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Research article

Exploring dynamics in RLC circuits: a novel approach utilizing the (k, ϕ)-Hilfer proportional fractional operator

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Abstract: In this study, the theory of fractional calculus is applied to the electrical circuits. In this work, we investigated the Langevin-type differential equations under the (k, ϕ)-Hilfer proportional fractional derivative. By utilizing the bivariate Mittag-Leffler function and the ψ-Laplace transform, we designed a representation of an explicit analytical solution for the linear system corresponding to the considered model. We explored Ulam–Hyers stability results with the Mittag-Leffler function and their generalizations to confirm Ulam stability by applying the extended Gronwall inequality under the context of the (k, ϕ)-proportional fractional operators. Finally, the RLC electrical circuit model was chosen as the application's agent to validate the accuracy of our theoretical results. Our results offer additional analytical choices due to the wider range of parameter values compared to earlier studies.

Keywords: electrical circuit; fractional differential equation; ϕ-Laplace transform; fractional RLC circuit; (k, ϕ)-Hilfer proportional derivative
Mathematics Subject Classification: 00A71, 34A05, 34A08, 34B15, 47H10

1. Introduction

Our main goal of this work is to achieve a representation of solutions for the Langevin-type differential equations under the (k, ϕ)-Hilfer proportional fractional derivative operator ((k, ϕ)-Hilfer-

PFDO) as follows:

$$\begin{cases} {}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}v(t) - \lambda\, {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t) = \mu\, v(t) + g(t, v(t)), \quad 0 < \rho \leq 1, \quad a < t \leq b, \\ \lim\limits_{t\to a^+} {}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}v(t) = c_0, \quad \lim\limits_{t\to a^+} {}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t)\Big) = c_1, \end{cases} \tag{1.1}$$

where ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}$ is the (k,ϕ)-Hilfer-PFDO of order $1/2 < \alpha \leq 1$, and type $0 \leq \beta \leq 1$, $k > 0$, $g \in C([a,b] \times \mathbb{R}, \mathbb{R})$, where $0 \leq a \leq t \leq b < \infty$, and $c_i \in \mathbb{R}$ for $i = 0, 1$. The proposed model (1.1) was created to fill the gap of research that has never been done and was inspired by the following stories: Calculus with non-integer-order (Fractional Calculus or FC) is powerful knowledge extended from classical calculus. It involves the concepts of integration and differentiation of non-integer orders, in contrast to classical calculus, where the order is always an integer. In the last few decades, FC has gained a lot of interest from researchers since it offers greater degrees of freedom compared to the integer order. It can express memory and hereditary features, resulting in more accurate and practical solutions. Many researchers have contributed to and published extensive literature across applied sciences, engineering disciplines, and related fields. Fractional derivative operators (FDOs) are defined differently, each with its own properties and characteristics. The most common FDOs depend on the Euler-type gamma function, including the Riemann–Liouville (RL) and Caputo operators, whose foundations are presented in [1, 2]. Other classical developments, such as the Hadamard and Katugampola operators, can be found in [3, 4]. Further systematic treatments are provided by Diethelm [5], Zhou [6], and Hilfer [7]. In 2012, the k-RL fractional integral operator (FIO) was introduced in [8] by helping the Euler-type k-gamma function, while the k-RL-FDO was studied by the researchers in [9] in 2015. Moreover, the ϕ-RL fractional integral and derivative operators [2] were studied by several researchers. Next, the (k,ϕ)-RL fractional integral and derivative operators were initiated by the researchers in [9, 10]. The Hilfer-type fractional derivative operator was constructed in [7], which is the generalized the sense for RL and Caputo fraction derivative operators. In 2018, the ϕ-Hilfer-FDO was constructed in [11]. The ϕ-proportional-FDO and the ϕ-proportional-FIO were introduced in [12, 13]. In 2012, the (k,ϕ)-Hilfer-FDO was investigated by the researchers in [14]. The (k,ϕ)-Hilfer-PFDO was formed by combining two operators of the (k,ϕ)-RL proportional and (k,ϕ)-Caputo proportional derivative operators. The (k,ϕ)-Hilfer-PFDO is an example of a broad fractional derivative framework, separated by the parameter values β, ρ, k, and another function $\phi(t)$ (see [15, 16] for more details). Of all the operators mentioned above, we can call them the generalized operators of fractional operators or *generalized fractional operators.* The first benefit of fractional derivatives is that they take memory into account. The memory effect is an essential characteristic of a differential equation. Another advantage is that fractional derivatives created a large number of diffusion processes. For some applications of fractional derivative operators, see [17–19] and references therein.

Merging differential equations with fractional order, so-called fractional differential equations (FDEs), can be encountered in many great works designed to understand the interplay of FDEs dealing with initial/boundary value problems in the context of various FDOs. Researchers can practically apply fixed-point theorems and numerical methods to analyze and solve these equations, which have been increasingly studied for their qualitative properties, like the existence and stability results. A Langevin equation, introduced by Paul Langevin, is the well-known differential equation that finds practical application in describing Brownian motion. This motion, the random movement of a small particle in a fluid, is a key concept in physics and scientific phenomena [20]. The classical Langevin equation is defined by

$$m\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}=(2k_bT\gamma)^{\frac{1}{2}}\xi(t),$$

where m denotes the particle mass, γ denotes the friction coefficient, k_b denotes the Boltzmann constant, T denotes the temperature, $\xi(t)$ denotes the Gaussian white noise with $(\mu,\sigma^2)=(0,1)$, and $x(t)$ denotes the position of a particle. Moreover, a fractional Langevin equation extends the classical Langevin equation by using a FDO to account for a stochastic differential equation that describes the motion of a particle undergoing random fluctuations incorporated with memory impacts. Due to the complexity of this equation and the difficulty in obtaining analytical solutions, several powerful numerical methods have been developed and are commonly used for their simulation and analysis. For example, in 2017, Li et al. [21] used the fixed-point theory of Leray-Schauder's types to analyze a class of two fractional-orders Langevin equations. In 2019, Baghani and Nieto [22] established sufficient conditions for the existence and uniqueness of the results of a two fractional-order Langevin equation. In [23], Aydin and Mahmudov studied and analyzed the sequential conformable derivative of the Langevin-type differential equation. They presented a representation of a solution by the conformable Laplace transform techniques. The existence and uniqueness of solutions are investigated. For more modern works, we offer [24–26] and references therein.

Research trends on the existence results of the fractional differential system are focusing increasingly on practical applications to achieve effective results for real-world problems in various fields, including engineering, mathematics, physics, electrical, finance, and medicine. Numerous pieces of research employ applications for real-life data in many types of issues, such as disease transmission, financial crisis, blood ethanol concentration, drug concentration levels in the blood, and RLC electric circuits. For example, Aydin and Mahmudov [23] studied the Langevin-type differential equation under conformable derivatives and used LRC circuits as an application. Norouzi and N'Guérékata [27] applied the financial crisis model for the ϕ-Hilfer fractional differential equation. Prakash et al. [28] analyzed the model of blood alcohol concentration corresponding to the fractional system under singular and non-singular derivatives. Hatime et al. [29] utilized pharmacokinetics in the case of drug concentration prediction in plasma clinical trials as an application example of their analysis for the generalized fractional form of Newton's Cooling Law. The solutions of fractional nonlinear electrical transmission lines and the perturbed nonlinear Schroedinger equation with the Kerr law nonlinearity term were determined by Fendzi-DonfackWe et al. [30]. Later, they [31] studied the soliton solutions for an intrinsic fractional discrete nonlinear electrical transmission lattice. Over the years, Ulam stability has become an effective instrument in stability strategies. It ensures robustness of solutions to functional equations by assuring that approximate solutions are close to exact solutions. Several varieties of Ulam stability and their applications have been studied, each corresponding to a distinct sort of functional equation under various conditions, like as Ulam-Hyers (UH) stability [32–34], Ulam-Hyers-Rassias (UHR) stability [35, 36], Ulam-Hyers-Mittag-Leffler (UHML) stability [37], Ulam-Hyers-Rassias-Mittag-Leffler (UHRML) stability [38], and their generalization. In addition, many researchers employ the Gronwall inequality to analyze Ulam stability . It is an effective method to provide bounds on functions, especially differential and integral equations. Some researchers have presented work on the fractional differential equations under fractional operators and utilize the Gronwall inequality to establish the Ulam stability [39–41].

One explanation is because a fractional order permits the system to display appropriate degrees of freedom, which contributes to the *memory* effect seen in physical systems. Furthermore, the fractional-order derivative demonstrates superior accuracy and efficiency in simulating real-world problems. This is especially true in models where memory or the features of genetic qualities play a significant role or are stressed. In this regard, it has been demonstrated that fractional calculus offers the benefit of assisting in accurately modeling natural processes. Revisiting the motivation, the main contribution of this novelty may be summarized as follows:

(1) We apply the bivariate Mittag-Leffler function to bring novel properties to the ϕ-Laplace transform.
(2) A description of the solutions for the linear problem in relation to the nonlinear problem is offered.
(3) The fixed-point theorem has established the existence, uniqueness, and stability of the solution of the (k,ϕ)-Hilfer proportional differential equation for the Langevin-type model.
(4) RLC circuit models are redesigned as an application to customize our problem.

This work is structured as follows: We provide some definitions and some materials of the (k,ϕ)-PFOs in Section 2. We give the necessary lemmas used throughout this work and analyze an generalized Gronwall inequality under the mentioned operator. In addition, the definition of ϕ-Laplace transform and their properties, as well as the Mittag-Leffler functions, are also expressed in this section. In Section 3, we investigate a representation of solutions of the linear system associated with the nonlinear one using the ϕ-Laplace transform. In Section 4, we establish the major theories, separated into two subsections. First, we analyze the uniqueness result by utilizing the Banach's contraction mapping principle. Second, we examine the UHRML stability and its generalization of solutions. In Section 5, we look into the numerical applications to confirm the precision of our findings. One example illustrates the representation of the proposed system (1.1). The other one is provided by modifying the proposed system (1.1), also known as RLC circuits, to customize our system. The overall results of our discussion are concluded in the last section.

2. Preliminaries

Assume $C([a,b],\mathbb{R})$ and $C^2([a,b],\mathbb{R})$ are Banach spaces of all continuous function v on $[a,b]$ equipped with the norm $\|v\| = \sup_{t\in[a,b]}\{|v(t)|\}$. The space $\mathcal{AC}^n$ is the n-times absolutely continuous function v on $[a,b]$, which is provided by $\mathcal{AC}^n := \mathcal{AC}^n([a,b],\mathbb{R}) = \{v : [a,b] \to \mathbb{R}; v^{(n-1)} \in \mathcal{AC}([a,b],\mathbb{R})\}$, and $L^q([a,b],\mathbb{R})$ is a Banach space of all Lebesgue measurable $f : [a,b] \to \mathbb{R}$ equipped with $\|f\|_{L^q} < +\infty$. Let $\phi : [a,b] \to \mathbb{R}$ be a strictly increasing continuous function via $\phi'(t) \neq 0$. To facilitate calculations in this work, we have provided the following symbol for your convenience:

$$ {}^{\rho}_{k}\Psi^{\frac{\alpha}{k}-1}_{\phi}(t,s) := (\phi(t)-\phi(s))^{\frac{\alpha}{k}-1}\, e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(s))}. \tag{2.1} $$

Now, we present some definitions and materials corresponding to the (k,ϕ)-proportional fractional derivative and integral operators ((k,ϕ)-PFDO/(k,ϕ)-PFIO), which will be employed throughout this study.

Definition 2.1 ([15]). *Assume $\alpha > 0$, $k > 0$, $0 < \rho \leq 1$, and $g \in L^1([a,b],\mathbb{R})$. The (k,ϕ)-RL-PFIO of* *order α of g is given by*

$$ {}_{a,k}\mathcal{I}^{\alpha,\rho;\phi}g(t) := \frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^t {}^{\rho}_{k}\Psi_{\phi}^{\frac{\alpha}{k}-1}(t,s)g(s)\phi'(s)ds, \quad \Gamma_k(\alpha) := \int_0^{\infty} s^{\alpha-1}e^{-s^k/k}ds, $$

where $\Gamma(z) = \lim_{k\to 1}\Gamma_k(z)$, $\Gamma_k(k) = 1$, $\Gamma_k(z+k) = z\Gamma_k(z)$ *and* $\Gamma_k(z) = (k)^{z/k-1}\Gamma(z/k)$.

Definition 2.2 ([15]). *Assume* $\alpha > 0$, $k > 0$, $0 < \rho \le 1$, $g \in C([a,b],\mathbb{R})$, $\phi(t) \in C^n([a,b],\mathbb{R})$ *via* $\phi'(t) \ne 0$, *and* $n \in \mathbb{N}$ *so that* $n = \lfloor \alpha/k \rfloor + 1$. *The* (k,ϕ)*-RL-PFDO of* α *of* g *is given by,*

$$ {}^{\mathbb{RL}}_{a,k}\mathfrak{D}^{\alpha,\rho;\phi}g(t) := {}_{k}\mathfrak{D}^{n,\rho;\phi}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\phi}g(t)\Big) := \frac{{}_{k}\mathfrak{D}^{n,\rho;\phi}}{\rho^{\frac{nk-\alpha}{k}}k\Gamma_k(nk-\alpha)}\int_a^t {}^{\rho}_{k}\Psi_{\phi}^{\frac{nk-\alpha}{k}-1}(t,s)g(s)\phi'(t)ds, $$

where ${}_{k}\mathfrak{D}^{n,\rho;\phi} := \underbrace{{}_{k}\mathfrak{D}^{\rho;\phi}{}_{k}\mathfrak{D}^{\rho;\phi}\cdots{}_{k}\mathfrak{D}^{\rho;\phi}}_{n\ time}$ *and* ${}_{k}\mathfrak{D}^{1,\rho;\phi}g(t) := {}_{k}\mathfrak{D}^{\rho;\phi}g(t) := (1-\rho)g(t) + k\rho g'(t)/\phi'(t)$.

Definition 2.3 ([15]). *Assume* $\alpha > 0$, $k > 0$, $0 < \rho \le 1$, $g \in C^n([a,b],\mathbb{R})$, $\phi(t) \in C^n([a,b],\mathbb{R})$ *with* $\phi'(t) \ne 0$, *and* $n \in \mathbb{N}$ *so that* $n = \lfloor \alpha/k \rfloor + 1$. *The* (k,ϕ)*-Caputo-PFDO of* α *of* g *is given by*

$$ {}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\phi}g(t) := {}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\phi}\Big({}_{k}\mathfrak{D}^{n,\rho;\phi}g(t)\Big) := \frac{1}{\rho^{\frac{nk-\alpha}{k}}k\Gamma_k(nk-\alpha)}\int_a^t {}^{\rho}_{k}\Psi_{\phi}^{\frac{nk-\alpha}{k}-1}(t,s)\Big({}_{k}\mathfrak{D}^{n,\rho;\phi}g(s)\Big)\phi'(s)ds. $$

Definition 2.4 ([15]). *Assume* $\alpha > 0$, $k > 0$, $0 < \rho \le 1$, $0 \le \beta \le 1$, $g \in C^n([a,b],\mathbb{R})$, $\phi(t) \in C^n([a,b],\mathbb{R})$ *with* $\phi'(t) \ne 0$, *and* $n \in \mathbb{N}$ *so that* $n = \lfloor \alpha/k \rfloor + 1$. *The* (k,ϕ)*-Hilfer-PFDO of* α *and* β *of* g *is given by*

$$ {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}g(t) := {}_{a,k}\mathcal{I}^{\beta(nk-\alpha),\rho;\phi}\Big({}_{k}\mathfrak{D}^{n,\rho;\phi}\Big({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\phi}g(t)\Big)\Big). $$

Next, we list some key properties used in this work.

Lemma 2.5 ([15]). *Assume* $\alpha \ge 0$, $\delta \ge 0$, $k > 0$, $\eta > 0$, $0 < \rho \le 1$, $\omega \in \mathbb{R}$, $\omega/k > -1$, *and* $n = \lfloor \omega/k \rfloor + 1$. *We have*

(i) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\phi}\Big[{}^{\rho}_{k}\Psi_{\phi}^{\frac{\omega}{k}-1}(t,a)\Big] = \frac{\Gamma_k(\omega)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega+\alpha)}\,{}^{\rho}_{k}\Psi_{\phi}^{\frac{\omega+\alpha}{k}-1}(t,a)$.

(ii) ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}\Big[{}^{\rho}_{k}\Psi_{\phi}^{\frac{\omega}{k}-1}(t,a)\Big] = \frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\,{}^{\rho}_{k}\Psi_{\phi}^{\frac{\omega-\alpha}{k}-1}(t,a)$. *In particular, for any* $m = 0,1,\ldots,n-1$, *we obtain* ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}\Big[{}^{\rho}_{k}\Psi_{\phi}^{m}(t,a)\Big] = 0$.

(iii) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\phi}\Big({}_{a,k}\mathcal{I}^{\delta,\rho;\phi}g(t)\Big) = {}_{a,k}\mathcal{I}^{\delta+\alpha,\rho;\phi}g(t) = {}_{a,k}\mathcal{I}^{\delta,\rho;\phi}\Big({}_{a,k}\mathcal{I}^{\alpha,\rho;\phi}g(t)\Big)$.

(iv) ${}^{H}_{a,k}\mathfrak{D}^{\omega,\beta,\rho;\phi}\Big({}_{a,k}\mathcal{I}^{\eta,\rho;\phi}g(t)\Big) = {}_{a,k}\mathcal{I}^{\eta-\omega,\rho;\phi}g(t)$, *where* $n = \lfloor \omega/k \rfloor + 1$, $\eta > nk$.

(v) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}g(t)\Big) = g(t) - \sum_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{\eta}{k}-i}(t,a)}{\rho^{\frac{\eta-ki}{k}}\Gamma_k(\eta+k-ki)}\Big[{}_{k}\mathfrak{D}^{n-i,\rho;\phi}\Big({}_{a,k}\mathcal{I}^{nk-\eta,\rho;\phi}g(a^+)\Big)\Big]$, $\eta = \alpha + \beta(nk-\alpha)$.

A generalized Gronwall inequality under the (k,ϕ)-PFDO and the (k,ϕ)-PFIO and its properties are provided.

Theorem 2.6 ([16]). *Assume* $\alpha > 0$, $k > 0$, $0 < \rho \le 1$, *and* $\phi \in C^1([a,b],\mathbb{R})$ *is an increasing function so that* $\phi'(t) \ne 0$ *for any* $t \in [a,b]$. *Let the following assumptions true:*

(H_1) *The two non-negative functions $x(t)$ and $y(t)$ are locally integrable on $[a,b]$;*
(H_2) *A function $z(t)$ is the non-negative, non-decreasing, and continuous functions provided on $[a,b]$ so that $z(t) \leq z^*$, where z^* is a real number.*

If

$$x(t) \leq y(t) + \frac{\Gamma_k(\alpha)}{k} z(t)_{a,k}\mathcal{I}^{\alpha,\rho;\phi} x(t), \tag{2.2}$$

then,

$$x(t) \leq y(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{[\Gamma_k(\alpha) z(t)]^n}{\rho^{\frac{n\alpha}{k}} k^{n+1} \Gamma_k(n\alpha)} {}^{\rho}_{k}\Psi_{\phi}^{\frac{n\alpha}{k}-1}(t,s) y(s) \right] \phi'(s) ds. \tag{2.3}$$

Corollary 2.7 ([16]). *Under the assumptions in Theorem* 2.6*, assume that $y(t)$ is a non-decreasing function on $t \in [a,b]$. Then,*

$$x(t) \leq y(t)\, \mathbb{E}_{k,\alpha,k} \left((\rho^{\frac{\alpha}{k}} k)^{-1} \Gamma_k(\alpha) z(t) \left(\phi(t) - \phi(s)\right)^{\frac{\alpha}{k}} \right), \tag{2.4}$$

where

$$\mathbb{E}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(n\alpha + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad Re(\alpha),\ k > 0. \tag{2.5}$$

Next, we recall some details the ϕ-Laplace transform and its materials that will be used in this work.

Definition 2.8 ([42]). *Assume $f : [a,\infty) \to \mathbb{R}$ and $\phi \in C([a,\infty),\mathbb{R})$ under $\phi'(t) > 0$. Then, the ϕ-Laplace transform of f is provided as*

$$\mathcal{L}_{\phi}\{f(t)\} = \int_a^{\infty} e^{-s(\phi(s)-\phi(a))} f(t) \phi'(t) dt, \quad \forall s \leq t, \tag{2.6}$$

where the right-sided of (2.6) *is valid.*

Definition 2.9 ([42]). *A function $f : [0,\infty) \to \mathbb{R}$ is said to be the ϕ-exponential-order if there are three constants $p \geq 0$, $M \geq 0$, $b \geq 0$ such that $|f(t)| \leq M \exp\left(p\phi(t)\right)$ for all $t \geq b$.*

The basic properties of Definition 2.8 were given as in Lemma 2.10.

Lemma 2.10 ([42]). *Let $\alpha > 0$, $\beta > 0$ and $|\lambda / s^{\alpha}| < 1$. Then,*

(i) $\mathcal{L}_{\phi}\{1\} = \frac{1}{s}$, $s > 0$.
(ii) $\mathcal{L}_{\phi}\left\{(\phi(t) - \phi(a))^{\beta}\right\} = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$, $s > 0$.
(iii) $\mathcal{L}_{\phi}\left\{e^{\lambda(\phi(t)-\phi(a))}\right\} = \frac{1}{s-\lambda}$, $s > \lambda$.
(iv) $\mathcal{L}_{\phi}\left\{e^{\lambda(\phi(t)-\phi(a))} g(t)\right\} = \mathcal{L}_{\phi}\{g(t)\}(s-\lambda)$.
(v) $\mathcal{L}_{\phi}\left\{(\phi(t) - \phi(a))^{\beta-1} \mathbb{E}_{\alpha,\beta}\left(\lambda(\phi(t) - \phi(a))^{\alpha}\right)\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}$ *where* $\mathbb{E}_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha+\beta)}$.

The ϕ–convolution integral operator of two functions is provided as in Definition 2.11:

Definition 2.11. *([42]). Assume that f and g are two piece-wise continuous functions on $[a,b]$ and of ϕ-exponential-order. Then, the ϕ–convolution of f and g is provided by*

$$\left(f *_{\phi} g\right)(t) = \int_a^t f(s) g\left(\phi^{-1}\left(\phi(t) + \phi(a) - \phi(s)\right)\right) \phi'(s) ds. \tag{2.7}$$

The Eq (2.7) *has a property* $(f *_\phi g)(t) = (g *_\phi f)(t)$. *In addition, we obtain*

$$\mathcal{L}_\phi\{f(t) *_\phi g(t)\} = \mathcal{L}_\phi\{f(t)\}\mathcal{L}_\phi\{g(t)\}.$$

Now, we prove the ϕ–Laplace transform of the (k,ϕ)-Hilfer-PFDO in Lemma 2.12.

Lemma 2.12. *Assume* $0 < \alpha \le 1$, $k > 0$, $0 < \rho \le 1$, $0 \le \beta \le 1$, $n \in \mathbb{N}$, $f \in \mathcal{AC}^n_{\alpha,\phi}([a,b],\mathbb{R})$, $\phi \in C^n([a,b],\mathbb{R})$ *such that* $\phi'(t) > 0$ *and* ${}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}f$ *is of* ϕ*-exponential-order. Then*

$$\begin{aligned}\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{n\alpha,\beta,\rho;\phi}f(t)\right\} &= [q(s)]^{\frac{n\alpha}{k}}\,\mathcal{L}_\phi\{f(t)\} - k\rho\sum_{i=0}^{n-1}[q(s)]^{\frac{\alpha(\beta+i)}{k}-\beta}\\ &\quad\times\Big[{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{\alpha(n-1-i),\beta,\rho;\phi}f(a^+)\Big)\Big], \qquad (2.8)\end{aligned}$$

where $q(s) = 1-\rho+k\rho s$. *Particularly, if* $n = 2$, *we achieve the following result*

$$\begin{aligned}\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}f(t)\right\} &= [q(s)]^{\frac{2\alpha}{k}}\,\mathcal{L}_\phi\{f(t)\} - k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}\Big[{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}f(a^+)\Big)\Big]\\ &\quad -k\rho[q(s)]^{\frac{\alpha(\beta+1)}{k}-\beta}\Big[{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}f(a^+)\Big]. \qquad (2.9)\end{aligned}$$

Proof. We are going to prove this lemma by utilizing mathematical induction technique. Assume that an Eq (2.8) holds for $n = 1$, it follows that

$$\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}f(t)\right\} = [q(s)]^{\frac{\alpha}{k}}\,\mathcal{L}_\phi\{f(t)\} - k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}\left[{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}f(a^+)\right]. \qquad (2.10)$$

Assume that an Eq (2.8) holds for $m \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{m\alpha,\beta,\rho;\phi}f(t)\right\} &= [q(s)]^{\frac{m\alpha}{k}}\,\mathcal{L}_\phi\{f(t)\} - k\rho\sum_{i=0}^{m-1}[q(s)]^{\frac{\alpha(\beta+i)}{k}-\beta}\\ &\quad\times\left[{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{\alpha(m-1-i),\beta,\rho;\phi}f(a^+)\Big)\right]. \qquad (2.11)\end{aligned}$$

Then,

$$\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{(m+1)\alpha,\beta,\rho;\phi}f(t)\right\} = \mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{m\alpha,\beta,\rho;\phi}f(t)\Big)\right\}. \qquad (2.12)$$

Using (2.10) and (2.11), the relation (2.12) can be obtained

$$\begin{aligned}\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{(m+1)\alpha,\beta,\rho;\phi}f(t)\right\} &= [q(s)]^{\frac{\alpha}{k}}\,\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{m\alpha,\beta,\rho;\phi}f(t)\right\}\\ &\quad -k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}\Big[{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{m\alpha,\beta,\rho;\phi}f(a^+)\Big)\Big]\\ &= [q(s)]^{\frac{(m+1)\alpha}{k}}\,\mathcal{L}_\phi\{f(t)\} - k\rho\sum_{i=0}^{m-1}[q(s)]^{\frac{\alpha(\beta+1+i)}{k}-\beta}\\ &\quad\times\Big[{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{\alpha(m-1-i),\beta,\rho;\phi}f(a^+)\Big)\Big]\\ &\quad -k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}\Big[{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{m\alpha,\beta,\rho;\phi}f(a^+)\Big)\Big]\end{aligned}$$

$$= \quad [q(s)]^{\frac{(m+1)\alpha}{k}} \mathcal{L}_{\phi}\{f(t)\} - k\rho \sum_{i=0}^{m} [q(s)]^{\frac{\alpha(\beta+i)}{k}-\beta} \times \Big[{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{\alpha(m-i),\beta,\rho;\phi} f(a^{+})\Big)\Big]. \tag{2.13}$$

Hence, by mathematical induction, Eq (2.8) holds. The proof is completed. □

Now, some important properties are analyzed in Lemma 2.13 that will be utilized in this work.

Lemma 2.13. *Let $\lambda, \mu \in \mathbb{R}$, $d > 0$, $[q(s)]^{\alpha/k} > 0$, $\left|\frac{\lambda}{[k\rho r(s)]^{\alpha/k}}\right| < 1$, and $\left|\frac{\mu}{[k\rho r(s)]^{2\alpha/k} - \lambda[k\rho r(s)]^{\alpha/k}}\right| < 1$. Then,*

$$\frac{1}{\left([q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}}\right)^{n+1}} = \sum_{m=0}^{\infty} \binom{n+m}{m} \frac{\lambda^m}{[k\rho r(s)]^{\frac{\alpha(2(n+1)+m)}{k}}}, \tag{2.14}$$

$$\frac{[q(s)]^{\frac{d\alpha}{k}}}{[q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+m}{m} \frac{\mu^n \lambda^m}{[k\rho r(s)]^{\frac{\alpha(2n+m+2-d)}{k}}}, \tag{2.15}$$

where

$$r(s) = s - \left(\frac{\rho - 1}{k\rho}\right) = \frac{q(s)}{k\rho} = \frac{1-\rho}{k\rho} + s \quad \textit{and} \quad q(s) = 1 - \rho + k\rho s. \tag{2.16}$$

Proof. First, we prove the relation (2.14). By using binomial theorem and $\left|\frac{\lambda}{[k\rho r(s)]^{\alpha/k}}\right| < 1$, we have

$$\begin{aligned} \frac{1}{\left([q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}}\right)^{n+1}} &= \frac{1}{\left([k\rho r(s)]^{\frac{2\alpha}{k}} - \lambda[k\rho r(s)]^{\frac{\alpha}{k}}\right)^{n+1}} \\ &= \frac{1}{[k\rho r(s)]^{\frac{2(n+1)\alpha}{k}}} \cdot \frac{1}{\left(1 - \frac{\lambda}{[k\rho r(s)]^{\frac{\alpha}{k}}}\right)^{n+1}} \\ &= \sum_{m=0}^{\infty} \binom{n+m}{m} \frac{\lambda^m}{[k\rho r(s)]^{\frac{\alpha(2(n+1)+m)}{k}}}. \end{aligned}$$

The relation (2.14) is achieved. Next, we prove the relation (2.15). By using geometric series with $\left|\frac{\mu}{[k\rho r(s)]^{2\alpha/k} - \lambda[k\rho r(s)]^{\alpha/k}}\right| < 1$ and $\left|\frac{\lambda}{[k\rho r(s)]^{\alpha/k}}\right| < 1$, which implies that

$$\begin{aligned} \frac{[q(s)]^{\frac{d\alpha}{k}}}{[q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu} &= \frac{[k\rho r(s)]^{\frac{\alpha d}{k}}}{[k\rho r(s)]^{\frac{2\alpha}{k}} - \lambda[k\rho r(s)]^{\frac{\alpha}{k}}} \cdot \frac{1}{1 - \frac{\mu}{[k\rho r(s)]^{\frac{2\alpha}{k}} - \lambda[k\rho r(s)]^{\frac{\alpha}{k}}}} \\ &= \sum_{n=0}^{\infty} \frac{[k\rho r(s)]^{\frac{\alpha d}{k}} \mu^n}{\left([k\rho r(s)]^{\frac{2\alpha}{k}} - \lambda[k\rho r(s)]^{\frac{\alpha}{k}}\right)^{n+1}} \\ &= \sum_{n=0}^{\infty} [k\rho r(s)]^{\frac{\alpha d}{k}} \mu^n \sum_{m=0}^{\infty} \binom{n+m}{m} \frac{\lambda^m}{[k\rho r(s)]^{\frac{\alpha(2(n+1)+m)}{k}}} \end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\frac{\mu^n\lambda^m}{[k\rho r(s)]^{\frac{\alpha(2n+m+2-d)}{k}}}.$$

The proof is completed. □

Lemma 2.14. *Let $\lambda,\mu\in\mathbb{R}$, $d>0$ and $(1-\rho+k\rho s)^{\alpha/k}>0$. Then,*

$$\begin{aligned}
&\mathcal{L}_\phi^{-1}\left\{\frac{[q(s)]^{\frac{d\alpha}{k}}}{[q(s)]^{\frac{2\alpha}{k}}-\lambda[q(s)]^{\frac{\alpha}{k}}-\mu}\right\}\\
=\ &(k\rho)^{\frac{\alpha}{k}(d-2)}{}_k^\rho\Psi_\phi^{\frac{\alpha(2-d)}{k}-1}(t,a)\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}(2-d)}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right), \qquad (2.17)\\
&\mathcal{L}_\phi^{-1}\left\{\frac{1}{[q(s)]^{\frac{2\alpha}{k}}-\lambda[q(s)]^{\frac{\alpha}{k}}-\mu}\mathcal{L}_\phi\{f(s)\}\right\}\\
=\ &(k\rho)^{-\frac{2\alpha}{k}}{}_k^\rho\Psi_\phi^{\frac{2\alpha}{k}-1}(t,a)\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)*_\phi f(t), \qquad (2.18)
\end{aligned}$$

where $q(s)$ and $r(s)$ are given in (2.16)*, and*

$$\mathbb{E}_{\alpha,\beta,\gamma}(u,v)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\frac{u^nv^m}{\Gamma(n\alpha+m\beta+\gamma)},\quad \alpha>0,\beta>0,\gamma>0,\quad u,v\in\mathbb{R}. \qquad (2.19)$$

Proof. Applying the relation (2.15), (2.19), and Lemma 2.10 yields that

$$\begin{aligned}
&\mathcal{L}_\phi^{-1}\left\{\frac{[q(s)]^{\frac{d\alpha}{k}}}{[q(s)]^{\frac{2\alpha}{k}}-\lambda[q(s)]^{\frac{\alpha}{k}}-\mu}\right\}\\
=\ &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\frac{\mu^n\lambda^m}{(k\rho)^{\frac{\alpha}{k}(2n+m+2-d)}}\mathcal{L}_\phi^{-1}\left\{\frac{1}{[r(s)]^{\frac{\alpha}{k}(2n+m+2-d)}}\right\}\\
=\ &e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(a))}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\frac{\mu^n\lambda^m}{(k\rho)^{\frac{\alpha}{k}(2n+m+2-d)}}\mathcal{L}_\phi^{-1}\left\{\frac{1}{s^{\frac{\alpha}{k}(2n+m+2-d)}}\right\}\\
=\ &\frac{{}_k^\rho\Psi_\phi^{\frac{\alpha(2-d)}{k}-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\frac{\mu^n\lambda^m(\phi(t)-\phi(a))^{\frac{\alpha}{k}(2n+m)}}{(k\rho)^{\frac{\alpha}{k}(2n+m)}\Gamma(\frac{\alpha}{k}(2n+m+2-d))}\\
=\ &\frac{{}_k^\rho\Psi_\phi^{\frac{\alpha(2-d)}{k}-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\frac{\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}\right)^n\left(\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)^m}{\Gamma(\frac{2\alpha}{k}n+\frac{\alpha}{k}m+\frac{\alpha}{k}(2-d))}\\
=\ &\frac{{}_k^\rho\Psi_\phi^{\frac{\alpha(2-d)}{k}-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}(2-d)}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right).
\end{aligned}$$

The relation (2.17) is obtained. Next, by taking $d = 0$ in the relation (2.17), we obtain

$$\mathcal{L}_{\phi}^{-1}\left\{\frac{1}{[q(s)]^{\frac{2\alpha}{k}}-\lambda[q(s)]^{\frac{\alpha}{k}}-\mu}\right\}=\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right). \tag{2.20}$$

Applying the relation (2.20), which yields that

$$\begin{aligned}&\mathcal{L}_{\phi}^{-1}\left\{\frac{1}{[q(s)]^{\frac{2\alpha}{k}}-\lambda[q(s)]^{\frac{\alpha}{k}}-\mu}\mathcal{L}_{\phi}\{f(s)\}\right\}\\ =\ &\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)*_{\phi}f(t).\end{aligned}$$

The relation (2.18) is achieved. □

3. Major results

3.1. Essential results

Lemma 3.1. *Assume that* $\mathcal{L}_{\phi}^{-1}\left\{{}_{a,k}^{H}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}v(t)\right\}$, $\mathcal{L}_{\phi}^{-1}\left\{{}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t)\right\}$, $\mathcal{L}_{\phi}^{-1}\{v(t)\}$, *and* $\mathcal{L}_{\phi}^{-1}\{h(t)\}$ *exist,* $1/2<\alpha\le 1$, $0\le\beta\le 1$, $0<\rho\le 1$, *and* $k>0$. *Then, the following linear initial value Langevin-type problem via the* (k,ϕ)*-Hilfer-PFDO as follows:*

$$\begin{cases}{}_{a,k}^{H}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}v(t)-\lambda\,{}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t)=\mu\,v(t)+h(t),\\ \lim\limits_{t\to a^{+}}{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}v(t)=c_{0},\quad \lim\limits_{t\to a^{+}}{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\left({}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t)\right)=c_{1},\end{cases} \tag{3.1}$$

where $c_i\in\mathbb{R}$ *for* $i=0,1$, *has the solution*

$$\begin{aligned}v(t)\ =\ &\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_{0}\\ &-\frac{\lambda{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_{0}\\ &+\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_{1}\\ &+\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)*_{\phi}h(t).\end{aligned} \tag{3.2}$$

Proof. By taking $\mathcal{L}_{\phi}$ to the first equation of (3.1), which implies that

$$\mathcal{L}_{\phi}\left\{{}_{a,k}^{H}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}v(t)\right\}-\lambda\mathcal{L}_{\phi}\left\{{}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t)\right\}=\mu\mathcal{L}_{\phi}\{v(t)\}+\mathcal{L}_{\phi}\{h(t)\}. \tag{3.3}$$

Applying Lemma 2.12, we have

$$\begin{aligned}\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}v(t)\right\} &= [q(s)]^{\frac{2\alpha}{k}}\mathcal{L}_\phi\{v(t)\} - k\rho[q(s)]^{\frac{\alpha(\beta+1)}{k}-\beta}c_0 - k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}c_1, \quad (3.4)\\ \mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t)\right\} &= [q(s)]^{\frac{\alpha}{k}}\mathcal{L}_\phi\{v(t)\} - k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}c_0, \quad (3.5)\end{aligned}$$

where $\lim\limits_{t\to a^+}{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}v(t) = c_0$ and $\lim\limits_{t\to a^+}{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\left({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t)\right) = c_1$.

Substituting (3.4)-(3.5) in (3.3), it follows form

$$\begin{aligned}\left([q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu\right)\mathcal{L}_\phi\{v(t)\} &= \left(k\rho[q(s)]^{\frac{\alpha(\beta+1)}{k}-\beta} - \lambda k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}\right)c_0\\ &\quad + k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}c_1 + \mathcal{L}_\phi\{h(t)\}.\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{L}_\phi\{v(t)\} &= \frac{[q(s)]^{\frac{\alpha(\beta+1)}{k}-\beta} - \lambda[q(s)]^{\frac{\alpha\beta}{k}-\beta}}{[q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu}k\rho c_0 + \frac{[q(s)]^{\frac{\alpha\beta}{k}-\beta}}{[q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu}k\rho c_1\\ &\quad + \frac{1}{[q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu}\mathcal{L}_\phi\{h(t)\}. \quad (3.6)\end{aligned}$$

Taking $\mathcal{L}_\phi^{-1}$ into the Eq (3.6), we get

$$\begin{aligned}v(t) &= \mathcal{L}_\phi^{-1}\left\{\frac{[q(s)]^{\frac{\alpha}{k}(\beta+1-\frac{k\beta}{\alpha})}}{[q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu}\right\}k\rho c_0 - \mathcal{L}_\phi^{-1}\left\{\frac{[q(s)]^{\frac{\alpha}{k}(\beta-\frac{k\beta}{\alpha})}}{[q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu}\right\}\lambda k\rho c_0\\ &\quad + \mathcal{L}_\phi^{-1}\left\{\frac{[q(s)]^{\frac{\alpha}{k}(\beta-\frac{k\beta}{\alpha})}}{[q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu}\right\}k\rho c_1 + \mathcal{L}_\phi^{-1}\left\{\frac{1}{[q(s)]^{\frac{2\alpha}{k}} - \lambda[q(s)]^{\frac{\alpha}{k}} - \mu}\mathcal{L}_\phi\{h(t)\}\right\}.\end{aligned}$$

Using the relations (2.17) and (2.18) in Lemma 2.14, the required result (3.2) is obtained. □

In light of Lemma 3.1, we give the operator $Q : C^2([a,b],\mathbb{R}) \to C^2([a,b],\mathbb{R})$ as follows:

$$\begin{aligned}(Qv)(t) &= \frac{{}^{\rho}_{k}\Psi_\phi^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}, \frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0\\ &\quad - \frac{\lambda{}^{\rho}_{k}\Psi_\phi^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}, \frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0\\ &\quad + \frac{{}^{\rho}_{k}\Psi_\phi^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}, \frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_1\\ &\quad + \int_a^t \frac{{}^{\rho}_{k}\Psi_\phi^{\frac{2\alpha}{k}-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}, \frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\end{aligned}$$

$$\times f\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v(s)\right)\phi'(s)ds. \tag{3.7}$$

For the ease of computation, we give some symbols

$$\mathcal{E}_{\mu,\lambda} := e^{\frac{|\mu|(\phi(b)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}+\frac{|\lambda|(\phi(b)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}}, \tag{3.8}$$

$$\Lambda_i := \frac{(\phi(b)-\phi(a))^{\frac{i\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}}{(k\rho)^{\frac{i\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}, \quad i=1,2, \tag{3.9}$$

$$\Lambda_3 := \frac{(\phi(b)-\phi(a))^{\frac{2\alpha}{k}}}{\frac{2\alpha}{k}(k\rho)^{\frac{2\alpha}{k}}}. \tag{3.10}$$

Next, we show some estimation functions that were applied in this work.

Lemma 3.2. *Assume* $1/2<\alpha\le 1$, $0\le\beta\le 1$, $0<\rho\le 1$, $k>0$, *and* n, $m\in\mathbb{N}\cup\{0\}$. *If*

$$d<2n+m+2-\frac{k}{\alpha}(n+m+1), \tag{3.11}$$

then an estimation of the function is as follows:

$$\begin{aligned}&\left|\frac{{}_k^\rho\Psi_\phi^{\frac{\alpha}{k}(2-d)-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}(2-d)}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\right|\\ \le\;& \frac{{}_k^\rho\Psi_\phi^{\frac{\alpha}{k}(2-d)-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}}e^{\frac{|\mu|(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}+\frac{|\lambda|(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}}.\end{aligned} \tag{3.12}$$

Proof. Using the properties of the Euler-type gamma function with the inequality (3.11), we get

$$\Gamma\left(\frac{\alpha}{k}(2n+m+2-d)\right)>\Gamma(n+m+1). \tag{3.13}$$

Then,

$$\begin{aligned}&\left|\frac{{}_k^\rho\Psi_\phi^{\frac{\alpha}{k}(2-d)-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}(2-d)}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\right|\\
\le\;& \frac{{}_k^\rho\Psi_\phi^{\frac{\alpha}{k}(2-d)-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\binom{n+m}{m}\frac{|\mu|^n|\lambda|^m(\phi(t)-\phi(a))^{\frac{\alpha}{k}(2n+m)}}{(k\rho)^{\frac{\alpha}{k}(2n+m)}\Gamma(\frac{\alpha}{k}(2n+m+2-d))}\\
\le\;& \frac{{}_k^\rho\Psi_\phi^{\frac{\alpha}{k}(2-d)-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(n+m)!}{n!m!}\cdot\frac{|\mu|^n|\lambda|^m(\phi(t)-\phi(a))^{\frac{\alpha}{k}(2n+m)}}{(k\rho)^{\frac{\alpha}{k}(2n+m)}(n+m)!}\\
=\;& \frac{{}_k^\rho\Psi_\phi^{\frac{\alpha}{k}(2-d)-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{|\mu|(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}\right)^n\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{|\lambda|(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)^m\end{aligned}$$

$$= \frac{{}_k^\rho\Psi_\phi^{\frac{\alpha}{k}(2-d)-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}(2-d)}} e^{\frac{|\mu|(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}+\frac{|\lambda|(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}}.$$

This completes the proof. □

3.2. Existence and uniqueness property

Theorem 3.3. *Assume* $0<\alpha\le 1$, $0\le\beta\le 1$, $0<\rho\le 1$, $k>0$, *and* $f\in C([a,b]\times\mathbb{R},\mathbb{R})$. *Let*

(H_1) *There exists a constant* $\mathcal{L}>0$ *such that*

$$\left|f(t,u)-f(t,v)\right|\le\mathcal{L}\left|u(t)-v(t)\right|,\quad t\in[a,b],\quad u,v\in\mathbb{R}. \tag{3.14}$$

If $\Lambda_3\mathcal{E}_{\mu,\lambda}\mathcal{L}<1$, *then there is a unique solution for the proposed model* (1.1).

Proof. In view of Lemma 3.1, $v\in C([a,b],\mathbb{R})$ is a solution to the proposed model (1.1) if v corresponds the Eq (3.2). Setting $\mathcal{F}^*=\sup_{t\in[a,b]}\left|f(t,0)\right|<+\infty$, we provide $\mathcal{B}_r:=\{v\in C([a,b],\mathbb{R}):\|v\|\le r\}$, where

$$r\ge\frac{[\Lambda_1c_0+\Lambda_2(\lambda c_0+c_1)+\Lambda_3\mathcal{F}^*]\mathcal{E}_{\mu,\lambda}}{1-\Lambda_3\mathcal{E}_{\mu,\lambda}\mathcal{L}}.$$

Next, we prove that the set $\mathcal{B}_r$ is an invariant with respect to Q, which is, $Q\mathcal{B}_r\subset\mathcal{B}_r$. For any $v\in\mathcal{B}_r$ with the result of $0<e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(a))}\le 1$ for any $0<a\le s<t\le b<+\infty$, we have

$$\begin{aligned}
\|Qv\| &= \sup_{t\in[a,b]}\Bigg\{\Bigg|\frac{{}_k^\rho\Psi_\phi^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0\Bigg|\\
&\quad+\Bigg|\frac{\lambda{}_k^\rho\Psi_\phi^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0\Bigg|\\
&\quad+\Bigg|\frac{{}_k^\rho\Psi_\phi^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_1\Bigg|\\
&\quad+\Bigg|\int_a^t\frac{{}_k^\rho\Psi_\phi^{\frac{2\alpha}{k}-1}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\\
&\qquad\times f\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v(s)\right)\phi'(s)ds\Bigg|\Bigg\}\\
&\le \Bigg[\frac{(\phi(b)-\phi(a))^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}c_0+\frac{(\phi(b)-\phi(a))^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\lambda c_0\\
&\quad+\frac{(\phi(b)-\phi(a))^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}c_1\Bigg]e^{\frac{|\mu|(\phi(b)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}+\frac{|\lambda|(\phi(b)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}}
\end{aligned}$$

$$
\begin{aligned}
&+(\mathcal{L}\|v\|+\mathcal{F}^*)\int_a^b \frac{(\phi(s)-\phi(a))^{\frac{2\alpha}{k}-1}}{(k\rho)^{\frac{2\alpha}{k}}} e^{\frac{|\mu|(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}}+\frac{|\lambda|(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}}\phi'(s)ds \\
\leq\ & \Bigg[\frac{(\phi(b)-\phi(a))^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}c_0+\frac{(\phi(b)-\phi(a))^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\lambda c_0+\frac{(\phi(b)-\phi(a))^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}c_1 \\
&+\frac{(\phi(b)-\phi(a))^{\frac{2\alpha}{k}}}{\frac{2\alpha}{k}(k\rho)^{\frac{2\alpha}{k}}}(\mathcal{L}\|v\|+\mathcal{F}^*)\Bigg]\mathcal{E}_{\mu,\lambda} \\
=\ & \Big[\Lambda_1 c_0+\Lambda_2(\lambda c_0+c_1)+\Lambda_3(\mathcal{L}\|v\|+\mathcal{F}^*)\Big]\mathcal{E}_{\mu,\lambda}\leq r,
\end{aligned}
$$

which yields that $Q\mathcal{B}_r \subset \mathcal{B}_r$. Next, we prove that Q is a contraction. Suppose that $v_1, v_2 \in C([a,b],\mathbb{R})$. Thus,

$$
\begin{aligned}
&\left|(Qv_1)(t)-(Qv_2)(t)\right| \\
\leq\ & \sup_{t\in[a,b]}\Bigg\{\int_a^t \frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right) \\
&\times\Big|f\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v_1(s)\right)-f\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v_2(s)\right)\Big|ds\Bigg\} \\
\leq\ & \frac{(\phi(b)-\phi(a))^{\frac{2\alpha}{k}}}{\frac{2\alpha}{k}(k\rho)^{\frac{2\alpha}{k}}}\mathcal{E}_{\mu,\lambda}\mathcal{L}\|v_1-v_2\| \\
=\ & \Lambda_3\mathcal{E}_{\mu,\lambda}\mathcal{L}\|v_1-v_2\|.
\end{aligned}
$$

Since $\Lambda_3\mathcal{E}_{\mu,\lambda}\mathcal{L} < 1$, we can conclude that Q is a contraction. Therefore, by Banach's contraction mapping principle, Q has a fixed-point that corresponds to the unique solution of the model (1.1). □

3.3. Ulam-Hyers-Mittag-Leffler stability properties

Ulam-Hyers-Mittag-Leffler stability results provide important insights into the robustness of solutions under perturbations, revealing the tolerance of the system to external effects. These stability-accurate analysis results give quantitative insight into the stability properties of solutions, giving insights on their long-term behavior. Now, we provide a variety of UHML stability results for the model (1.1). Moreover, let $g \in C([a,b]\times\mathbb{R},\mathbb{R})$ and $\phi \in C([a,b],\mathbb{R}^+)$. For the ease of computation, we provide some inequalities

$$
\begin{aligned}
\left|{}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}w(t)-\lambda\,{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}w(t)-\mu\,w(t)-g(t,w(t))\right| &\leq \epsilon, \quad t\in[a,b], && (3.15)\\
\left|{}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}w(t)-\lambda\,{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}w(t)-\mu\,w(t)-g(t,w(t))\right| &\leq \mathcal{G}_g(t), \quad t\in[a,b], && (3.16)\\
\left|{}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}w(t)-\lambda\,{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}w(t)-\mu\,w(t)-g(t,w(t))\right| &\leq \epsilon\Phi(t), \quad t\in[a,b], && (3.17)\\
\left|{}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}w(t)-\lambda\,{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}w(t)-\mu\,w(t)-g(t,w(t))\right| &\leq \Phi(t), \quad t\in[a,b]. && (3.18)
\end{aligned}
$$

Definition 3.4. *The model* (1.1) *is said to be UHML stable, if there exists a constant* $\mathfrak{C}_g > 0$ *such that* *for every $\epsilon > 0$ and $w \in C([a,b],\mathbb{R}^+)$ of* (3.15)*, there is $v \in C([a,b],\mathbb{R}^+)$ of the model* (1.1) *via*

$$\left|v(t) - w(t)\right| \leq \mathfrak{C}_g \epsilon \mathbb{E}_{k,2\alpha,k}\left(\kappa_g \left(\phi(t) - \phi(a)\right)^{\frac{2\alpha}{k}}\right), \quad \kappa_g \geq 0, \quad t \in [a,b]. \tag{3.19}$$

Definition 3.5. *The model* (1.1) *is said to be generalized UHML stable, if there exists a function $\mathcal{G}_g \in C(\mathbb{R}^+,\mathbb{R}^+)$ via $\mathcal{G}_g(0) = 0$, such that for every $\epsilon > 0$ and $w \in C([a,b],\mathbb{R}^+)$ of* (3.16)*, there is $v \in C([a,b],\mathbb{R}^+)$ of the model* (1.1) *via*

$$\left|v(t) - w(t)\right| \leq \mathcal{G}_g(\epsilon) \mathbb{E}_{k,2\alpha,k}\left(\kappa_g \left(\phi(t) - \phi(a)\right)^{\frac{2\alpha}{k}}\right), \quad \kappa_g \geq 0, \quad t \in [a,b]. \tag{3.20}$$

Definition 3.6. *The model* (1.1) *is said to be UHRML stable with respect to another function $\Phi(t)$, if there exists a constant $\mathfrak{C}_{g_\Phi} > 0$ such that for every $\epsilon > 0$ and $w \in C([a,b],\mathbb{R}^+)$ of* (3.17)*, there is $w \in C([a,b],\mathbb{R}^+)$ of the model* (1.1) *via*

$$\left|v(t) - w(t)\right| \leq \mathfrak{C}_{g_\Phi} \epsilon \Phi(t) \mathbb{E}_{k,2\alpha,k}\left(\kappa_{g_\Phi} \left(\phi(t) - \phi(a)\right)^{\frac{2\alpha}{k}}\right), \quad \kappa_{g_\Phi} \geq 0, \quad t \in [a,b]. \tag{3.21}$$

Definition 3.7. *The model* (1.1) *is said to be generalized UHRML stable with respect to another function $\Phi(t)$ such that for every $\epsilon > 0$ and $w \in C([a,b],\mathbb{R}^+)$ of* (3.18)*, there is $v \in C([a,b],\mathbb{R}^+)$ of the model* (1.1) *via*

$$\left|v(t) - w(t)\right| \leq \mathfrak{C}_{f_\Phi} \Phi(t) \mathbb{E}_{k,2\alpha,k}\left(\kappa_{g_\Phi} \left(\phi(t) - \phi(a)\right)^{\frac{2\alpha}{k}}\right), \quad \kappa_{g_\Phi} \geq 0, \quad t \in [a,b]. \tag{3.22}$$

Remark 3.8. *Suppose that $w \in C([a,b],\mathbb{R})$ is a solution of the model* (3.15) *if and only if there exits $x_w \in C([a,b],\mathbb{R})$, that is depends on w, such that $(\mathcal{R}_1)$. $\left|x_w(t)\right| \leq \epsilon$, $t \in [a,b]$; $(\mathcal{R}_2)$. ${}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi} w(t) - \lambda\, {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi} w(t) = \mu\, w(t) + g(t,w(t)) + x_w(t)$, $t \in [a,b]$.*

Remark 3.9. *Suppose that $w \in C([a,b],\mathbb{R})$ is a solution of the model* (3.17) *if and only if there exits $y_w \in C([a,b],\mathbb{R})$, that is depends on w, such that $(\mathcal{R}_1)$. $\left|y_w(t)\right| \leq \epsilon\Phi(t)$, $t \in [a,b]$; $(\mathcal{R}_2)$. ${}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi} w(t) - \lambda\, {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi} w(t) = \mu\, w(t) + g(t,w(t)) + y_w(t)$, $t \in [a,b]$.*

Theorem 3.10. *Assume that $g \in C([a,b] \times \mathbb{R},\mathbb{R})$, (H_1), and $\Lambda_3 \mathcal{E}_{\mu,\lambda} \mathcal{L} < 1$ hold. Then, the proposed model* (1.1) *is UHML stable and consequently generalized UHML stable on $[a,b]$.*

Proof. Given $\epsilon > 0$, $1/2 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 < \rho \leq 1$, $k > 0$, and $w \in C([a,b],\mathbb{R})$ is a solution of the model (3.15). From the assumption $(\mathcal{R}_2)$ in Remark 3.8, we obtain that

$$\begin{cases} {}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi} v(t) - \lambda\, {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi} v(t) = \mu\, v(t) + g(t,v(t)) + x_w(t), \quad t \in [a,b], \\ \lim\limits_{t \to a^+} {}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi} v(t) = c_0, \quad \lim\limits_{t \to a^+} {}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\left({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi} v(t)\right) = c_1. \end{cases} \tag{3.23}$$

Using Lemma 3.1, the model (3.23) has the solution

$$
\begin{aligned}
w(t) &= \frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0 \\
&-\frac{\lambda{}^{\rho}_{k}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0 \\
&+\frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_1 \\
&+\int_a^t\frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{\alpha}{k}-1}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right) \\
&\quad\times g\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v(s)\right)\phi'(s)ds \\
&+\int_a^t\frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{2\alpha}{k}}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right) \\
&\quad\times x_w\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right)\right)\phi'(s)ds. \qquad (3.24)
\end{aligned}
$$

Assume that $v\in C([a,b],\mathbb{R})$ is a solution of (1.1). Then,

$$
\begin{aligned}
v(t) &= \frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0 \\
&-\frac{\lambda{}^{\rho}_{k}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0 \\
&+\frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_1 \\
&+\int_a^t\frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{\alpha}{k}}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right) \\
&\quad\times g\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v(s)\right)\phi'(s)ds. \qquad (3.25)
\end{aligned}
$$

Utilizing $|a-b|\le|a|+|b|$ with (3.24)-(3.25), we get

$$
\begin{aligned}
&\left|v(t)-w(t)\right| \\
\le&\ \left|\int_a^t\frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{2\alpha}{k}}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\right.
\end{aligned}
$$

$$\times g\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v(s)\right)-g\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),w(s)\right)\phi'(s)ds\Bigg|$$

$$+\left|\int_a^t \frac{{}_k^\rho\Psi_\phi^{\frac{2\alpha}{k}}(t,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\right.$$

$$\left.\times x_w\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right)\right)\phi'(s)ds\right|. \tag{3.26}$$

By using the property $0<e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(a))}\leq 1$ for any $0<a\leq s<t\leq b<+\infty$, (H_1), and the assumption $(\mathcal{R}_1)$ in Remark 3.8, the result (3.26) can be provided

$$\begin{aligned}\left|v(t)-w(t)\right| &\leq \frac{\mathcal{E}_{\mu,\lambda}\mathcal{L}\Gamma_k(2\alpha)}{kk^{\frac{2\alpha}{k}-2}}\int_a^t\frac{{}_k^\rho\Psi_\phi^{\frac{2\alpha}{k}}(s,a)}{\rho^{\frac{2\alpha}{k}}k\Gamma_k(2\alpha)}\left|v(s)-w(s)\right|\phi'(s)ds+\frac{(\phi(b)-\phi(a))^{\frac{2\alpha}{k}}}{\frac{2\alpha}{k}(k\rho)^{\frac{2\alpha}{k}}}\mathcal{E}_{\mu,\lambda}\epsilon\\ &\leq \Lambda_3\mathcal{E}_{\mu,\lambda}\epsilon+\frac{\Gamma_k(2\alpha)}{k}\cdot\frac{\mathcal{E}_{\mu,\lambda}\mathcal{L}}{k^{\frac{2\alpha}{k}-2}}{}_{a,k}\mathcal{I}^{\alpha,\rho;\phi}\left|v(t)-w(t)\right|.\end{aligned}$$

Utilizing Theorem 2.6 and Corollary 2.7, yields

$$\left|v(t)-w(t)\right|\leq\Lambda_3\mathcal{E}_{\mu,\lambda}\epsilon\mathbb{E}_{k,2\alpha,k}\left((k\rho)^{-\frac{2\alpha}{k}}k\Gamma_k(2\alpha)\mathcal{E}_{\mu,\lambda}\mathcal{L}\left(\phi(t)-\phi(a)\right)^{\frac{2\alpha}{k}}\right).$$

By taking $\mathfrak{C}_f:=\Lambda_3\mathcal{E}_{\mu,\lambda}$ and $\kappa_f:=(k\rho)^{\frac{-2\alpha}{k}}k\Gamma_k(2\alpha)\mathcal{E}_{\mu,\lambda}\mathcal{L}$, implies that

$$\left|v(t)-w(t)\right|\leq\mathfrak{C}_f\epsilon\mathbb{E}_{k,2\alpha,k}\left(\kappa_f\left(\phi(t)-\phi(a)\right)^{\frac{2\alpha}{k}}\right). \tag{3.27}$$

Therefore, the model (1.1) is UHML stable. Additionally, by taking $\mathcal{G}_f(\epsilon)=\mathfrak{C}_f\epsilon$ via $\mathcal{G}_f(0)=0$, we have

$$\left|v(t)-w(t)\right|\leq\mathcal{G}_f(\epsilon)\mathbb{E}_{k,2\alpha,k}\left(\kappa_f\left(\phi(t)-\phi(a)\right)^{\frac{2\alpha}{k}}\right).$$

Then, a solution of the model (1.1) is generalized UHML stable. □

We now provide the necessary assumption used in Theorem 3.11.

(H_2) Let $\Phi\in C([a,b],\mathbb{R})$ be a non-decreasing function. There is $\chi_\Phi>0$ so that

$$\int_a^t\frac{{}_k^\rho\Psi_\phi^{\frac{2\alpha}{k}}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\Phi(s)\phi'(s)ds\leq\chi_\Phi\Phi(t),\ \ t\in[a,b].$$

Theorem 3.11. *Assume that $g\in C([a,b]\times\mathbb{R},\mathbb{R})$, (H_1), and $\Lambda_3\mathcal{E}_{\mu,\lambda}\mathcal{L}<1$ hold. Then, the model* (1.1) *is UHRML stable and consequently generalized UHRML stable on $[a,b]$.*

Proof. Assume that $\epsilon>0$ and $w\in C([a,b],\mathbb{R})$ is a solution of the model (3.15). From the assumption $(\mathcal{R}_2)$ in Remark 3.9, we obtain that

$$\begin{cases}{}_{a,k}^{H}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}v(t)-\lambda\,{}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t)=\mu\,v(t)+g(t,v(t))+y_w(t), & t\in(a,b],\\ \lim\limits_{t\to a^+}{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}v(t)=c_0, \quad \lim\limits_{t\to a^+}{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\left({}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\phi}v(t)\right)=c_1.\end{cases} \tag{3.28}$$

Using Lemma 3.1, a solution of (3.28) is

$$
\begin{aligned}
w(t) \;=\;& \frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0\\
&-\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\lambda c_0\\
&+\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_1\\
&+\int_a^t \frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\\
&\quad\times g\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v(s)\right)\phi'(s)ds\\
&+\int_a^t \frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\\
&\quad\times y_w\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right)\right)\phi'(s)ds.
\end{aligned}
\tag{3.29}
$$

Let $v \in C([a,b],\mathbb{R})$ be a solution of (1.1). Hence,

$$
\begin{aligned}
v(t) \;=\;& \frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_0\\
&-\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\lambda c_0\\
&+\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(\frac{\mu(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)c_1\\
&+\int_a^t \frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\\
&\quad\times g\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v(s)\right)\phi'(s)ds.
\end{aligned}
\tag{3.30}
$$

Utilizing $|a-b| \leq |a| + |b|$ with (3.29)-(3.30), we achieve

$$
\begin{aligned}
&\left|v(t)-w(t)\right|\\
\leq\;& \left|\int_a^t \frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\right.
\end{aligned}
$$

$$\times g\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),v(s)\right)-g\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right),w(s)\right)\phi'(s)ds\Bigg|$$

$$+\left|\int_a^t \frac{{}^{\rho}_{k}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(s,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(\frac{\mu(\phi(s)-\phi(a))^{\frac{2\alpha}{k}}}{(k\rho)^{\frac{2\alpha}{k}}},\frac{\lambda(\phi(s)-\phi(a))^{\frac{\alpha}{k}}}{(k\rho)^{\frac{\alpha}{k}}}\right)\right.$$

$$\left.\times y_w\left(\phi^{-1}\left(\phi(t)+\phi(a)-\phi(s)\right)\right)\phi'(s)ds\right|. \tag{3.31}$$

By using the property $0 < e^{\frac{\rho-1}{k\rho}(\phi(t)-\phi(a))} \le 1$ for any $0 < a \le s < t \le b < +\infty$, (H_1), (H_2), and $(\mathcal{R}_1)$ in Remark 3.9, the result (3.31) can be provided

$$\left|v(t)-w(t)\right| \le \epsilon\chi_{\Phi}\Phi(t)+\frac{\Gamma_k(2\alpha)}{k}\cdot\frac{\mathcal{E}_{\mu,\lambda}\mathcal{L}}{k^{\frac{2\alpha}{k}-2}}{}_{a,k}\mathcal{I}^{\alpha,\rho;\phi}\left|v(t)-w(t)\right|. \tag{3.32}$$

Utilizing Theorem 2.6 and Corollary 2.7, it follows that

$$\left|v(t)-w(t)\right| \le \epsilon\chi_{\Phi}\Phi(t)\mathbb{E}_{k,2\alpha,k}\left((k\rho)^{-\frac{2\alpha}{k}}k\Gamma_k(2\alpha)\mathcal{E}_{\mu,\lambda}\mathcal{L}\left(\phi(t)-\phi(a)\right)^{\frac{2\alpha}{k}}\right).$$

Taking $\mathfrak{C}_{f_\Phi} := \chi_\Phi$ and $\kappa_{f_\Phi} := (k\rho)^{-\frac{2\alpha}{k}}k\Gamma_k(2\alpha)\mathcal{E}_{\mu,\lambda}\mathcal{L}$, yields that

$$\left|v(t)-w(t)\right| \le \mathfrak{C}_{f_\Phi}\epsilon\Phi(t)\mathbb{E}_{k,2\alpha,k}\left(\kappa_{f_\Phi}\left(\phi(t)-\phi(a)\right)^{\frac{2\alpha}{k}}\right). \tag{3.33}$$

Therefore, the model (1.1) is UHRML stable. Additionally, by taking $\epsilon = 1$, we have

$$\left|v(t)-w(t)\right| \le \mathfrak{C}_{f_\Phi}\Phi(t)\mathbb{E}_{k,2\alpha,k}\left(\kappa_{f}\left(\phi(t)-\phi(s)\right)^{\frac{2\alpha}{k}}\right).$$

Hence, the solution of the model (1.1) is generalized UHRML stable. □

4. Some applications

We show two numerical examples to demonstrate the accuracy and validity of our major theoretical results in this section.

4.1. An example

Example 4.1. *Consider the Langevin-type differential equations with (k,ϕ)-Hilfer-PFDO of the form:*

$$\begin{cases} {}^{H}_{0,1}\mathfrak{D}^{\frac{3}{2},\frac{1}{2},1;\phi}v(t)-2\,{}^{H}_{0,1}\mathfrak{D}^{\frac{3}{4},\frac{1}{2},1;\phi}v(t)=3v(t)+\dfrac{4e^{-5t}}{(t+2)^2+6}\left(\dfrac{|v(t)|}{|v(t)|+1}\right), \quad t\in(0,1], \\ \lim\limits_{t\to0^+}{}_{0,1}\mathcal{I}^{\frac{1}{8},1;\phi}v(t)=1, \quad \lim\limits_{t\to0^+}{}_{0,1}\mathcal{I}^{\frac{1}{8},1;\phi}\left({}^{H}_{0,1}\mathfrak{D}^{\frac{3}{4},\frac{1}{2},1;\phi}v(t)\right)=2. \end{cases} \tag{4.1}$$

Here, $\alpha = 3/4$, $\beta = 1/2$, $\rho = 1$, $k = 1$, and $\phi(t) = (1 - e^{-2t})/3$ with $\lambda = 2$, $\mu = 3$, $a = 0$, $b = 1$, $c_0 = 1$, and $c_2 = 2$. Using all parameters, we get the results $\mathcal{E}_{\mu,\lambda} \approx 3.49359$, $\Lambda_1 \approx 1.16825$, $\Lambda_2 \approx 0.45954$, and $\Lambda_3 = 0.10315$. Consider the function

$$g(t, v(t)) = \frac{4e^{-5t}}{(t+2)^2 + 6}\left(\frac{|v(t)|}{|v(t)| + 1}\right). \tag{4.2}$$

For any $v_i \in \mathbb{R}$, $i = 1, 2$ and $t \in [0, 1]$, then

$$\left|g(t, v_1) - g(t, v_2)\right| \leq \frac{2}{5}|v_1 - v_2|.$$

(H_1) in Theorem 3.3 is satisfied via $\mathcal{L} = 2/5$, which implies that $\Lambda_3\mathcal{E}_{\mu,\lambda}\mathcal{L} \approx 0.14415 < 1$. Since all conditions in Theorem 3.3 are true, the model (4.1) has a unique solution on $[0, 1]$. Furthermore, from (3.27), we get $\mathfrak{C}_f := \Lambda_3\mathcal{E}_{\mu,\lambda} \approx 0.36038 > 0$ and $\kappa_f := (k\rho)^{-\frac{2\alpha}{k}} k\Gamma_k(2\alpha)\mathcal{E}_{\mu,\lambda}\mathcal{L} \approx 1.23845 > 0$. Then, the model (4.1) is UHML stable on $[0, 1]$. By taking $\mathcal{G}_f(\epsilon) = \mathfrak{C}_f\epsilon$ via $\mathcal{G}_f(0) = 0$, the model (4.1) is generalized UHML stable on $[0, 1]$. Moreover, by taking $\Phi(t) = (\phi(t) - \phi(a))^2$ into $(\mathcal{P}_1)$, we have $\chi_\Phi = \frac{2}{3}(\phi(1) - \phi(0))^{\frac{3}{2}}\mathcal{E}_{\mu,\lambda} \approx 0.36038$. By applying (3.33), we have $\mathfrak{C}_{f_\Phi} \approx 0.36038$ and $\kappa_{f_\Phi} \approx 1.23845$. Therefore, by all conclusions in Theorem 3.11, the model (4.1) is UHRML stable on $[0, 1]$. Finally, if we set $\epsilon = 1$, the model (4.1) is generalized UHRML stable on $[0, 1]$.

4.2. Applications to an RLC electrical circuit model

In this part, we study an RLC electrical circuit model, which is commonly used in both physics and engineering disciplines. RLC circuit models are applied in a variety of signal processing and audio electronics applications, including filter construction. They can be configured as low-pass, high-pass, band-pass, or band-stop filters, and they are essential for frequency tuning in radio receivers. Furthermore, they are applied in control systems to efficiently reduce oscillations while retaining feedback loop stability. Let $V(t)$ be the voltage source, $Q(t)$ be the current in the RLC series electrical circuit at time t in which the oscillating electrical circuit consisting of a resistor (R), an inductance (L), and a capacitance (C) are connected in the series with a voltage source $V(t)$, as seen in the circuit diagram in Figure 1.

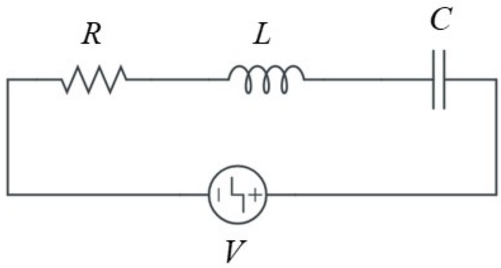

Figure 1. The RLC electrical circuit diagram.

The RLC electrical circuit diagram is widely utilized in physics and engineering, particularly to understand how electric currents behave in various systems. For a voltage source $V(t)$, which is measured in volts, Kirchhoff's voltage law [43] gives the below equation:

$$V(t) = V_L(t) + V_R(t) + V_C(t) = LQ'(t) + RQ(t) + \frac{1}{C}\int Q(s)ds, \tag{4.3}$$

where $V_L(t)$, $V_R(t)$, and $V_C(t)$ denote the voltage across the inductor, resistor, and capacitor at time t, respectively. Differentiating Eq (4.3) with respect to t, we obtain

$$V'(t) = LQ''(t) + RQ'(t) + \frac{1}{C}Q(t). \tag{4.4}$$

Our goal is to represent the physical problem (4.4) as an initial value problem. Assume that the constitutive equations related with three compartments, namely resistor $(V_R(t))$, inductor $(V_L(t))$, and capacitor $(V_C(t))$, are as follows:

$$V_L(t) = L\frac{dQ}{dt}, \qquad V_R(t) = RQ(t), \qquad V_C(t) = \frac{1}{C}\int_0^t Q(s)ds.$$

Using Kirchhoff's voltage law, the total of the voltage increases on every loop in a circuit model is equal to the voltage $V(t)$ absorbed in the system. Then,

$$V_L(t) + V_R(t) + V_C(t) = V(t)$$

or

$$L\frac{dQ}{dt} + RQ(t) + \frac{1}{C}\int Q(s)ds = V(t),$$

where $dV/dt = Q$ and $dV^2/dt^2 = dQ/dt$. Thus, the second-order inhomogeneous ordinary differential equation is given by

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q(t) = V(t).$$

In the particular example of fractional orders $\alpha \in (0,1]$, $\beta \in [0,1]$, $\rho \in (0,1]$, and $k > 0$, we look at an initial value problem for the fractional Langevin-type differential equations. Thus, the main objective is to create this problem using a model of the RLC series circuit, as shown in the example below.

Example 4.2. *Consider the RLC circuit model that is applied to the (k,ϕ)-Hilfer proportional fractional Langevin-type differential equations by substituting ordinary derivatives in this model. The considered model can be specified below:*

$$\begin{cases} L\,{}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}Q(t) + R\,{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}Q(t) + \dfrac{1}{C}\,Q(t) = V(t), \quad \alpha \in (0,1], \quad 2\alpha \in (1,2], \\ \lim\limits_{t\to a^+} {}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}Q(t) = Q_0, \quad \lim\limits_{t\to a^+} {}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\Big({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}Q(t)\Big) = Q_1, \end{cases} \tag{4.5}$$

where $I(t) = {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}Q(t)$.

Observe that for $\alpha = \beta = k = \rho = 1$ and $\phi(t) = t$, Eq (4.5) reduces to the classical RLC circuit

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q(t) = V(t),$$

which shows that the proposed fractional model is a direct generalization of the standard circuit law. The replacement of the first-order derivative by the (k,ϕ)-Hilfer proportional fractional derivative is further justified because such operators can describe hereditary effects and non-ideal behaviors of real circuit elements. In practice, resistors, capacitors, and inductors often deviate from their ideal models due to phenomena such as dielectric losses, constant-phase element behavior, and the skin effect. Fractional-order RLC models have therefore been investigated in the literature and applied successfully to capture such effects. For instance, Ahmadova and Mahmudov analyzed Langevin-type fractional differential equations in electric circuits [44]. Naveen et al. studied an RLC system under the Hilfer derivative with a numerical scheme [45]. Murugesan et al. considered Hadamard fractional operators for non-local RLC models, addressing existence and stability [46]. Thunibat et al. applied a Caputo-based Adomian decomposition method to nonlinear RLC circuits [47]. Murugesan et al. further extended the analysis to Hilfer–Hadamard fractional equations in RLC applications [48]. Shankar and Bora investigated stability issues of Caputo-type impulsive integro-differential RLC models [49].

To achieve the current $Q(t)$, we use the ϕ-Laplace transform and its properties. Applying $\mathcal{L}_\phi$ into the first equation of (4.5), we have

$$L\,\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}Q(t)\right\}+R\,\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}Q(t)\right\}+\frac{1}{C}\,\mathcal{L}_\phi\left\{Q(t)\right\}=\mathcal{L}_\phi\left\{V(t)\right\}. \tag{4.6}$$

Applying Lemma 2.12, we have

$$\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{2\alpha,\beta,\rho;\phi}Q(t)\right\} = [q(s)]^{\frac{2\alpha}{k}}\mathcal{L}_\phi\left\{Q(t)\right\}-k\rho[q(s)]^{\frac{\alpha(\beta+1)}{k}-\beta}c_0-k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}c_1, \tag{4.7}$$

$$\mathcal{L}_\phi\left\{{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}Q(t)\right\} = [q(s)]^{\frac{\alpha}{k}}\mathcal{L}_\phi\left\{Q(t)\right\}-k\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}c_0, \tag{4.8}$$

where $\lim_{t\to a^+}{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}Q(t)=c_0$ and $\lim_{t\to a^+}{}_{a,k}\mathcal{I}^{(1-\beta)(k-\alpha),\rho;\phi}\left({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\phi}Q(t)\right)=c_1$.

Substituting (4.7)-(4.8) into (4.6), it follows from

$$\begin{aligned}\left(L[q(s)]^{\frac{2\alpha}{k}}+R[q(s)]^{\frac{\alpha}{k}}-\frac{1}{C}\right)\mathcal{L}_\phi\left\{Q(t)\right\} &= \left(Lk\rho[q(s)]^{\frac{\alpha(\beta+1)}{k}-\beta}+Rk\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}\right)c_0\\ &\quad +Lk\rho[q(s)]^{\frac{\alpha\beta}{k}-\beta}c_1+\mathcal{L}_\phi\left\{V(t)\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{L}_\phi\left\{Q(t)\right\} &= \frac{L[q(s)]^{\frac{\alpha(\beta+1)}{k}-\beta}+R[q(s)]^{\frac{\alpha\beta}{k}-\beta}}{L[q(s)]^{\frac{2\alpha}{k}}+R[q(s)]^{\frac{\alpha}{k}}-\frac{1}{C}}k\rho c_0+\frac{L[q(s)]^{\frac{\alpha\beta}{k}-\beta}}{L[q(s)]^{\frac{2\alpha}{k}}+R[q(s)]^{\frac{\alpha}{k}}-\frac{1}{C}}k\rho c_1\\ &\quad +\frac{1}{L[q(s)]^{\frac{2\alpha}{k}}+R[q(s)]^{\frac{\alpha}{k}}-\frac{1}{C}}\mathcal{L}_\phi\left\{V(t)\right\}.\end{aligned} \tag{4.9}$$

Taking $\mathcal{L}_\phi^{-1}$ into the equation (4.9), we get

$$\begin{aligned}Q(t) &= L\mathcal{L}_\phi^{-1}\left\{\frac{[q(s)]^{\frac{\alpha}{k}(\beta+1-\frac{k\beta}{\alpha})}}{L[q(s)]^{\frac{2\alpha}{k}}+R[q(s)]^{\frac{\alpha}{k}}-\frac{1}{C}}\right\}k\rho c_0+R\mathcal{L}_\phi^{-1}\left\{\frac{[q(s)]^{\frac{\alpha}{k}(\beta-\frac{k\beta}{\alpha})}}{L[q(s)]^{\frac{2\alpha}{k}}+R[q(s)]^{\frac{\alpha}{k}}-\frac{1}{C}}\right\}k\rho c_0\\ &\quad +L\mathcal{L}_\phi^{-1}\left\{\frac{[q(s)]^{\frac{\alpha}{k}(\beta-\frac{k\beta}{\alpha})}}{L[q(s)]^{\frac{2\alpha}{k}}+R[q(s)]^{\frac{\alpha}{k}}-\frac{1}{C}}\right\}k\rho c_1+\mathcal{L}_\phi^{-1}\left\{\frac{1}{L[q(s)]^{\frac{2\alpha}{k}}+R[q(s)]^{\frac{\alpha}{k}}-\frac{1}{C}}\mathcal{L}_\phi\left\{V(t)\right\}\right\}.\end{aligned}$$

Applying the inverse Laplace transform (2.17) and (2.18) in Lemma 2.14, we achieve the explicit solution of the model (4.5) as follows:

$$\begin{aligned}
&Q(t)\\
=\;& \frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(-\frac{(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{LC(k\rho)^{\frac{2\alpha}{k}}},-\frac{R(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{L(k\rho)^{\frac{\alpha}{k}}}\right)Q_0\\
&+\frac{R{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{L(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(-\frac{(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{LC(k\rho)^{\frac{2\alpha}{k}}},-\frac{R(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{L(k\rho)^{\frac{\alpha}{k}}}\right)Q_0\\
&+\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(-\frac{(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{LC(k\rho)^{\frac{2\alpha}{k}}},-\frac{R(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{L(k\rho)^{\frac{\alpha}{k}}}\right)Q_1\\
&+\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(-\frac{(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{LC(k\rho)^{\frac{2\alpha}{k}}},-\frac{R(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{L(k\rho)^{\frac{\alpha}{k}}}\right)*_{\phi}E(t).
\end{aligned}\tag{4.10}$$

Applying the definition of ϕ-convolution (Definition 2.11), which implies that

$$\begin{aligned}
&Q(t)\\
=\;& \frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(-\frac{(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{LC(k\rho)^{\frac{2\alpha}{k}}},-\frac{R(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{L(k\rho)^{\frac{\alpha}{k}}}\right)Q_0\\
&+\frac{R{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{L(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(-\frac{(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{LC(k\rho)^{\frac{2\alpha}{k}}},-\frac{R(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{L(k\rho)^{\frac{\alpha}{k}}}\right)Q_0\\
&+\frac{{}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta-1}(t,a)}{(k\rho)^{\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}}\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}-\frac{\alpha\beta}{k}+\beta}\left(-\frac{(\phi(t)-\phi(a))^{\frac{2\alpha}{k}}}{LC(k\rho)^{\frac{2\alpha}{k}}},-\frac{R(\phi(t)-\phi(a))^{\frac{\alpha}{k}}}{L(k\rho)^{\frac{\alpha}{k}}}\right)Q_1\\
&+\frac{1}{(k\rho)^{\frac{2\alpha}{k}}}\int_a^t {}_{k}^{\rho}\Psi_{\phi}^{\frac{2\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{2\alpha}{k},\frac{\alpha}{k},\frac{2\alpha}{k}}\left(-\frac{(\phi(t)-\phi(s))^{\frac{2\alpha}{k}}}{LC(k\rho)^{\frac{2\alpha}{k}}},-\frac{R(\phi(t)-\phi(s))^{\frac{\alpha}{k}}}{L(k\rho)^{\frac{\alpha}{k}}}\right)E(s)ds.
\end{aligned}\tag{4.11}$$

To compare graphical results for various voltage functions $V(t)$ in the exact solution (4.10), we consider three special cases for $Q(t)$ with $R = 20$, $L = 8$, $C = 1/200$, $Q_0 = 1$, and $Q_1 = 0.5$. All graphical results in Figures 2–6 are plotted directly from the explicit solution (4.11).

Case I: We consider the voltage function $V(t) = 20$ and fixed $\rho = 1.00$ via varied $\alpha \in \{0.96, 0.97, 0.98, 0.99, 1.00\}$, $\beta \in \{0.20, 0.40, 0.60, 0.80, 1.00\}$, and $k \in \{0.82, 0.84, 0.86, 0.88\}$. The graphical results of $Q(t)$ is displayed as in Figures 2–4. It can be observed that as the time period increases, all three graphical simulations exhibit increased oscillations in their solutions. The amplitude length of current $Q(t)$ increases with increasing values of fractional-order α (Figure 2) and fractal-dimension β (Figure 3), respectively. Moreover, the amplitude length of the current $Q(t)$ is inversely proportional to the value of k, as shown in Figure 4. These observations indicate that the fractional parameters α and β play the role of tuning the memory and damping properties of the RLC circuit.

Larger values of α and β correspond to stronger memory effects, resulting in higher oscillation amplitudes of $Q(t)$, whereas the parameter k acts oppositely by reducing the amplitude. Such fractional-order behaviors are consistent with experimental evidence that real capacitors and inductors deviate from their ideal integer-order models due to dielectric losses, skin effect, and constant-phase element behavior.

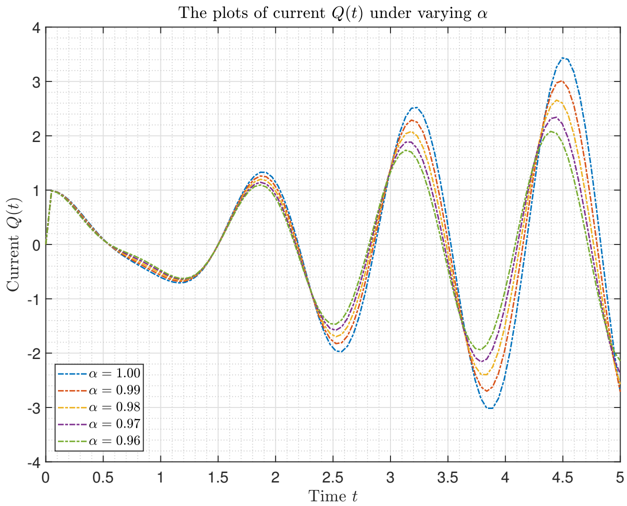

Figure 2. The simulation of current $Q(t)$ under varying α and fixed $\beta = 0.95$, $\rho = 1.00$, and $k = 0.88$ for the proposed model (4.5), obtained from the explicit solution of (4.11).

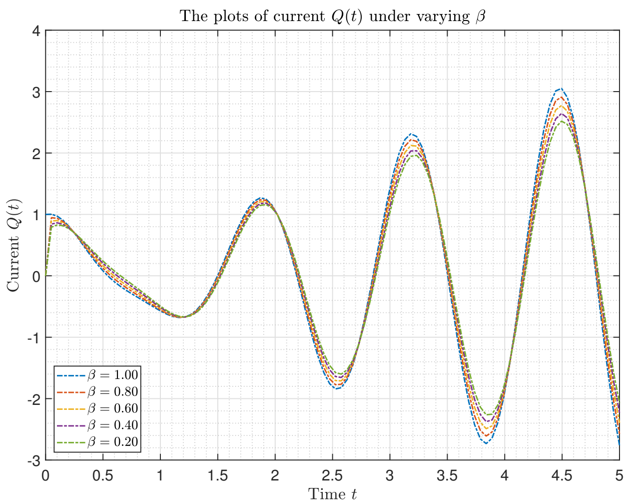

Figure 3. The simulation of current $Q(t)$ under varying β and fixed $\alpha = 0.99$, $\rho = 1.00$, and $k = 0.88$ for the proposed model (4.5), obtained from the explicit solution of (4.11).

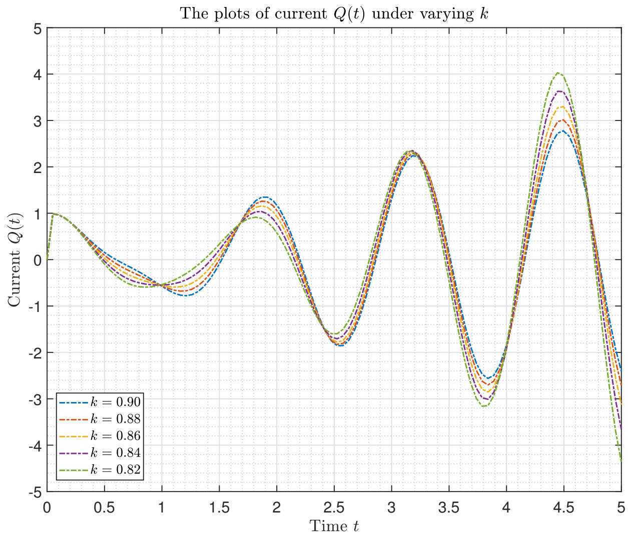

Figure 4. The simulation of current $Q(t)$ under varying k and fixed $\alpha = 0.99, \beta = 0.95$, and $\rho = 1.00$ for the proposed model (4.5), obtained from the explicit solution of (4.11).

Case II: We consider voltage functions $V(t) = V_0 \sin(\theta_j t)$ and $V(t) = V_0 \cos(\theta_j t)$ for $j = 1, 2, 3, 4, 5$, and $V_0 = 20$ with $\theta_j = \frac{1}{\sqrt{L_j C_j}}$ and $(L_1, C_1, \theta_1) = (4, 0.0625, 2)$, $(L_2, C_2, \theta_2) = (5, 0.008, 5)$, $(L_3, C_3, \theta_3) = (2, 0.0078125, 8)$, $(L_4, C_4, \theta_4) = (10, 0.001, 10)$, and $(L_5, C_5, \theta_5) = (3, 0.0015, 15)$. It is noticed that the wavelengths of the current value $Q(t)$ decrease as frequency values θ_j increase, as shown in Figures 5 and 6.

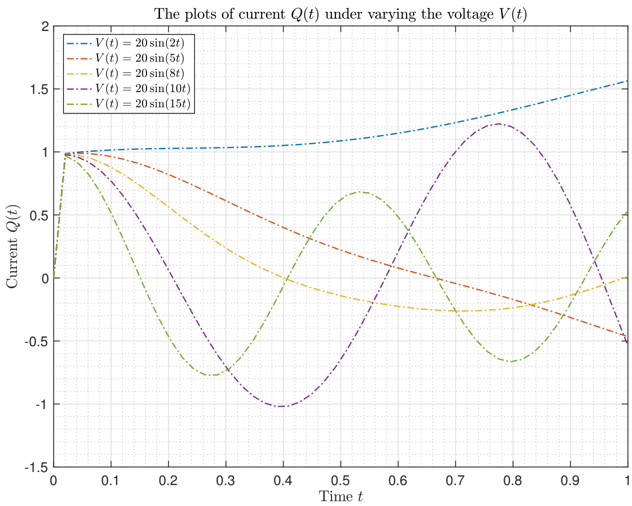

Figure 5. The simulation of current $Q(t)$ under varying $V(t) = V_0 \sin(\theta_j t)$ and fixed $\alpha = 0.99$, $\beta = 0.95$, $\rho = 1.00$, and $k = 0.88$ for the proposed model (4.5), obtained from the explicit solution of (4.11).

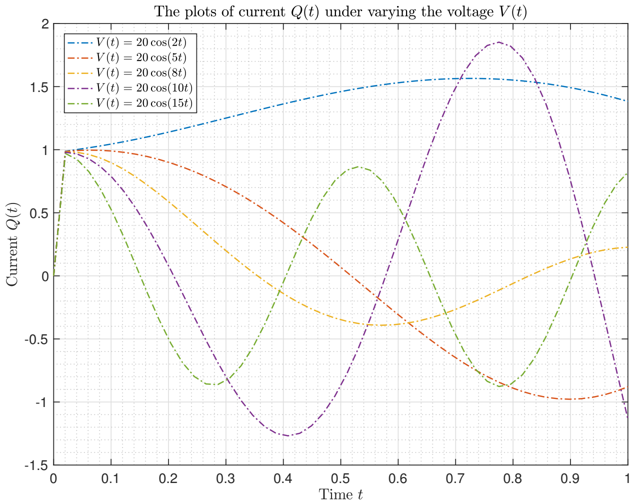

Figure 6. The simulation of current $Q(t)$ under varying $V(t) = V_0 \cos(\theta_j t)$ and fixed $\alpha = 0.99, \beta = 0.95, \rho = 1.00$, and $k = 0.88$ for the proposed model (4.5).

5. Conclusions

In this work, we propose the Langevin-type differential equation in the sense of the (k, ϕ)–Hilfer–PFDO. The ϕ–Laplace transform techniques derive an exact analytical solution to the proposed linear problem by utilizing the bivariate Mittag–Leffler function. Its demonstration is expanded to a global solution for the proposed nonlinear problem. Next, the uniqueness of the global analytical solution is proved by applying Banach's contraction mapping principle. Various of UHML stability results for the model (1.1) are discussed by helping functional analysis technique. The results indicate that the solution to the problem exhibits robustness, supported by numerical applications that demonstrate the accuracy and practical relevance of the theoretical results. These applications are confirmed with examples, including RLC circuit models, showcasing how fractional calculus can be utilized to real-life engineering problems like circuit analysis. We provide an exact solution to the proposed model in the special case and compare the obtained results for the current $Q(t)$ with various functions $V(t)$ and parameters, α, β, ρ, and k, in the applicability of electric circuit theory. Finally, we give examples to back up our major results. The work also provides useful techniques for solving nonlinear Langevin-type fractional differential equations, which are valuable for understanding complex physical and biological systems. Additionally, we explore novel avenues for using fractional calculus to describe RLC circuit models and lays the groundwork for applying these results to other engineering systems and initial/boundary value problems. The primary innovations of our paper are the following:

- We use the bivariate Mittag-Leffler function to bring novel properties to the ϕ-Laplace transform.
- We apply a bivariate Mittag-Leffler function with the properties of the ϕ-Laplace transform to achieve exact explicit solutions to the problem (1.1).
- We study the existence, uniqueness, and stability of the solution of the (k, ϕ)-Hilfer proportional differential equation for the Langevin-type model.
- We apply the problem (1.1) to RLC circuit models.

Our result can be extended to a variety of potential directions by changing fractional order α, fractal dimension β, proportional constant ρ, constant k, and another function, $\phi(t)$. We refer the reader to some works [23, 44–49]. In future work, we intend to address the problems of approximation controllability and asymptotic stability. The Langevin-type differential equation is discussed in various ways, including stochastic, delay, and variable-coefficient differential equations, among others.

Author contributions

Weerawat Sudsutad: Conceptualization, methodology, software, writing-original draft preparation, writing-review suggestions and editing, formal analysis; Aphirak Aphithana: Conceptualization, methodology, writing-original draft preparation, writing-review suggestions and editing, formal analysis; Chatthai Thaiprayoon: Conceptualization, methodology, software, writing-original draft preparation, writing-review suggestions and editing, supervision; Jutarat Kongson: Conceptualization, methodology, writing-original draft preparation, writing-review suggestions and editing, formal analysis, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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