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Research article

Analytical solutions of a modified Taylor–Goldstein equation modeling linearized gravity waves influenced by multiple chemicals in the atmosphere

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Abstract: This work provided analytical investigations of internal gravity waves affected by the presence of two localized chemicals. Based on the coupling between gravity wave equations under the Boussinesq approximation and the continuity equations for two chemicals, the study presented a mathematical model modeling gravity-wave interactions with the two atmospheric chemicals. The linearized version of this model was analytically investigated for two cases: one with constant mean flow and the other one with a nonconstant mean flow, resulting in a critical level effect. It was observed that in both cases the presence of the two chemicals led to a significant impact on gravity waves. Analytical investigations of the vertically varying mean flow case reveal that the wave-reduction behavior was observed in the vicinity of the critical layer.

Keywords: Taylor–Goldstein equation; mathematical modelling; fluid mechanics; geophysical flows; computational fluid dynamics; Heating; gravity waves; atmospheric chemistry; partial differential equations; wave type equations
Mathematics Subject Classification: 76B15

1. Introduction

In fluid mechanics, internal waves are defined as waves that occur within the fluid. Internal gravity waves can be formed in the atmosphere due to gravitational and buoyancy forces. Internal gravity waves are recognized to travel either upward or downward inside the atmosphere, perhaps dissipating and resulting in turbulent flow.

The meteorological conditions and atmospheric patterns are greatly affected by internal gravity waves. The exchanges between wave types, as well as the couplings between mean flow and waves, lead to variations in wind speed [1]. At the time where the phase speed of gravity waves is equivalent to the mean flow, a critical level occurs. This phenomenon of a critical level aligns mathematically with a singularity in the linearized version of the gravity wave equations represented by the Taylor-Goldstein equation [2, 3].

Photochemical reactions are chemical processes that occur when certain atmospheric substances receive sunlight. Ozone is well recognized as the main collector of UV rays in the middle atmosphere, whereas in the lower atmosphere it is water vapor. Consequently, such chemicals are vital elements of the atmospheric energetic system [4–6].

Transport profoundly influences the geographical spread of chemical compounds in the intermediate atmosphere. Numerical modeling, data simulations, measurements, and satellite and aircraft observations of chemical species have improved our understanding of the relationship between atmospheric chemistry and dynamics [7].

Gravity-wave interactions with atmospheric chemical components mutually influence each other. Gravity waves are either dampened or amplified by chemical processes, such as photochemical heating of atmospheric components [8, 9]. In contrast, gravitational waves influence the amounts of atmospheric substances. One effect is the enhancement in wind speed resulting from gravity -wave interactions. This influences the advection rate of atmospheric chemical components.

Early investigations of linearized equations of gravity waves, with nonconstant mean flow configuration, have been carried out by [10], where they found that wave amplitude is reduced when they propagate through the critical layer. To study the behavior of gravity waves at the critical level, investigations carried out by [11–13] considered the nonlinear terms in the equations of internal gravity waves. They observed that waves are absorbed at the critical layer at the beginning, whereas they get reflected at later times due to the nonlinearity effect.

Lacis and Hansen [14] introduced an approximation of the absorbed solar radiation. Heating rates resulting from the absorption of solar energy by ozone and oxygen were calculated by [15]. The heating rates of water vapor, ozone, and other chemicals were investigated by [16–18]. Lindzen and Will [19] and Luther [20] introduced an analytical approach to the calculation of heating caused by ozone and other compounds. Subsequently, this approach was parametrized by [21]. Xu [9] investigated the impact of chemical processes on linear equations of waves considering multiple cases. In 2003, Xu et al. [22] examined the influence of non-breaking nonlinear gravitational waves on chemical substances in the middle atmosphere.

The impact of the heating released by chemicals on gravity waves needs to be investigated. Analytical investigations of the gravity wave equations affected by the chemicals would help to understand the effect of the chemicals on gravity waves.

The influence of chemical heating induced by a chemical on internal gravity waves as well as the influence of these waves on the distribution of a single chemical was considered in [23]. Almohaimeed [24] extended the investigation considering a configuration in which multiple chemicals are present in the model and presented numerical simulations showing the mutual effect.

In this work, we present analytical solutions to investigate the impact of chemical heating caused by several chemicals on gravity waves. We analyze a modified Taylor–Goldstein equation, modeling linearized waves influenced by two chemicals. We consider several configurations and ultimately provide asymptotic approximations for gravity waves influenced by two localized chemicals.

We examine a model which includes equations for gravity waves, in conjunction with equations that represent the transit of two chemicals. Gravity wave equations are formulated using the Boussinesq approach, which is often considered in the field, and was adopted to formulate such equations by [25–27].

We consider a linearized version of gravity wave equations along with steady-state chemicals. We derive an equation representing linearized gravity waves affected by the chemicals; and take Fourier representation of this equation. We carry out analytical investigations and find analytical solutions for constant, as well as nonconstant mean flow cases.

We find that gravity waves are greatly affected by the chemicals. Our analytical investigations show that the heating released by the chemicals causes a huge impact on the wave amplitude, which is seen in both cases of mean flow. In the nonconstant mean flow case, we find that this impact of the chemical is apparent around the critical level, and wave reduction behavior is observed as waves cross the critical layer.

2. The model

In this work, we consider a configuration in which equations are represented in terms of time t as well as two spatial dimensions: x in the horizontal direction and z in the vertical direction. Let $\mathcal{U}$ be the horizontal velocity of the fluid and $\mathcal{W}$ be the vertical velocity of the fluid. The stream function of the fluid (air), denoted by Ψ, is defined in terms of fluid velocities as $\mathcal{U} = -\frac{\partial \Psi}{\partial z}$ and $\mathcal{W} = \frac{\partial \Psi}{\partial x}$. The air density is denoted by ρ. When the air density is vertically stratified, changes in the fluid occur. As can be seen, for instance, in [2], it is more appropriate to express the dimensional density ρ in the form of background and perturbation quantities as

$$\rho = \bar{\rho}^{\star} + \rho^{\star},$$

and, in a similar approach, express dimensional stream function Ψ as

$$\Psi(x, z, t) = \bar{\psi}^{\star} + \psi^{\star},$$

where $\bar{\rho}^{\star}$ and $\bar{\rho}(z)$ depend only on z. Such quantities can be expressed in scaled forms as

$$\rho(x, z, t) = \bar{\rho}(z) + \epsilon \rho(x, z, t) \ ,$$

and

$$\Psi(x, z, t) = \bar{\psi}(z) + \epsilon \psi(x, z, t),$$

where $\epsilon \ll 1$ is the ratio of the perturbations with respect to the background flow. The background flow is considered to be a horizontal flow with vertical shear, which means that $\bar{u}(z) = -\bar{\psi}'(z)$ and $\bar{w} = 0$.

Following the discussion and derivation steps from the equations of a stratified fluid flow shown in [23, 24, 26], the scaled equations of gravity waves affected by the two chemicals are given in the form

$$\begin{aligned} &\nabla^2 \frac{\partial \psi}{\partial t} - \bar{u}'' \frac{\partial \psi}{\partial x} + \bar{u} \nabla^2 \frac{\partial \psi}{\partial x} + \epsilon \left(\frac{\partial \psi}{\partial x} \nabla^2 \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial z} \nabla^2 \frac{\partial \psi}{\partial x} \right) = -\frac{g}{\bar{\rho}} \frac{\partial \rho}{\partial x}, \\ &\frac{\partial \rho}{\partial t} + \bar{\rho}' \frac{\partial \psi}{\partial x} + \bar{u} \frac{\partial \rho}{\partial x} + \epsilon \left(\frac{\partial \psi}{\partial x} \frac{\partial \rho}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \rho}{\partial x} \right) = -k_1 \mathcal{J}_1 - k_2 \mathcal{J}_2. \end{aligned} \tag{2.1}$$

Here, $\bar{u}$ represents the background mean flow, $\bar{\rho}$ is the the background mean density, and g is the gravitational force; k_1, k_2 are the scaled coefficients of the chemicals involved; $\mathcal{J}_1, \mathcal{J}_2$ represent the heat rate released from the chemicals. The differential operator parameter ∇^2 is defined as $\nabla^2 = (\delta \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2})$, where δ is called the aspect ratio. In the physical context, the quantity $\nabla^2\psi$ can be defined as $\nabla^2\psi = \zeta$, where ζ is the vorticity perturbation of the flow.

We next introduce an equation that simulates the distribution of atmospheric chemical species. Following, for example [28–30], a scaled equation that represents chemical species in the atmosphere in terms of the mixing ratio C, has the form

$$\frac{\partial C}{\partial t} + U\,\frac{\partial C}{\partial x} + W\,\frac{\partial C}{\partial z} = D\,\nabla^2 C + S(C), \tag{2.2}$$

where U and W represent the velocity in horizontal and vertical directions, respectively; D is the diffusion coefficient; and $S(C)$ is the reaction-processes term. In a combined form, (2.2) can be represented as

$$\frac{DC}{Dt} = D\nabla^2 C + S(C). \tag{2.3}$$

We consider a combined model of Eq (2.1) representing gravity waves impacted by the two chemicals, and equations of the form (2.2) for the two chemicals involved in our configuration. The rate of heating released by the chemical is commonly represented, as can be seen, for example, in [31], in a form that is proportional to $(-\frac{DC}{Dt})$. As explained previously, the components of the velocity of the fluid in Eq (2.2), are expressed in terms of background and perturbations, as $U = \bar{u} + \epsilon u$ and $W = \epsilon w$, where $\bar{u}$ is the background mean flow, which could be a constant or $\bar{u}(z)$; and $u = -\psi_z$ and $w = \psi_x$ are the perturbed velocities of the fluid.

In this work, we consider a configuration in which there are two chemicals interacting with gravity waves in a localized region; yet these two chemicals do not interact with each other. Thus, the full model has the form

$$\nabla^2\frac{\partial\psi}{\partial t} - \bar{u}''\frac{\partial\psi}{\partial x} + \bar{u}\nabla^2\frac{\partial\psi}{\partial x} + \epsilon\left(\frac{\partial\psi}{\partial x}\nabla^2\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial z}\nabla^2\frac{\partial\psi}{\partial x}\right) = -\frac{g}{\bar{\rho}}\frac{\partial\rho}{\partial x}, \tag{2.4}$$

$$\frac{\partial\rho}{\partial t} + \bar{\rho}'\frac{\partial\psi}{\partial x} + \bar{u}\frac{\partial\rho}{\partial x} + \epsilon\left(\frac{\partial\psi}{\partial x}\frac{\partial\rho}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\rho}{\partial x}\right) = \gamma_1\frac{DC_1}{Dt} + \gamma_2\frac{DC_2}{Dt}, \tag{2.5}$$

$$\frac{DC_1}{Dt} = \frac{\partial C_1}{\partial t} + (\bar{u} + \epsilon u)\frac{\partial C_1}{\partial x} + \epsilon w\frac{\partial C_1}{\partial z} = D_1\nabla^2 C_1 + S(C_1), \tag{2.6}$$

$$\frac{DC_2}{Dt} = \frac{\partial C_2}{\partial t} + (\bar{u} + \epsilon u)\frac{\partial C_2}{\partial x} + \epsilon w\frac{\partial C_2}{\partial z} = D_2\nabla^2 C_2 + S(C_2), \tag{2.7}$$

where γ_1, γ_2 is the heating coefficient of the chemicals C_1 and C_2, respectively.

3. The analytical solutions

In Eqs (2.4)–(2.7), the magnitude of the perturbation is considered to be small relative to the background flow; that is, we are considering a configuration in which $\epsilon \ll 1$.

Previous studies of gravity waves concluded that the effect of nonlinearity starts at later times. By considering the nonlinear effects to the gravity wave equations, [11–13] expanded on the work of [10] to examine the behavior at the critical layer. They discovered that waves are absorbed at early times and reflected at later times. A thorough numerical study of the evolution of nonlinear gravity waves at the critical level for was conducted by [26]. In agreement with the linear critical theory of [10], it was found that waves are absorbed at early times and the effect of nonlinear terms becomes apparent at later times where waves can be reflected.

Provided that we are considering the case where $\epsilon \ll 1$, terms involving this parameter in our model (2.4)–(2.7) are neglected. Thus, we have

$$\nabla^2 \frac{\partial \psi}{\partial t} - \bar{u}'' \frac{\partial \psi}{\partial x} + \bar{u} \nabla^2 \frac{\partial \psi}{\partial x} = -\frac{g}{\bar{\rho}} \frac{\partial \rho}{\partial x}, \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + \bar{\rho}' \frac{\partial \psi}{\partial x} + \bar{u} \frac{\partial \rho}{\partial x} = \gamma_1 \frac{DC_1}{Dt} + \gamma_2 \frac{DC_2}{Dt}, \tag{3.2}$$

$$\frac{DC_1}{Dt} = \frac{\partial C_1}{\partial t} + \bar{u} \frac{\partial C_1}{\partial x}. \tag{3.3}$$

$$\frac{DC_2}{Dt} = \frac{\partial C_2}{\partial t} + \bar{u} \frac{\partial C_2}{\partial x}. \tag{3.4}$$

We study these linearized equations on $-\infty < x < \infty$ and $0 \le z < \infty$. At $z = 0$, we impose

$$\psi(x, 0, t) = \mathrm{e}^{-\alpha^2 x^2} \left(\mathrm{e}^{ik_0 x} + \mathrm{e}^{-ik_0 x} \right), \tag{3.5}$$

which leads to a wave-pocket solution with wave numbers k centred at k_0, where α is a constant and $\alpha \ll 1$. At the upper boundary, we consider that waves propagate continuously upward, meaning the presence of positive group velocity waves. We consider $\bar{\rho} = \mathrm{e}^{-\frac{z}{h}}$, and $\bar{u}$ is considered to be constant in one case and to be $u(z)$ in the other case. As $x \to \pm\infty$, we consider the conditions $\psi \to 0$ and $\rho \to 0$.

For the consideration of the chemical initial condition, it can be observed from several studies in localized cases that the chemical has a peak around a specific level horizontally and vertically and approaches zero at the boundaries, as can be seen in [18], for example. This means that it can be represented mathematically in a Gaussian or exponential form. We follow this configuration and consider the initial profiles of the two chemicals to be in the forms

$$C_1(x, z, 0) = \mathrm{e}^{-a_1^2 x^2} \mathrm{e}^{-b_1^2 (z - z_1)^2}, \; C_2(x, z, 0) = \mathrm{e}^{-a_2^2 x^2} \mathrm{e}^{-b_2^2 (z - z_2)^2}, \tag{3.6}$$

where a_1, a_2, b_1, b_2, and z_1, z_2 are constants.

For the analysis in this work, steady-state representation is considered for the chemicals as well as for ψ and ρ. We consider a Fourier representation in the horizontal direction for the functions ψ, ρ and C. Based on the boundary conditions above, these functions are smooth and absolutely integrable and their Fourier transforms exist [32]. The continuous Fourier transformation and inverse have the form

$$\begin{aligned} \hat{\psi}(k, z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(x, z) \, \mathrm{e}^{-ikx} \, dx, \\ \psi(x, z) &= \int_{-\infty}^{\infty} \hat{\psi}(k, z) \, \mathrm{e}^{ikx} \, dk, \end{aligned} \tag{3.7}$$

where $\hat{\psi}$ is the stream function in the Fourier form. A similar representation of (3.7) is considered to define the Fourier forms $\hat{\rho}(k, z)$ and $\hat{C}(k, z)$.

The lower boundary (3.5) in Fourier representation has the form

$$\hat{\psi}(k, z=0) = \frac{1}{2\sqrt{\pi}\alpha}\left(e^{-\frac{(k-k_0)^2}{4\alpha^2}} + e^{-\frac{(k+k_0)^2}{4\alpha^2}}\right). \tag{3.8}$$

Since $\alpha \ll 1$, as mentioned above, both $e^{-\frac{(k+k_0)^2}{4\alpha^2}}$ and $e^{-\frac{(k-k_0)^2}{4\alpha^2}}$ approach zero at $k = 0$ so that they can be set to zero; hence, there is no overlap where $e^{-\frac{(k-k_0)^2}{4\alpha^2}}$ is considered for $k > 0$ while $e^{-\frac{(k+k_0)^2}{4\alpha^2}}$ is considered for $k < 0$.

With the consideration of Fourier representation, our models (3.1)–(3.4) simplify to become

$$\bar{u}\hat{\psi}_{zz} - \bar{u}''\hat{\psi} - \delta k^2\bar{u}\hat{\psi} = -\frac{g}{\bar{\rho}}\hat{\rho} \tag{3.9}$$

$$\bar{u}\hat{\rho} + \bar{\rho}'\hat{\psi} = \bar{u}\,(\gamma_1\hat{C}_1 + \gamma_2\hat{C}_2). \tag{3.10}$$

Solving (3.10) for ρ and substituting into Eq (3.9) and rearranging we obtain

$$\hat{\psi}_{zz} + \left(\frac{N^2}{\bar{u}^2} - \delta k^2 - \frac{\bar{u}''}{\bar{u}}\right)\hat{\psi} = G(k, z), \tag{3.11}$$

where $N^2 = \frac{-g\bar{\rho}'}{\bar{\rho}}$ represents the buoyancy frequency and is constant in this work due to the choice of $\bar{\rho}$; $G(k, z) = \frac{-g(\gamma_1\hat{C}_1+\gamma_2\hat{C}_2)}{\bar{u}\bar{\rho}}$. The homogenous part of the Eq (3.11) is the well-known Taylor–Goldstein equation for which the solution can be obtained using the Frobenius series.

The investigation of the solution of Eq (3.11) is our main focus in this work. The solution of Eq (3.11) is of the form

$$\hat{\psi} = \hat{\psi}_h + \hat{\psi}_p,$$

where $\hat{\psi}_p$ is a particular solution and $\hat{\psi}_h$ is a combined solution of the homogenous part.

In the following discussion, we start with the consideration of a constant $\bar{u}$. We then extend our discussion by considering the configuration in which the mean flow $\bar{u}(z)$ is not constant.

3.1. Constant mean flow case

When the mean flow $\bar{u}$ is constant, Eq (3.11) has the form

$$\hat{\psi}_{zz} + \left(\frac{N^2}{\bar{u}^2} - \delta k^2\right)\hat{\psi} = G(k, z). \tag{3.12}$$

We start with the homogenous part of (3.12). When $\frac{N^2}{\bar{u}^2} < \delta k^2$, we have exponential solutions; meanwhile, for the case $\frac{N^2}{\bar{u}^2} > \delta k^2$, we have wave-type solutions. The aspect ratio is $\delta = \frac{L_z}{L_x}$ and is usually considered to be $\delta \ll 1$ in the atmosphere. Taking into account that $\delta \ll 1$ and $\alpha \ll 1$, it is clear that (3.5) approaches zero for wave modes when $\frac{N^2}{\bar{u}^2} < \delta k^2$ is satisfied. Thus, we consider wave modes k satisfying $\frac{N^2}{\bar{u}^2} > \delta k^2$. Therefore, we have solutions of wave-type for the homogeneous part of (3.12) of the form

$$\hat{\psi}_h(k, z) = F_1(k)e^{imz} + F_2(k)e^{-imz}, \tag{3.13}$$

where $m(k) = \mathrm{sgn}(\bar{u})\sqrt{\frac{N^2}{\bar{u}^2} - \delta k^2}$, in accordance with the discussion of group velocity given in [10].

We apply the parameter variation method to obtain the particular solution. Based on the lower boundary condition, the limit is chosen such that $\hat{\psi}_p(k,0) = 0$. The particular solution is then given as

$$\hat{\psi}_p(k,z) = -\mathrm{e}^{imz}\left(\int_0^z \frac{\mathrm{e}^{-im\tilde{z}}G(k,\tilde{z})}{W(\mathrm{e}^{im\tilde{z}},\mathrm{e}^{-im\tilde{z}})}d\tilde{z}\right) + \mathrm{e}^{-imz}\left(\int_0^z \frac{\mathrm{e}^{im\tilde{z}}G(k,\tilde{z})}{W(\mathrm{e}^{im\tilde{z}},\mathrm{e}^{-im\tilde{z}})}d\tilde{z}\right), \tag{3.14}$$

where $G(k,z) = \frac{-g}{\bar{u}\bar{\rho}(z)}\left(\gamma_1\hat{C}_1(k,z) + \gamma_2\hat{C}_2(k,z)\right)$ and the Wronskian is $W(\mathrm{e}^{im\tilde{z}},\mathrm{e}^{-im\tilde{z}}) = -2im$. Substituting for the Fourier transforms $\hat{C}_1(k,z)$ and $\hat{C}_2(k,z)$ of the initial profiles of the chemicals given in (3.6), we find that

$$G(k,z) = \frac{-g}{\bar{u}\bar{\rho}(z)}\left(\gamma_1\hat{C}_1(k,z) + \gamma_2\hat{C}_2(k,z)\right) = \frac{-g}{\bar{u}\bar{\rho}(z)}\left(\gamma_1\mathrm{e}^{-b_1^2(z-z_1)^2}\frac{\sqrt{\pi}}{2\pi a_1}\mathrm{e}^{\frac{-k^2}{4a_1^2}} + \gamma_2\mathrm{e}^{-b_2^2(z-z_2)^2}\frac{\sqrt{\pi}}{2\pi a_2}\mathrm{e}^{\frac{-k^2}{4a_2^2}}\right).$$

We evaluate the integrals in the righthand side of (3.14) using a change of variables and completing square techniques, eventually expressing in terms of error functions as the following:

$$\begin{aligned}
\int_0^z \frac{e^{-im\tilde{z}}G(k,\tilde{z})}{W(e^{im\tilde{z}},e^{-im\tilde{z}})}d\tilde{z} &= \frac{g\sqrt{\pi}}{4\pi im\bar{u}}\left(\frac{\gamma_1}{a_1}e^{\frac{-k^2}{4a_1^2}}\int_0^z e^{-im\tilde{z}}e^{\frac{\tilde{z}}{h}}e^{-b_1^2(\tilde{z}-z_1)^2}d\tilde{z} + \frac{\gamma_2}{a_2}e^{\frac{-k^2}{4a_2^2}}\int_0^z e^{-im\tilde{z}}e^{\frac{\tilde{z}}{h}}e^{-b_2^2(\tilde{z}-z_2)^2}d\tilde{z}\right)\\
&= \frac{g\sqrt{\pi}}{4\pi im\bar{u}}\left[\frac{\gamma_1}{a_1}e^{\frac{-k^2}{4a_1^2}}e^{-imz_1}e^{\frac{z_1}{h}}e^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_1^2}}\left(\int_0^{z-z_1} e^{-b_1^2\left(\bar{z}_1-\frac{\frac{1}{h}-im}{2b_1^2}\right)^2}d\bar{z}_1 - \int_0^{-z_1} e^{-b_1^2\left(\bar{z}_1-\frac{\frac{1}{h}-im}{2b_1^2}\right)^2}d\bar{z}_1\right)\right.\\
&\quad\left. + \frac{\gamma_2}{a_2}e^{\frac{-k^2}{4a_2^2}}e^{-imz_2}e^{\frac{z_2}{h}}e^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_2^2}}\left(\int_0^{z-z_2} e^{-b_2^2\left(\bar{z}_2-\frac{\frac{1}{h}-im}{2b_2^2}\right)^2}d\bar{z}_2 - \int_0^{-z_2} e^{-b_2^2\left(\bar{z}_2-\frac{\frac{1}{h}-im}{2b_2^2}\right)^2}d\bar{z}_2\right)\right]\\
&= \frac{g}{8im\bar{u}}\left[\frac{\gamma_1}{a_1b_1}e^{\frac{-k^2}{4a_1^2}}e^{-imz_1}e^{\frac{z_1}{h}}e^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_1^2}}\left(\mathrm{erf}\left(\frac{2b_1^2h(z-z_1)+imh-1}{2hb_1}\right) - \mathrm{erf}\left(\frac{imh+2b_1^2hz_1-1}{2hb_1}\right)\right)\right.\\
&\quad\left. + \frac{\gamma_2}{a_2b_2}e^{\frac{-k^2}{4a_2^2}}e^{-imz_2}e^{\frac{z_2}{h}}e^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_2^2}}\left(\mathrm{erf}\left(\frac{2b_2^2h(z-z_2)+imh-1}{2hb_2}\right) - \mathrm{erf}\left(\frac{imh+2b_2^2hz_2-1}{2hb_2}\right)\right)\right]
\end{aligned} \tag{3.15}$$

In a similar approach, we find that

$$\begin{aligned}
\int_0^z \frac{e^{im\tilde{z}}G(k,\tilde{z})}{W(e^{im\tilde{z}},e^{-im\tilde{z}})}d\tilde{z} = \frac{g}{8im\bar{u}}&\left[\frac{\gamma_1}{a_1b_1}e^{\frac{-k^2}{4a_1^2}}e^{imz_1}e^{\frac{z_1}{h}}e^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_1^2}}\left(\mathrm{erf}\left(\frac{imh-2b_1^2hz_1+1}{2hb_1}\right)\right.\right.\\
&\left. - \mathrm{erf}\left(\frac{2b_1^2h(z_1-z)+imh+1}{2hb_1}\right)\right)\\
&+ \frac{\gamma_2}{a_2b_2}e^{\frac{-k^2}{4a_2^2}}e^{imz_2}e^{\frac{z_2}{h}}e^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_2^2}}\left(\mathrm{erf}\left(\frac{imh-2b_2^2hz_2+1}{2hb_2}\right)\right.\\
&\left.\left. - \mathrm{erf}\left(\frac{2b_2^2h(z_2-z)+imh+1}{2hb_2}\right)\right)\right]
\end{aligned} \tag{3.16}$$

We substitute (3.15) and (3.16) into (3.14) to obtain

$$\begin{aligned}\hat{\psi}_p(k,z) = &-\frac{g}{8im\bar{u}}e^{imz}\left[\frac{\gamma_1}{a_1b_1}e^{\frac{-k^2}{4a_1^2}}e^{-imz_1}e^{\frac{z_1}{h}}e^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_1^2}}\left(\operatorname{erf}\left(\frac{2b_1^2h(z-z_1)+imh-1}{2hb_1}\right)\right.\right.\\ &\left.-\operatorname{erf}\left(\frac{imh+2b_1^2hz_1-1}{2hb_1}\right)\right)+\frac{\gamma_2}{a_2b_2}e^{\frac{-k^2}{4a_2^2}}e^{-imz_2}e^{\frac{z_2}{h}}e^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_2^2}}\left(\operatorname{erf}\left(\frac{2b_2^2h(z-z_2)+imh-1}{2hb_2}\right)\right.\\ &\left.\left.-\operatorname{erf}\left(\frac{imh+2b_2^2hz_2-1}{2hb_2}\right)\right)\right]+\frac{g}{8im\bar{u}}e^{-imz}\left[\frac{\gamma_1}{a_1b_1}e^{\frac{-k^2}{4a_1^2}}e^{imz_1}e^{\frac{z_1}{h}}e^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_1^2}}\left(\operatorname{erf}\left(\frac{imh-2b_1^2hz_1+1}{2hb_1}\right)\right.\right.\\ &\left.-\operatorname{erf}\left(\frac{2b_1^2h(z_1-z)+imh+1}{2hb_1}\right)\right)+\frac{\gamma_2}{a_2b_2}e^{\frac{-k^2}{4a_2^2}}e^{imz_2}e^{\frac{z_2}{h}}e^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_2^2}}\left(\operatorname{erf}\left(\frac{imh-2b_2^2hz_2+1}{2hb_2}\right)\right.\\ &\left.\left.-\operatorname{erf}\left(\frac{2b_2^2h(z_2-z)+imh+1}{2hb_2}\right)\right)\right]\end{aligned}\tag{3.17}$$

The general solution of (3.12) is

$$\hat{\psi}(k,z) = F_1(k)\mathrm{e}^{imz} + F_2(k)\mathrm{e}^{-imz} + \hat{\psi}_p(k,z). \tag{3.18}$$

In the case where $k > 0$, the upward-group velocity wave correlates with the function e^{imz} in (3.18), while the downward-group velocity wave results from the function e^{-imz}. Based on the upper boundary condition, there can be no waves with downward group velocity. Hence, we need to have $F_2(k) = 0$ in (3.18) for $k > 0$. For the case where $k < 0$ and following the same reasoning, we need to have $F_1(k) = 0$. Based on the condition (3.8), when $k > 0$, we get

$$F_1(k) = \frac{\sqrt{\pi}}{2\pi\alpha}\mathrm{e}^{-\frac{(k-k_0)^2}{4\alpha^2}},$$

whereas when $k < 0$, we get

$$F_2(k) = \frac{\sqrt{\pi}}{2\pi\alpha}\mathrm{e}^{-\frac{(k+k_0)^2}{4\alpha^2}}.$$

Next, we find the solution $\psi(x,z)$

$$\begin{aligned}\psi(x,z) &= \int_{-\infty}^{\infty}\hat{\psi}(k,z)\mathrm{e}^{ikx}\,dk\\ &= \frac{\sqrt{\pi}}{2\pi\alpha}\left[\int_{-\infty}^{0}\mathrm{e}^{-\frac{(k+k_0)^2}{4\alpha^2}}\mathrm{e}^{-imz}\mathrm{e}^{ikx}\,dk + \int_{0}^{\infty}\mathrm{e}^{-\frac{(k-k_0)^2}{4\alpha^2}}\mathrm{e}^{imz}\mathrm{e}^{ikx}\,dk\right] + \int_{-\infty}^{\infty}\hat{\psi}_p(k,z)\mathrm{e}^{ikx}\,dk.\end{aligned}\tag{3.19}$$

Given that $\alpha \ll 1$, the predominant influence in the first integral is almost at $(k+k_0)$. Expanding $\mathrm{e}^{-im(k)z}$ around $(k+k_0)$ and integrating about $\bar{k} = k + k_0$, we obtain

$$\int_{-\infty}^{0}\mathrm{e}^{-\frac{(k+k_0)^2}{4\alpha^2}}\mathrm{e}^{-imz}\mathrm{e}^{ikx}\,dk \sim \mathrm{e}^{-ik_0x}\left(\mathrm{e}^{-im(-k_0)z}\int_{-\infty}^{k_0}\mathrm{e}^{-\frac{(\bar{k})^2}{4\alpha^2}}\mathrm{e}^{i\bar{k}x}\,d\bar{k} - im'(-k_0)\,z\,\mathrm{e}^{-im(-k_0)z}\int_{-\infty}^{k_0}\mathrm{e}^{-\frac{(\bar{k})^2}{4\alpha^2}}\mathrm{e}^{i\bar{k}x}\bar{k}\,d\bar{k}\right), \tag{3.20}$$

where $m'(k) = \frac{d}{dk}m(k)$. As $\bar{k} \to k_0$, the function $\mathrm{e}^{-\frac{(\bar{k})^2}{4\alpha^2}}$ approaches zero;hence, the integration limits can be expanded as

$$\int_{-\infty}^{0} \mathrm{e}^{-\frac{(k+k_0)^2}{4\alpha^2}} \mathrm{e}^{-imz} \mathrm{e}^{ikx}\, dk \sim \mathrm{e}^{-ik_0 x}\Big(\mathrm{e}^{-im(-k_0)z} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(\bar{k})^2}{4\alpha^2}} \mathrm{e}^{i\bar{k}x}\, d\bar{k} - im'(-k_0)\, z\, \mathrm{e}^{-im(-k_0)z} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(\bar{k})^2}{4\alpha^2}} \mathrm{e}^{i\bar{k}x} \bar{k}\, d\bar{k}\Big)$$
$$\sim \mathrm{e}^{-ik_0 x}\Big(2\alpha\sqrt{\pi}\mathrm{e}^{-\alpha^2 x^2} \mathrm{e}^{-im(-k_0)z} + 4\alpha^3\sqrt{\pi}\, m'(-k_0) x\, \mathrm{e}^{-\alpha^2 x^2}\, z\, \mathrm{e}^{-im(-k_0)z}\Big), \tag{3.21}$$

In a similar approach, the second integration in (3.19) can be approximated as

$$\int_{0}^{\infty} \mathrm{e}^{-\frac{(k-k_0)^2}{4\alpha^2}} \mathrm{e}^{imz} \mathrm{e}^{ikx}\, dk \sim \mathrm{e}^{ik_0 x}\Big(2\alpha\sqrt{\pi}\mathrm{e}^{-\alpha^2 x^2} \mathrm{e}^{im(k_0)z} - 4\alpha^3\sqrt{\pi}\, m'(k_0)\, x\, \mathrm{e}^{-\alpha^2 x^2}\, z\, \mathrm{e}^{im(k_0)z}\Big), \tag{3.22}$$

The particular solution $\psi_p(x,z)$ corresponding to the last integral in (3.19) can be found as

$$\begin{aligned}
\psi_p(x,z) = -\frac{g}{8\bar{u}} \int_{-\infty}^{\infty} \Bigg[&\frac{\gamma_1}{a_1 b_1\, i\, m} e^{\frac{-k^2}{4a_1^2}} e^{-imz_1} e^{\frac{z_1}{h}} e^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_1^2}} \left(\mathrm{erf}\left(\frac{2b_1^2 h(z-z_1)+imh-1}{2hb_1}\right)\right.\\
&\left.-\,\mathrm{erf}\left(\frac{imh+2b_1^2 hz_1-1}{2hb_1}\right)\right) + \frac{\gamma_2}{a_2 b_2} e^{\frac{-k^2}{4a_2^2}} e^{-imz_2} e^{\frac{z_2}{h}} e^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_2^2}} \left(\mathrm{erf}\left(\frac{2b_2^2 h(z-z_2)+imh-1}{2hb_2}\right)\right.\\
&\left.-\,\mathrm{erf}\left(\frac{imh+2b_2^2 hz_2-1}{2hb_2}\right)\right)\Bigg] e^{imz+ikx}\, dk\\
+\frac{g}{8\bar{u}} \int_{-\infty}^{\infty} \Bigg[&\frac{\gamma_1}{a_1 b_1\, i\, m} e^{\frac{-k^2}{4a_1^2}} e^{imz_1} e^{\frac{z_1}{h}} e^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_1^2}} \left(\mathrm{erf}\left(\frac{imh-2b_1^2 hz_1+1}{2hb_1}\right)\right.\\
&\left.-\,\mathrm{erf}\left(\frac{2b_1^2 h(z_1-z)+imh+1}{2hb_1}\right)\right) + \frac{\gamma_2}{a_2 b_2} e^{\frac{-k^2}{4a_2^2}} e^{imz_2} e^{\frac{z_2}{h}} e^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_2^2}} \left(\mathrm{erf}\left(\frac{imh-2b_2^2 hz_2+1}{2hb_2}\right)\right.\\
&\left.-\,\mathrm{erf}\left(\frac{2b_2^2 h(z_2-z)+imh+1}{2hb_2}\right)\right)\Bigg] e^{-imz+ikx}\, dk
\end{aligned} \tag{3.23}$$

Rearranging, we obtain

$$\begin{aligned}
\psi_p(x,z) = -\frac{\gamma_1 g\, \mathrm{e}^{\frac{z_1}{h}}}{8a_1 b_1 \bar{u}} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{-k^2}{4a_1^2}} \Bigg[&\frac{\mathrm{e}^{imz}}{im} \mathrm{e}^{-imz_1} \mathrm{e}^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_1^2}} \left(\mathrm{erf}\left(\frac{2b_1^2 h(z-z_1)+imh-1}{2hb_1}\right)\right.\\
&\left.-\mathrm{erf}\left(\frac{imh+2b_1^2 hz_1-1}{2hb_1}\right)\right) + \frac{\mathrm{e}^{-imz}}{im} \mathrm{e}^{imz_1} \mathrm{e}^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_1^2}} \left(\mathrm{erf}\left(\frac{imh-2b_1^2 hz_1+1}{2hb_1}\right)\right.\\
&\left.-\mathrm{erf}\left(\frac{2b_1^2 h(z_1-z)+imh+1}{2hb_1}\right)\right)\Bigg] \mathrm{e}^{ikx}\, dk\\
+\frac{\gamma_2 g\, \mathrm{e}^{\frac{z_2}{h}}}{8a_2 b_2 \bar{u}} \int_{-\infty}^{\infty} \mathrm{e}^{\frac{-k^2}{4a_2^2}} \Bigg[&\frac{\mathrm{e}^{imz}}{im} \mathrm{e}^{-imz_2} \mathrm{e}^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_2^2}} \left(\mathrm{erf}\left(\frac{2b_2^2 h(z-z_2)+imh-1}{2hb_2}\right)\right.
\end{aligned} \tag{3.24}$$

$$-\text{erf}\left(\frac{imh + 2b_2^2hz_2 - 1}{2hb_2}\right)\right) + \frac{\mathrm{e}^{-imz}}{im}\mathrm{e}^{imz_2}\mathrm{e}^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_2^2}}\left(\text{erf}\left(\frac{imh - 2b_2^2hz_2 + 1}{2hb_2}\right)\right.$$
$$\left.\left.-\text{erf}\left(\frac{2b_2^2h(z_2 - z) + imh + 1}{2hb_2}\right)\right)\right]\mathrm{e}^{ikx}\, dk,$$

which can be written as

$$\psi_p(x,z) = -\frac{\gamma_1 g\,\mathrm{e}^{\frac{z_1}{h}}}{8a_1b_1\bar{u}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{-k^2}{4a_1^2}}M_1(k,z)\mathrm{e}^{ikx}\, dk + \frac{\gamma_2 g\,\mathrm{e}^{\frac{z_2}{h}}}{8a_2b_2\bar{u}}\int_{-\infty}^{\infty}\mathrm{e}^{\frac{-k^2}{4a_2^2}}M_2(k,z)\mathrm{e}^{ikx}\, dk. \tag{3.25}$$

Here,

$$\begin{aligned} M_1(k,z) = \frac{1}{im}\Big[&e^{imz}e^{-imz_1}e^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_1^2}}\left(\text{erf}\left(\frac{2b_1^2h(z-z_1) + imh - 1}{2hb_1}\right) - \text{erf}\left(\frac{imh + 2b_1^2hz_1 - 1}{2hb_1}\right)\right) \\ &+e^{-imz}e^{imz_1}e^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_1^2}}\left(\text{erf}\left(\frac{imh - 2b_1^2hz_1 + 1}{2hb_1}\right) - \text{erf}\left(\frac{2b_1^2h(z_1 - z) + imh + 1}{2hb_1}\right)\right)\Big] \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} M_2(k,z) = \Bigg[&\frac{\mathrm{e}^{imz}}{im}\mathrm{e}^{-imz_2}\mathrm{e}^{\frac{\left(\frac{1}{h}-im\right)^2}{4b_2^2}}\left(\text{erf}\left(\frac{2b_2^2h(z-z_2) + imh - 1}{2hb_2}\right) - \text{erf}\left(\frac{imh + 2b_2^2hz_2 - 1}{2hb_2}\right)\right) \\ &+\frac{\mathrm{e}^{-imz}}{im}\mathrm{e}^{imz_2}\mathrm{e}^{\frac{\left(\frac{1}{h}+im\right)^2}{4b_2^2}}\left(\text{erf}\left(\frac{imh - 2b_2^2hz_2 + 1}{2hb_2}\right) - \text{erf}\left(\frac{2b_2^2h(z_2 - z) + imh + 1}{2hb_2}\right)\right)\Bigg]. \end{aligned} \tag{3.27}$$

Starting with the first integral in (3.25), the predominant influence is around $k = 0$ given that $a_1 \ll 1$. Thus, we approximate the integral by considering Taylor expansion of $M_1(k,z)$ around $k = 0$ to get

$$\int_{-\infty}^{\infty}\mathrm{e}^{\frac{-k^2}{4a_1^2}}M_1(k,z)\mathrm{e}^{ikx}\, dk \sim \left(2a_1\sqrt{\pi}\mathrm{e}^{-a_1^2x^2}M_1(0,z) + 4a_1^3ix\sqrt{\pi}\mathrm{e}^{-a_1^2x^2}\frac{\partial M_1}{\partial k}(0,z)\right). \tag{3.28}$$

Similarly, we have

$$\int_{-\infty}^{\infty}\mathrm{e}^{\frac{-k^2}{4a_2^2}}M_2(k,z)\mathrm{e}^{ikx}\, dk \sim \left(2a_2\sqrt{\pi}\mathrm{e}^{-a_2^2x^2}M_2(0,z) + 4a_2^3ix\sqrt{\pi}\mathrm{e}^{-a_2^2x^2}\frac{\partial M_2}{\partial k}(0,z)\right). \tag{3.29}$$

The general solution $\psi(x,z)$ can then be given as

$$\begin{aligned} \psi(x,z) \sim\ &\mathrm{e}^{-ik_0x}\left(\mathrm{e}^{-\alpha^2x^2}\mathrm{e}^{-im(-k_0)z} + 2\alpha^2\, m'(-k_0)x\,\mathrm{e}^{-\alpha^2x^2}\, z\,\mathrm{e}^{-im(-k_0)z}\right) \\ &+\mathrm{e}^{ik_0x}\left(\mathrm{e}^{-\alpha^2x^2}\mathrm{e}^{im(k_0)z} - 2\alpha^2\, m'(k_0)\, x\,\mathrm{e}^{-\alpha^2x^2}\, z\,\mathrm{e}^{im(k_0)z}\right) \\ &-\frac{\gamma_1 g\,\mathrm{e}^{\frac{z_1}{h}}}{8a_1b_1\bar{u}}\left(2a_1\sqrt{\pi}\mathrm{e}^{-a_1^2x^2}M_1(0,z) + 4a_1^3ix\sqrt{\pi}\mathrm{e}^{-a_1^2x^2}\frac{\partial M_1}{\partial k}(0,z)\right) \\ &+\frac{\gamma_2 g\,\mathrm{e}^{\frac{z_2}{h}}}{8a_2b_2\bar{u}}\left(2a_2\sqrt{\pi}\mathrm{e}^{-a_2^2x^2}M_2(0,z) + 4a_2^3ix\sqrt{\pi}\mathrm{e}^{-a_2^2x^2}\frac{\partial M_2}{\partial k}(0,z)\right). \end{aligned} \tag{3.30}$$

Equation (3.30) represents the stream function of gravity waves influenced by the two chemicals with constant mean flow. We note that the third and fourth terms in Eq (3.30) correspond to the influence of the chemical heating induced by the two chemicals. This shows that the heating induced by the two chemicals has a substantial influence on gravity waves they propagate upward. The greatest effect occurs around the level of z_1 and z_2, where the chemicals are centered.

The following figures illustrate the analytical solution (3.30) with the parameters $\bar{u} = -1$, $k_0 = 2$, $\alpha = 0.1$, $\gamma_1 = 0.05$, and $\gamma_2 = 0.03$. The mixing ratio profiles of the two chemicals considered are shown in Figure 1. Figure 2 shows the stream function $\psi(x,z)$ without a chemical effect (homogeneous solution), as well as the stream function $\psi(x,z)$ under the effect of the two chemicals given by the solution (3.30), with small modifications of the third and fourth terms corresponding to the shift in horizontal localization $(x - x_1)$ and $(x - x_2)$. We note that gravity waves are greatly affected by the heating released from the two chemicals. This effect can be clearly seen in the vicinity of the chemicals.

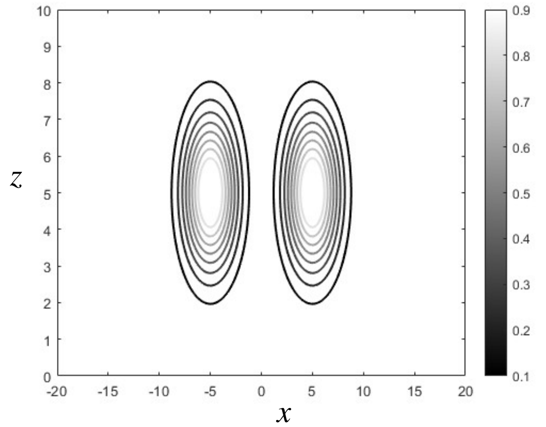


Figure 1. The contour lines of the two chemical concentrations, which show the localization of $C_1(x,z) = \mathrm{e}^{-a_1^2(x-x_1)^2}\mathrm{e}^{-b^2(z-z_1)^2}$ in the left and the localization of $C_2(x,z) = \mathrm{e}^{-a_2^2(x-x_1)^2}\mathrm{e}^{-b_2^2(z-z_2)^2}$ in the right, where $a_1 = 0.4$, $a_2 = 0.4$, $b_1 = 0.5$, $b_2 = 0.5$, and $x_1 = -5$, $x_2 = 5$, $z_1 = z_2 = 5$.

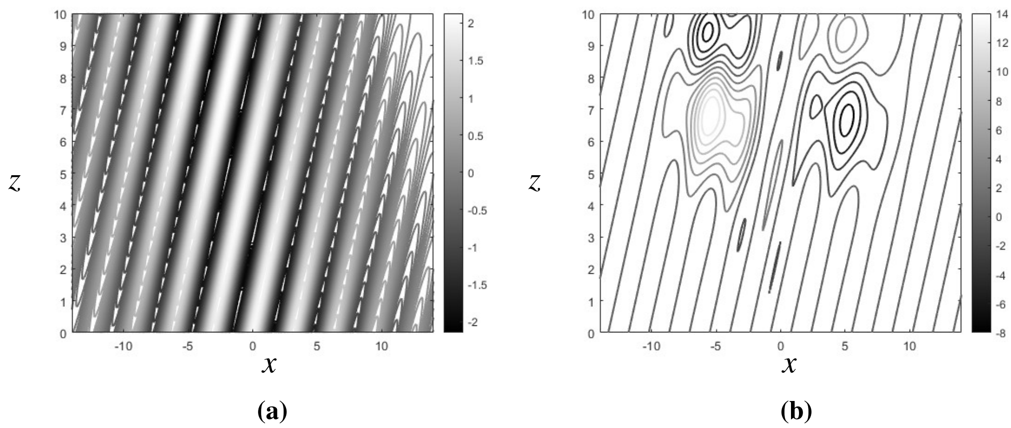
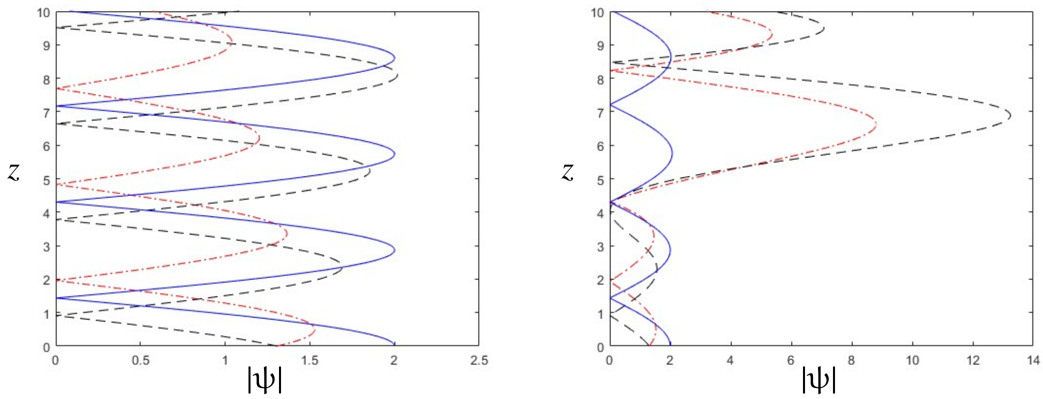


Figure 2. The contour lines of the stream function $\psi(x,z)$, where (a) represents the stream function $\psi(x,z)$ with no chemical effects (homogeneous solution) and (b) represents $\psi(x,z)$ with the chemical effect given by (3.30). These chemical effects on waves can be seen in (b) in the vicinity where the chemicals are centered.

The magnitudes of gravity waves at different horizontal points are shown in Figure 3. The figure compares the amplitude of gravity waves with and without the chemical effect. In agreement with Figure 2, the effect becomes apparent in the vicinity of the chemicals. We note that the waves at $x = -5$, where C_1 is centered, are affected more than other cases. This behavior occurs due to the effect of chemical heating, as γ_1 is considered greater than γ_2 associated with the other chemical.



(a) **(b)**

Figure 3. This figure shows $|\psi(x, z)|$ of solution (3.30) at $x = -5$ (dashed line), $x = 0$ (solid line), and $x = 5$ (the dashed-dotted line), where (a) represents these magnitudes with no chemical effects (homogeneous solution) and (b) represents $|\psi(x, z)|$ (of (3.30)) with the two chemical effect at the same horizontal points given in (a).

3.2. Nonconstant mean flow case

In this subsection, we consider Eq (3.11) with the more general case, a nonconstant mean flow of the form $\bar{u} = \bar{u}(z)$. We consider a configuration where $\bar{u}(z)$ is zero at $z = z_{cr}$, leading to a critical level at this point since Eq (3.11) is singular at this level. In this case, Eq (3.11) takes the form of a modified Taylor–Goldstein equation as

$$\hat{\psi}_{zz} + \left(\frac{N^2}{\bar{u}(z)^2} - \delta k^2 - \frac{\bar{u}(z)''}{\bar{u}(z)}\right)\hat{\psi} = G(k, z), \tag{3.31}$$

with $G(k, z) = \frac{-g(\gamma_1 \hat{C}_1 + \gamma_2 \hat{C}_2)}{\bar{u}\bar{\rho}}$ as before. As in the previous case, we consider that only waves with upward group velocity are present at the upper boundary. The general solution of the modified Taylor–Goldstein Eq (3.31) is given as

$$\hat{\psi}(k, z) = \hat{\psi}_h(k, z) + \hat{\psi}_p(k, z).$$

As can be seen in [10, 33], the solution of the homogeneous part of (3.11), known as the Taylor–Goldstein equation, is commonly approximated using the Frobenius series. Following this approach, the homogeneous solution as $z \to z_{cr}r$ is given as

$$\hat{\psi}_h(k, z) \sim \alpha_0(k)\phi(k, z) + \tilde{\alpha}_0(k)\tilde{\phi}(k, z) \tag{3.32}$$

where

$$\phi(k, z) = (z - z_{cr})^{\frac{1}{2}+i\beta} + \alpha_1(k)(z - z_{cr})^{\frac{3}{2}+i\beta} + \alpha_2(k)(z - z_{cr})^{\frac{5}{2}+i\beta}$$

and

$$\tilde{\phi}(k,z) = (z-z_{cr})^{\frac{1}{2}-i\beta} + \tilde{\alpha}_1(k)(z-z_{cr})^{\frac{3}{2}-i\beta} + \tilde{\alpha}_2(k)(z-z_{cr})^{\frac{5}{2}-i\beta}$$

where $\beta = (Ri_c - \frac{1}{4})^{\frac{1}{2}}$ and Ri_c is the Richardson number at $z = z_{cr}$; and

$$\alpha_1(k) = \frac{(1+Ri)\bar{u}_c''}{2\mu\bar{u}_c'},$$

$\bar{u}_c' = \bar{u}'(z_{cr})$, $\bar{u}_c'' = \bar{u}''(z_{cr})$,where $\mu = \frac{1}{2} + i\beta$ and

$$\alpha_2(k) = \frac{\left(\frac{\bar{u_c''}^2}{2} + \bar{u}_c'\bar{u}_c''' + \delta k^2\bar{u_c'}^2\right) - \alpha_1\bar{u}_c\bar{u}_c''(\mu^2+\mu-1)}{(\mu^2+3\mu+2)\bar{u_c'}^2 + N^2}.$$

The definitions of $\tilde{\alpha}_1(k)$ and $\tilde{\alpha}_2(k)$ are similar to $\alpha_1(k)$ and $\alpha_2(k)$, with μ defined as $(\frac{1}{2} - i\beta)$. Based on the boundary conditions at the lower and upper boundary, $\alpha_0(k)$ and $\tilde{\alpha}_0(k)$ can be specified.

The specific solution $\hat{\psi}_p(k,z)$ of (3.31) has the form

$$\begin{aligned}\hat{\psi}_p(k,z) \sim &-\frac{g(z-z_{cr})^{\frac{1}{2}+i\beta}}{4\sqrt{\pi}\,i\beta}\left(\frac{\gamma_1 e^{\frac{-k^2}{4a_1^2}}}{a_1}\int_0^z \frac{(\tilde{z}-z_{cr})^{\frac{1}{2}-i\beta}}{\bar{u}(\tilde{z})} e^{\frac{\tilde{z}}{h}-b_1^2(\tilde{z}-z_1)^2}d\tilde{z} + \frac{\gamma_2 e^{\frac{-k^2}{4a_2^2}}}{a_2}\int_0^z \frac{(\tilde{z}-z_{cr})^{\frac{1}{2}-i\beta}}{\bar{u}(\tilde{z})} e^{\frac{\tilde{z}}{h}-b_2^2(\tilde{z}-z_2)^2}d\tilde{z}\right)\\ &+\frac{g(z-z_{cr})^{\frac{1}{2}-i\beta}}{4\sqrt{\pi}\,i\beta}\left(\frac{\gamma_1 e^{\frac{-k^2}{4a_1^2}}}{a_1}\int_0^z \frac{(\tilde{z}-z_{cr})^{\frac{1}{2}+i\beta}}{\bar{u}(\tilde{z})} e^{\frac{\tilde{z}}{h}-b_1^2(\tilde{z}-z_1)^2}d\tilde{z} + \frac{\gamma_2 e^{\frac{-k^2}{4a_2^2}}}{a_2}\int_0^z \frac{(\tilde{z}-z_{cr})^{\frac{1}{2}+i\beta}}{\bar{u}(\tilde{z})} e^{\frac{\tilde{z}}{h}-b_2^2(\tilde{z}-z_2)^2}d\tilde{z}\right).\end{aligned} \tag{3.33}$$

Analogous to the approach used to approximate the homogeneous solution, we adopt a similar methodology and consider approximations at the critical level z_{cr}. The mean flow $\bar{u}(z)$ is approximated by considering its Taylor approximation as $\bar{u}(\tilde{z}) \sim (\tilde{z}-z_{cr})\bar{u}'_{z_{cr}}$. In addition, we consider a configuration in which the chemicals are centered at the critical level and approximated by the Taylor expansion around the critical level. Therefore, the first term in (3.33) is approximated as

$$\begin{aligned}&\int_0^z \frac{(\tilde{z}-z_{cr})^{\frac{1}{2}-i\beta}}{\bar{u}(\tilde{z})}\exp\left(\frac{\tilde{z}}{h} - b_1^2(\tilde{z}-z_{cr})^2\right)d\tilde{z}\\ &\sim \int_0^z \frac{(\tilde{z}-z_{cr})^{\frac{1}{2}-i\beta}}{(\tilde{z}-z_{cr})\bar{u}'_{z_{cr}}}\left\{\exp\left(\frac{z_{cr}}{h}\right) + (\tilde{z}-z_{cr})\left(\frac{1}{h}\right)\exp\left(\frac{z_{cr}}{h}\right)\right.\\ &\left.-\frac{1}{2}(\tilde{z}-z_{cr})^2\left(\frac{1}{h}\right)^2(2b_1^2h^2-1)\exp\left(\frac{z_{cr}}{h}\right)\right\}d\tilde{z}\\ &\sim \frac{\exp\left(\frac{z_{cr}}{h}\right)}{\bar{u}'_{z_{cr}}}\left\{\frac{(z-z_{cr})^{\frac{1}{2}-i\beta}}{\left(\frac{1}{2}-i\beta\right)} + \frac{(z-z_{cr})^{\frac{3}{2}-i\beta}}{h\left(\frac{3}{2}-i\beta\right)}\right.\\ &-(2b_1^2h^2-1)\frac{(z-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2\left(\frac{5}{2}-i\beta\right)} - \frac{(-z_{cr})^{\frac{1}{2}-i\beta}}{\left(\frac{1}{2}-i\beta\right)}\\ &\left.-\frac{(-z_{cr})^{\frac{3}{2}-i\beta}}{h\left(\frac{3}{2}-i\beta\right)} + (2b_1^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2\left(\frac{5}{2}-i\beta\right)}\right\},\end{aligned} \tag{3.34}$$

as $z \to z_{cr}$, and it follows that the second integration in (3.33) can be approximated as

$$\begin{aligned}
&\int_0^z \frac{(\tilde{z}-z_{cr})^{\frac{1}{2}-i\beta}}{\bar{u}(\tilde{z})} e^{\frac{\tilde{z}}{h}-b_2^2(\tilde{z}-z_{cr})^2} d\tilde{z} \sim \frac{e^{\frac{z_{cr}}{h}}}{\bar{u}'_{z_{cr}}} \Big\{ \frac{(z-z_{cr})^{\frac{1}{2}-i\beta}}{(\frac{1}{2}-i\beta)} + \frac{(z-z_{cr})^{\frac{3}{2}-i\beta}}{h(\frac{3}{2}-i\beta)} \\
&-(2b_2^2h^2-1)\frac{(z-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2(\frac{5}{2}-i\beta)} - \frac{(-z_{cr})^{\frac{1}{2}-i\beta}}{(\frac{1}{2}-i\beta)} - \frac{(-z_{cr})^{\frac{3}{2}-i\beta}}{h(\frac{3}{2}-i\beta)} + (2b_2^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2(\frac{5}{2}-i\beta)} \Big\},
\end{aligned} \tag{3.35}$$

as $z \to z_{cr}$. Similarly, we find that

$$\begin{aligned}
&\int_0^z \frac{(\tilde{z}-z_{cr})^{\frac{1}{2}+i\beta}}{\bar{u}(\tilde{z})} e^{\frac{\tilde{z}}{h}-b_1^2(\tilde{z}-z_{cr})^2} d\tilde{z} \sim \frac{e^{\frac{z_{cr}}{h}}}{\bar{u}'_{z_{cr}}} \Big\{ \frac{(z-z_{cr})^{\frac{1}{2}+i\beta}}{(\frac{1}{2}+i\beta)} + \frac{(z-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)} \\
&-(2b_1^2h^2-1)\frac{(z-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)} - \frac{(-z_{cr})^{\frac{1}{2}+i\beta}}{(\frac{1}{2}+i\beta)} - \frac{(-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)} + (2b_1^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)} \Big\},
\end{aligned} \tag{3.36}$$

and it follows that

$$\begin{aligned}
&\int_0^z \frac{(\tilde{z}-z_{cr})^{\frac{1}{2}+i\beta}}{\bar{u}(\tilde{z})} e^{\frac{\tilde{z}}{h}-b_2^2(\tilde{z}-z_{cr})^2} d\tilde{z} \sim \frac{e^{\frac{z_{cr}}{h}}}{\bar{u}'_{z_{cr}}} \Big\{ \frac{(z-z_{cr})^{\frac{1}{2}+i\beta}}{(\frac{1}{2}+i\beta)} + \frac{(z-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)} \\
&-(2b_2^2h^2-1)\frac{(z-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)} - \frac{(-z_{cr})^{\frac{1}{2}+i\beta}}{(\frac{1}{2}+i\beta)} - \frac{(-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)} + (2b_2^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)} \Big\},
\end{aligned} \tag{3.37}$$

as $z \to z_{cr}$. Substituting (3.37)–(3.34) into (3.33) we obtain

$$\begin{aligned}
\hat{\psi}_p(k,z) \sim &-\frac{g(z-z_{cr})^{\frac{1}{2}+i\beta} e^{\frac{z_{cr}}{h}}}{4\sqrt{\pi}\, i\beta\, \bar{u}'_{z_{cr}}} \Bigg[\frac{\gamma_1 e^{\frac{-k^2}{4a_1^2}}}{a_1} \Bigg(\frac{(z-z_{cr})^{\frac{1}{2}-i\beta}}{\frac{1}{2}-i\beta} + \frac{(z-z_{cr})^{\frac{3}{2}-i\beta}}{h(\frac{3}{2}-i\beta)} - (2b_1^2h^2-1)\frac{(z-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2(\frac{5}{2}-i\beta)} \\
&-\frac{(-z_{cr})^{\frac{1}{2}-i\beta}}{\frac{1}{2}-i\beta} - \frac{(-z_{cr})^{\frac{3}{2}-i\beta}}{h(\frac{3}{2}-i\beta)} + (2b_1^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2(\frac{5}{2}-i\beta)} \Bigg) + \frac{\gamma_2 e^{\frac{-k^2}{4a_2^2}}}{a_2} \Bigg(\frac{(z-z_{cr})^{\frac{1}{2}-i\beta}}{\frac{1}{2}-i\beta} \\
&+\frac{(z-z_{cr})^{\frac{3}{2}-i\beta}}{h(\frac{3}{2}-i\beta)} - (2b_2^2h^2-1)\frac{(z-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2(\frac{5}{2}-i\beta)} - \frac{(-z_{cr})^{\frac{1}{2}-i\beta}}{\frac{1}{2}-i\beta} - \frac{(-z_{cr})^{\frac{3}{2}-i\beta}}{h(\frac{3}{2}-i\beta)} + (2b_2^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2(\frac{5}{2}-i\beta)} \Bigg) \Bigg] \\
&+\frac{g(z-z_{cr})^{\frac{1}{2}-i\beta} e^{\frac{z_{cr}}{h}}}{4\sqrt{\pi}\, i\beta\, \bar{u}'_{z_{cr}}} \Bigg[\frac{\gamma_1 e^{\frac{-k^2}{4a_1^2}}}{a_1} \Bigg(\frac{(z-z_{cr})^{\frac{1}{2}+i\beta}}{\frac{1}{2}+i\beta} + \frac{(z-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)} - (2b_1^2h^2-1)\frac{(z-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)} \\
&-\frac{(-z_{cr})^{\frac{1}{2}+i\beta}}{\frac{1}{2}+i\beta} - \frac{(-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)} + (2b_1^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)} \Bigg) + \frac{\gamma_2 e^{\frac{-k^2}{4a_2^2}}}{a_2} \Bigg(\frac{(z-z_{cr})^{\frac{1}{2}+i\beta}}{\frac{1}{2}+i\beta} + \frac{(z-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)} \\
&-(2b_2^2h^2-1)\frac{(z-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)} - \frac{(-z_{cr})^{\frac{1}{2}+i\beta}}{\frac{1}{2}+i\beta} - \frac{(-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)} + (2b_2^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)} \Bigg) \Bigg].
\end{aligned} \tag{3.38}$$

To meet the requirement that only waves with upward group velocity are present according to the upper boundary condition, we should have $\tilde{\alpha}_0(k) = 0$ for $k > 0$ and $\alpha_0(k) = 0$ for $k < 0$ in the general solution. Considering condition (3.8) and given that as

$$z \to 0, \hat{\psi}_p(k,z) \to 0,$$

we obtain

$$\alpha_0(k) \sim \frac{\sqrt{\pi}}{2\pi\alpha\,\phi(k,0)} \mathrm{e}^{-\frac{(k-k_0)^2}{4\alpha^2}},$$

for $k > 0$; and

$$\tilde{\alpha}_0(k) \sim \frac{\sqrt{\pi}}{2\pi\alpha\,\tilde{\phi}(k,0)} \mathrm{e}^{-\frac{(k+k_0)^2}{4\alpha^2}},$$

for $k < 0$. Thus, the solution $\hat{\psi}(k,z)$ becomes

$$\hat{\psi}(k,z) \sim \frac{\sqrt{\pi}}{2\pi\alpha} \mathrm{e}^{-\frac{(k-k_0)^2}{4\alpha^2}} \frac{\phi(k,z)}{\phi(k,0)} + \hat{\psi}_p(k,z), \tag{3.39}$$

for $k > 0$, and

$$\hat{\psi}(k,z) \sim \frac{\sqrt{\pi}}{2\pi\alpha} \mathrm{e}^{-\frac{(k+k_0)^2}{4\alpha^2}} \frac{\tilde{\phi}(k,z)}{\tilde{\phi}(k,0)} + \hat{\psi}_p(k,z), \tag{3.40}$$

for $k < 0$.

Next, we apply the inverse Fourier to obtain the solution $\psi(x,z)$,

$$\begin{aligned} \psi(x,z) = \int_{-\infty}^{\infty} \hat{\psi}(k,z) \mathrm{e}^{ikx}\, dk &\sim \frac{\sqrt{\pi}}{2\pi\alpha} \int_{-\infty}^{0} \mathrm{e}^{-\frac{(k+k_0)^2}{4\alpha^2}} \frac{\tilde{\phi}(k,z)}{\tilde{\phi}(k,0)} \mathrm{e}^{ikx}\, dk \\ &+ \frac{\sqrt{\pi}}{2\pi\alpha} \int_{0}^{\infty} \mathrm{e}^{-\frac{(k-k_0)^2}{4\alpha^2}} \frac{\phi(k,z)}{\phi(k,0)} \mathrm{e}^{ikx}\, dk + \int_{-\infty}^{\infty} \hat{\psi}_p(k,z) \mathrm{e}^{ikx}\, dk. \end{aligned} \tag{3.41}$$

In a similar approach to the approximation considered in the previous section, we approximate the first two integrals as

$$\begin{aligned} \int_{-\infty}^{0} \mathrm{e}^{-\frac{(k+k_0)^2}{4\alpha^2}} \frac{\tilde{\phi}(k,z)}{\tilde{\phi}(k,0)} \mathrm{e}^{ikx}\, dk \sim \mathrm{e}^{-ik_0 x} \Big(&(2\alpha\sqrt{\pi} \mathrm{e}^{-\alpha^2 x^2}) \frac{\tilde{\phi}(-k_0,z)}{\tilde{\phi}(-k_0,0)} \\ &+ 4\alpha^3 ix\sqrt{\pi} \mathrm{e}^{-\alpha^2 x^2} \frac{\partial}{\partial k} \left(\frac{\tilde{\phi}(k,z)}{\tilde{\phi}(k,0)} \right) (-k_0,z) \Big), \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} \int_{0}^{\infty} \mathrm{e}^{-\frac{(k-k_0)^2}{4\alpha^2}} \frac{\phi(k,z)}{\phi(k,0)} \mathrm{e}^{ikx}\, dk \sim \mathrm{e}^{ik_0 x} \Big(&(2\alpha\sqrt{\pi} \mathrm{e}^{-\alpha^2 x^2}) \frac{\phi(k_0,z)}{\phi(k_0,0)} \\ &+ 4\alpha^3 ix\sqrt{\pi} \mathrm{e}^{-\alpha^2 x^2} \frac{\partial}{\partial k} \left(\frac{\phi(k,z)}{\phi(k,0)} \right) (k_0,z) \Big). \end{aligned} \tag{3.43}$$

For the third integral in (3.41), we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{\psi}_p(k,z)\mathrm{e}^{ikx}\,dk \sim{}& \frac{\gamma_1 g \mathrm{e}^{-a_1^2x^2}\,\mathrm{e}^{\frac{z_{cr}}{h}}}{2\,i\beta\,\bar{u}'_{z_{cr}}}\Big[(z-z_{cr})^{\frac{1}{2}+i\beta}\Big(\frac{(-z_{cr})^{\frac{1}{2}-i\beta}}{\frac{1}{2}-i\beta}+\frac{(-z_{cr})^{\frac{3}{2}-i\beta}}{h(\frac{3}{2}-i\beta)}\\
&-(2b_1^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2(\frac{5}{2}-i\beta)}\Big)+(z-z_{cr})^{\frac{1}{2}-i\beta}\Big(-\frac{(-z_{cr})^{\frac{1}{2}+i\beta}}{\frac{1}{2}+i\beta}-\frac{(-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)}\\
&+(2b_1^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)}\Big)-\frac{8i\beta(z-z_{cr})}{4\beta^2+1}+\frac{8i\beta(z-z_{cr})^2}{h(4\beta^2+9)}\\
&+(2b_1^2h^2-1)\frac{(6i\beta-5)(z-z_{cr})^3}{h^2(4\beta^2+25)}\Big]+\frac{\gamma_2 g \mathrm{e}^{-a_2^2x^2}\,\mathrm{e}^{\frac{z_{cr}}{h}}}{2\,i\beta\,\bar{u}'_{z_{cr}}}\Big[(z-z_{cr})^{\frac{1}{2}+i\beta}\Big(\frac{(-z_{cr})^{\frac{1}{2}-i\beta}}{\frac{1}{2}-i\beta}\\
&+\frac{(-z_{cr})^{\frac{3}{2}-i\beta}}{h(\frac{3}{2}-i\beta)}-(2b_2^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2(\frac{5}{2}-i\beta)}\Big)+(z-z_{cr})^{\frac{1}{2}-i\beta}\Big(-\frac{(-z_{cr})^{\frac{1}{2}+i\beta}}{\frac{1}{2}+i\beta}-\frac{(-z_{cr})^{\frac{3}{2}+i\beta}}{h(\frac{3}{2}+i\beta)}\\
&+(2b_2^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2(\frac{5}{2}+i\beta)}\Big)-\frac{8i\beta(z-z_{cr})}{4\beta^2+1}+\frac{8i\beta(z-z_{cr})^2}{h(4\beta^2+9)}+(2b_2^2h^2-1)\frac{(6i\beta-5)(z-z_{cr})^3}{h^2(4\beta^2+25)}\Big]
\end{aligned} \tag{3.44}$$

which can be substituted along with (3.44) and (3.43) into Eq (3.41) to obtain

$$\begin{aligned}
\psi(x,z) \sim{}& \mathrm{e}^{-\alpha^2x^2}\mathrm{e}^{-ik_0x}\Big(\frac{\tilde{\phi}(-k_0,z)}{\tilde{\phi}(-k_0,0)}+2\alpha^2 ix\frac{\partial}{\partial k}\left(\frac{\tilde{\phi}(k,z)}{\tilde{\phi}(k,0)}\right)\Big|_{-k_0,z}\Big)\\
&+\mathrm{e}^{-\alpha^2x^2}\mathrm{e}^{ik_0x}\Big(\frac{\phi(k_0,z)}{\phi(k_0,0)}+2\alpha^2 ix\sqrt{\pi}\frac{\partial}{\partial k}\left(\frac{\phi(k,z)}{\phi(k,0)}\right)\Big|_{k_0,z}\Big)\\
&+\frac{\gamma_1 g\mathrm{e}^{-a_1^2x^2}\,\mathrm{e}^{\frac{z_{cr}}{h}}}{2i\beta\,\bar{u}'_{z_{cr}}}\left[(z-z_{cr})^{\frac{1}{2}+i\beta}\left(\frac{(-z_{cr})^{\frac{1}{2}-i\beta}}{\left(\frac{1}{2}-i\beta\right)}+\frac{(-z_{cr})^{\frac{3}{2}-i\beta}}{h\left(\frac{3}{2}-i\beta\right)}-(2b_1^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2\left(\frac{5}{2}-i\beta\right)}\right)\right.\\
&+(z-z_{cr})^{\frac{1}{2}-i\beta}\left(-\frac{(-z_{cr})^{\frac{1}{2}+i\beta}}{\left(\frac{1}{2}+i\beta\right)}-\frac{(-z_{cr})^{\frac{3}{2}+i\beta}}{h\left(\frac{3}{2}+i\beta\right)}+(2b_1^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2\left(\frac{5}{2}+i\beta\right)}\right)\\
&\left.-\frac{8i\beta(z-z_{cr})}{(4\beta^2+1)}+\frac{8i\beta(z-z_{cr})^2}{h(4\beta^2+9)}+(2b_1^2h^2-1)\frac{(6i\beta-5)(z-z_{cr})^3}{h^2(4\beta^2+25)}\right]\\
&+\frac{\gamma_2 g\mathrm{e}^{-a_2^2x^2}\,\mathrm{e}^{\frac{z_{cr}}{h}}}{2i\beta\,\bar{u}'_{z_{cr}}}\left[(z-z_{cr})^{\frac{1}{2}+i\beta}\left(\frac{(-z_{cr})^{\frac{1}{2}-i\beta}}{\left(\frac{1}{2}-i\beta\right)}+\frac{(-z_{cr})^{\frac{3}{2}-i\beta}}{h\left(\frac{3}{2}-i\beta\right)}-(2b_2^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}-i\beta}}{2h^2\left(\frac{5}{2}-i\beta\right)}\right)\right.\\
&+(z-z_{cr})^{\frac{1}{2}-i\beta}\left(-\frac{(-z_{cr})^{\frac{1}{2}+i\beta}}{\left(\frac{1}{2}+i\beta\right)}-\frac{(-z_{cr})^{\frac{3}{2}+i\beta}}{h\left(\frac{3}{2}+i\beta\right)}+(2b_2^2h^2-1)\frac{(-z_{cr})^{\frac{5}{2}+i\beta}}{2h^2\left(\frac{5}{2}+i\beta\right)}\right)\\
&\left.-\frac{8i\beta(z-z_{cr})}{(4\beta^2+1)}+\frac{8i\beta(z-z_{cr})^2}{h(4\beta^2+9)}+(2b_2^2h^2-1)\frac{(6i\beta-5)(z-z_{cr})^3}{h^2(4\beta^2+25)}\right]
\end{aligned} \tag{3.45}$$

as $z \to z_{cr}$. Equation (3.45) represents an asymptotic approximation to the linearized gravity waves affected by two chemicals. In this nonconstant mean flow case where we have a critical level at z_{cr}, we need to investigate the behavior of the solution $\psi(x,z)$ as waves propagate across the critical level. Studying the expressions $(z-z_{cr})^{\frac{1}{2}\pm i\beta}$ included in the homogeneous and particular parts of the solution (3.45), we recall that

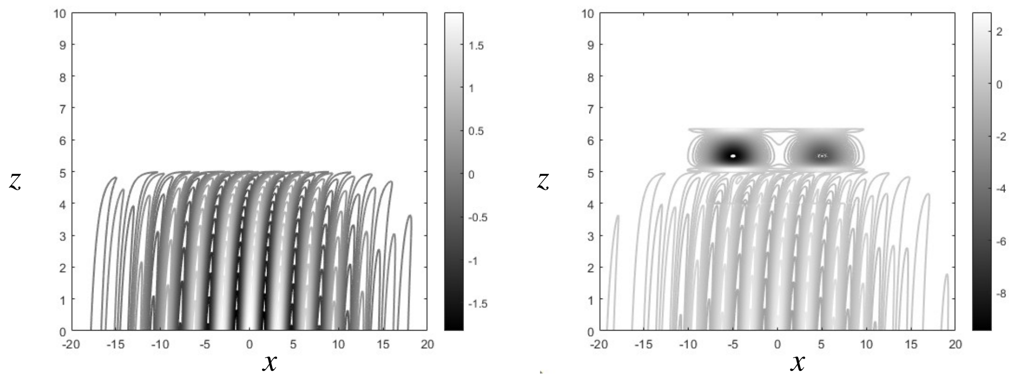
$$(z - z_{cr})^{\frac{1}{2}+i\beta} = \begin{cases} (z - z_{cr})^{\frac{1}{2}} \mathrm{e}^{i\beta \log (z-z_{cr})}, & \text{when } z > z_{cr} \\ i|z - z_{cr}|^{\frac{1}{2}} \mathrm{e}^{i\beta(\log |z-z_{cr}| \pm i\pi)}, & \text{when } z < z_{cr}\,. \end{cases} \tag{3.46}$$

Based on the direction of wave propagation and following the discussion given by [33], the branch $(-i\pi)$ is considered. A similar approach can be applied to $(z - z_{cr})^{\frac{1}{2}-i\beta}$.

Therefore, taking into account the representation shown in (3.46) and looking at the first two lines of (3.45) representing the homogeneous solution , we observe that the amplitude of the waves is reduced by a factor of $\mathrm{e}^{-2\pi\beta}$ as they propagate across the critical layer. This aligns with the results obtained by [10], where they studied the homogeneous Taylor–Goldstein equation and concluded that the magnitude of the waves is diminished by $\mathrm{e}^{-2\pi\sqrt{\mathrm{Ri}-\frac{1}{4}}}$ as they propagate through the critical layer.

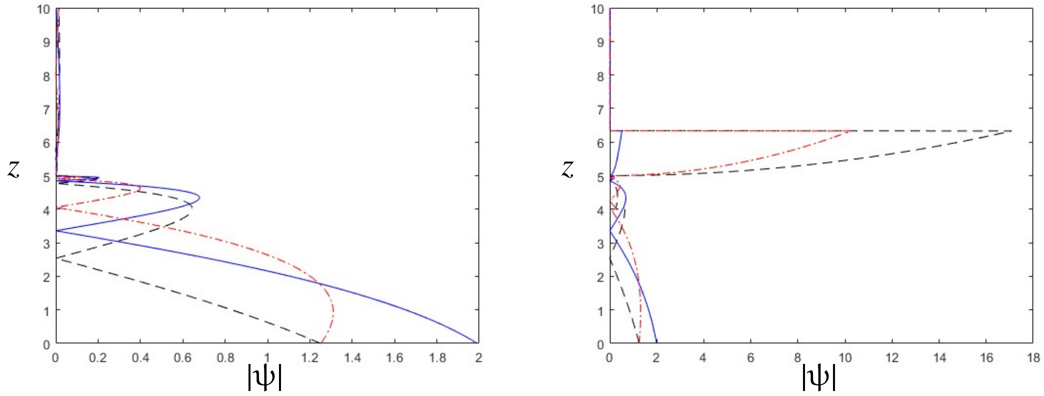
The interaction of the two chemicals clearly has an impact on gravity waves as can be seen in (3.45). Analyzing the part corresponding to the particular solution in (3.45), we also find that the wave reduction behavior occurs in the particular case. This indicates that despite the influence of the two chemicals on gravity waves, gravity waves still preserve this wave reduction around the critical level as they eventually get absorbed at the critical layer.

The upcoming figures are produced with the same set of parameters as in the previous subsection except that the background mean flow is considered as $\bar{u} = \tanh(z - 5)$, so we have a critical level at $z = 5$. Figure 4 illustrates the stream function $\psi(x, z)$ given in Eq (3.45), with a minor modification corresponding to the change in the horizontal localization of chemicals $(x - x_1)$ and $(x - x_2)$. The upper part of this figure corresponds to $\psi(x, z)$ the homogeneous solution without chemical effect. Part (b) of the figure shows the total solution (3.45) which represents the stream function $\psi(x, z)$ with the chemical effect. We note that the heating released by the chemicals has a great impact on gravity waves, especially around the critical layer. This impact is more clarified in Figure 5, which shows the magnitudes of $\psi(x, z)$ at specific horizontal points. Both figures illustrate the wave-reduction behavior of gravity waves as they cross the critical layer.



(a) (b)

Figure 4. The contour lines of the stream function $\psi(x, z)$, where, (a) represents the stream function $\psi(x, z)$ with no chemical effects (homogeneous solution) and (b) represents $\psi(x, z)$ with the chemical effect given by (3.45). These chemical effects on waves can be seen in (b) in the vicinity where the chemicals are centered. Due to the presence of critical level, wave reduction behavior is seen around the critical layer as in (a).



(a) **(b)**

Figure 5. This figure shows $|\psi(x, z)|$ of solution (3.45) at $x = -5$ (dashed line), $x = 0$ (solid line), and $x = 5$ (the dashed-dotted line), where (a) represents these magnitudes with no chemical effects (homogeneous solution) and (b) represents $|\psi(x, z)|$ (of (3.45)) with the two chemical effect at the same horizontal points given in (a).

4. Conclusions

In this work, we investigated the analytical solutions for the linearized equations of gravity waves influenced by two localized chemicals. We started our investigations with a constant mean flow; and then considered the more general case with a nonconstant mean flow.

In both cases, we found that the effect of the chemical heating induced by the two chemicals is clearly evident. In the case where $\bar{u}$ is constant, there is no critical layer where gravity waves propagate upward, and the impact of the two chemicals can be observed mainly in the vicinity of localized chemicals.

In the nonconstant mean flow case and with the impact of the two chemicals, gravity waves were found to have a similar behavior of wave reduction through the critical level as found in previous studies [10]. Asymptotic solutions of linearized equations showed that the amplitude of gravity waves is reduced by a factor of $e^{-2\pi\beta}$ as the waves travel throughout the critical level. It was found from the particular solution of the modified Taylor–Goldstein equation that gravity heating induced by the two chemicals has a significant impact on the gravity waves, yet the amplitude of gravity waves is reduced as waves cross the critical level.

In both cases of mean flow, we found that the wave amplitude is effectively modified by the heating released by the two chemicals. This can be clearly seen in the vicinity of the two chemicals. In the nonconstant mean flow case, the wave amplitude is greatly influenced around the critical level. In both cases, there is a higher impact on the wave amplitude when the heating parameter is greater, as can be seen in the figures, where the effect is greater around the chemical that produces more heating.

The heating released by the chemicals varies from one chemical to another. For example, ozone is a major absorber of solar radiation in the middle atmosphere, and hence releases more heating compared to water vapor. The mixing ratio and localization of the chemicals are also important factors that need to be taken into consideration when comparing the effect of the two chemicals on gravity waves.

The findings of this work can be considered with specified chemicals in the atmosphere. As an example of potential chemicals, ozone and water vapor can be considered to study their effect on gravity waves. Numerical simulations of the effect of these chemicals on gravity waves, as well as the effect of gravity waves on the chemicals, are illustrated in [24].

Our analytical approximations for the solutions of linearized gravity waves affected by chemicals found in this work align with the numerical solutions of the full model of gravity waves affected by chemicals given in [24]. These analytical approximations confirm the corresponding results in [24], where we find that gravity waves are greatly affected by the chemicals; and wave-amplitude reduction behavior occurs at the critical layer as in the early time cases of the full model.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest

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