1. *[4] Assume that $\alpha > 0$, and let ψ be an integrable real-valued function defined on the closed interval $[a,b]$. Let Ω be a function belonging to $C^1([a,b],\mathbb{R})$, which is both differentiable and strictly increasing, with the additional condition that $\Omega'(t) \neq 0$ holds throughout $[a,b]$. The left-hand-side Ω-Riemann–Liouville fractional integral of order α of the function ψ is defined as follows:*

$$(I_{a^+}^{\alpha,\Omega}\psi)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (\Omega(t)-\Omega(s))^{\alpha-1}\Omega'(s)\psi(s)\,ds.$$

Definition 2.2. *[4] Assume that $\alpha > 0$, and let ψ be an integrable real-valued function defined on the closed interval $[a,b]$. Let Ω be a function belonging to $C^m([a,b],\mathbb{R})$, which is both differentiable and strictly increasing, with the additional condition that $\Omega'(t) \neq 0$ holds throughout $[a,b]$. The left-hand-side Ω-Riemann–Liouville fractional derivative of order α of the function ψ is defined as follows:*

$$(D_{a^+}^{\alpha,\Omega}\psi)(t) = \left[\frac{1}{\Omega'(t)}\frac{d}{dt}\right]^m I_{a^+}^{m-\alpha,\Omega}\psi(t).$$

Definition 2.3. *[14] Assume that $\alpha > 0$, and let Ω and ψ be two functions belonging to $C^m([a,b],\mathbb{R})$, with Ω increasing and $\Omega'(t) \neq 0$ for every $t \in [a,b]$. The left-hand-side Ω-Caputo fractional derivative of order α of the function ψ is defined as follows:*

$$({}^C D_{a^+}^{\alpha,\Omega}\psi)(t) = I_{a^+}^{m-\alpha,\Omega}\left[\frac{1}{\Omega'(t)}\frac{d}{dt}\right]^m \psi(t).$$

Lemma 2.4. *[13] Let $\psi \in C^{m-1}([a,b],\mathbb{R})$ and $\alpha > 0$. We then have*

$$I_{a^+}^{\alpha,\Omega}\,{}^C D_{a^+}^{\alpha,\Omega}\psi(t) = \psi(t) - \sum_{k=0}^{m-1}\frac{\psi_\Omega^{[k]}(a)}{k!}(\Omega(t)-\Omega(a))^k,$$

where $\psi_\Omega^{[m]}(t) = \left[\frac{1}{\Omega'(t)}\frac{d}{dt}\right]^m \psi(t)$. In particular, if $0 < \alpha < 1$, we have $I_{a^+}^{\alpha,\Omega}\,{}^C D_{a^+}^{\alpha,\Omega}\psi(t) = \psi(t) - \psi(a)$.

Lemma 2.5. *[4] Let $\alpha > 0$, $\psi \in C([a,b],\mathbb{R})$, and $\Omega \in C^1([a,b],\mathbb{R})$. Then for all $t \in [a,b]$,*

(1) $I_{a^+}^{\alpha,\Omega}(.)$ maps $C([a,b],\mathbb{R})$ into $C([a,b],\mathbb{R})$;

(2) $I_{a^+}^{\alpha,\Omega}\psi(a) = lim_{t\to a^+} I_{a^+}^{\alpha,\Omega}\psi(t) = 0.$

Theorem 2.6. *[13] Let $\alpha > 0$ and $\psi_1, \psi_2 \in C^m([a, b], \mathbb{R})$. Then*

$$^{C}D_{a^+}^{\alpha,\Omega}\psi_1(t) = {}^{C}D_{a^+}^{\alpha,\Omega}\psi_2(t) \Leftrightarrow \psi_1(t) = \psi_2(t) + \sum_{k=0}^{m-1} c_k(\Omega(t) - \Omega(a))^k,$$

where c_k is an arbitrary constant.

Lemma 2.7. *[4, 13, 25] Suppose that $\alpha, \beta > 0$ and let $\psi \in L^1[a, b]$ and $\Omega \in C^1([a, b], \mathbb{R})$. Then*

(1) $I_{a^+}^{\alpha,\Omega}(\Omega(t) - \Omega(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(\Omega(t) - \Omega(a))^{\alpha+\beta-1}$;
(2) $^{C}D_{a^+}^{\alpha,\Omega}(\Omega(t) - \Omega(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\Omega(t) - \Omega(a))^{\beta-\alpha-1}$;
(3) $(I_{a^+}^{\alpha,\Omega} I_{a^+}^{\beta,\Omega}\psi)(t) = (I_{a^+}^{\alpha+\beta,\Omega}\psi)(t)$.

Lemma 2.8. *[25] Let α be a positive number and $\psi \in C^{n-1}([a, b], \mathbb{R})$. Then $^{C}D_{a^+}^{\alpha,\Omega} I_{a^+}^{\alpha,\Omega}\psi(t) = \psi(t)$.*

The existence and uniqueness of the solutions are examined with selected fp theorems. Further details on fp theory can be found in [26–28].

Theorem 2.9. *[29] (Banach fp theorem) Let $\mathbb{B}$ be a Banach space and let K be a closed subset of $\mathbb{B}$. If $\Upsilon : K \longrightarrow K$ is a contraction mapping, then a unique fp in K exists such that $\Upsilon u = u$.*

Theorem 2.10. *[30] (Schaefer fp theorem) Consider a Banach space $\mathbb{B}$ and a compact continuous operator $\Upsilon : \mathbb{B} \longrightarrow \mathbb{B}$. If the set $\{u \in \mathbb{B} : u = \mu\Upsilon u\}$ for some $\mu \in (0, 1)$ is bounded, then Υ has a fp in $\mathbb{B}$.*

Theorem 2.11. *[31] (Schauder–Tychonoff fp theorem) Let $\mathbb{B}$ be a Banach space and K be a nonempty, closed, convex subset of $\mathbb{B}$. If $\Upsilon : K \to K$ is continuous and compact, then Υ has an fp in K.*

3. Main results

The purpose of this section is to analyze the existence and uniqueness of solutions for the Caputo fractional boundary value problems (CFBVPs). For this, we define the space

$$\mathbb{B} = \{u : u \in C([0, 1], \mathbb{R}), {}^{C}D_{0^+}^{\beta_i,\Omega}u \in C([0, 1], \mathbb{R}), i = 1, 2, ..., n - 2\}, \tag{3.1}$$

which is equipped with the norm

$$\|u\| = \max\Big\{ \max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |{}^{C}D_{0^+}^{\beta_1,\Omega}u(t)|, ..., \max_{t\in[0,1]} |{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t)|\Big\}.$$

Lemma 3.1. $(\mathbb{B}, \|.\|)$ *is a Banach space.*

Proof. Let $\{u_k(t)\}_{k=1}^{\infty}$ be a Cauchy sequence in the space $(\mathbb{B}, \|.\|)$. Clearly, $\{u_k(t)\}_{k=1}^{\infty}$ and $\{{}^{C}D_{0^+}^{\beta_i,\Omega}u_k(t)\}_{k=1}^{\infty}$, $i = 1, 2, ..., n - 2$, are Cauchy sequences in $C([0, 1], \mathbb{R})$. Accordingly, $\{u_k(t)\}_{k=1}^{\infty}$ and $\{{}^{C}D_{0^+}^{\beta_i,\Omega}u_k(t)\}_{k=1}^{\infty}$ converge to some v and w_i, $i = 1, 2, ..., n - 2$, on $[0, 1]$ uniformly and $v, w_i \in C([0, 1], \mathbb{R})$. We need to prove that $w_i = {}^{C}D_{0^+}^{\beta_i,\Omega}v$. Observe that

$$|I_{0^+}^{\beta_i,\Omega}\, {}^{C}D_{0^+}^{\beta_i,\Omega}u_k(t) - I_{0^+}^{\beta_i,\Omega}w_i(t)| \;\leq\; \frac{1}{\Gamma(\beta_i)}\int_0^t (\Omega(t) - \Omega(s))^{\beta_i-1}\Omega'(s)\, |{}^{C}D_{0^+}^{\beta_i,\Omega}u_k(s) - w_i(s)|ds$$

$$\leq \frac{(\Omega(1)-\Omega(0))^{\beta_i}}{\Gamma(\beta_i+1)} \max_{t\in[0,1]} |{}^C D_{0^+}^{\beta_i,\Omega} u_k(t) - w_i(t)|.$$

By the convergence of $\{{}^C D_{0^+}^{\beta_i,\Omega} u_k(t)\}_{k=1}^{\infty}$, we have $\lim_{k\to\infty} I_{0^+}^{\beta_i,\Omega}\,{}^C D_{0^+}^{\beta_i,\Omega} u_k(t) = I_{0^+}^{\beta_i,\Omega} w_i(t)$ uniformly for $t \in [0,1]$. Conversely, by Lemma 2.4, one has

$$I_{0^+}^{\beta_i,\Omega}\,{}^C D_{0^+}^{\beta_i,\Omega} u_k(t) = u_k(t) - \sum_{j=0}^{[\beta_i]} \frac{(u_k)_{\Omega}^{[j]}(0)}{j!} (\Omega(t)-\Omega(0))^j.$$

Consider the sequences of the initial values $\{(u_k)_{\Omega}^{[j]}(0)\}$. Subtracting this identity for indices k and ℓ gives a Vandermonde-type linear system for the differences

$$\Delta_j^{k,\ell} := (u_k)_{\Omega}^{[j]}(0) - (u_\ell)_{\Omega}^{[j]}(0).$$

Since $u_k - u_\ell \to 0$ uniformly and likewise, $I_{0^+}^{\beta_i,\Omega}({}^C D_{0^+}^{\beta_i,\Omega} u_k - {}^C D_{0^+}^{\beta_i,\Omega} u_\ell) \to 0$, it follows that $\Delta_j^{k,\ell} \to 0$ for each j. Thus, every sequence $\{(u_k)_{\Omega}^{[j]}(0)\}$ is a Cauchy sequence in $\mathbb{R}$ and converges; we denote its limit by L_j. Finally, by the uniqueness of polynomial representation, these limiting coefficients must coincide with the generalized initial values of the limit function v. More precisely, the representation for v can be written as

$$v(t) = I_{0^+}^{\beta_i,\Omega} w_i(t) + \sum_{j=0}^{[\beta_i]} \frac{v_{\Omega}^{[j]}(0)}{j!} (\Omega(t)-\Omega(0))^j,$$

while passing to the limit in the sequence $\{u_k\}$ yields

$$v(t) = I_{0^+}^{\beta_i,\Omega} w_i(t) + \sum_{j=0}^{[\beta_i]} \frac{L_j}{j!} (\Omega(t)-\Omega(0))^j.$$

By the uniqueness of this representation, we obtain $L_j = v_{\Omega}^{[j]}(0)$ for each j. On the other hand, applying Lemma 2.4 to v yields

$$I_{0^+}^{\beta_i,\Omega}\,{}^C D_{0^+}^{\beta_i,\Omega} v(t) = v(t) - \sum_{j=0}^{[\beta_i]} \frac{v_{\Omega}^{[j]}(0)}{j!} (\Omega(t)-\Omega(0))^j.$$

Therefore, we get $I_{0^+}^{\beta_i,\Omega} w_i(t) = I_{0^+}^{\beta_i,\Omega}\,{}^C D_{0^+}^{\beta_i,\Omega} v(t)$ and so $w_i(t) = {}^C D_{0^+}^{\beta_i,\Omega} v(t)$, which proves that $(\mathbb{B}, \|.\|)$ is a Banach space. □

We now examine the following BVP:

$$\begin{aligned} &{}^C D_{0^+}^{\alpha,\Omega} u(t) + k(t) = 0, \quad t \in (0,1), \\ &u(0) = u'(0) = u''(0) = ... = u^{(n-2)}(0) = 0, \\ &{}^C D_{0^+}^{\gamma_1,\Omega} u(1) = \textstyle\sum_{j=1}^{\infty} \mu_j\, {}^C D_{0^+}^{\gamma_2,\Omega} u(\xi_j). \end{aligned} \tag{3.2}$$

Lemma 3.2. *If $k \in C([0,1],\mathbb{R})$ is α-times integrable function, then $u = u(t)$ is a solution of (3.2) if and only if*

$$u(t) = \int_0^1 \Omega'(s)\mathbb{G}_*(t,s)k(s)\,ds, \tag{3.3}$$

where $\mathbb{G}_*$ is defined by

$$\mathbb{G}_*(t,s) = \begin{cases} -\frac{1}{\Gamma(\alpha)}(\Omega(t)-\Omega(s))^{\alpha-1} + (\Omega(t)-\Omega(0))^{n-1}\frac{(\Omega(1)-\Omega(s))^{\alpha-\gamma_1-1}P(s)}{\mathcal{A}}, & s \le t, \\ \\ (\Omega(t)-\Omega(0))^{n-1}\frac{(\Omega(1)-\Omega(s))^{\alpha-\gamma_1-1}P(s)}{\mathcal{A}}, & s \ge t, \end{cases} \tag{3.4}$$

such that

$$\mathcal{A} = \frac{\Gamma(n)}{\Gamma(n-\gamma_1)}(\Omega(1)-\Omega(0))^{n-\gamma_1-1} - \frac{\Gamma(n)}{\Gamma(n-\gamma_2)}\sum_{j=1}^{\infty}\mu_j(\Omega(\xi_j)-\Omega(0))^{n-\gamma_2-1} \tag{3.5}$$

and

$$\mathcal{P}(s) = \frac{1}{\Gamma(\alpha-\gamma_1)} - \frac{1}{\Gamma(\alpha-\gamma_2)}\sum_{s\le\xi_j}\mu_j\frac{(\Omega(\xi_j)-\Omega(s))^{\alpha-\gamma_2-1}}{(\Omega(1)-\Omega(s))^{\alpha-\gamma_2-1}}(\Omega(1)-\Omega(s))^{\gamma_1-\gamma_2}. \tag{3.6}$$

Proof. By applying $I_{0^+}^{\alpha,\Omega}$ on (3.2) and using Theorem 2.6 and Lemma 2.5, we have

$$\begin{aligned} u(t) &= -I_{0^+}^{\alpha,\Omega}k(t) + c_0 + c_1(\Omega(t)-\Omega(0)) + ... + c_{n-1}(\Omega(t)-\Omega(0))^{n-1} \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t \Omega'(s)(\Omega(t)-\Omega(s))^{\alpha-1}k(s)\,ds + \sum_{m=0}^{n-1} c_m(\Omega(t)-\Omega(0))^m \end{aligned}$$

where c_m and $m = 0,1,2,...,n-1$, are constants. By applying the initial boundary conditions, it follows that $c_0 = c_1 = ... = c_{n-2} = 0$ and

$$u(t) = -I_{0^+}^{\alpha,\Omega}k(t) + c_{n-1}(\Omega(t)-\Omega(0))^{n-1} \tag{3.7}$$

$$= -\frac{1}{\Gamma(\alpha)}\int_0^t \Omega'(s)(\Omega(t)-\Omega(s))^{\alpha-1}k(s)\,ds + c_{n-1}(\Omega(t)-\Omega(0))^{n-1}. \tag{3.8}$$

On the other hand, since

$${}^{C}D_{0^+}^{\gamma_1,\Omega}u(1) = -I_{0^+}^{\alpha-\gamma_1,\Omega}k(1) + c_{n-1}\frac{\Gamma(n)}{\Gamma(n-\gamma_1)}(\Omega(1)-\Omega(0))^{n-\gamma_1-1},$$

and

$${}^{C}D_{0^+}^{\gamma_2,\Omega}u(\xi_j) = -I_{0^+}^{\alpha-\gamma_2,\Omega}k(\xi_j) + c_{n-1}\frac{\Gamma(n)}{\Gamma(n-\gamma_2)}(\Omega(\xi_j)-\Omega(0))^{n-\gamma_2-1},$$

from the second boundary condition, we get

$$c_{n-1}\Big[\frac{\Gamma(n)}{\Gamma(n-\gamma_1)}(\Omega(1)-\Omega(0))^{n-\gamma_1-1}-\frac{\Gamma(n)}{\Gamma(n-\gamma_2)}\sum_{j=1}^{\infty}\mu_j(\Omega(\xi_j)-\Omega(0))^{n-\gamma_2-1}\Big]$$
$$=I_{0^+}^{\alpha-\gamma_1,\Omega}k(1)-\sum_{j=1}^{\infty}\mu_j I_{0^+}^{\alpha-\gamma_2,\Omega}k(\xi_j)$$
$$=\frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^1\Omega'(s)(\Omega(1)-\Omega(s))^{\alpha-\gamma_1-1}k(s)\,ds$$
$$-\frac{1}{\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\int_0^{\xi_j}\Omega'(s)(\Omega(\xi_j)-\Omega(s))^{\alpha-\gamma_2-1}k(s)\,ds$$

and also

$$c_{n-1}=\frac{1}{\mathcal{A}\Gamma(\alpha-\gamma_1)}\int_0^1\Omega'(s)(\Omega(1)-\Omega(s))^{\alpha-\gamma_1-1}k(s)\,ds$$
$$-\frac{1}{\mathcal{A}\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\int_0^{\xi_j}\Omega'(s)(\Omega(\xi_j)-\Omega(s))^{\alpha-\gamma_2-1}k(s)\,ds, \tag{3.9}$$

so that

$$c_{n-1}=\frac{1}{\mathcal{A}}\int_0^1\Omega'(s)(\Omega(1)-\Omega(s))^{\alpha-\gamma_1-1}\mathcal{P}(s)k(s)\,ds, \tag{3.10}$$

where $\mathcal{A}$ and $\mathcal{P}(s)$ are defined in (3.5) and (3.6), respectively. By substituting (3.10) into (3.8), the solution of (3.2) is obtained as follows:

$$u(t)=-I_{0^+}^{\alpha,\Omega}k(t)+(\Omega(t)-\Omega(0))^{n-1}\frac{1}{\mathcal{A}}[I_{0^+}^{\alpha-\gamma_1,\Omega}k(1)-\sum_{j=1}^{\infty}\mu_j I_{0^+}^{\alpha-\gamma_2,\Omega}k(\xi_j)]$$
$$=-\frac{1}{\Gamma(\alpha)}\int_0^t\Omega'(s)(\Omega(t)-\Omega(s))^{\alpha-1}k(s)\,ds$$
$$+(\Omega(t)-\Omega(0))^{n-1}\frac{1}{\mathcal{A}}\int_0^1\Omega'(s)(\Omega(1)-\Omega(s))^{\alpha-\gamma_1-1}\mathcal{P}(s)k(s)\,ds$$
$$=\int_0^1\Omega'(s)\mathbb{G}_*(t,s)k(s)\,ds,$$

where $\mathbb{G}_*(t,s)$ is given by (3.4). Consequently, the BVP (3.2) is equivalent to (3.3). □

Lemma 3.3. *Green's function $\mathbb{G}_*(t,s)$ given by (3.4) is continuous function on $s,t\in[0,1]$ and satisfies the following inequality:*

$$|\mathbb{G}_*(t,s)| \le \frac{1}{\Gamma(\alpha)}(\Omega(1)-\Omega(0))^{\alpha-1}+\frac{\mathcal{P}(\Omega(1)-\Omega(0))^{\alpha+n-\gamma_1-2}}{\mathcal{A}}:=\mathcal{G}_1,$$

where $\mathcal{P}:=\max_{s\in[0,1]}|\mathcal{P}(s)|$.

Proof. By simple computation, we can prove that

$$\begin{aligned}
\mathcal{P}'(s) &= \frac{1}{\Gamma(\alpha-\gamma_2)}\sum_{s\le\xi_j}\mu_j\{(\alpha-\gamma_2-1)(\Omega(\xi_j)-\Omega(s))^{\alpha-\gamma_2-2}(\Omega(1)-\Omega(s))^{\gamma_1-\alpha+1}\Omega'(s)\\
&\quad+(\gamma_1-\alpha+1)(\Omega(\xi_j)-\Omega(s))^{\alpha-\gamma_2-1}(\Omega(1)-\Omega(s))^{\gamma_1-\alpha}\Omega'(s)\}\\
&= \frac{1}{\Gamma(\alpha-\gamma_2)}\sum_{s\le\xi_j}\mu_j(\Omega(\xi_j)-\Omega(s))^{\alpha-\gamma_2-2}(\Omega(1)-\Omega(s))^{\gamma_1-\alpha}\Omega'(s)[(\alpha-\gamma_2-1)(\Omega(1)-\Omega(s))\\
&\quad+(\gamma_1-\alpha+1)(\Omega(\xi_j)-\Omega(s))]\\
&\ge \frac{1}{\Gamma(\alpha-\gamma_2)}\sum_{s\le\xi_j}\mu_j(\Omega(\xi_j)-\Omega(s))^{\alpha-\gamma_2-2}(\Omega(1)-\Omega(s))^{\gamma_1-\alpha}\Omega'(s)[(\alpha-\gamma_2-1)(\Omega(\xi_j)-\Omega(s))\\
&\quad-(\alpha-\gamma_1-1)(\Omega(\xi_j)-\Omega(s))]\\
&> 0,
\end{aligned}$$

and so $\mathcal{P}(s)$ is a nondecreasing continuous function on $[0,1]$. We can then easily see that Green's function $\mathbb{G}_*(t,s)$ is continuous and the following inequality is satisfied:

$$\begin{aligned}
|\mathbb{G}_*(t,s)| &\le \left|\frac{1}{\Gamma(\alpha)}(\Omega(t)-\Omega(s))^{\alpha-1}+(\Omega(t)-\Omega(0))^{n-1}\frac{(\Omega(1)-\Omega(s))^{\alpha-\gamma_1-1}\mathcal{P}(s)}{\mathcal{A}}\right|\\
&\le \frac{1}{\Gamma(\alpha)}(\Omega(1)-\Omega(0))^{\alpha-1}+\frac{\mathcal{P}(\Omega(1)-\Omega(0))^{\alpha+n-\gamma_1-2}}{\mathcal{A}}.
\end{aligned}$$

□

The analysis that follows is based on imposing suitable growth assumptions on the function f, under which we are able to prove the existence and uniqueness of solutions to the nonlinear BVP (1.1).

(C_1) Constants are such that $L_1, L_2, ..., L_{n-1} \in (0,\infty)$ and

$$|f(t,u_1,u_2,...,u_{n-1}) - f(t,v_1,v_2,...,v_{n-1})| \le L_1|u_1-v_1| + ... + L_{n-1}|u_{n-1}-v_{n-1}|,$$

where $u_i, v_i \in \mathbb{R}$, $i = 1,2,...,n-1$.

(C_2) For all $u_i \in \mathbb{R}$, $h_1, h_2 \in C([0,1],\mathbb{R}_0^+)$ exist such that

$$|f(t,u_1,u_2,...,u_{n-1})| \le h_1(t) + h_2(t)\,[|u_1| + |u_2| + ... + |u_{n-1}|]\,.$$

For convenience, we define the following constants:

$$\mathcal{H} := \max_{t\in[0,1]}|h(t)|, \tag{3.11}$$

$$\mathcal{H}_1 := \max_{t\in[0,1]}h_1(t), \tag{3.12}$$

$$\mathcal{H}_2 := \max_{t\in[0,1]}h_2(t), \tag{3.13}$$

$$\mathcal{L} := \max\{L_1, L_2, ..., L_{n-1}\} \text{ with } L_1, L_2, ..., L_{n-1} \in (0,\infty), \tag{3.14}$$

and

$$\begin{aligned}
\mathcal{M} := \max_{1\le i\le n-2}\Big\{&\frac{1}{\Gamma(\alpha)}+\frac{(\Omega(1)-\Omega(0))^{n-\gamma_1-1}}{\mathcal{A}\Gamma(\alpha-\gamma_1)}+\frac{(\Omega(1)-\Omega(0))^{n-\gamma_2-1}}{\mathcal{A}\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j,\\
&\frac{(\Omega(1)-\Omega(0))^{-\beta_i}}{\Gamma(\alpha-\beta_i)}+\frac{\Gamma(n)(\Omega(1)-\Omega(0))^{n-\beta_i-\gamma_1-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_1)}+\frac{\Gamma(n)(\Omega(1)-\Omega(0))^{n-\beta_i-\gamma_2-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\Big\}.
\end{aligned} \tag{3.15}$$

Lemma 3.4. *Assume that f is a continuous function on $[0,1]\times\mathbb{R}^{n-1}$. Then $u\in\mathbb{B}$ is a solution of (1.1) if and only if $u\in\mathbb{B}$ is a solution of the following integral equation:*

$$u(t)=\int_0^1 \Omega^{'}(s)\mathbb{G}_*(t,s)\lambda h(s)f(s,u(s),{}^{C}D_{0^+}^{\beta_1,\Omega}u(s),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(s))\,ds.$$

Proof. Let $u\in\mathbb{B}$ be a solution of (1.1). We can prove that $u\in\mathbb{B}$ is a solution of (3.3) from Lemma 3.2. In contrast, if $u\in\mathbb{B}$ is a solution of (3.3), we define $z(t)$ via Lemma 3.2, from Eqs (3.7), (3.9), and $k(t)=\lambda h(t)f(t,u(t),{}^{C}D_{0^+}^{\beta_1,\Omega}u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t))$; that is

$$\begin{aligned}z(t)=&-I_{0^+}^{\alpha,\Omega}\lambda h(t)f(t,u(t),{}^{C}D_{0^+}^{\beta_1,\Omega}u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t))\\&+(\Omega(t)-\Omega(0))^{n-1}\frac{1}{\mathcal{A}}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}\lambda h(1)f(1,u(1),{}^{C}D_{0^+}^{\beta_1,\Omega}u(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(1))\\&-\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}\lambda h(\xi_j)f(\xi_j,u(\xi_j),{}^{C}D_{0^+}^{\beta_1,\Omega}u(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(\xi_j)))\Big].\end{aligned}\tag{3.16}$$

According to Lemmas 2.7 and 2.8, we have

$${}^{C}D_{0^+}^{\alpha,\Omega}z(t)=-\lambda h(t)f(t,u(t),{}^{C}D_{0^+}^{\beta_1,\Omega}u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t)).$$

That is,

$${}^{C}D_{0^+}^{\alpha,\Omega}u(t)=-\lambda h(t)f(t,u(t),{}^{C}D_{0^+}^{\beta_1,\Omega}u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t)).$$

Also $u(0)=u'(0)=...u^{(n-2)}(0)=0$ is satisfied. From (3.5), we get

$$1=\frac{\Gamma(n)}{\Gamma(n-\gamma_1)}\frac{1}{\mathcal{A}}(\Omega(1)-\Omega(0))^{n-\gamma_1-1}-\frac{\Gamma(n)}{\Gamma(n-\gamma_2)}\frac{1}{\mathcal{A}}\sum_{j=1}^{\infty}\mu_j(\Omega(\xi_j)-\Omega(0))^{n-\gamma_2-1}.$$

We then have

$$\begin{aligned}&{}^{C}D_{0^+}^{\gamma_1,\Omega}u(1)-\sum_{j=1}^{\infty}\mu_j{}^{C}D_{0^+}^{\gamma_2,\Omega}u(\xi_j)\\=&-I_{0^+}^{\alpha-\gamma_1,\Omega}k(1)+\frac{\Gamma(n)}{\Gamma(n-\gamma_1)}(\Omega(1)-\Omega(0))^{n-\gamma_1-1}\frac{1}{\mathcal{A}}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}k(1)-\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}k(\xi_j))\Big]\\&+\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}k(\xi_j))\\&-\sum_{j=1}^{\infty}\mu_j\frac{\Gamma(n)}{\Gamma(n-\gamma_2)}(\Omega(\xi_j)-\Omega(0))^{n-\gamma_2-1}\frac{1}{\mathcal{A}}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}k(1)-\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}k(\xi_j))\Big]\\=&\Big(-1+\frac{\Gamma(n)}{\Gamma(n-\gamma_1)}(\Omega(1)-\Omega(0))^{n-\gamma_1-1}\frac{1}{\mathcal{A}}\Big)I_{0^+}^{\alpha-\gamma_1,\Omega}k(1)\\&-\frac{\Gamma(n)}{\Gamma(n-\gamma_1)}(\Omega(1)-\Omega(0))^{n-\gamma_1-1}\frac{1}{\mathcal{A}}\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}k(\xi_j))\end{aligned}$$

$$
\begin{aligned}
&-\Big(-1-\sum_{j=1}^{\infty}\mu_j\frac{\Gamma(n)}{\Gamma(n-\gamma_2)}(\Omega(\xi_j)-\Omega(0))^{n-\gamma_2-1}\frac{1}{\mathcal{A}}\Big)\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}k(\xi_j))\\
&-\sum_{j=1}^{\infty}\mu_j\frac{\Gamma(n)}{\Gamma(n-\gamma_2)}(\Omega(\xi_j)-\Omega(0))^{n-\gamma_2-1}\frac{1}{\mathcal{A}}I_{0^+}^{\alpha-\gamma_1,\Omega}k(1)\\
=\;&0,
\end{aligned}
$$

which implies that $u\in\mathbb{B}$ is a solution of the BVP (1.1). □

In view of Eq (3.16) in Lemma 3.4, $u\in\mathbb{B}$ such that

$$
\begin{aligned}
u(t)=&-I_{0^+}^{\alpha,\Omega}\lambda h(t)f(t,u(t),{}^{C}D_{0^+}^{\beta_1,\Omega}u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t))\\
&+(\Omega(t)-\Omega(0))^{n-1}\frac{1}{\mathcal{A}}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}\lambda h(1)f(1,u(1),{}^{C}D_{0^+}^{\beta_1,\Omega}u(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(1))\\
&-\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}\lambda h(\xi_j)f(\xi_j,u(\xi_j),{}^{C}D_{0^+}^{\beta_1,\Omega}u(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(\xi_j)))\Big]
\end{aligned}
\tag{3.17}
$$

is a solution of the BVP (1.1). Let $\Upsilon:\mathbb{B}\longrightarrow\mathbb{B}$ be defined by

$$
\begin{aligned}
\Upsilon u(t)=&-I_{0^+}^{\alpha,\Omega}\lambda h(t)f(t,u(t),{}^{C}D_{0^+}^{\beta_1,\Omega}u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t))\\
&+(\Omega(t)-\Omega(0))^{n-1}\frac{1}{\mathcal{A}}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}\lambda h(1)f(1,u(1),{}^{C}D_{0^+}^{\beta_1,\Omega}u(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(1))\\
&-\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}\lambda h(\xi_j)f(\xi_j,u(\xi_j),{}^{C}D_{0^+}^{\beta_1,\Omega}u(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(\xi_j)))\Big].
\end{aligned}
\tag{3.18}
$$

Then finding the solution $u(t)$ given by (3.17) becomes the problem of finding the fp of the mapping Υ given in (3.18).

Lemma 3.5. *The operator Υ defined in (3.18) is well-defined.*

Proof. In view of the continuity of f, h, and Ω, the operator Υ is continuous such that $\Upsilon u\in C([0,1],\mathbb{R})$. We also have

$$
\begin{aligned}
&{}^{C}D_{0^+}^{\beta_i,\Omega}(\Upsilon u(t))\\
=&{}^{C}D_{0^+}^{\beta_i,\Omega}\Big(-I_{0^+}^{\alpha,\Omega}\lambda h(t)f(t,u(t),{}^{C}D_{0^+}^{\beta_1,\Omega}u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t))\\
&+(\Omega(t)-\Omega(0))^{n-1}\frac{1}{\mathcal{A}}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}\lambda h(1)f(1,u(1),{}^{C}D_{0^+}^{\beta_1,\Omega}u(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(1))\\
&-\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}\lambda h(\xi_j)f(\xi_j,u(\xi_j),{}^{C}D_{0^+}^{\beta_1,\Omega}u(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(\xi_j)))\Big]\Big)\\
=&-I_{0^+}^{\alpha-\beta_i,\Omega}\lambda h(t)f(t,u(t),{}^{C}D_{0^+}^{\beta_1,\Omega}u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t))\\
&+\frac{\Gamma(n)}{\mathcal{A}\Gamma(n-\beta_i)}(\Omega(t)-\Omega(0))^{n-\beta_i-1}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}\lambda h(1)f(1,u(1),{}^{C}D_{0^+}^{\beta_1,\Omega}u(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(1))\\
&-\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}\lambda h(\xi_j)f(\xi_j,u(\xi_j),{}^{C}D_{0^+}^{\beta_1,\Omega}u(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(\xi_j)))\Big].
\end{aligned}
$$

We get

$$
\begin{aligned}
&{}^{C}D_{0^+}^{\beta_i,\Omega}(\Upsilon u(t))\\
= &-\frac{1}{\Gamma(\alpha-\beta_i)}\int_0^t \Omega'(s)(\Omega(t)-\Omega(s))^{\alpha-\beta_i-1}\lambda h(s)f(s,u(s),{}^{C}D_{0^+}^{\beta_1,\Omega}u(s),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(s))\,ds\\
&+\frac{\Gamma(n)}{\mathcal{A}\Gamma(n-\beta_i)}(\Omega(t)-\Omega(0))^{n-\beta_i-1}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}\lambda h(1)f(1,u(1),{}^{C}D_{0^+}^{\beta_1,\Omega}u(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(1))\\
&-\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}\lambda h(\xi_j)f(\xi_j,u(\xi_j),{}^{C}D_{0^+}^{\beta_1,\Omega}u(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(\xi_j)))\Big].
\end{aligned}
\tag{3.19}
$$

Therefore, ${}^{C}D_{0^+}^{\beta_i,\Omega}(\Upsilon u)\in C([0,1],\mathbb{R})$, $i=1,2,...,n-2$. □

Theorem 3.6. *Let $f\in C([0,1]\times\mathbb{R}^{n-1},\mathbb{R})$ and assume that (C_1) and (C_2) are hold. If*

$$|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\mathcal{H}_2(n-1)<1, \tag{3.20}$$

where $\mathcal{H}$, $\mathcal{H}_2$, and $\mathcal{M}$ are defined in (3.11), (3.13), and (3.15), respectively, then the BVP (1.1) has at least one solution.

Proof. We utilize the Schaefer fp theorem to demonstrate the existence of an fp for the operator Υ given in (3.18). The argument proceeds through four distinct steps.
Step 1. We aim to show that Υ is a continuous operator. Let $\{u_n\}$, $n\in\mathbb{N}$ be a sequence in the Banach space $\mathbb{B}$ defined by (3.1) such that $u_n\to u$ as $n\to\infty$. By applying (3.18) and (3.19), we derive the following:

$$
\begin{aligned}
&|(\Upsilon u_n)(t)-(\Upsilon u)(t)|\\
=&\Bigg|-I_{0^+}^{\alpha,\Omega}\lambda h(t)[f(t,u_n(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u_n(t))-f(t,u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t))]\\
&+(\Omega(t)-\Omega(0))^{n-1}\frac{1}{\mathcal{A}}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}\lambda h(1)[f(1,u_n(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u_n(1))-f(1,u(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(1))]\\
&-\sum_{j=1}^{\infty}\mu_jI_{0^+}^{\alpha-\gamma_2,\Omega}\lambda h(\xi_j)[f(\xi_j,u_n(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u_n(\xi_j)))-f(\xi_j,u(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(\xi_j)))]\Big]\Bigg|\\
\le&I_{0^+}^{\alpha,\Omega}|\lambda||h(t)||f(t,u_n(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u_n(t))-f(t,u(t),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(t))|\\
&+\frac{(\Omega(t)-\Omega(0))^{n-1}}{\mathcal{A}}\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}|\lambda||h(1)||f(1,u_n(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u_n(1))-f(1,u(1),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(1))|\\
&+\sum_{j=1}^{\infty}\mu_jI_{0^+}^{\alpha-\gamma_2,\Omega}|\lambda||h(\xi_j)||f(\xi_j,u_n(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u_n(\xi_j)))-f(\xi_j,u(\xi_j),...,{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(\xi_j)))|\Big]\\
\le&\frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha-1}}{\Gamma(\alpha)}\int_0^t\Omega'(s)\{L_1|u_n(s)-u(s)|+...+L_{n-1}|{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u_n(s)-{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(s)|\}\,ds\\
&+\frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{n+\alpha-\gamma_1-2}}{\mathcal{A}\Gamma(\alpha-\gamma_1)}\int_0^1\Omega'(s)\{L_1|u_n(s)-u(s)|+...+L_{n-1}|{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u_n(s)-{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(s)|\}\,ds\\
&+\frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{n+\alpha-\gamma_2-2}}{\mathcal{A}\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\int_0^{\xi_j}\Omega'(s)\{L_1|u_n(s)-u(s)|+...
\end{aligned}
$$

$$+ L_{n-1}|{}^{C}D_{0^+}^{\beta_{n-2},\Omega}u_n(s) - {}^{C}D_{0^+}^{\beta_{n-2},\Omega}u(s)|\}\,ds$$

$$\leq \frac{|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha}}{\Gamma(\alpha)}\|u_n - u\| + \frac{|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{n+\alpha-\gamma_1-1}}{\mathcal{A}\Gamma(\alpha-\gamma_1)}\|u_n - u\|$$

$$+ \frac{|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{n+\alpha-\gamma_2-1}}{\mathcal{A}\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\|u_n - u\|$$

$$= |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha}$$

$$\times\left\{\frac{1}{\Gamma(\alpha)} + \frac{(\Omega(1)-\Omega(0))^{n-\gamma_1-1}}{\mathcal{A}\Gamma(\alpha-\gamma_1)} + \frac{(\Omega(1)-\Omega(0))^{n-\gamma_2-1}}{\mathcal{A}\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\right\}\|u_n - u\|.$$

Thus we have $|(\Upsilon u_n)(t) - (\Upsilon u)(t)| \leq |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\|u_n - u\|$. We also get

$$|D_{0^+}^{\beta_i,\Omega}(\Upsilon u_n)(t) - D_{0^+}^{\beta_i,\Omega}(\Upsilon u)(t)|$$

$$\leq \frac{|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i)}\|u - v\| + \frac{\Gamma(n)|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{n+\alpha-\beta_i-\gamma_1-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_1)}\|u_n - u\|$$

$$+ \frac{\Gamma(n)|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{n+\alpha-\beta_i-\gamma_2-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\|u_n - u\|$$

$$\leq |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha}$$

$$\times\left\{\frac{(\Omega(1)-\Omega(0))^{-\beta_i}}{\Gamma(\alpha-\beta_i)} + \frac{\Gamma(n)(\Omega(1)-\Omega(0))^{n-\beta_i-\gamma_1-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_1)} + \frac{\Gamma(n)(\Omega(1)-\Omega(0))^{n-\beta_i-\gamma_2-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\right\}\|u_n - u\|$$

$$\leq |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\|u_n - u\|.$$

Thus, we find that $\|\Upsilon u_n - \Upsilon u\| \leq |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\|u_n - u\|$. Since $u_n \to u$, this implies that Υ is a continuous operator.

Step 2. We see that Υ maps bounded sets into uniform sets in $\mathbb{B}$. For this reason, we show that for all $\mathcal{R}_1 > 0$, some $\mathcal{R}_2 > 0$ exists such that for all

$$u \in \mathcal{B}_{\mathcal{R}_1} := \{u \in \mathbb{B} : \|u\| \leq \mathcal{R}_1\},$$

$\|\Upsilon u\| \leq \mathcal{R}_2$ is satisfied. Indeed, let $u \in \mathcal{B}_{\mathcal{R}_1}$ for all $t \in [0,1]$, and thus we have

$$|\Upsilon u(t)|$$

$$\leq \frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \Omega'(s)\{\mathcal{H}_1 + \mathcal{H}_2[\max_{s\in[0,1]}|u(s)| + ... + \max_{s\in[0,1]}|D_{0^+}^{\beta_{n-2},\Omega}u(s)|]\}\,ds$$

$$+ \frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{n+\alpha-\gamma_1-2}}{\mathcal{A}\Gamma(\alpha-\gamma_1)}\int_0^1 \Omega'(s)\{\mathcal{H}_1 + \mathcal{H}_2[\max_{s\in[0,1]}|u(s)| + ... + \max_{s\in[0,1]}|D_{0^+}^{\beta_{n-2},\Omega}u(s)|]\}\,ds$$

$$+ \frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{n+\alpha-\gamma_2-2}}{\mathcal{A}\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\int_0^{\xi_j} \Omega'(s)\{\mathcal{H}_1 + \mathcal{H}_2[\max_{s\in[0,1]}|u(s)| + ... + \max_{s\in[0,1]}|D_{0^+}^{\beta_{n-2},\Omega}u(s)|]\}\,ds$$

$$\leq \frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \Omega'(s)\{\mathcal{H}_1 + \mathcal{H}_2(n-1)\|u\|\,\}\,ds$$

$$+ \frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{n+\alpha-\gamma_1-2}}{\mathcal{A}\Gamma(\alpha-\gamma_1)}\int_0^1 \Omega'(s)\{\mathcal{H}_1 + \mathcal{H}_2(n-1)\|u\|\}\,ds$$

$$+\frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{n+\alpha-\gamma_2-2}}{\mathcal{A}\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\int_0^{\xi_j}\Omega'(s)\{\mathcal{H}_1+\mathcal{H}_2(n-1)\|u\|\}\,ds$$
$$\leq|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\{\mathcal{H}_1+\mathcal{H}_2(n-1)\mathcal{R}_1\},$$

and

$$\begin{aligned}
&|D_{0^+}^{\beta_i,\Omega}\Upsilon u(t)|\\
\leq&\frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha-\beta_i}}{\Gamma(\alpha-\beta_i)}\{\mathcal{H}_1+\mathcal{H}_2(n-1)\|u\|\}\\
&+\frac{\Gamma(n)|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{n+\alpha-\beta_i-\gamma_1-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_1)}\{\mathcal{H}_1+\mathcal{H}_2(n-1)\|u\|\}\\
&+\frac{\Gamma(n)|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{n+\alpha-\beta_i-\gamma_2-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_2)}\sum_{j=1}^{\infty}\mu_j\{\mathcal{H}_1+\mathcal{H}_2(n-1)\|u\|\}\\
\leq&|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\{\mathcal{H}_1+\mathcal{H}_2(n-1)\mathcal{R}_1\}.
\end{aligned}$$

Hence, an $\mathcal{R}_2:=|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\{\mathcal{H}_1+\mathcal{H}_2(n-1)\mathcal{R}_1\}$ exists such that $\|\Upsilon u\|\leq\mathcal{R}_2$. Thus $\{\Upsilon u\}$ is a uniformly bounded set.

Step 3. We show that Υ maps bounded sets into equicontinuous sets of $\mathbb{B}$. Let $\mathcal{B}_{\mathcal{R}_1}$ be a bounded set of $\mathbb{B}$ as in Step 2 and $u\in\mathcal{B}_{\mathcal{R}_1}$. Consequently for $t_1,t_2\in[0,1]$ with $t_1<t_2$, we have

$$\begin{aligned}
|(\Upsilon u)(t_2)-(\Upsilon u)(t_1)|\leq&\int_0^1\Omega'(s)|\mathbb{G}_*(t_2,s)-\mathbb{G}_*(t_1,s)||\lambda||h(s)||f(s,u(s),...,D_{0^+}^{\beta_{n-2},\Omega}u(s))|\,ds\\
\leq&|\lambda||\mathcal{H}|\int_0^1\Omega'(s)|\mathbb{G}_*(t_2,s)-\mathbb{G}_*(t_1,s)||f(s,u(s),...,D_{0^+}^{\beta_{n-2},\Omega}u(s))|\,ds\\
\leq&|\lambda||\mathcal{H}|\{\mathcal{H}_1+\mathcal{H}_2(n-1)\mathcal{R}_1\}\int_0^1\Omega'(s)|\mathbb{G}_*(t_2,s)-\mathbb{G}_*(t_1,s)|\,ds,
\end{aligned}$$

and from the continuity of $\mathbb{G}_*(t,s)$, it follows that $|(\Upsilon u)(t_2)-(\Upsilon u)(t_1)|\to 0$ as $t_2\to t_1$. Moreover,

$$\begin{aligned}
&|D_{0^+}^{\beta_i,\Omega}\Upsilon u(t_2)-D_{0^+}^{\beta_i,\Omega}\Upsilon u(t_1)|\\
\leq&\frac{|\lambda|\mathcal{H}\{\mathcal{H}_1+\mathcal{H}_2(n-1)\mathcal{R}_1\}}{\Gamma(\alpha-\beta_i)}\Big[\int_0^{t_1}\Omega'(s)|(\Omega(t_2)-\Omega(s))^{\alpha-\beta_i-1}-(\Omega(t_1)-\Omega(s))^{\alpha-\beta_i-1}|\,ds\\
&+\int_{t_1}^{t_2}\Omega'(s)|(\Omega(t_2)-\Omega(s))^{\alpha-\beta_i-1}|\,ds\Big]\\
&+\frac{\Gamma(n)}{\mathcal{A}\Gamma(n-\beta_i)}|(\Omega(t_2)-\Omega(0))^{n-\beta_i-1}-(\Omega(t_1)-\Omega(0))^{n-\beta_i-1}|\\
&\times\Big|\Big[I_{0^+}^{\alpha-\gamma_1,\Omega}\lambda h(1)f(1,u(1),D_{0^+}^{\beta_1,\Omega}u(1),...,D_{0^+}^{\beta_{n-2},\Omega}u(1))\\
&-\sum_{j=1}^{\infty}\mu_j(I_{0^+}^{\alpha-\gamma_2,\Omega}\lambda h(\xi_j)f(\xi_j,u(\xi_j),D_{0^+}^{\beta_1,\Omega}u(\xi_j),...,D_{0^+}^{\beta_{n-2},\Omega}u(\xi_j)))\Big]\Big|.
\end{aligned}$$

Using the continuity of the function Ω, the right-hand side of the inequality above tends to zero. This shows that the set $\Upsilon(\mathcal{B}_{\mathcal{R}_1})$ is an equicontinuous set. Therefore, from Step 1 to Step 3, we can conclude that $\Upsilon : \mathbb{B} \longrightarrow \mathbb{B}$ is a completely continuous operator with the Arzela–Ascoli theorem.
Step 4. We see that the $\mathcal{S} = \{u \in \mathbb{B} : u = \mu\Upsilon u, \text{ for some } \mu \in (0,1)\}$, is bounded. Let $u \in \mathcal{S}$ be a solution of (1.1) such that $u(t) = \mu(\Upsilon u)(t)$ for some $\mu \in (0,1)$. By Step 2, for all $t \in [0,1]$, we have

$$\|u\| \leq \|\Upsilon u\| \leq |\lambda|\mathcal{H}(\Omega(1) - \Omega(0))^{\alpha}\mathcal{M}\{\mathcal{H}_1 + \mathcal{H}_2(n-1)\|u\|\}.$$

We obtain

$$\|u\| \leq \frac{|\lambda|\mathcal{H}(\Omega(1) - \Omega(0))^{\alpha}\mathcal{M}\mathcal{H}_1}{1 - |\lambda|\mathcal{H}(\Omega(1) - \Omega(0))^{\alpha}\mathcal{M}\mathcal{H}_2(n-1)}.$$

By (3.20), this implies that the set $\mathcal{S}$ is bounded. By applying Theorem 2.10, we find that Υ has at least one fp, which is a solution of the problem (1.1). The proof is completed. □

Example 3.7. *We consider the following BVP for $t \in (0,1)$:*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{7}{2},t}u(t) + \frac{1}{3}e^{-t}\left(\frac{u(t)+D_{0^+}^{\frac{1}{2},t}u(t)+D_{0^+}^{\frac{3}{2},t}u(t)}{e^{3+t^2}\left(1+|u(t)|+|D_{0^+}^{\frac{1}{2},t}u(t)|+|D_{0^+}^{\frac{3}{2},t}u(t)|\right)}\right) = 0, \\ u(0) = u'(0) = u''(0) = 0, \\ {}^{C}D_{0^+}^{1,t}u(1) = \sum_{j=1}^{\infty} \frac{1}{j^2}{}^{C}D_{0^+}^{1,t}u(1 - \frac{1}{j+1}). \end{cases} \tag{3.21}$$

Note (3.21) is a particular case of (1.1) with $\Omega(t) = t$, where $\lambda = \frac{1}{3}$, $h(t) = e^{-t}$, $\alpha = \frac{7}{2}$, $n = 4$,

$$f(t,u_1,u_2,u_3) = \frac{1}{e^{3+t^2}}\left(\frac{u_1(t)+u_2(t)+u_3(t)}{1+|u_1(t)|+|u_2(t)|+|u_3(t)|}\right) \in C([0,1]\times\mathbb{R}^3,\mathbb{R}),$$

$\gamma_1 = \gamma_2 = 1$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{3}{2}$, *and* $\mu_j = \frac{1}{j^2}$, $\xi_j = 1 - \frac{1}{j+1}$. *Moreover, the Banach space is*

$$\mathbb{B} = \{u : u, D_{0^+}^{\frac{1}{2},t}u, D_{0^+}^{\frac{3}{2},t}u \in C([0,1],\mathbb{R})\}$$

with the norm

$$\|u\| = \max\{\max_{t\in[0,1]}|u(t)|, \max_{t\in[0,1]}|D_{0^+}^{\frac{1}{2},t}u(t)|, \max_{t\in[0,1]}|D_{0^+}^{\frac{3}{2},t}u(t)|\}.$$

From the given data, we have

$$\mathbb{G}_*(t,s) = \begin{cases} -\frac{8}{15\sqrt{\pi}}(t-s)^{\frac{5}{2}} + \frac{t^3}{\mathcal{A}}(1-s)^{\frac{3}{2}}\mathcal{P}(s), & s \leq t, \\ \frac{t^3}{\mathcal{A}}(1-s)^{\frac{3}{2}}\mathcal{P}(s), & s \geq t, \end{cases}$$

$$\begin{gathered} \mathcal{A} = 3 - 3\sum_{j=1}^{\infty}\frac{1}{(j+1)^2} \geq S_4 + \int_4^{\infty}\frac{1}{(x+1)^2}dx \approx 1.008, \\ \max_{t,s\in[0,1]}|\mathbb{G}_*(t,s)| \leq 1.543, \\ \max_{s\in[0,1]}|\mathcal{P}(s)| = \left|\frac{4}{3\sqrt{\pi}} - \frac{4}{3\sqrt{\pi}}\sum_{s\leq 1-\frac{1}{j+1}}\frac{1}{j^2}\frac{(1-\frac{1}{j+1}-s)^{\frac{3}{2}}}{(1-s)^{\frac{3}{2}}}\right| \leq 1.252, \\ \mathcal{M} \approx \max\{1.543, 2.742, 6.605\} = 6.605. \end{gathered}$$

First, we will show that the conditions (C_1) *and* (C_2) *in Theorem 3.6 are satisfied. Since*

$$\begin{aligned} |f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)| &\le \frac{1}{e^3}\left|\frac{u_1(t)+u_2(t)+u_3(t)}{1+|u_1(t)|+|u_2(t)|+|u_3(t)|}-\frac{v_1(t)+v_2(t)+v_3(t)}{1+|v_1(t)|+|v_2(t)|+|v_3(t)|}\right| \\ &\le \frac{1}{e^3}|u_1-v_1|+\frac{1}{e^3}|u_2-v_2|+\frac{1}{e^3}|u_3-v_3|, \end{aligned}$$

where $L=\frac{1}{e^3}$, C_1 *is satisfied. Since*

$$|f(t,u_1,u_2,u_3)| \le \frac{1}{e^3}(|u_1|+|u_2|+|u_3|)$$

with $h_1(t)=0,\ \ h_2(t)=\frac{1}{e^{3+t^2}} \in C([0,1],\mathbb{R}_0^+)$, C_2 *is satisfied. We can easily calculate*

$$|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\mathcal{H}_2(n-1) \approx 0.329 < 1.$$

Thus, the assumptions of Theorem 3.6 hold. This implies that the problem (3.21) has at least one solution.

Theorem 3.8. *Let* $f \in C([0,1]\times\mathbb{R}^{n-1},\mathbb{R})$ *and assume that* (C_2) *is held. Then the problem (1.1) has at least one solution, provided that*

$$r \ge \frac{|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\mathcal{H}_1}{1-|\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\mathcal{H}_2(n-1)} \tag{3.22}$$

for a constant $r>0$, *where* $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$, *and* $\mathcal{M}$ *are defined in (3.11), (3.12), (3.13), and (3.15), respectively.*

Proof. Consider

$$\mathcal{B}_r := \{u \in \mathbb{B} : \|u\| \le r\}.$$

Clearly, $\mathcal{B}_r$ is a nonempty, compact, and convex subset of $\mathbb{B}$. The operator Υ is continuous by Step 2 in the proof of Theorem 3.6. Similarly, by Step 2 in the proof of Theorem 3.6 and (3.22), we get

$$\|\Upsilon u\| \le |\lambda|\mathcal{H}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\{\mathcal{H}_1+\mathcal{H}_2(n-1)\|u\|\} \le r. \tag{3.23}$$

Thus, by (3.23), we find that $\Upsilon : \mathcal{B}_r \longrightarrow \mathcal{B}_r$. It follows by Theorem 2.11, that a fp exists such that $\|u\| \le r$. The proof is completed. □

Theorem 3.9. *Assume that* $f \in C([0,1]\times\mathbb{R}^{n-1},\mathbb{R})$ *and the condition* (C_1) *is satisfied. If*

$$|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M} < 1, \tag{3.24}$$

then the BVP (1.1) has a unique solution on $[0,1]$, *where* $\mathcal{H}$, $\mathcal{L}$, *and* $\mathcal{M}$ *are defined in (3.11), (3.14), and (3.15), respectively.*

Proof. In view of Lemma 3.5, the operator Υ is well-defined. We first show that the operator Υ is a contraction mapping from $\mathbb{B}$ to $\mathbb{B}$. Let $u, v \in \mathbb{B}$. By applying the identities provided in Lemma 3.5, we deduce that

$$
\begin{aligned}
&|\Upsilon u(t) - \Upsilon v(t)| \\
&= \Big| - I_{0^+}^{\alpha,\Omega} \lambda h(t)[f(t, u(t), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} u(t)) - f(t, v(t), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} v(t))] \\
&\quad + (\Omega(t) - \Omega(0))^{n-1} \frac{1}{\mathcal{A}} \Big[I_{0^+}^{\alpha-\gamma_1,\Omega} \lambda h(1)[f(1, u(1), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} u(1)) - f(1, v(1), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} v(1))] \\
&\quad - \sum_{j=1}^{\infty} \mu_j I_{0^+}^{\alpha-\gamma_2,\Omega} \lambda h(\xi_j)[f(\xi_j, u(\xi_j), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} u(\xi_j))) - f(\xi_j, v(\xi_j), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} v(\xi_j)))] \Big] \Big| \\
&\leq I_{0^+}^{\alpha,\Omega} |\lambda||h(t)||f(t, u(t), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} u(t)) - f(t, v(t), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} v(t))| \\
&\quad + \frac{(\Omega(t) - \Omega(0))^{n-1}}{\mathcal{A}} \Big[I_{0^+}^{\alpha-\gamma_1,\Omega} |\lambda||h(1)||f(1, u(1), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} u(1)) - f(1, v(1), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} v(1))| \\
&\quad + \sum_{j=1}^{\infty} \mu_j I_{0^+}^{\alpha-\gamma_2,\Omega} |\lambda||h(\xi_j)||f(\xi_j, u(\xi_j), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} u(\xi_j))) - f(\xi_j, v(\xi_j), ..., {}^{C}D_{0^+}^{\beta_{n-2},\Omega} v(\xi_j)))| \Big] \\
&\leq \frac{|\lambda|\mathcal{H}(\Omega(1) - \Omega(0))^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \Omega'(s)\{L_1|u(s) - v(s)| + ... + L_{n-1}|{}^{C}D_{0^+}^{\beta_{n-2},\Omega} u(s) - {}^{C}D_{0^+}^{\beta_{n-2},\Omega} v(s)|\}\, ds \\
&\quad + \frac{|\lambda|\mathcal{H}(\Omega(1) - \Omega(0))^{n+\alpha-\gamma_1-2}}{\mathcal{A}\Gamma(\alpha - \gamma_1)} \int_0^1 \Omega'(s)\{L_1|u(s) - v(s)| + ... + L_{n-1}|{}^{C}D_{0^+}^{\beta_{n-2},\Omega} u(s) - {}^{C}D_{0^+}^{\beta_{n-2},\Omega} v(s)|\}\, ds \\
&\quad + \frac{|\lambda|\mathcal{H}(\Omega(1) - \Omega(0))^{n+\alpha-\gamma_2-2}}{\mathcal{A}\Gamma(\alpha - \gamma_2)} \sum_{j=1}^{\infty} \mu_j \int_0^{\xi_j} \Omega'(s)\{L_1|u(s) - v(s)| + ... \\
&\quad + L_{n-1}|{}^{C}D_{0^+}^{\beta_{n-2},\Omega} u(s) - {}^{C}D_{0^+}^{\beta_{n-2},\Omega} v(s)|\}\, ds \\
&\leq \frac{|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1) - \Omega(0))^{\alpha}}{\Gamma(\alpha)} \|u - v\| + \frac{|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1) - \Omega(0))^{n+\alpha-\gamma_1-1}}{\mathcal{A}\Gamma(\alpha - \gamma_1)} \|u - v\| \\
&\quad + \frac{|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1) - \Omega(0))^{n+\alpha-\gamma_2-1}}{\mathcal{A}\Gamma(\alpha - \gamma_2)} \sum_{j=1}^{\infty} \mu_j \|u - v\| \\
&= |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1) - \Omega(0))^{\alpha} \\
&\quad \times \left\{ \frac{1}{\Gamma(\alpha)} + \frac{(\Omega(1) - \Omega(0))^{n-\gamma_1-1}}{\mathcal{A}\Gamma(\alpha - \gamma_1)} + \frac{(\Omega(1) - \Omega(0))^{n-\gamma_2-1}}{\mathcal{A}\Gamma(\alpha - \gamma_2)} \sum_{j=1}^{\infty} \mu_j \right\} \|u - v\|.
\end{aligned}
$$

Therefore, we have

$$
|\Upsilon u(t) - \Upsilon v(t)| \leq |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1) - \Omega(0))^{\alpha} \mathcal{M} \|u - v\| \quad \text{for all} \quad t \in [0, 1].
$$

We also get

$$
\begin{aligned}
&|D_{0^+}^{\beta_i,\Omega} \Upsilon u(t) - D_{0^+}^{\beta_i,\Omega} \Upsilon v(t)| \\
&\leq \frac{|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1) - \Omega(0))^{\alpha-\beta_i}}{\Gamma(\alpha - \beta_i)} \|u - v\| + \frac{\Gamma(n)|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1) - \Omega(0))^{n+\alpha-\beta_i-\gamma_1-1}}{\mathcal{A}\Gamma(n - \beta_i)\Gamma(\alpha - \gamma_1)} \|u - v\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\Gamma(n)|\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{n+\alpha-\beta_i-\gamma_2-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_2)} \sum_{j=1}^{\infty} \mu_j \|u-v\| \\
&\le |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha} \\
&\quad \times \left\{ \frac{(\Omega(1)-\Omega(0))^{-\beta_i}}{\Gamma(\alpha-\beta_i)} + \frac{\Gamma(n)(\Omega(1)-\Omega(0))^{n-\beta_i-\gamma_1-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_1)} + \frac{\Gamma(n)(\Omega(1)-\Omega(0))^{n-\beta_i-\gamma_2-1}}{\mathcal{A}\Gamma(n-\beta_i)\Gamma(\alpha-\gamma_2)} \sum_{j=1}^{\infty} \mu_j \right\} \|u-v\| \\
&\le |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha} \mathcal{M} \|u-v\|.
\end{aligned}
$$

Thus, we have

$$
|D_{0^+}^{\beta_i,\Omega} \Upsilon u(t) - D_{0^+}^{\beta_i,\Omega} \Upsilon v(t)| \le |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha} \mathcal{M} \|u-v\|, \quad \text{for all} \quad t \in [0,1].
$$

It is clear that $\|\Upsilon u - \Upsilon v\| \le |\lambda|\mathcal{H}(n-1)\mathcal{L}(\Omega(1)-\Omega(0))^{\alpha}\mathcal{M}\|u-v\|$ and the operator Υ is a contraction mapping from $\mathbb{B}$ to $\mathbb{B}$ with (3.24). If we use Theorem 2.9, then a unique fp $u \in \mathbb{B}$ exists which is the solution of the BVP (1.1). □

4. Conclusions

In this study, we investigated a class of nonlinear Ω-CFDEs subject to infinite-point boundary conditions. By employing suitable fp theorems, we established sufficient conditions for the existence of solutions. The results obtained contribute to the growing theory of FDEs by extending classical approaches to more general boundary settings. Future work may include numerical methods and applications to real-world problems modeled by such equations.

Recent advances in FC and nonlocal differential equations include the development of robust numerical schemes that preserve important properties such as positivity and conservation. For example, Yang and Zhang [32] proposed a conservative, positivity-preserving, nonlinear FV scheme for the multiterm nonlocal Nagumo-type equations on distorted meshes. Incorporating such recent studies strengthens the theoretical and practical context of our work.

FDEs with infinite-point boundary conditions play a significant role in modeling complex systems with memory and hereditary properties, which commonly arise in fields such as physics, engineering, biology, and control theory. In this context, the theoretical results obtained in this study provide a solid foundation for the analysis of phenomena such as heat conduction, viscoelasticity, anomalous diffusion, and population dynamics. Our work aims to contribute to the mathematical modeling and the solutions of problems encountered in these application areas.

Author contributions

Özlem Batıt Özen: Writing–original draft; Aynur Şahin: Writing–review and editing. All authors have read and agreed to the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools to create this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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