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Research article

Complexiton and interaction solutions of the (1+1)-dimensional sixth-order Ramani equation

Sukri Khareng and Ömer Ünsal*

Eskişehir Osmangazi University, Department of Mathematics and Computer Sciences, Eskişehir, 26480, TÜRKİYE

* **Correspondence:** Email: ounsal@ogu.edu.tr; Tel: +90 222 239 37 50 - 2209.

Abstract: This study shows the attainability of solutions to the sixth-order Ramani equation through a procedure called the simplified Hirota method, which can be thought of as a special case of the Hirota direct method. This special case comes into sight by virtue of the transition between real and complex parameters. In this procedure, dispersion relations and phase shifts play a decisive role in finding solutions. Particularly, chosen cases of phase shifts provide us different kinds of solutions, such as soliton, complexiton, and interaction solutions.

Keywords: complexiton solution; interaction solution; simplified Hirota method; the (1+1)-dimensional sixth-order Ramani equation
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1. Introduction

In the nonlinear world, a special type of exact solutions, namely interaction solution, has occupied an important place. What makes researchers want to study this topic is the difficulty of obtaining interaction solutions of nonlinear equations and provision convenience of understanding about the dynamic behavior of obtained interaction wave solutions. Interaction solutions arise from the existence of two different wave types in the referred solution. Interaction may be between soliton and periodic or soliton and complexiton. In order to better understand the different types of exact solutions, there are many papers published in literature [1–7]. Interaction solutions occur as a consequence of various exact solution techniques. These miscellaneous procedures also provide different kinds of wave propagations. The newest one of the mostly known waves is complexiton, which differs from other waves in having a different shape. This newest class of nonlinear waves has a brief history can be summarized as given in the following passage.

Researcher Ma introduced the latest kind of wave, which was derived from the Korteweg-de Vries equation with the help of the bilinear concept [8–10], and then presented various studies [11, 12]. Afterwards, some researchers have paid attention to studies on this topic. Eventually, some complexiton solution methods have been established to contribute to the science world. In literature, complexiton is also named a periodic-soliton. Here are some periodic-soliton solution techniques that are composed of wave transformations and sub-equations according to the form of the solutions intended to be obtained in [13–16].

In the current study, we utilize the modification of the simplified Hirota method introduced in [17, 18]. We adapted the referred method to the sixth-order Ramani equation to derive periodic-soliton solutions. Besides, interaction and soliton solutions are also gained at the end of the procedure. All these types of solutions are constructed for three different dimensions, $N = 1, 2, 3$.

In the next section, we give the utilization of the method to the sixth-order Ramani equation. Gained solutions are enriched with plotted graphs for some specific parameters. The last section provides some outcomes related to this study.

2. Derivation of solutions

This part of the paper provides soliton, complexiton, and interaction solutions to the (1+1)-dimensional sixth-order Ramani equation via a kind of modification of the simplified Hirota method [17–20]. The (1+1)-dimensional sixth-order Ramani equation arises in various branches of applied sciences, especially in wave theory in physics. This equation appears as

$$u_{xxxxxx} + 15u_x u_{xxxx} + 15u_{xx}u_{xxx} + 45u_x^2 u_{xx} - 5u_{xxxt} - 15u_x u_{xt} - 15u_t u_{xx} - 5u_{tt} = 0, \tag{2.1}$$

in previous studies [21, 22]. In former studies [21–30], Eq (2.1) was studied to obtain its Lax pair, Bäcklund self-transformation, multiple soliton solutions, multiple singular soliton solutions, exact traveling wave solutions, rogue wave solutions, solitary solutions, interaction solutions, and N-soliton solutions and also examined in the sense of its Painleve test, symmetry analysis, conservation laws, and bilinear form. Through the transformation

$$u = 2(\ln f)_x, \tag{2.2}$$

Eq (2.1) turns to

$$\left(D_x^6 - 5D_x^3 D_t - 5D_t^2\right) f \cdot f = 0, \tag{2.3}$$

which is known to be a bilinear form. D_t and D_x correspond to Hirota derivatives in [31]. Hirota's bilinear method [31] tells us that Eq (2.1) has $2N$-soliton solutions that can be derived via the formula

$$u = 2\left(\ln\left(1 + \sum_{i=1}^{2N} e^{\eta_i} + \sum_{i<j}^{2N} e^{A_{ij}} e^{\eta_i+\eta_j} + \sum_{i<j<k}^{2N} e^{A_{ij}} e^{A_{ik}} e^{A_{jk}} e^{\eta_i+\eta_j+\eta_k} + \cdots + \Big(\prod_{i<j} e^{A_{ij}}\Big) e^{\sum_{i=1}^{2N}\eta_i}\right)\right)_x. \tag{2.4}$$

According to nonlinear theory, Eq (2.1) has dispersion relations and phase shifts in the following forms:

$$\begin{aligned} \eta_j &= k_j' x + w_j' t, \\ w_j' &= \left(-\tfrac{1}{2} \pm \tfrac{3\sqrt{5}}{10}\right) k_j'^3, \\ e^{A_{jl}} &= -\frac{(k_j' - k_l')^6 - 5(k_j' - k_l')^3(w_j' - w_l') - 5(w_j' - w_l')^2}{(k_j' + k_l')^6 - 5(k_j' + k_l')^3(w_j' + w_l') - 5(w_j' + w_l')^2}, \quad (1 \le j < l \le 2N). \end{aligned} \tag{2.5}$$

With the choice $w'_j = \left(-\frac{1}{2} \pm \frac{3\sqrt{5}}{10}\right) k'^3_j$, phase shifts can be expressed as

$$e^{A_{jl}} = \frac{\sqrt{5}k'^4_j - \sqrt{5}k'^3_j k'_l - \sqrt{5}k'_j k'^3_l + \sqrt{5}k'^4_l \mp 3k'^4_j \pm 5k'^3_j k'_l \mp 4k'^2_j k'^2_l \pm 5k'_j k'^3_l \mp 3k'^4_l}{\sqrt{5}k'^4_j + \sqrt{5}k'^3_j k'_l + \sqrt{5}k'_j k'^3_l + \sqrt{5}k'^4_l \mp 3k'^4_j \mp 5k'^3_j k'_l \mp 4k'^2_j k'^2_l \mp 5k'_j k'^3_l \mp 3k'^4_l}, \quad (1 \le j < l \le 2N).$$

The transition

$$k'_{2m-1} = a_m + ib_m, \qquad k'_{2m} = a_m - ib_m, \qquad (1 \le m \le N),$$

between real and complex parameters enables us to build complexiton solutions, where a_m and b_m $(1 \le m \le N)$ are real parameters.

With appropriate choices of a_m and b_m, we give interaction solutions, soliton solutions, and complexiton solutions. Specifically, if $e^{A_{2m-1,2m}} > 1$, $1 \le m \le N$ nonsingular N-complexiton solution is derived, generally.

Case I: With the choices

$$\begin{aligned} \Omega_1 &= xa_1 + \tfrac{3}{2}ta_1b_1^2 - \tfrac{1}{2}ta_1^3 \mp \tfrac{9}{10}t\sqrt{5}\,a_1b_1^2 \pm \tfrac{3}{10}t\sqrt{5}\,a_1^3, \\ \theta_1 &= xb_1 + \tfrac{1}{2}tb_1^3 - \tfrac{3}{2}ta_1^2b_1 \mp \tfrac{3}{10}t\sqrt{5}\,b_1^3 \pm \tfrac{9}{10}t\sqrt{5}\,a_1^2b_1, \end{aligned}$$

and

$$\eta_1 = \eta_2^* = \Omega_1 + i\theta_1, \tag{2.6}$$

with

$$e^{A_{12}} = \frac{b_1^2(\sqrt{5}b_1^2 \mp 5b_1^2 - 3\sqrt{5}a_1^2 \pm 7a_1^2)}{a_1^2(-3\sqrt{5}b_1^2 \pm 7b_1^2 + \sqrt{5}a_1^2 \mp 5a_1^2)}, \tag{2.7}$$

we derive solution of (2.1) as

$$u = \frac{2\left(a_1\cos\theta_1 - b_1\sin\theta_1 + a_1e^{A_{12}}e^{\Omega_1}\right)}{\sqrt{e^{A_{12}}}\cosh\left(\Omega_1 + \ln(\sqrt{e^{A_{12}}})\right) + \cos\theta_1}. \tag{2.8}$$

Solution (2.8) causes two different kinds of solutions, such as

- **(i)** If $\sqrt{5}b_1^2 \mp 5b_1^2 - 3\sqrt{5}a_1^2 \pm 7a_1^2 < 0$ and $-3\sqrt{5}b_1^2 \pm 7b_1^2 + \sqrt{5}a_1^2 \mp 5a_1^2 < 0$ are satisfied; (2.8) gives a one-wave complexiton.
- **(ii)** Equation (2.8) turns to a one-wave soliton if $b_1 = 0$ when $a_1 \neq 0$.

The following illustrations are given to prove that the inequalities $\sqrt{5}b_1^2 \mp 5b_1^2 - 3\sqrt{5}a_1^2 \pm 7a_1^2 < 0$ and $-3\sqrt{5}b_1^2 \pm 7b_1^2 + \sqrt{5}a_1^2 \mp 5a_1^2 < 0$ have a mutual solution that consists of at least one point. Figure 1 shows that these inequalities actually have a mutual solution area, which consists of infinitely many points in the (a1, b1) plane.

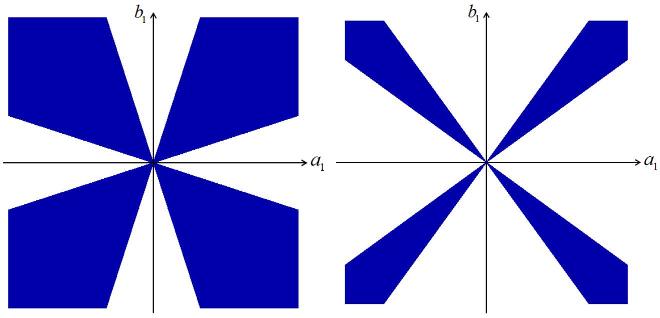

Figure 1. Left: Solution of inequalities $\sqrt{5}b_1^2 - 5b_1^2 - 3\sqrt{5}a_1^2 + 7a_1^2 < 0$ and $-3\sqrt{5}b_1^2 + 7b_1^2 + \sqrt{5}a_1^2 - 5a_1^2 < 0$. **Right:** Solution of inequalities $\sqrt{5}b_1^2 + 5b_1^2 - 3\sqrt{5}a_1^2 - 7a_1^2 < 0$ and $-3\sqrt{5}b_1^2 - 7b_1^2 + \sqrt{5}a_1^2 + 5a_1^2 < 0$.

Case II: ($N = 2$) Taking into consideration

$$\eta_{2m-1} = \eta_{2m}^* = \Omega_m + i\theta_m, \tag{2.9}$$

and

$$\begin{aligned} \Omega_m &= xa_m + \tfrac{3}{2}ta_mb_m^2 - \tfrac{1}{2}ta_m^3 \mp \tfrac{9}{10}t\sqrt{5}\,a_mb_m^2 \pm \tfrac{3}{10}t\sqrt{5}\,a_m^3, \\ \theta_m &= xb_m + \tfrac{1}{2}tb_m^3 - \tfrac{3}{2}ta_m^2b_m \mp \tfrac{3}{10}t\sqrt{5}\,b_m^3 \pm \tfrac{9}{10}t\sqrt{5}\,a_m^2b_m, \quad (m = 1, 2), \end{aligned}$$

with phase shifts such that

$$\begin{aligned} (e^{A_{13}})^* &= e^{A_{24}}, \\ (e^{A_{14}})^* &= e^{A_{23}}, \\ e^{A_{2m-1,2m}} &= \frac{b_m^2(\sqrt{5}b_m^2 \mp 5b_m^2 - 3\sqrt{5}a_m^2 \pm 7a_m^2)}{a_m^2(-3\sqrt{5}b_m^2 \pm 7b_m^2 + \sqrt{5}a_m^2 \mp 5a_m^2)}, \quad (m = 1, 2). \end{aligned}$$

$$\begin{aligned} e^{A_{13}} = -\Big(&\big(\mathrm{i}\sqrt{5}a_1b_2 + \mathrm{i}\sqrt{5}a_2b_1 \mp 4\mathrm{i}a_1b_1 \pm \mathrm{i}a_1b_2 \pm \mathrm{i}a_2b_1 \mp 4\mathrm{i}a_2b_2 + \sqrt{5}a_1a_2 \\ &- \sqrt{5}b_1b_2 \mp 2a_1^2 \pm a_1a_2 \mp 2a_2^2 \pm 2b_1^2 \mp b_1b_2 \pm 2b_2^2\big)(a_1 + \mathrm{i}b_1 - a_2 - \mathrm{i}b_2)^2\Big) \\ &\Big/\Big(\big(\mathrm{i}\sqrt{5}a_2b_1 - 2b_2^2 \pm \mathrm{i}a_1b_2 - \sqrt{5}b_1b_2 \pm a_1a_2 + \mathrm{i}\sqrt{5}a_1b_2 \mp b_1b_2 \pm 4\mathrm{i}a_1b_1 \\ &\pm \mathrm{i}a_2b_1 \pm 4\mathrm{i}a_2b_2 + \sqrt{5}a_1a_2 \pm 2a_2^2 \mp 2b_1^2 \pm 2a_1^2\big)(a_1 + \mathrm{i}b_1 + a_2 + \mathrm{i}b_2)^2\Big), \end{aligned}$$

$$\begin{aligned} e^{A_{14}} = -\Big(&\big(-\mathrm{i}\sqrt{5}a_2b_1 \mp 2b_2^2 \pm \mathrm{i}a_1b_2 - \sqrt{5}b_1b_2 \mp a_1a_2 + \mathrm{i}\sqrt{5}a_1b_2 \mp b_1b_2 \pm 4\mathrm{i}a_1b_1 \\ &\mp \mathrm{i}a_2b_1 \mp 4\mathrm{i}a_2b_2 - \sqrt{5}a_1a_2 \pm 2a_2^2 \mp 2b_1^2 \pm 2a_1^2\big)(a_1 + \mathrm{i}b_1 - a_2 + \mathrm{i}b_2)^2\Big) \\ &\Big/\Big(\big(\mathrm{i}\sqrt{5}a_1b_2 - \mathrm{i}\sqrt{5}a_2b_1 \mp 4\mathrm{i}a_1b_1 \pm \mathrm{i}a_1b_2 \mp \mathrm{i}a_2b_1 \pm 4\mathrm{i}a_2b_2 - \sqrt{5}a_1a_2 \\ &- \sqrt{5}b_1b_2 \mp 2a_1^2 \mp a_1a_2 \mp 2a_2^2 \pm 2b_1^2 \mp b_1b_2 \pm 2b_2^2\big)(a_1 + \mathrm{i}b_1 + a_2 - \mathrm{i}b_2)^2\Big). \end{aligned}$$

The solution for the case ($N = 2$) is obtained as

$$
\begin{aligned}
u = 2\Bigg(\ln \Big(& 1 + 2e^{\Omega_1}\cos\theta_1 + 2e^{\Omega_2}\cos\theta_2 + e^{A_{12}}e^{2\Omega_1} + e^{A_{34}}e^{2\Omega_2} \\
& + 2e^{\Omega_1+\Omega_2}\big[\operatorname{Re}(e^{A_{13}})\cos(\theta_1+\theta_2) - \operatorname{Im}(e^{A_{13}})\sin(\theta_1+\theta_2)\big] \\
& + 2e^{\Omega_1+\Omega_2}\big[\operatorname{Re}(e^{A_{14}})\cos(\theta_1-\theta_2) - \operatorname{Im}(e^{A_{14}})\sin(\theta_1-\theta_2)\big] \\
& + 2e^{A_{12}}e^{2\Omega_1+\Omega_2}\big[(\operatorname{Re}(e^{A_{13}})\operatorname{Re}(e^{A_{14}}) + \operatorname{Im}(e^{A_{13}})\operatorname{Im}(e^{A_{14}}))\cos\theta_2 \\
& - (\operatorname{Im}(e^{A_{13}})\operatorname{Re}(e^{A_{14}}) - \operatorname{Re}(e^{A_{13}})\operatorname{Im}(e^{A_{14}}))\sin\theta_2\big] \\
& + 2e^{A_{34}}e^{\Omega_1+2\Omega_2}\big[(\operatorname{Re}(e^{A_{13}})\operatorname{Re}(e^{A_{14}}) - \operatorname{Im}(e^{A_{13}})\operatorname{Im}(e^{A_{14}}))\cos\theta_1 \\
& - (\operatorname{Re}(e^{A_{13}})\operatorname{Im}(e^{A_{14}}) + \operatorname{Im}(e^{A_{13}})\operatorname{Re}(e^{A_{14}}))\sin\theta_1\big] \\
& + e^{A_{12}}e^{A_{34}}|e^{A_{13}}|^2|e^{A_{14}}|^2e^{2\Omega_1+2\Omega_2}\Big)\Bigg)_x .
\end{aligned}
\tag{2.10}
$$

The following three different types of solutions are derived from solution (2.10) depending on the choices of a_1, b_1, a_2, b_2. They can be summarized as

- **(i)** For $m = 1, 2$; if inequalities $\sqrt{5}\, b_m^2 \mp 5b_m^2 - 3\sqrt{5}\, a_m^2 \pm 7a_m^2 < 0$ and $-3\sqrt{5}\, b_m^2 \pm 7b_m^2 + \sqrt{5}\, a_m^2 \mp 5a_m^2 < 0$, are satisfied, (2.10) provides two-wave complexiton.
- **(ii)** If inequalities $\sqrt{5}\, b_1^2 \mp 5b_1^2 - 3\sqrt{5}\, a_1^2 \pm 7a_1^2 < 0$ and $-3\sqrt{5}\, b_1^2 \pm 7b_1^2 + \sqrt{5}\, a_1^2 \mp 5a_1^2 < 0$, are satisfied while $b_2 = 0$ and $a_2 \neq 0$, (2.10) turns to an interaction solution of one-wave complexiton and one-wave soliton.
- **(iii)** For $m = 1, 2$; if $b_m = 0$ while $a_m \neq 0$ (2.10) gives us a two-wave soliton.

Case III: ($N = 3$) With similar and generalized choices

$$
\eta_{2m-1} = \eta_{2m}^* = \Omega_m + i\theta_m, \tag{2.11}
$$

and

$$
\begin{aligned}
\Omega_m &= xa_m + \tfrac{3}{2}ta_mb_m^2 - \tfrac{1}{2}ta_m^3 \mp \tfrac{9}{10}t\sqrt{5}\, a_mb_m^2 \pm \tfrac{3}{10}t\sqrt{5}\, a_m^3, \\
\theta_m &= xb_m + \tfrac{1}{2}tb_m^3 - \tfrac{3}{2}ta_m^2b_m \mp \tfrac{3}{10}t\sqrt{5}\, b_m^3 \pm \tfrac{9}{10}t\sqrt{5}\, a_m^2b_m, \quad (m = 1, 2, 3),
\end{aligned}
$$

with phase shifts

$$
\begin{aligned}
&(e^{A_{13}})^* = e^{A_{24}}, \quad (e^{A_{14}})^* = (e^{A_{23}}), \quad (e^{A_{15}})^* = e^{A_{26}}, \\
&(e^{A_{16}})^* = e^{A_{25}}, \quad (e^{A_{35}})^* = e^{A_{46}}, \quad (e^{A_{36}})^* = e^{A_{45}},
\end{aligned}
$$

$$
e^{A_{2m-1,2m}} = \frac{b_m^2(\sqrt{5}\, b_m^2 \mp 5b_m^2 - 3\sqrt{5}\, a_m^2 \pm 7a_m^2)}{a_m^2(-3\sqrt{5}\, b_m^2 \pm 7b_m^2 + \sqrt{5}\, a_m^2 \mp 5a_m^2)}, \qquad (m = 1, 2, 3).
$$

$$
\begin{aligned}
e^{A_{15}} = -\Big(\Big(& \mathrm{i}\sqrt{5}a_1b_3 + \mathrm{i}\sqrt{5}a_3b_1 \mp 4\mathrm{i}a_1b_1 \pm \mathrm{i}a_1b_3 \pm \mathrm{i}a_3b_1 \mp 4\mathrm{i}a_3b_3 + \sqrt{5}a_1a_3 - \sqrt{5}b_1b_3 \\
& \mp 2a_1^2 \pm a_1a_3 \mp 2a_3^2 \pm 2b_1^3 \mp b_1b_3 \pm 2b_3^2\Big)(a_1 + \mathrm{i}b_1 - a_3 - \mathrm{i}b_3)^2\Big) \\
\Big/\Big(\Big(& \pm \mathrm{i}a_3b_1 \pm 4\mathrm{i}a_3b_3 - \sqrt{5}b_1b_3 + \mathrm{i}\sqrt{5}a_1b_3 \pm a_1a_3 + \sqrt{5}a_1a_3 + \mathrm{i}\sqrt{5}a_3b_1 \\
& \pm 4\mathrm{i}a_1b_1 \pm \mathrm{i}a_1b_3 \mp 2b_3^2 \pm 2a_3^2 \mp b_1b_3 \mp 2b_1^2 \pm 2a_1^2\Big)(a_1 + \mathrm{i}b_1 + a_3 + \mathrm{i}b_3)^2\Big),
\end{aligned}
$$

$$e^{A_{16}} = -\Big(\big(\mathrm{i}\sqrt{5}a_1b_3 - \mathrm{i}\sqrt{5}a_3b_1 \pm 4\mathrm{i}a_1b_1 \pm \mathrm{i}a_1b_3 \mp \mathrm{i}a_3b_1 \mp 4\mathrm{i}a_3b_3 - \sqrt{5}a_1a_3 - \sqrt{5}b_1b_3 \pm 2a_1^2 \mp a_1a_3 \pm 2a_3^2 \mp 2b_1^2 \mp b_1b_3 \mp 2b_3^2\big)(a_1 + \mathrm{i}b_1 - a_3 + \mathrm{i}b_3)^2\Big) \Big/ \Big(\big(\mathrm{i}\sqrt{5}a_1b_3 - \mathrm{i}\sqrt{5}a_3b_1 \mp 4\mathrm{i}a_1b_1 \pm \mathrm{i}a_1b_3 \mp \mathrm{i}a_3b_1 \pm 4\mathrm{i}a_3b_3 - \sqrt{5}a_1a_3 - \sqrt{5}b_1b_3 \mp 2a_1^2 \mp a_1a_3 \mp 2a_3^2 \pm 2b_1^2 \mp b_1b_3 \pm 2b_3^2\big)(a_1 + \mathrm{i}b_1 + a_3 - \mathrm{i}b_3)^2\Big),$$

$$e^{A_{35}} = -\Big(\big(\mathrm{i}\sqrt{5}a_2b_3 + \mathrm{i}\sqrt{5}a_3b_2 \mp 4\mathrm{i}a_2b_2 \pm \mathrm{i}a_2b_3 \pm \mathrm{i}a_3b_2 \mp 4\mathrm{i}a_3b_3 + \sqrt{5}a_2a_3 - \sqrt{5}b_2b_3 \mp 2a_2^2 \pm a_2a_3 \mp 2a_3^2 \pm 2b_2^2 \mp b_2b_3 \pm 2b_3^2\big)(a_2 + \mathrm{i}b_2 - a_3 - \mathrm{i}b_3)^2\Big) \Big/ \Big(\big(\mathrm{i}\sqrt{5}a_2b_3 \mp 2b_2^2 - \sqrt{5}b_2b_3 \pm \mathrm{i}a_3b_2 \pm 4\mathrm{i}a_3b_3 + \mathrm{i}\sqrt{5}a_3b_2 \pm a_2a_3 + \sqrt{5}a_2a_3 \mp b_2b_3 \pm 4\mathrm{i}a_2b_2 \pm \mathrm{i}a_2b_3 \pm 2a_2^2 \mp 2b_3^2 \pm 2a_3^2\big)(a_2 + \mathrm{i}b_2 + a_3 + \mathrm{i}b_3)^2\Big),$$

$$e^{A_{36}} = -\Big(\big(\mathrm{i}\sqrt{5}a_2b_3 - \mathrm{i}\sqrt{5}a_3b_2 \mp 2b_2^2 - \sqrt{5}a_2a_3 \mp a_2a_3 \mp b_2b_3 \pm 4\mathrm{i}a_2b_2 \pm \mathrm{i}a_2b_3 \pm 2a_2^2 \mp 2b_3^2 \pm 2a_3^2 - \sqrt{5}b_2b_3 \mp \mathrm{i}a_3b_2 \mp 4\mathrm{i}a_3b_3\big)(a_2 + \mathrm{i}b_2 - a_3 + \mathrm{i}b_3)^2\Big) \Big/ \Big(\big(\mathrm{i}\sqrt{5}a_2b_3 - \mathrm{i}\sqrt{5}a_3b_2 \mp 4\mathrm{i}a_2b_2 \pm \mathrm{i}a_2b_3 \mp \mathrm{i}a_3b_2 \pm 4\mathrm{i}a_3b_3 - \sqrt{5}a_2a_3 - \sqrt{5}b_2b_3 \mp 2a_2^2 \mp a_2a_3 \mp 2a_3^2 \pm 2b_2^2 \mp b_2b_3 \pm 2b_3^2\big)(a_2 + \mathrm{i}b_2 + a_3 - \mathrm{i}b_3)^2\Big),$$

The solution for the case ($N = 3$) is then obtained as

$$\begin{aligned} u = 2\Big(\ln\Big[\, & 1 + 2e^{\Omega_1}\cos\theta_1 + 2e^{\Omega_2}\cos\theta_2 + 2e^{\Omega_3}\cos\theta_3 + e^{A_{12}}e^{2\Omega_1} + e^{A_{34}}e^{2\Omega_2} + e^{A_{56}}e^{2\Omega_3} \\ & + 2e^{\Omega_1+\Omega_2}(\mathrm{Re}(e^{A_{13}})\cos(\theta_1+\theta_2) - \mathrm{Im}(e^{A_{13}})\sin(\theta_1+\theta_2) + \mathrm{Re}(e^{A_{14}})\cos(\theta_1-\theta_2) - \mathrm{Im}(e^{A_{14}})\sin(\theta_1-\theta_2)) \\ & + 2e^{\Omega_1+\Omega_3}(\mathrm{Re}(e^{A_{15}})\cos(\theta_1+\theta_3) - \mathrm{Im}(e^{A_{15}})\sin(\theta_1+\theta_3) + \mathrm{Re}(e^{A_{16}})\cos(\theta_1-\theta_3) - \mathrm{Im}(e^{A_{16}})\sin(\theta_1-\theta_3)) \\ & + 2e^{\Omega_2+\Omega_3}(\mathrm{Re}(e^{A_{35}})\cos(\theta_2+\theta_3) - \mathrm{Im}(e^{A_{35}})\sin(\theta_2+\theta_3) + \mathrm{Re}(e^{A_{36}})\cos(\theta_2-\theta_3) - \mathrm{Im}(e^{A_{36}})\sin(\theta_2-\theta_3)) \\ & + 2e^{A_{12}}e^{2\Omega_1+\Omega_2}[(\mathrm{Re}(e^{A_{13}})\mathrm{Re}(e^{A_{14}}) + \mathrm{Im}(e^{A_{13}})\mathrm{Im}(e^{A_{14}}))\cos\theta_2 - (\mathrm{Im}(e^{A_{13}})\mathrm{Re}(e^{A_{14}}) - \mathrm{Re}(e^{A_{13}})\mathrm{Im}(e^{A_{14}}))\sin\theta_2] \\ & + 2e^{A_{34}}e^{2\Omega_2+\Omega_1}[(\mathrm{Re}(e^{A_{13}})\mathrm{Re}(e^{A_{14}}) - \mathrm{Im}(e^{A_{13}})\mathrm{Im}(e^{A_{14}}))\cos\theta_1 - (\mathrm{Im}(e^{A_{13}})\mathrm{Re}(e^{A_{14}}) + \mathrm{Re}(e^{A_{13}})\mathrm{Im}(e^{A_{14}}))\sin\theta_1] \\ & + 2e^{A_{12}}e^{2\Omega_1+\Omega_3}[(\mathrm{Re}(e^{A_{15}})\mathrm{Re}(e^{A_{16}}) + \mathrm{Im}(e^{A_{15}})\mathrm{Im}(e^{A_{16}}))\cos\theta_3 - (\mathrm{Im}(e^{A_{15}})\mathrm{Re}(e^{A_{16}}) - \mathrm{Re}(e^{A_{15}})\mathrm{Im}(e^{A_{16}}))\sin\theta_3] \\ & + 2e^{A_{56}}e^{2\Omega_3+\Omega_1}[(\mathrm{Re}(e^{A_{15}})\mathrm{Re}(e^{A_{16}}) - \mathrm{Im}(e^{A_{15}})\mathrm{Im}(e^{A_{16}}))\cos\theta_1 - (\mathrm{Im}(e^{A_{15}})\mathrm{Re}(e^{A_{16}}) + \mathrm{Re}(e^{A_{15}})\mathrm{Im}(e^{A_{16}}))\sin\theta_1] \\ & + 2e^{A_{34}}e^{2\Omega_2+\Omega_3}[(\mathrm{Re}(e^{A_{35}})\mathrm{Re}(e^{A_{36}}) + \mathrm{Im}(e^{A_{35}})\mathrm{Im}(e^{A_{36}}))\cos\theta_3 - (\mathrm{Im}(e^{A_{35}})\mathrm{Re}(e^{A_{36}}) - \mathrm{Re}(e^{A_{35}})\mathrm{Im}(e^{A_{36}}))\sin\theta_3] \\ & + 2e^{A_{56}}e^{2\Omega_3+\Omega_2}[(\mathrm{Re}(e^{A_{35}})\mathrm{Re}(e^{A_{36}}) - \mathrm{Im}(e^{A_{35}})\mathrm{Im}(e^{A_{36}}))\cos\theta_2 - (\mathrm{Im}(e^{A_{35}})\mathrm{Re}(e^{A_{36}}) + \mathrm{Re}(e^{A_{35}})\mathrm{Im}(e^{A_{36}}))\sin\theta_2] \\ & + \ldots + e^{A_{12}}e^{A_{34}}e^{A_{56}}|e^{A_{13}}|^2|e^{A_{14}}|^2|e^{A_{15}}|^2|e^{A_{16}}|^2|e^{A_{35}}|^2|e^{A_{36}}|^2e^{2\Omega_1+2\Omega_2+2\Omega_3}\Big]\Big)_x. \end{aligned} \tag{2.12}$$

As you can easily see through the process applied during Cases I and II, the more parameters are employed, the more conditions we get from the solution. If parameters a_m and b_m ($m = 1, 2, 3$) are chosen accordingly, solution (2.12) can be considered in four different conditions as follows:

(i) For $m = 1, 2, 3$, if the inequalities $\sqrt{5}b_m^2 \mp 5b_m^2 - 3\sqrt{5}a_m^2 \pm 7a_m^2 < 0$ and $-3\sqrt{5}b_m^2 \pm 7b_m^2 + \sqrt{5}a_m^2 \mp 5a_m^2 < 0$ are satisfied, (2.12) gives a three-wave complexiton.

(ii) For $m = 1, 2$; if $\sqrt{5}b_m^2 \mp 5b_m^2 - 3\sqrt{5}a_m^2 \pm 7a_m^2 < 0$ and $-3\sqrt{5}b_m^2 \pm 7b_m^2 + \sqrt{5}a_m^2 \mp 5a_m^2 < 0$ are satisfied while $b_3 = 0$ and $a_3 \neq 0$, then (2.12) represents an interaction solution of a two-wave complexiton and a one-wave soliton.

(iii) If the inequalities $\sqrt{5}b_1^2 \mp 5b_1^2 - 3\sqrt{5}a_1^2 \pm 7a_1^2 < 0$ and $-3\sqrt{5}b_1^2 \pm 7b_1^2 + \sqrt{5}a_1^2 \mp 5a_1^2 < 0$ are satisfied while $a_2 \neq 0$, $b_2 = 0$, $a_3 \neq 0$, $b_3 = 0$, then (2.12) gives an interaction solution of a two-wave soliton and a one-wave complexiton.

(iv) The choices $a_m \neq 0$, $b_m = 0$ ($m = 1, 2, 3$) transform (2.12) to a three-wave soliton.

Figure 2 displays the graphical illustrations of one-complexiton and one-soliton solutions of (2.8) for some specific choices, respectively. Figure 3 displays the graphical illustrations of two-complexiton and soliton-complexiton solutions of (2.10) for some specific choices, respectively. Figure 4 displays the graphical illustration of two-soliton solution of (2.10) for some specific choices.

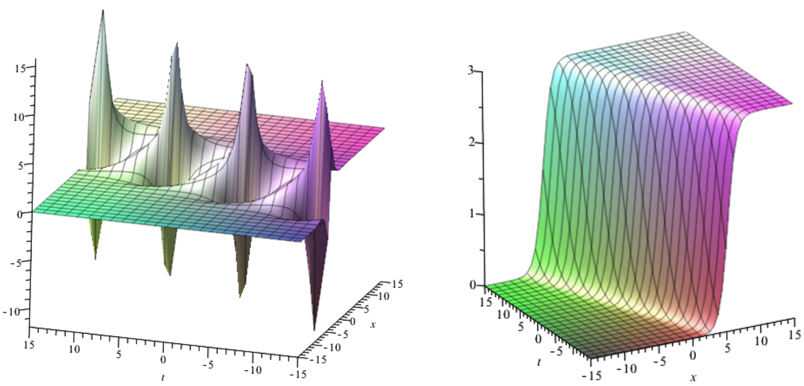

Figure 2. Left: One-complexiton solution of (2.8) with $a_1 = 1$, $b_1 = 1$. **Right:** One-soliton solution of (2.8) with $a_1 = 1.5$, $b_1 = 0$.

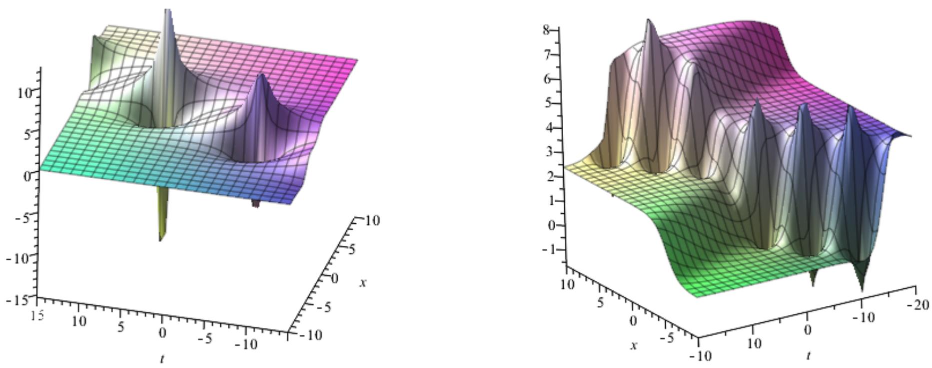

Figure 3. Left: Two-complexiton solution of (2.10) with $a_1 = 0.1$, $b_1 = 0.1$, $a_2 = 0.9$, $b_2 = 0.9$. **Right:** Soliton–complexiton solution of (2.10) with $a_1 = 1.1$, $b_1 = 1.3$, $a_2 = 1.2$, $b_2 = 0$.

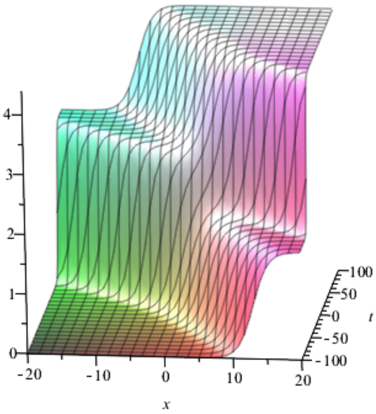

Figure 4. Two-soliton solution of (2.10) with $a_1 = 0.9$, $b_1 = 0$, $a_2 = 1.3$, $b_2 = 0$.

3. Conclusions

This work was carried out to derive complexiton waves and interaction waves of the (1+1)-dimensional sixth-order Ramani equation. Our motivation factor to make such a paper is the application area of partial differential equations solved in this paper. It has some beautiful real-world applications in physics. Since the classical simplified Hirota method and the Hirota direct method do not let us obtain nonsingular waves, we used a modification of the simplified Hirota method introduced by Wazwaz and Zhaqilao [17]. What makes this method interesting is the attainability of real solutions in spite of using complex parameters. Our derived real solutions have been presented in detail by giving necessary conditions to differentiate different sorts of solutions. These were sorted by their names: one-wave soliton, one-wave complexiton, interaction solution of one-wave complexiton and one-wave soliton, and extended ones. The necessary condition for nonsingularity of obtained complexiton solutions was given as a mathematical expression. Derived complexiton solutions have periodicity, which can be observed via sketched graphics. For some specific choices of parameters, some sketched graphs were illustrated to enlighten interested researchers about wave propagation and the dynamical behavior of our derived solutions.

Author contributions

Sukri Khareng: Investigation, visualization, writing–original draft; Ömer Ünsal: Methodology, supervision, writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflict of interest.

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