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Research article

An exponentiated Haq distribution: Theoretical aspects, estimation, and modeling of radiation and COVID-19 datasets

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Abstract: In this study, we proposed a new extension of the Haq distribution using an exponentiated-G family of distributions. The new distribution is named the exponentiated Haq distribution. The exponentiated Haq distribution improves the adaptability and flexibility of the underlying distribution and is thus applicable to a variety of datasets with such features as skewness, heavy tails, and high variability. Important properties, such as quantile function, mode, moments, incomplete moments, entropy measure, as well as order statistics of the proposed distribution have been obtained. The estimation of parameters of the exponentiated Haq distribution was conducted using six classical parameter estimation methods and an extensive simulation study was conducted to determine how the estimators performed. Real-world applications to radiation and COVID-19 datasets demonstrate that the proposed distribution outperforms several existing distributions, establishing it as a powerful tool for modeling complex data in diverse fields.

Keywords: Haq distribution; exponentiated-G family; statistical inference; simulation; data analysis
Mathematics Subject Classification: 60E05, 62F10

1. Introduction

Lifetime data has always been of great importance in the past and present, and has a wide application that cuts across different fields like health, management, engineering, and metrology. The ability to represent and analyze time-to-event data through proper and adequate characterization and interpretation is vital for meaningful results and guiding decision-making in different areas. The theoretical consistency and functionality of the frameworks of probabilistic modeling is an invaluable aid in the effort. The widely used distributions, however, are inadequate in capturing the intricate nature of empirical data, including skewness, large tails, and non-homogeneous dynamics of the hazard rate. Historically, restrictions on classical distributions (e.g., small parameter spaces and restrictive moment structures) have been a barrier to their use. In turn, scholars have focused their attention on development of augmented distributions within which structural benefits of well-known families are coupled with increased theory and data flexibility. A particularly prominent example is the use of techniques of random variable power transformation, where extra parameters of the transform have been added to control the shape and tail of the underlying distribution. A variety of extended distribution families have emerged from this line of research. Notable examples include the Marshall-Olkin generated family (MO-G) [1], exponentiated distributions [2], McDonald-G (Mc-G) [3], Frechet-G [4], generalized exponential-G [5], inverted Nadarajah–Haghighi power series [6], exponentiated generalized alpha power exponential [7], length-biased weighted exponentiated inverted exponential [8], exponentiated truncated inverse Weibull-G [9], exponentiated power generalized Weibull power series [10], exponentiated inverted Topp-Leone [11], new odd reparameterized exponential transformed-x family [12], and exponentiated generalized Weibull exponential [13].

The more recent utilization of the exponentiated family of distributions involves the exponentiated Weibull-Weibull [14], exponentiated Weibull power function [15], exponentiated XLindley [16], and some other generalizations, e.g., exponentiated Pareto [17], exponentiated Lomax [18], exponentiated power Lindley [19], exponentiated Frechet [20], exponentiated quasi-Lindley [21], exponentiated Lindley geometric [22], and exponentiated ZLindley [23].

The one-parameter Haq distribution was introduced in [24], combining exponential and xgamma distributions. The probability density function (PDF) of the Haq distribution is

$$g(x;\tau)=\left(\frac{\tau}{\tau+1}\right)^2\left(2+\tau+\frac{\tau}{2}x^2\right)e^{-\tau x},\quad \tau>0\ and\ x>0. \tag{1}$$

The corresponding cumulative distribution function (CDF) is given as:

$$G(x;\tau)=1-\left(\frac{(1+\tau)^2+\tau x+0.5(\tau x)^2}{(1+\tau)^2}\right)e^{-\tau x}. \tag{2}$$

The key aims for conducting the present study are:

- To come up with a more versatile probability distribution characterized as the exponentiated Haq (EHaq) by adding a shape parameter to the power of the CDF of the Haq distribution. The new model is developed to enhance the existing model's shapes regarding its density function and failure rate function.
- To derive its various theoretical characteristics, including moments, reliability measures, and order statistics.
- To estimate the parameters of the EHaq distribution utilizing classical estimation approaches, including maximum likelihood, Anderson Darling, Cramer von Mises, maximum product spacing, ordinary least squares, and weighted least squares. The evaluation of these estimation approaches is assessed via a detailed numerical study.
- Two datasets related to radiation and COVID-19 are utilized to show the applicability and flexibility of the EHaq distribution.

The rest of the study is organized as follows. The new probability distribution is introduced, and its shape analysis is presented in Section 2. Section 3 is based on the derivation of important statistical characteristics. Point parameter estimation, along with a detailed simulation study, is performed in Section 4. Radiation and COVID-19 datasets are analyzed in Section 5. We conclude our study in Section 6.

2. The exponentiated Haq distribution and its shape analysis

In this section, an extended form of the Haq distribution is proposed by inducting a shape parameter using the exponentiation to the CDF function technique $F(x) = [G(x)]^{\delta}$, where $G(x)$ is the CDF of the Haq distribution with scale parameter (τ). The derived CDF of the EHaq distribution with parameters δ and τ is as follows:

$$F(x;\delta,\tau) = \left[1 - \left(\frac{(1+\tau)^2+\tau x+0.5(\tau x)^2}{(1+\tau)^2}\right) e^{-\tau x}\right]^{\delta}, \tag{3}$$

where $\delta > 0, \tau > 0,$ and $x \geq 0.$

The PDF is derived from Eq (3) and is given as:

$$f(x;\delta,\tau) = \frac{\delta\tau^2(\tau x^2+2\tau+4)e^{-\tau x}}{2(1+\tau)^2}\left[1 - \left(1 + \frac{2\tau x+\tau^2 x^2}{2(1+\tau)^2}\right) e^{-\tau x}\right]^{\delta-1}. \tag{4}$$

2.1. Mixture representation of the density function

An important mixture representation of the density function is obtained in this subsection. Start with the PDF given in Eq (4) and use the following binomial expansion for the last term.

$$\left[1 - \left(1 + \frac{2\tau x+\tau^2 x^2}{2(1+\tau)^2}\right) e^{-\tau x}\right]^{\delta-1} = \sum_{i=0}^{\infty}(-1)^i \binom{\delta-1}{i}\left(1 + \frac{2\tau x \,+\, \tau^2 x^2}{2(1\,+\,\tau)^2}\right)^i e^{-i\tau x}.$$

Putting in Eq (4), we have

$$f(x;\delta,\tau) = \frac{\delta\tau^2(\tau x^2+2\tau+4)e^{-\tau x}}{2(1+\tau)^2}\sum_{i=0}^{\infty}(-1)^i \binom{\delta-1}{i}\left(1 + \frac{2\tau x + \tau^2 x^2}{2(1+\tau)^2}\right)^i e^{-i\tau x}. \tag{5}$$

Now again use $(1+z)^i = \sum_{j=0}^{i}\binom{i}{j}(z)^j$ in the last term:

$$\left(1 + \frac{2\tau x \,+\, \tau^2 x^2}{2(1\,+\,\tau)^2}\right)^i = \sum_{j=0}^{i}\binom{i}{j}\left(\frac{2\tau x \,+\, \tau^2 x^2}{2(1\,+\,\tau)^2}\right)^j,$$

and put in Eq (5):

$$f(x;\delta,\tau) = \frac{\delta\tau^2(\tau x^2+2\tau+4)e^{-\tau x}}{2(1+\tau)^2}\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i \binom{\delta-1}{i}\binom{i}{j}\left(\frac{2\tau x + \tau^2 x^2}{2(1+\tau)^2}\right)^j e^{-i\tau x}. \tag{6}$$

Further, expand

$$\left(\frac{2\tau x + \tau^2 x^2}{2(1+\tau)^2}\right)^j = \sum_{k=0}^{j}\binom{j}{k}\left(\frac{\tau x}{(1+\tau)^2}\right)^{j-k}\left(\frac{\tau^2 x^2}{2(1+\tau)^2}\right)^k = \sum_{k=0}^{j}\binom{j}{k}\frac{\tau^{j+k}}{2^k(1+\tau)^{2j}}x^{j+k}.$$

Equation (6) is written as

$$f(x;\delta,\tau) = \frac{\delta\tau^2(\tau x^2 + 2\tau + 4)e^{-\tau x}}{2(1+\tau)^2}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i\binom{\delta-1}{i}\binom{i}{j}\binom{j}{k}\frac{\tau^{j+k}}{2^k(1+\tau)^{2j}}x^{j+k}e^{-i\tau x},$$

and

$$f(x;\delta,\tau) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i\binom{\delta-1}{i}\binom{i}{j}\binom{j}{k}\frac{\delta\tau^{j+k+2}}{2^k(1+\tau)^{2j+2}}\left[(4+2\tau)x^{j+k} + \tau x^{j+k+2}\right]e^{-(i+1)\tau x}. \tag{7}$$

We explore the limiting behavior of the proposed distribution. Mathematically, it is observed at the lower limit (0) of a variable that the PDF of the EHaq distribution starts from zero for all values of δ other than 1. For $\delta = 1$, its starting value is $\frac{\tau^2(2+\tau)}{(1+\tau)^2}$. On the other hand, at the upper limit (∞), it attains the value 0 for all values of δ. We also illustrate it graphically based on some choices of parameters presented in Figure 1.

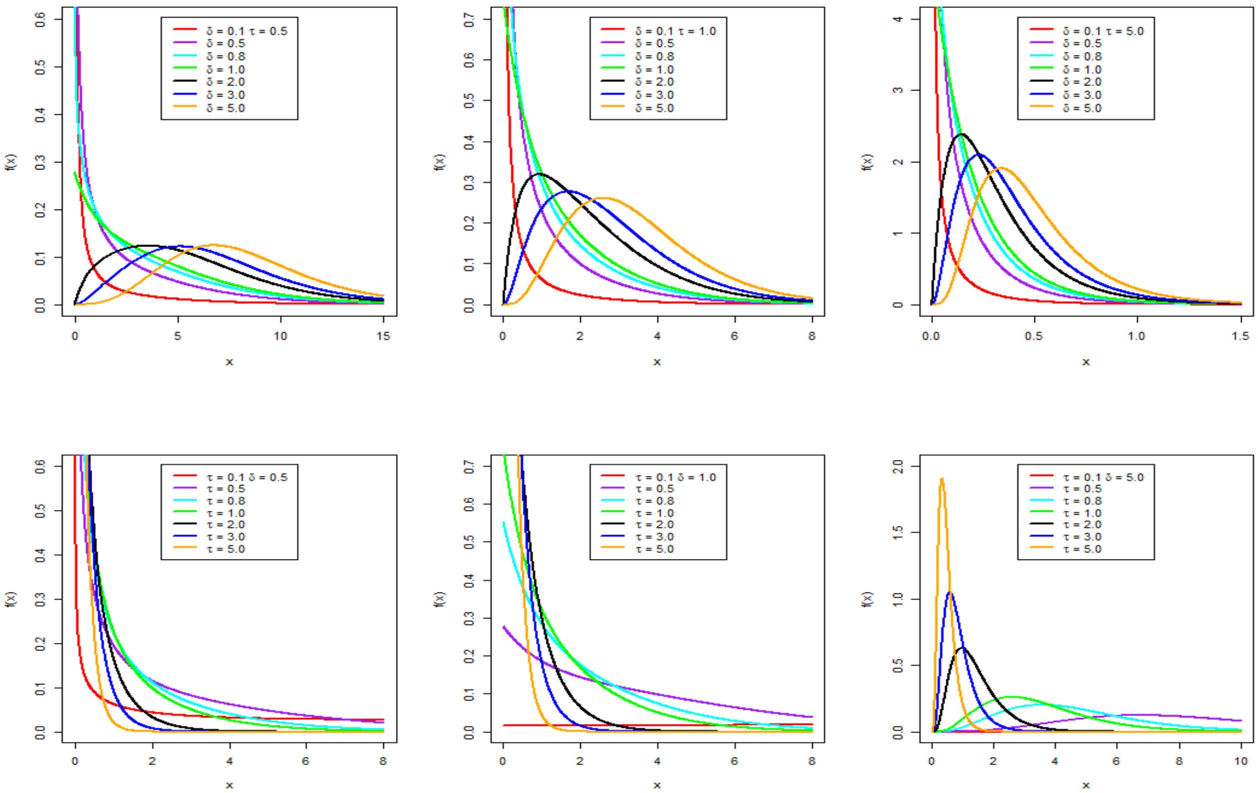

Figure 1. Plots of the EHaq density function for various sets of values for δ and τ.

The influence of the parameters on the shape of the EHaq distribution:

- Effect of shape parameter δ: The density is heavily skewed to the right, with a high peak near the origin and a heavy tail (long tail) for lower values of parameter δ (e.g., 0.1, 0.5). The peakedness decreases and the model spreads out more evenly as the parameter δ increases (e.g., 1.0→5.0). Thus, the shape parameter controls the peakedness and tail thickness of the probability distribution.
- Effect of scale parameter τ: The density function presents decay slowly, producing long heavy tails based on small values of τ. As the parameter τ increases, the density function shifts toward the left and decays much faster, concentrating probability near zero. Thus, the scale parameter τ controls the rate of decay and spread of the probability distribution.

Practical implications:

- The proposed distribution captures rate extreme events with high probability near zero and long right tails when the parameter δ is small. This case of distribution can analyze the failure times of mechanical systems where early failure dominates or finance/assurance data with occasional very large claims.
- The distribution becomes more balanced between peak and spread when the parameter is moderate and approximately equal to 1. This subfamily applies to data observation without heavy extremes, general lifetime modeling, or reliability studies.
- In the third subfamily, when the shape parameter $\delta > 1$, the probability model flattens with lower skewness. The extreme observations are less likely.
- When the scale parameter of the EHaq distribution is small (e.g., 0.1-0.5), the tail of the EHaq distribution is heavy. This situation applies to overdispersed lifetime data and survival modeling.
- The distribution is steep near zero and declines quickly based on large values of parameter τ. This case is applicable in analyzing short survival times, waiting times, or system failures where many events occur very early.

3. Some useful measures of the EHaq distribution

In this section, some theoretical properties of the EHaq distribution, such as survival, hazard, quantile function, moments with associated measures, stress-strength reliability, Renyi entropy, and order statistics, are presented to check the significance of the proposed model.

3.1. Survival and hazard functions of the EHaq distribution

The survival function is defined as

$$S(x;\delta,\tau) = 1 - F(x;\delta,\tau).$$

The survival function of the EHaq distribution is

$$S(x;\delta,\tau) = 1 - \left[1 - \left(\frac{(1+\tau)^2+\tau x+0.5(\tau x)^2}{(1+\tau)^2}\right)e^{-\tau x}\right]^{\delta}. \tag{8}$$

The hazard rate function is given by

$$h(x;\delta,\tau)=\frac{f(x;\delta,\tau)}{S(x;\delta,\tau)}.$$

The hazard function of the EHaq distribution is

$$h(x;\delta,\tau)=\frac{\frac{\delta\tau^2(\tau x^2+2\tau+4)e^{-\tau x}}{2(1+\tau)^2}\left[1-\left(1+\frac{2\tau x+\tau^2x^2}{2(1+\tau)^2}\right)e^{-\tau x}\right]^{\delta-1}}{1-\left[1-\left(\frac{(1+\tau)^2+\tau x+0.5(\tau x)^2}{(1+\tau)^2}\right)e^{-\tau x}\right]^{\delta}}. \tag{9}$$

We also illustrate the hazard function graphically based on some choices of parameters presented in Figure 2.

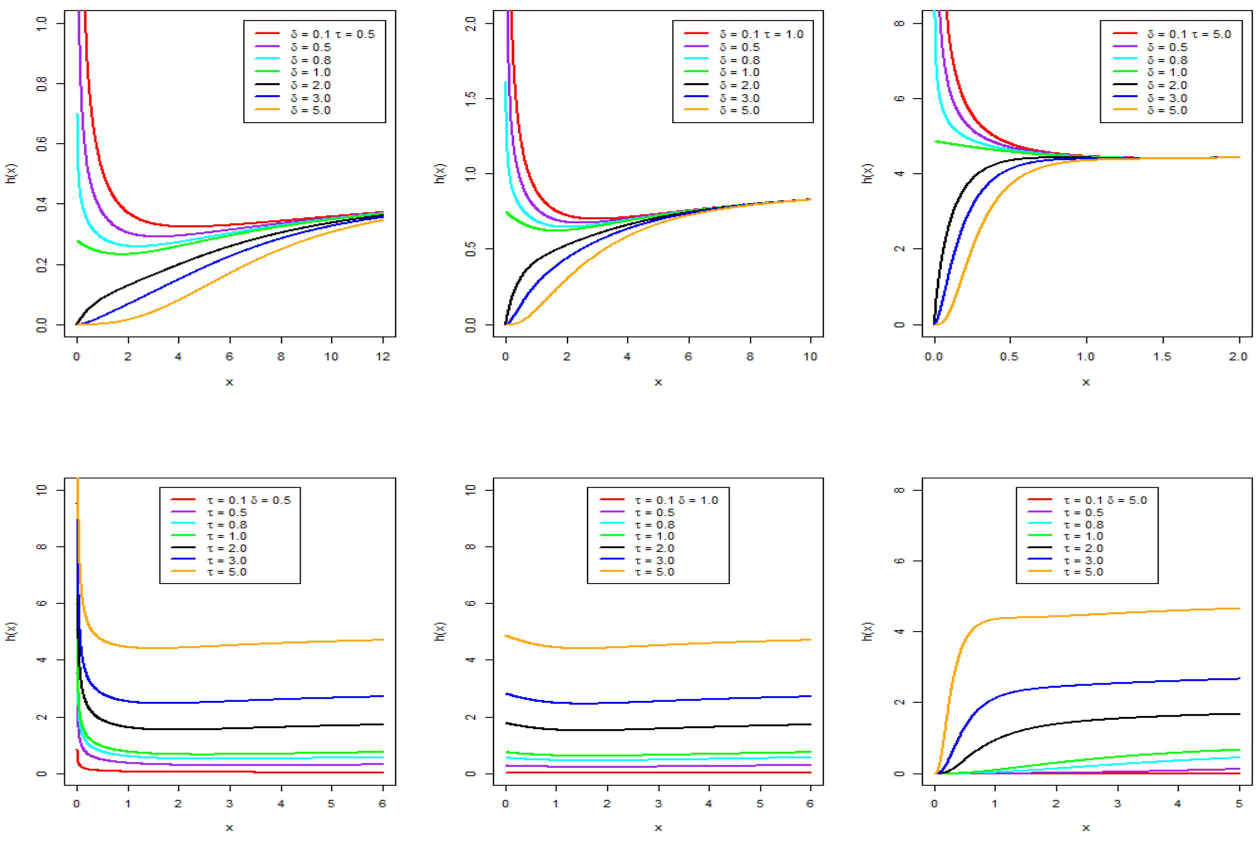

Figure 2. Plots of the EHaq hazard rate function for various sets of values for δ and τ.

The impact of distribution parameters on the failure rate shape of the EHaq distribution:

- Effect of shape parameter δ for fixed τ: The failure rate starts low, sometimes rises slowly, and can remain approximately constant for small values of parameter δ. Also, τ is fixed. The failure rate increases, and represents accelerated failure for moderate values of δ (e.g., 0.8–2). For higher values of δ, the failure rate of the EHaq distribution steeply increases, often resembling early-life reliability weakness followed by rapidly higher risk.
- Effect of scale parameter τ for fixed δ: The hazard rate of the EHaq model tends to rise more slowly; the failure rate curve seems nearly constant or softly increasing, indicating long

survival times with small risk for small values of τ. The hazard rate increases near its origin and peaks early, then stabilizes.

3.2. *Quantile function and random number generator*

It is observed from Eq (1) that the CDF is increasing, so the quantile function can be obtained using the following relation:

$$u_z = G_X^{-1}(z),$$

where z is a random number generated from a uniform distribution with parameters 0 and 1.

$$\left[1-\left(\frac{(1+\tau)^2+\tau x+0.5(\tau x)^2}{(1+\tau)^2}\right)e^{-\tau x}\right]^{\delta}=z,$$

and

$$\left(\frac{(1+\tau)^2+\tau x+0.5(\tau x)^2}{(1+\tau)^2}\right)e^{-\tau x}=1+z^{\frac{1}{\delta}}.$$

The final equation is

$$((1+\tau)^2+\tau x+0.5(\tau x)^2)e^{-\tau x}=(1+\tau)^2\left(1+z^{\frac{1}{\delta}}\right). \quad (10)$$

As the left side of Eq (10) is a multiple of polynomials in terms of variable x and its exponent form, x cannot be separated, and the solution for this equation can be obtained numerically.

In addition to moment-based characteristics, the indices using the quantile function provide replacements for understanding the shape of the probability model, especially when the data observations contain extreme values or outliers. The commonly used procedures include Bowley's coefficient of skewness (BSK) and Moors' coefficient of kurtosis (MKUR).

The BSK is defined as

$$BSK=\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1},$$

where Q_1, Q_2, and Q_3 represent the first quartile, median, and third quartile, respectively.

The MKUR is defined as

$$MKUR=\frac{(Q_7-Q_5)-(Q_3-Q_1)}{Q_6-Q_2}.$$

Numerical values of quartiles, BSK, and MKUR based on parameter choices are attained and presented in Table 1.

Table 1. First three quantiles, BSK, and MKUR values for some selected values of parameters.

δ	τ	Q_1	Q_2	Q_3	BSK	MKUR
0.5	0.5	0.2382	1.1379	3.4049	0.4318	0.5595
	1.0	0.0870	0.4015	1.2298	0.4496	0.5175
	1.5	0.0515ig	0.2345	0.7043	0.4394	0.5142
	2.0	0.0365	0.1647	0.4882	0.4322	0.5170
	3.0	0.0230	0.1033	0.3019	0.4243	0.5222
	4.0	0.0169	0.0754	0.2190	0.4210	0.5252
	5.0	0.0133	0.0594	0.1721	0.4189	0.5270
	6.0	0.0110	0.0491	0.1419	0.4177	0.5280
	7.0	0.0094	0.0419	0.1207	0.4164	0.5286
	8.0	0.0082	0.0365	0.1052	0.4162	0.5294
1.0	0.5	0.5520	1.4097	2.9023	0.2701	0.5808
	1.0	0.4015	1.0182	2.1269	0.2851	0.5705
	1.5	0.2345	0.5847	1.2214	0.2903	0.5581
	2.0	0.1647	0.4063	0.8410	0.2854	0.5557
	3.0	0.1033	0.2521	0.5147	0.2768	0.5573
	4.0	0.0754	0.1831	0.3712	0.2717	0.5594
	5.0	0.0594	0.1440	0.2908	0.2689	0.5608
	5.5	0.0538	0.1302	0.2624	0.2677	0.5612
	6.0	0.0491	0.1188	0.2392	0.2672	0.5616
	7.0	0.0419	0.1011	0.2034	0.2660	0.5624
	8.0	0.0365	0.0881	0.1770	0.2650	0.5627
1.5	0.5	1.0053	2.0654	3.6384	0.1948	0.5892
	1.0	0.7270	1.4979	2.6937	0.2160	0.5822
	1.5	0.4201	0.8567	1.5604	0.2341	0.5698
	2.0	0.2933	0.5922	1.0739	0.2341	0.5653
	3.0	0.1828	0.3649	0.6545	0.2277	0.5645
	4.0	0.1330	0.2642	0.4705	0.2227	0.5657
	5.0	0.1048	0.2074	0.3679	0.2199	0.5667
	6.0	0.0865	0.1709	0.3023	0.2180	0.5674
	7.0	0.0737	0.1454	0.2568	0.2164	0.5680
	8.0	0.0642	0.1266	0.2233	0.2153	0.5684

The findings presented in Table 1 show the behavior of the first quartile, median, third quartile, BSK, and MKUR for different choices of parameters. All three quartiles consistently show a decreasing pattern as parameter τ increases. This indicates that the probability model becomes more concentrated around smaller values. Bowley's coefficient of skewness values are all positive but smaller than 0.5, indicating a moderate right skewness that gradually shows a decreasing pattern with higher values of parameter τ, indicating that the EHaq distribution becomes more symmetric. On the contrary, the Moors' coefficient of kurtosis values range between 0.51 and 0.59 for all considered parameter choices, showing a platykurtic distribution with lighter tails compared to the normal distribution.

3.3. Non-central moments

In this subsection, we derive the rth non-central moments of the EHaq distribution. The rth moment is a key statistical measure that captures key characteristics of the probability model. The main descriptive characteristics of a probability distribution are its central tendency, dispersion, skewness, as well as kurtosis. These parameters are essential for the theoretical background of study and practical implementation. Compliantly with that, the mathematical formula for the rth non-central moment of the EHaq distribution is explained as follows.

$$\mu_r' = \mathbb{E}[X^r] = \int_0^\infty x^r f(x; \delta, \tau) dx.$$

Substitute the density function (7) into the moment definition.

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} (-1)^i \binom{\delta - 1}{i} \binom{i}{j} \binom{j}{k} \frac{\delta \tau^{j+k+2}}{2^k (1 + \tau)^{2j+2}} \left[(4 + 2\tau) \int_0^\infty x^{r+j+k} e^{-(i+1)\tau x} dx + \tau \int_0^\infty x^{r+j+k+2} e^{-(i+1)\tau x} dx \right].$$

Use the gamma integral identity $\int_0^\infty x^a e^{-bx} dx = \frac{\Gamma(a+1)}{b^{a+1}}$, for $a > -1, b > 0$. Apply it to both integrals:

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} (-1)^i \binom{\delta - 1}{i} \binom{i}{j} \binom{j}{k} \frac{\delta \tau^{j+k+2}}{2^k (1 + \tau)^{2j+2}} \left[\frac{(4 + 2\tau)\Gamma(r + j + k + 1)}{[(i + 1)\tau]^{r+j+k+1}} + \frac{\tau \Gamma(r + j + k + 3)}{[(i + 1)\tau]^{r+j+k+3}} \right].$$

The first four non-central moments are obtained by applying the above equation.

Table 2 and Figure 3 summarize how the distribution's statistical properties vary with parameters. The central tendency shows a decreasing pattern as the parameter τ increases, depicting that the distribution moves closer to zero for larger values of parameter τ, whereas the increase in parameter δ commonly results in a higher mean. Similarly, variance shows decreasing behavior with an increase in parameter τ, indicating that the distribution becomes more concentrated around the central value. The variance increases with large values of parameter δ, illustrating a greater spread in the data. The coefficient of skewness values are all positive, indicating that the proposed distribution is right-skewed, with heavier right tails. The coefficient of kurtosis is consistently higher than 3, confirming that the probability model is leptokurtic with heavy tails and sharp peaks. Higher values of τ and δ decrease kurtosis, resulting in less tail heaviness and a milder peak. With increasing τ and δ, the distribution becomes more concentrated, symmetric, and light-tailed.

Table 2. Some computational statistics of the EHaq distribution for some choices of parameters.

δ	τ	Mean	μ_2'	μ_3'	μ_4'	Variance	Skewness	Kurtosis
0.5	0.5	2.3395	14.2331	126.41	1426.9	8.7596	2.0106	8.0995
	1.0	0.9150	2.4444	10.147	55.125	1.6071	2.4389	10.9037
	1.5	0.5357	0.8653	2.2479	7.8161	0.5783	2.6482	12.6837
	2.0	0.3723	0.4203	0.7762	1.9482	0.2817	2.7429	13.6649
	3.0	0.2289	0.1577	0.1796	0.2822	0.1053	2.7890	14.3417
	4.0	0.1651	0.0812	0.0660	0.0743	0.0539	2.7752	14.3496
	5.0	0.1292	0.0494	0.0311	0.0271	0.0327	2.7518	14.1922
	6.0	0.1062	0.0332	0.0170	0.0121	0.0219	2.7304	14.0180
	7.0	0.0903	0.0239	0.0103	0.0062	0.0157	2.7130	13.8652
1.0	0.5	3.7778	25.778	240.00	2773.3	11.506	1.4266	5.6119
	1.0	1.5000	4.5000	19.5000	108.00	2.2500	1.7778	7.2222
	1.5	0.8800	1.6000	4.3378	15.3600	0.8256	1.9685	8.4009
	2.0	0.6111	0.7778	1.5000	3.8333	0.4043	2.0635	9.1210
	3.0	0.3750	0.2917	0.3472	0.5556	0.1510	2.1220	9.7087
	4.0	0.2700	0.1500	0.1275	0.1462	0.0771	2.1191	9.7935
	5.0	0.2111	0.0911	0.0600	0.0533	0.0465	2.1027	9.7271
	5.5	0.1735	0.0612	0.0329	0.0238	0.0311	2.0858	9.6285
	6.0	0.1473	0.0440	0.0200	0.0122	0.0223	2.0714	9.5342
	7.0	0.1281	0.0332	0.0130	0.0069	0.0168	2.0597	9.4533
1.5	0.5	1.9270	6.2832	28.206	158.89	2.5699	1.5034	6.0731
	1.0	1.1330	2.2429	6.2978	22.662	0.9591	1.6853	7.0128
	1.5	0.7867	1.0914	2.1807	5.6624	0.4724	1.7823	7.6326
	2.0	0.4821	0.4091	0.5050	0.8212	0.1767	1.8503	8.1941
	3.0	0.3467	0.2102	0.1854	0.2162	0.0900	1.8545	8.3155
	4.0	0.2709	0.1276	0.0872	0.0788	0.0542	1.8425	8.2879
	5.0	0.2225	0.0857	0.0478	0.0352	0.0362	1.8281	8.2200
	6.0	0.1889	0.0616	0.0290	0.0180	0.0259	1.8153	8.1485
	7.0	0.1642	0.0464	0.0189	0.0101	0.0195	1.8046	8.0846

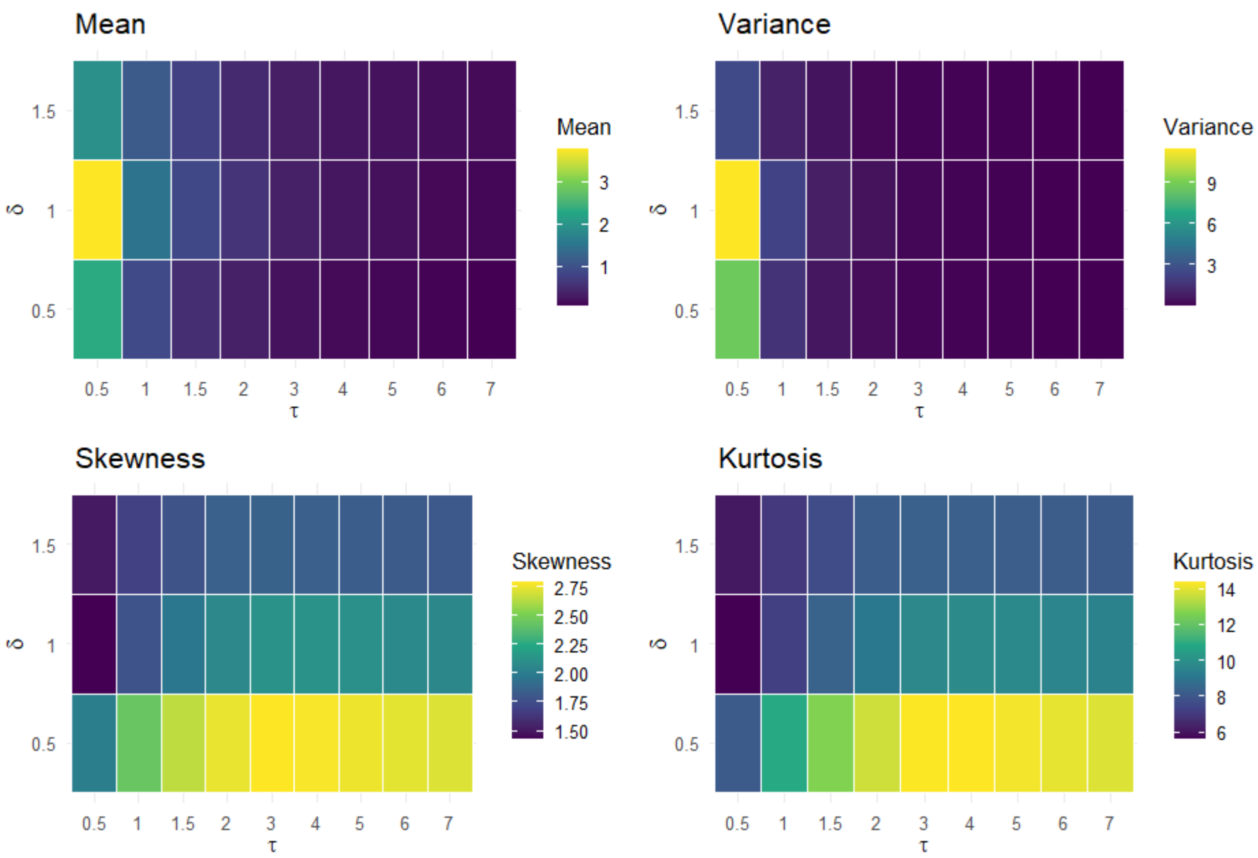

Figure 3. Heatmaps of mean, variance, skewness, and kurtosis for different combinations of parameters.

3.4. Incomplete moments

This subsection is based on the computation of incomplete moments. Incomplete moments serve as an essential analytical tool in economics, especially in the study of income distribution. Incomplete moments allow for computation of important characteristics such as mean deviation, Lorenz and Bonferroni curves, and various inequality indices, such as Pietra and Gini coefficients. A lower incomplete rth moment of the EHaq distribution is defined as

$$\mu_r(t) = \int_0^t x^r f(x)dx.$$

Plug into the definition:

$$\mu_r(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i \binom{\delta-1}{i}\binom{i}{j}\binom{j}{k}\frac{\delta\tau^{j+k+2}}{2^k(1+\tau)^{2j+2}}\left[(4+2\tau)\int_0^t x^{r+j+k}e^{-(i+1)\tau x}dx \right.$$
$$\left. + \tau\int_0^t x^{r+j+k+2}e^{-(i+1)\tau x}dx\right].$$

Utilize the lower incomplete gamma function:

$$\int_0^t x^a e^{-bx} dx = \frac{\gamma(a+1,bt)}{b^{a+1}}, for\ a > -1, b > 0.$$

Apply this to both integrals:

$$\mu_r(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i\binom{\delta-1}{i}\binom{i}{j}\binom{j}{k}\frac{\delta\tau^{j+k+2}}{2^k(1\ +\ \tau)^{2j+2}}\left[\frac{(4+2\tau)\gamma(r+j+k+1,(i+1)\tau t)}{[(i+1)\tau]^{r+j+k+1}} + \frac{\tau\gamma(r+j+k+3,(i+1)\tau t)}{[(i+1)\tau]^{r+j+k+3}}\right].$$

3.5. Lorenz and Bonferroni curves

In this subsection, we compute Lorenz and Bonferroni curves. The Lorenz curve $(L(t))$ shows a graphical representation of the cumulative share of income, or wealth, thus allowing a qualitative evaluation of the inequality. The Bonferroni curve $(B(t))$ is an extension of the Lorenz curve, which provides a more thorough view by including additional characteristics of discrepancy. Theoretical definitions of both curves are given by

$$L(t) = \frac{\mu_1(t)}{\mu_1'}\ and\ B(t) = \frac{L(t)}{F(t)},$$

where $\mu_1(t)$ is the first incomplete moment and μ_1 is the mean of distribution.

$$\mu_1(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i\binom{\delta-1}{i}\binom{i}{j}\binom{j}{k}\frac{\delta\tau^{j+k+2}}{2^k(1\ +\ \tau)^{2j+2}}\left[\frac{(4+2\tau)\gamma(j+k+2,(i+1)\tau t)}{[(i+1)\tau]^{j+k+2}} + \frac{\tau\gamma(j+k+4,(i+1)\tau t)}{[(i+1)\tau]^{j+k+4}}\right],$$

$$\mu_1 = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i\binom{\delta-1}{i}\binom{i}{j}\binom{j}{k}\frac{\delta\tau^{j+k+2}}{2^k(1\ +\ \tau)^{2j+2}}\left[\frac{(4+2\tau)\Gamma(j+k+2)}{[(i+1)\tau]^{j+k+2}} + \frac{\tau\Gamma(j+k+4)}{[(i+1)\tau]^{j+k+4}}\right],$$

where $\gamma(s,x) = \int_0^x t^{s-1}\, e^{-t} dt$ is the lower incomplete gamma function.

3.6. Mean deviation

Mean deviation (MD) is a statistical characteristic utilized to quantify dispersion or variability within a probability model. In the context of probability models, it represents average absolute deviation of data points from the mean [25]. The MD is mathematically defined as

$$MD(\mu) = 2\mu\, F(\mu) - 2\int_0^{\mu} x f(x;\delta,\tau)\, dx.$$

For the EHaq distribution, the expression of MD is

$$MD(\mu) = 2\mu\, F(\mu)$$
$$-2\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^i\binom{\delta-1}{i}\binom{i}{j}\binom{j}{k}\frac{\delta\tau^{j+k+2}}{2^k(1+\tau)^{2j+2}}\left[\frac{(4+2\tau)\gamma(j+k+2,(i+1)\tau\mu)}{[(i+1)\tau]^{j+k+2}}\right.$$
$$\left.+\frac{\tau\gamma(j+k+4,(i+1)\tau\mu)}{[(i+1)\tau]^{j+k+4}}\right].$$

3.7. Stress-strength reliability

The idea of stress-strength reliability (SSR) plays a main role in survival and reliability analysis. It is defined as the probability that a component or system with random strength U_1 can withstand an applied random stress U_2. The SSR for the EHaq distribution is derived below.

Suppose $U_1 \sim EHaq(\tau,\delta_1)$ and $U_2 \sim EHaq(\tau,\delta_2)$, and then the SSR parameter R can be obtained as

$$R = P(U_1 > X) = \int_0^{\infty} P(U_1 > X|X = x) f_x(x) dx,$$

$$R = \int_0^{\infty} S_{U_1}(x) f_{U_2}(x) dx,$$

$$R = \int_0^{\infty}\left[1-\left[1-\left(1+\frac{2\tau x+\tau^2x^2}{2(1+\tau)^2}\right)e^{-\tau x}\right]^{\delta_1}\right]\frac{\delta_2\tau^2(4+2\tau+\tau x^2)e^{-\tau x}}{2(1+\tau)^2}\left[1\right.$$
$$\left.-\left(1+\frac{2\tau x+\tau^2x^2}{2(1+\tau)^2}\right)e^{-\tau x}\right]^{\delta_2-1} dx,$$

$$R = \int_0^{\infty}\frac{\delta_2\tau^2(4+2\tau+\tau x^2)e^{-\tau x}}{2(1+\tau)^2}\left[1-\left(1+\frac{2\tau x+\tau^2x^2}{2(1+\tau)^2}\right)e^{-\tau x}\right]^{\delta_2-1} dx$$
$$-\int_0^{\infty}\frac{\delta_2\tau^2(4+2\tau+\tau x^2)e^{-\tau x}}{2(1+\tau)^2}\left[1-\left(1+\frac{2\tau x+\tau^2x^2}{2(1+\tau)^2}\right)e^{-\tau x}\right]^{\delta_2-1}\left[1\right.$$
$$\left.-\left(1+\frac{2\tau x+\tau^2x^2}{2(1+\tau)^2}\right)e^{-\tau x}\right]^{\delta_1} dx.$$

As the total area under the EHaq is unity,

$$R = 1 - \int_0^\infty \frac{\delta_2 \tau^2 (4 + 2\tau + \tau x^2) e^{-\tau x}}{2(1+\tau)^2} \left[1 - \left(1 + \frac{2\tau x + \tau^2 x^2}{2(1+\tau)^2}\right) e^{-\tau x}\right]^{\delta_2 - 1} \left[1 - \left(1 + \frac{2\tau x + \tau^2 x^2}{2(1+\tau)^2}\right) e^{-\tau x}\right]^{\delta_1} dx,$$

$$R = 1 - \int_0^\infty \frac{\delta_2 \tau^2 (4 + 2\tau + \tau x^2) e^{-\tau x}}{2(1+\tau)^2} \left[1 - \left(1 + \frac{2\tau x + \tau^2 x^2}{2(1+\tau)^2}\right) e^{-\tau x}\right]^{\delta_1 + \delta_2 - 1} dx.$$

Now by multiplying with $\delta_1 + \delta_2$, we get

$$R = 1 - \int_0^\infty (\delta_1 + \delta_2) \frac{\delta_2 \tau^2 (4 + 2\tau + \tau x^2) e^{-\tau x}}{(\delta_1 + \delta_2) 2(1+\tau)^2} \left[1 - \left(1 + \frac{2\tau x + \tau^2 x^2}{2(1+\tau)^2}\right) e^{-\tau x}\right]^{\delta_1 + \delta_2 - 1} dx,$$

$$R = 1 - \frac{\delta_2}{\delta_1 + \delta_2} \int_0^\infty \frac{(\delta_1 + \delta_2) \tau^2 (4 + 2\tau + \tau x^2) e^{-\tau x}}{2(1+\tau)^2} \left[1 - \left(1 + \frac{2\tau x + \tau^2 x^2}{2(1+\tau)^2}\right) e^{-\tau x}\right]^{\delta_1 + \delta_2 - 1} dx.$$

As the total area under the EHaq is unity,

$$R = \frac{\delta_1}{\delta_1 + \delta_2}.$$

3.8. Mean residual life

The mean residual life (MRL) function is a vital idea in the field of survival analysis and reliability studies. The MRL provides the expected remaining lifetime of a component or system, given that it has already survived up to a specific time t.

The MRL function for the EHaq distribution at time t is defined as

$$m(t) = \mathbb{E}[X - t | X > t] = \frac{1}{S(t)} \int_t^\infty x g(x) dx - t \quad for\ t > 0,$$

where $\int_t^\infty x g(x) dx$ is the first incomplete moment.

$$\int_t^\infty x g(x) dx = \sum_{i=0}^\infty \sum_{j=0}^i \sum_{k=0}^j (-1)^i \binom{\delta - 1}{i} \binom{i}{j} \binom{j}{k} \frac{\delta \tau^{j+k+2}}{2^k (1 + \tau)^{2j+2}} \left[\frac{(4 + 2\tau) \gamma(j + k + 2, (i+1)\tau t)}{[(i+1)\tau]^{j+k+2}} + \frac{\tau \gamma(j + k + 4, (i+1)\tau t)}{[(i+1)\tau]^{j+k+4}}\right],$$

and

$$S(t) = 1 - \left[1 - \left(1 + \frac{2\tau t + \tau^2 t^2}{2(1+\tau)^2}\right) e^{-\tau t}\right]^{\delta}.$$

So, the final expression of the mean residual life function is obtained by substituting $\int_t^\infty x g(x) dx$ and $S(t)$ into the definition.

4. Estimation of parameters

In this section, we discuss parameter estimation of the EHaq distribution. Six different methods are utilized to obtain point estimates of the EHaq distribution, including maximum likelihood (ML), maximum product spacing (MPS), Anderson-Darling (AD), Cramer-von Mises (CVM), least squares, and weighted least squares estimation. These optimization tasks were carried out using the quasi-Newton approach developed in [26] and implemented in the R software (4.5.1 version). The *optim* and *nlminb* functions are utilized.

4.1. Method of ML estimation

In this subsection, we look at the ML estimation approach, one of the most often used estimation techniques that deals with estimating the EHaq distribution's τ and δ parameters.

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from the EHaq distribution and $x_1, x_2, x_3, \ldots, x_n$ represent the observed values of the sample. Then the log-likelihood function $l(\delta, \tau)$ is given by

$$l(\delta,\tau) = n\ln(\delta) + 2n\ln(\tau) - 2\ln(2) - 2n\ln(1+\tau) - \tau\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln(\tau x_i^2 + 2\tau + 4)$$
$$+ (\delta - 1)\sum_{i=1}^{n}\ln\left[1 - \left(1 + \frac{2\tau x_i + \tau^2 x_i^2}{2(1+\tau)^2}\right)e^{-\tau x_i}\right].$$

To obtain estimates of the parameters, we differentiate the above equation with respect to parameters and set them equal to zero.

$$\frac{\partial l(\delta,\tau)}{\partial\tau} = \frac{2n}{\tau} - \frac{2n}{1+\tau} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{2 + x_i^2}{\tau x_i^2 + 2\tau + 4}$$
$$+ (\delta - 1)\sum_{i=1}^{n}\frac{\left(\frac{\tau(2(4 + 3\tau + \tau^2) + 2\tau x_i + \tau(1+\tau)x_i^2)}{2(1+\tau)^3}\right)x_i e^{-\tau x_i}}{1 - \left(1 + \frac{2\tau x_i + \tau^2 x_i^2}{2(1+\tau)^2}\right)e^{-\tau x_i}},$$

and

$$\frac{\partial l(\delta,\tau)}{\partial\delta}=\frac{n}{\delta}+\sum_{i=1}^{n}\ln\left[1-\left(1+\frac{2\tau x_i+\tau^2x_i^2}{2(1+\tau)^2}\right)e^{-\tau x_i}\right].$$

The parameter estimates $(\hat{\delta},\hat{\tau})$ of $\omega=(\delta,\tau)$ are obtained by solving these equations numerically. From the methodology of maximum likelihood estimation, the joint distribution of the estimations $(\hat{\delta},\hat{\tau})$ is asymptotically bivariate normal.

$$\sqrt{n}\begin{pmatrix}\hat{\delta}-\delta\\ \hat{\tau}-\tau\end{pmatrix}\sim N_2\left[\begin{pmatrix}\hat{\delta}-\delta\\ \hat{\tau}-\tau\end{pmatrix},\frac{1}{n}K(\omega)^{-1}\right],$$

where $K(\omega)$ is the expected Fisher information (FI) matrix computed as follows:

$$K(\omega)=\begin{bmatrix}K_{\delta\delta} & K_{\delta\tau}\\ K_{\tau\delta} & K_{\tau\tau}\end{bmatrix}=\begin{bmatrix}-E\left(\frac{\partial^2 l(\delta,\tau)}{\partial\delta^2}\right) & -E\left(\frac{\partial^2 l(\delta,\tau)}{\partial\delta\partial\tau}\right)\\ -E\left(\frac{\partial^2 l(\delta,\tau)}{\partial\tau\partial\delta}\right) & -E\left(\frac{\partial^2 l(\delta,\tau)}{\partial\tau^2}\right)\end{bmatrix}.$$

4.2. Method of MPS estimation

The MPS estimation approach introduced in [27] serves as an alternative to the ML estimation technique. Estimates based on MPS estimation techniques are obtained by maximizing the geometric mean of spacing between successive distribution functions. MPS estimates are obtained by maximizing the following function.

$$\varphi(x_i)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(x_i),$$

where $D_i(x_i)=F(x_{i:n})-F(x_{i-1:n})$, $F(x_{0:n})=0$, and $F(x_{n+1:1})=1$.

4.3. Method of AD estimation

This subsection is based on estimation of EHaq distribution parameters using the AD estimation approach. We present the AD estimation method for estimating parameters δ and τ of the EHaq distribution. The AD estimation is based on the Anderson-Darling goodness-of-fit statistic introduced in [28]. The AD estimates $\hat{\delta}_{AD}$, $\hat{\tau}_{AD}$ of the parameters δ and τ are obtained by minimizing

$$AD(\delta,\tau)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_{i:n})+\log S(x_{n+1-i:n})].$$

4.4. *Method of CVM estimation*

In this subsection, we present the CVM estimators for parameters of the EHaq distribution. The CVM approach was introduced in [29] and is based on minimizing the distance between the distribution CDF and empirical distribution function. The CVM estimates are obtained by the maximizing function defined below.

$$CVM(\delta,\tau)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(x_{i:n})-\frac{2i-1}{2n}\right)^2.$$

4.5. *Method of ordinary least squares estimation*

In this subsection, ordinary least squares (OLS) estimation was discussed in terms of parameter estimation on an EHaq distribution. The method, introduced in [30], served as an alternative to the maximum likelihood method. The OLS estimates are determined with the objective of minimizing the following objective function.

$$OLS(\delta,\tau)=\sum_{i=1}^{n}\left[F(x_{i:n})-\frac{i}{n+1}\right]^2.$$

4.6. *Method of weighted least squares estimation*

This subsection introduces weighted least squares (WLS) estimators for parameters δ and τ of the EHaq distribution, based on the method proposed by [30]. The WLS estimates $\hat{\delta}$ and $\hat{\tau}$ are obtained minimizing the distance given below.

$$WLS(\delta,\tau)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{(n+1-i)i}\left[F(x_{i:n})-\frac{i}{n+1}\right]^2.$$

4.7. *Numerical simulation*

This subsection presents a detailed simulation study to illustrate estimation performance of the suggested estimators (ML, AD, CVM, OLS, WLS, and MPS) for point estimation of the EHaq distribution. The estimation performance of these considered estimators is evaluated based on absolute bias (AB), mean relative error (MRE), and mean squared error (MSE), computed from 10,000 replications of the simulation across different parameter settings and sample sizes. The formulas utilized to compute AB, MRE, and MSE are presented below.

$$\text{AB}=\frac{1}{10{,}000}\sum_{i=1}^{10{,}000}|\hat{\varphi}_i-\varphi|,\ \text{MRE}=\frac{1}{10{,}000}\sum_{i=1}^{10{,}000}\frac{|\hat{\varphi}_i-\varphi|}{\varphi},\text{and MSE}=\frac{1}{10{,}000}\sum_{i=1}^{10{,}000}(\hat{\varphi}_i-\varphi)^2,$$

where $\varphi=(\delta,\tau)$. The following settings are considered for the simulation study. The random samples are selected from the EHaq distribution using the quantile function provided in Eq (10). The sample sizes are $n=$ 25, 50, 100, 200, and 300. The initial parameter settings are: Setting I:

$(\delta = 0.5, \tau = 0.5)$, Setting II: $(\delta = 0.5, \tau = 1.0)$, Setting III: $(\delta = 0.5, \tau = 1.5)$, Setting IV: $(\delta = 1.0, \tau = 0.5)$, Setting V: $(\delta = 1.0, \tau = 1.0)$, Setting VI: $(\delta = 1.0, \tau = 1.5)$, Setting VII: $(\delta = 15, \tau = 0.5)$, Setting VIII: $(\delta = 1.5, \tau = 1.0)$, and Setting IX: $(\delta = 1.5, \tau = 1.5)$. The results of the simulation study are presented in Tables 3–11.

Table 3. Estimation of the EHaq distribution parameters for $\delta = 0.5$ and $\tau = 0.5$.

n	Method	$\hat{\delta}$				$\hat{\tau}$			
		Estimate	Bias	MRE	MSE	Estimate	Bias	MRE	MSE
25	ML	0.5525	0.0525	0.1050	0.0225	0.5439	0.0439	0.0878	0.0174
	AD	0.5249	0.0249	0.0498	0.0200	0.5192	0.0192	0.0385	0.0161
	CVM	0.5582	0.0582	0.1165	0.0408	0.5503	0.0503	0.1006	0.0300
	OLS	0.5110	0.0110	0.0221	0.0283	0.5039	0.0039	0.0079	0.0225
	WLS	0.5156	0.0156	0.0312	0.0244	0.5102	0.0102	0.0203	0.0199
	MPS	0.4740	0.0260	0.0520	0.0137	0.4678	0.0322	0.0644	0.0126
50	ML	0.5280	0.0280	0.0560	0.0084	0.5230	0.0230	0.0461	0.0074
	AD	0.5140	0.0140	0.0281	0.0085	0.5114	0.0114	0.0228	0.0077
	CVM	0.5261	0.0261	0.0522	0.0135	0.5252	0.0252	0.0503	0.0124
	OLS	0.5043	0.0043	0.0086	0.0114	0.5029	0.0029	0.0059	0.0107
	WLS	0.5082	0.0082	0.0165	0.0098	0.5075	0.0075	0.0150	0.0090
	MPS	0.4848	0.0152	0.0303	0.0062	0.4808	0.0192	0.0385	0.0061
100	ML	0.5164	0.0164	0.0328	0.0035	0.5121	0.0121	0.0243	0.0030
	AD	0.5080	0.0080	0.0161	0.0035	0.5058	0.0058	0.0117	0.0031
	CVM	0.5139	0.0139	0.0278	0.0057	0.5121	0.0121	0.0241	0.0051
	OLS	0.5034	0.0034	0.0067	0.0052	0.5012	0.0012	0.0025	0.0048
	WLS	0.5063	0.0063	0.0127	0.0044	0.5049	0.0049	0.0097	0.0040
	MPS	0.4941	0.0059	0.0118	0.0028	0.4902	0.0098	0.0196	0.0027
200	ML	0.5103	0.0103	0.0206	0.0014	0.5066	0.0066	0.0133	0.0011
	AD	0.5054	0.0054	0.0108	0.0014	0.5036	0.0036	0.0073	0.0012
	CVM	0.5070	0.0070	0.0140	0.0026	0.5066	0.0066	0.0132	0.0024
	OLS	0.5018	0.0018	0.0036	0.0025	0.5012	0.0012	0.0024	0.0023
	WLS	0.5038	0.0038	0.0076	0.0021	0.5035	0.0035	0.0070	0.0019
	MPS	0.5002	0.0002	0.0004	0.0011	0.4966	0.0034	0.0068	0.0010
300	ML	0.5070	0.0070	0.0141	0.0007	0.5038	0.0038	0.0076	0.0006
	AD	0.5032	0.0032	0.0064	0.0007	0.5017	0.0017	0.0035	0.0006
	CVM	0.5042	0.0042	0.0084	0.0016	0.5037	0.0037	0.0074	0.0016
	OLS	0.5008	0.0008	0.0015	0.0016	0.5001	0.0001	0.0002	0.0015
	WLS	0.5024	0.0024	0.0048	0.0013	0.5019	0.0019	0.0037	0.0012
	MPS	0.5013	0.0013	0.0026	0.0006	0.4981	0.0019	0.0038	0.0005

Table 4. Estimation of the EHaq distribution for $\delta = 0.5$ and $\tau = 1.0$.

n	Method	$\hat{\delta}$				$\hat{\tau}$			
		Estimate	Bias	MRE	MSE	Estimate	Bias	MRE	MSE
25	ML	0.5577	0.0577	0.1154	0.0234	1.1105	0.1105	0.1105	0.0978
	AD	0.5298	0.0298	0.0596	0.0209	1.0545	0.0545	0.0545	0.0906
	CVM	0.5591	0.0591	0.1182	0.0404	1.1242	0.1242	0.1242	0.1747
	OLS	0.5126	0.0126	0.0252	0.0284	1.0193	0.0193	0.0193	0.1274
	WLS	0.5178	0.0178	0.0356	0.0251	1.0343	0.0343	0.0343	0.1143
	MPS	0.4816	0.0184	0.0367	0.0134	0.9452	0.0548	0.0548	0.0630
50	ML	0.5299	0.0299	0.0597	0.0081	1.0516	0.0516	0.0516	0.0334
	AD	0.5140	0.0140	0.0279	0.0078	1.0240	0.0240	0.0240	0.0333
	CVM	0.5252	0.0252	0.0503	0.0130	1.0543	0.0543	0.0543	0.0587
	OLS	0.5038	0.0038	0.0076	0.0110	1.0049	0.0049	0.0049	0.0501
	WLS	0.5086	0.0086	0.0171	0.0095	1.0166	0.0166	0.0166	0.0424
	MPS	0.4897	0.0103	0.0207	0.0056	0.9632	0.0368	0.0368	0.0263
100	ML	0.5185	0.0185	0.0370	0.0031	1.0279	0.0279	0.0279	0.0128
	AD	0.5099	0.0099	0.0198	0.0031	1.0162	0.0162	0.0162	0.0137
	CVM	0.5141	0.0141	0.0282	0.0056	1.0320	0.0320	0.0320	0.0267
	OLS	0.5037	0.0037	0.0074	0.0051	1.0078	0.0078	0.0078	0.0244
	WLS	0.5068	0.0068	0.0137	0.0044	1.0150	0.0150	0.0150	0.0201
	MPS	0.4992	0.0008	0.0016	0.0023	0.9852	0.0148	0.0148	0.0107
200	ML	0.5091	0.0091	0.0181	0.0010	1.0118	0.0118	0.0118	0.0045
	AD	0.5041	0.0041	0.0081	0.0010	1.0060	0.0060	0.0060	0.0045
	CVM	0.5055	0.0055	0.0110	0.0025	1.0121	0.0121	0.0121	0.0117
	OLS	0.5004	0.0004	0.0009	0.0024	1.0002	0.0002	0.0002	0.0113
	WLS	0.5026	0.0026	0.0053	0.0020	1.0054	0.0054	0.0054	0.0092
	MPS	0.5016	0.0016	0.0031	0.0008	0.9952	0.0048	0.0048	0.0037
300	ML	0.5061	0.0061	0.0122	0.0005	1.0069	0.0069	0.0069	0.0019
	AD	0.5030	0.0030	0.0060	0.0005	1.0034	0.0034	0.0034	0.0021
	CVM	0.5034	0.0034	0.0069	0.0017	1.0067	0.0067	0.0067	0.0077
	OLS	0.5001	0.0001	0.0001	0.0016	0.9988	0.0012	0.0012	0.0075
	WLS	0.5018	0.0018	0.0036	0.0013	1.0027	0.0027	0.0027	0.0059
	MPS	0.5024	0.0024	0.0049	0.0004	0.9985	0.0015	0.0015	0.0017

Table 5. Estimation of the EHaq distribution for $\delta = 0.5$ and $\tau = 1.5$.

n	Method	$\hat{\delta}$				$\hat{\tau}$			
		Estimate	Bias	MRE	MSE	Estimate	Bias	MRE	MSE
25	ML	0.5589	0.0589	0.1178	0.0233	1.6736	0.1736	0.1157	0.2417
	AD	0.5294	0.0294	0.0587	0.0201	1.5833	0.0833	0.0555	0.2116
	CVM	0.5565	0.0565	0.1129	0.0373	1.6967	0.1967	0.1312	0.4362
	OLS	0.5104	0.0104	0.0207	0.0261	1.5269	0.0269	0.0179	0.3086
	WLS	0.5163	0.0163	0.0326	0.0236	1.5492	0.0492	0.0328	0.2695
	MPS	0.4849	0.0151	0.0303	0.0131	1.4151	0.0849	0.0566	0.1520
50	ML	0.5332	0.0332	0.0663	0.0081	1.5881	0.0881	0.0587	0.0875
	AD	0.5178	0.0178	0.0356	0.0077	1.5478	0.0478	0.0319	0.0877
	CVM	0.5289	0.0289	0.0577	0.0138	1.5984	0.0984	0.0656	0.1619
	OLS	0.5075	0.0075	0.0149	0.0115	1.5185	0.0185	0.0124	0.1361
	WLS	0.5122	0.0122	0.0244	0.0100	1.5369	0.0369	0.0246	0.1165
	MPS	0.4949	0.0051	0.0102	0.0053	1.4531	0.0469	0.0313	0.0651
100	ML	0.5171	0.0171	0.0341	0.0027	1.5388	0.0388	0.0259	0.0287
	AD	0.5085	0.0085	0.0170	0.0027	1.5204	0.0204	0.0136	0.0306
	CVM	0.5120	0.0120	0.0241	0.0054	1.5431	0.0431	0.0287	0.0653
	OLS	0.5018	0.0018	0.0036	0.0050	1.5043	0.0043	0.0029	0.0599
	WLS	0.5050	0.0050	0.0100	0.0042	1.5168	0.0168	0.0112	0.0485
	MPS	0.4999	0.0001	0.0002	0.0019	1.4773	0.0227	0.0151	0.0239
200	ML	0.5096	0.0096	0.0192	0.0009	1.5200	0.0200	0.0134	0.0091
	AD	0.5049	0.0049	0.0098	0.0008	1.5113	0.0113	0.0075	0.0096
	CVM	0.5065	0.0065	0.0131	0.0025	1.5245	0.0245	0.0163	0.0305
	OLS	0.5015	0.0015	0.0030	0.0023	1.5053	0.0053	0.0036	0.0291
	WLS	0.5038	0.0038	0.0076	0.0020	1.5139	0.0139	0.0093	0.0235
	MPS	0.5033	0.0033	0.0067	0.0007	1.4973	0.0027	0.0018	0.0076
300	ML	0.5054	0.0054	0.0108	0.0004	1.5099	0.0099	0.0066	0.0040
	AD	0.5027	0.0027	0.0053	0.0004	1.5060	0.0060	0.0040	0.0043
	CVM	0.5047	0.0047	0.0094	0.0017	1.5166	0.0166	0.0111	0.0201
	OLS	0.5013	0.0013	0.0027	0.0016	1.5039	0.0039	0.0026	0.0195
	WLS	0.5031	0.0031	0.0061	0.0013	1.5099	0.0099	0.0066	0.0154
	MPS	0.5026	0.0026	0.0051	0.0003	1.4994	0.0006	0.0004	0.0034

Table 6. Estimation of the EHaq distribution for $\delta = 1.0$ and $\tau = 0.5$.

n	Method	$\hat{\delta}$				$\hat{\tau}$			
		Estimate	Bias	MRE	MSE	Estimate	Bias	MRE	MSE
25	ML	1.1270	0.1270	0.1270	0.1429	0.5315	0.0315	0.0629	0.0111
	AD	1.0664	0.0664	0.0664	0.1296	0.5115	0.0115	0.0231	0.0107
	CVM	1.1649	0.1649	0.1649	0.3023	0.5341	0.0341	0.0681	0.0167
	OLS	1.0427	0.0427	0.0427	0.1934	0.4987	0.0013	0.0025	0.0135
	WLS	1.0523	0.0523	0.0523	0.1667	0.5043	0.0043	0.0086	0.0119
	MPS	0.9267	0.0733	0.0733	0.0811	0.4674	0.0326	0.0652	0.0092
50	ML	1.0569	0.0569	0.0569	0.0488	0.5152	0.0152	0.0304	0.0048
	AD	1.0283	0.0283	0.0283	0.0480	0.5054	0.0054	0.0108	0.0049
	CVM	1.0680	0.0680	0.0680	0.0754	0.5161	0.0161	0.0323	0.0069
	OLS	1.0138	0.0138	0.0138	0.0608	0.4990	0.0010	0.0020	0.0062
	WLS	1.0234	0.0234	0.0234	0.0518	0.5034	0.0034	0.0068	0.0053
	MPS	0.9430	0.0570	0.0570	0.0374	0.4776	0.0224	0.0449	0.0045
100	ML	1.0295	0.0295	0.0295	0.0214	0.5076	0.0076	0.0152	0.0023
	AD	1.0153	0.0153	0.0153	0.0222	0.5029	0.0029	0.0058	0.0024
	CVM	1.0331	0.0331	0.0331	0.0304	0.5080	0.0080	0.0159	0.0031
	OLS	1.0072	0.0072	0.0072	0.0273	0.4995	0.0005	0.0009	0.0029
	WLS	1.0150	0.0150	0.0150	0.0231	0.5027	0.0027	0.0054	0.0025
	MPS	0.9638	0.0362	0.0362	0.0187	0.4858	0.0142	0.0284	0.0023
200	ML	1.0141	0.0141	0.0141	0.0094	0.5040	0.0040	0.0080	0.0010
	AD	1.0076	0.0076	0.0076	0.0105	0.5018	0.0018	0.0035	0.0012
	CVM	1.0169	0.0169	0.0169	0.0139	0.5045	0.0045	0.0090	0.0015
	OLS	1.0042	0.0042	0.0042	0.0132	0.5003	0.0003	0.0006	0.0015
	WLS	1.0083	0.0083	0.0083	0.0109	0.5020	0.0020	0.0039	0.0012
	MPS	0.9767	0.0233	0.0233	0.0089	0.4915	0.0085	0.0169	0.0010
300	ML	1.0079	0.0079	0.0079	0.0061	0.5025	0.0025	0.0051	0.0007
	AD	1.0034	0.0034	0.0034	0.0068	0.5011	0.0011	0.0021	0.0008
	CVM	1.0091	0.0091	0.0091	0.0088	0.5028	0.0028	0.0055	0.0010
	OLS	1.0007	0.0007	0.0007	0.0085	0.5000	0.0000	0.0001	0.0009
	WLS	1.0040	0.0040	0.0040	0.0069	0.5013	0.0013	0.0025	0.0008
	MPS	0.9812	0.0188	0.0188	0.0060	0.4936	0.0064	0.0128	0.0007

Table 7. Estimation of the EHaq distribution for $\delta = 1.0$ and $\tau = 1.0$.

n	Method	$\hat{\delta}$				$\hat{\tau}$			
		Estimate	Bias	MRE	MSE	Estimate	Bias	MRE	MSE
25	ML	1.1191	0.1191	0.1191	0.1238	1.0748	0.0748	0.0748	0.0553
	AD	1.0569	0.0569	0.0569	0.1076	1.0301	0.0301	0.0301	0.0512
	CVM	1.1466	0.1466	0.1466	0.2152	1.0830	0.0830	0.0830	0.0835
	OLS	1.0294	0.0294	0.0294	0.1390	1.0037	0.0037	0.0037	0.0644
	WLS	1.0402	0.0402	0.0402	0.1243	1.0151	0.0151	0.0151	0.0571
	MPS	0.9239	0.0761	0.0761	0.0726	0.9330	0.0670	0.0670	0.0429
50	ML	1.0533	0.0533	0.0533	0.0459	1.0328	0.0328	0.0328	0.0229
	AD	1.0251	0.0251	0.0251	0.0456	1.0120	0.0120	0.0120	0.0230
	CVM	1.0623	0.0623	0.0623	0.0704	1.0359	0.0359	0.0359	0.0324
	OLS	1.0096	0.0096	0.0096	0.0574	0.9981	0.0019	0.0019	0.0287
	WLS	1.0206	0.0206	0.0206	0.0490	1.0081	0.0081	0.0081	0.0246
	MPS	0.9416	0.0584	0.0584	0.0359	0.9502	0.0498	0.0498	0.0212
100	ML	1.0251	0.0251	0.0251	0.0197	1.0153	0.0153	0.0153	0.0104
	AD	1.0113	0.0113	0.0113	0.0211	1.0049	0.0049	0.0049	0.0109
	CVM	1.0291	0.0291	0.0291	0.0295	1.0166	0.0166	0.0166	0.0145
	OLS	1.0037	0.0037	0.0037	0.0267	0.9979	0.0021	0.0021	0.0137
	WLS	1.0109	0.0109	0.0109	0.0225	1.0044	0.0044	0.0044	0.0115
	MPS	0.9608	0.0392	0.0392	0.0178	0.9676	0.0324	0.0324	0.0102
200	ML	1.0130	0.0130	0.0130	0.0090	1.0085	0.0085	0.0085	0.0049
	AD	1.0061	0.0061	0.0061	0.0098	1.0031	0.0031	0.0031	0.0054
	CVM	1.0146	0.0146	0.0146	0.0130	1.0087	0.0087	0.0087	0.0071
	OLS	1.0022	0.0022	0.0022	0.0124	0.9994	0.0006	0.0006	0.0069
	WLS	1.0066	0.0066	0.0066	0.0101	1.0035	0.0035	0.0035	0.0056
	MPS	0.9763	0.0237	0.0237	0.0086	0.9812	0.0188	0.0188	0.0049
300	ML	1.0094	0.0094	0.0094	0.0060	1.0066	0.0066	0.0066	0.0034
	AD	1.0049	0.0049	0.0049	0.0067	1.0030	0.0030	0.0030	0.0037
	CVM	1.0108	0.0108	0.0108	0.0087	1.0070	0.0070	0.0070	0.0047
	OLS	1.0025	0.0025	0.0025	0.0084	1.0008	0.0008	0.0008	0.0046
	WLS	1.0056	0.0056	0.0056	0.0069	1.0036	0.0036	0.0036	0.0038
	MPS	0.9831	0.0169	0.0169	0.0058	0.9871	0.0129	0.0129	0.0034

Table 8. Estimation of the EHaq distribution for $\delta = 1.0$ and $\tau = 1.5$.

n	Method	$\hat{\delta}$				$\hat{\tau}$			
		Estimate	Bias	MRE	MSE	Estimate	Bias	MRE	MSE
25	ML	1.1212	0.1212	0.1212	0.1316	1.6164	0.1164	0.0776	0.1396
	AD	1.0592	0.0592	0.0592	0.1146	1.5440	0.0440	0.0293	0.1319
	CVM	1.1500	0.1500	0.1500	0.2328	1.6285	0.1285	0.0857	0.2202
	OLS	1.0330	0.0330	0.0330	0.1506	1.5020	0.0020	0.0014	0.1694
	WLS	1.0441	0.0441	0.0441	0.1329	1.5208	0.0208	0.0139	0.1504
	MPS	0.9266	0.0734	0.0734	0.0767	1.3905	0.1095	0.0730	0.1071
50	ML	1.0562	0.0562	0.0562	0.0459	1.5569	0.0569	0.0379	0.0593
	AD	1.0275	0.0275	0.0275	0.0443	1.5224	0.0224	0.0149	0.0599
	CVM	1.0658	0.0658	0.0658	0.0685	1.5623	0.0623	0.0415	0.0864
	OLS	1.0130	0.0130	0.0130	0.0554	1.5014	0.0014	0.0009	0.0755
	WLS	1.0232	0.0232	0.0232	0.0477	1.5162	0.0162	0.0108	0.0646
	MPS	0.9447	0.0553	0.0553	0.0354	1.4240	0.0760	0.0506	0.0534
100	ML	1.0286	0.0286	0.0286	0.0202	1.5280	0.0280	0.0187	0.0265
	AD	1.0152	0.0152	0.0152	0.0212	1.5116	0.0116	0.0077	0.0286
	CVM	1.0339	0.0339	0.0339	0.0295	1.5318	0.0318	0.0212	0.0389
	OLS	1.0086	0.0086	0.0086	0.0265	1.5019	0.0019	0.0013	0.0363
	WLS	1.0147	0.0147	0.0147	0.0223	1.5107	0.0107	0.0071	0.0302
	MPS	0.9644	0.0356	0.0356	0.0178	1.4512	0.0488	0.0325	0.0258
200	ML	1.0137	0.0137	0.0137	0.0092	1.5160	0.0160	0.0106	0.0130
	AD	1.0065	0.0065	0.0065	0.0100	1.5071	0.0071	0.0047	0.0144
	CVM	1.0146	0.0146	0.0146	0.0131	1.5159	0.0159	0.0106	0.0187
	OLS	1.0022	0.0022	0.0022	0.0124	1.5011	0.0011	0.0008	0.0180
	WLS	1.0072	0.0072	0.0072	0.0103	1.5080	0.0080	0.0053	0.0148
	MPS	0.9771	0.0229	0.0229	0.0087	1.4720	0.0280	0.0186	0.0129
300	ML	1.0089	0.0089	0.0089	0.0058	1.5092	0.0092	0.0062	0.0081
	AD	1.0041	0.0041	0.0041	0.0064	1.5033	0.0033	0.0022	0.0090
	CVM	1.0095	0.0095	0.0095	0.0084	1.5092	0.0092	0.0062	0.0115
	OLS	1.0013	0.0013	0.0013	0.0081	1.4994	0.0006	0.0004	0.0113
	WLS	1.0048	0.0048	0.0048	0.0066	1.5041	0.0041	0.0028	0.0092
	MPS	0.9829	0.0171	0.0171	0.0057	1.4780	0.0220	0.0146	0.0082

Table 9. Estimation of the EHaq distribution for $\delta = 1.5$ and $\tau = 0.5$.

n	Method	$\hat{\delta}$				$\hat{\tau}$			
		Estimate	Bias	MRE	MSE	Estimate	Bias	MRE	MSE
25	ML	1.7263	0.2263	0.1509	0.4245	0.5271	0.0271	0.0542	0.0088
	AD	1.6186	0.1186	0.0791	0.3642	0.5089	0.0089	0.0177	0.0083
	CVM	1.7864	0.2864	0.1909	0.8069	0.5286	0.0286	0.0573	0.0125
	OLS	1.5744	0.0744	0.0496	0.4936	0.4969	0.0031	0.0062	0.0103
	WLS	1.5930	0.0930	0.0620	0.4382	0.5023	0.0023	0.0045	0.0091
	MPS	1.3838	0.1162	0.0775	0.2204	0.4686	0.0314	0.0627	0.0076
50	ML	1.6040	0.1040	0.0693	0.1425	0.5144	0.0144	0.0288	0.0040
	AD	1.5565	0.0565	0.0376	0.1376	0.5058	0.0058	0.0116	0.0041
	CVM	1.6286	0.1286	0.0858	0.2238	0.5158	0.0158	0.0316	0.0057
	OLS	1.5346	0.0346	0.0231	0.1748	0.5003	0.0003	0.0006	0.0051
	WLS	1.5498	0.0498	0.0332	0.1504	0.5042	0.0042	0.0083	0.0044
	MPS	1.4105	0.0895	0.0596	0.1027	0.4799	0.0201	0.0402	0.0038
100	ML	1.5495	0.0495	0.0330	0.0554	0.5068	0.0068	0.0137	0.0018
	AD	1.5267	0.0267	0.0178	0.0594	0.5025	0.0025	0.0051	0.0019
	CVM	1.5590	0.0590	0.0393	0.0864	0.5072	0.0072	0.0143	0.0025
	OLS	1.5146	0.0146	0.0098	0.0765	0.4995	0.0005	0.0009	0.0024
	WLS	1.5260	0.0260	0.0174	0.0624	0.5023	0.0023	0.0047	0.0020
	MPS	1.4388	0.0612	0.0408	0.0476	0.4868	0.0132	0.0264	0.0018
200	ML	1.5245	0.0245	0.0163	0.0259	0.5031	0.0031	0.0061	0.0009
	AD	1.5135	0.0135	0.0090	0.0285	0.5009	0.0009	0.0017	0.0010
	CVM	1.5286	0.0286	0.0191	0.0380	0.5031	0.0031	0.0061	0.0012
	OLS	1.5070	0.0070	0.0046	0.0358	0.4993	0.0007	0.0015	0.0012
	WLS	1.5147	0.0147	0.0098	0.0293	0.5011	0.0011	0.0021	0.0010
	MPS	1.4614	0.0386	0.0258	0.0243	0.4916	0.0084	0.0167	0.0009
300	ML	1.5152	0.0152	0.0102	0.0165	0.5022	0.0022	0.0045	0.0006
	AD	1.5080	0.0080	0.0053	0.0182	0.5008	0.0008	0.0016	0.0006
	CVM	1.5184	0.0184	0.0123	0.0240	0.5024	0.0024	0.0048	0.0008
	OLS	1.5041	0.0041	0.0027	0.0230	0.4999	0.0001	0.0003	0.0008
	WLS	1.5092	0.0092	0.0061	0.0186	0.5010	0.0010	0.0020	0.0006
	MPS	1.4700	0.0300	0.0200	0.0160	0.4940	0.0060	0.0119	0.0006

Table 10. Estimation of the EHaq distribution for $\delta = 1.5$ and $\tau = 1.0$.

n	Method	$\hat{\delta}$				$\hat{\tau}$			
		Estimate	Bias	MRE	MSE	Estimate	Bias	MRE	MSE
25	ML	1.7120	0.2120	0.1413	0.3811	1.0606	0.0606	0.0606	0.0424
	AD	1.6107	0.1107	0.0738	0.3298	1.0218	0.0218	0.0218	0.0398
	CVM	1.7976	0.2976	0.1984	0.9775	1.0710	0.0710	0.0710	0.0655
	OLS	1.5868	0.0868	0.0579	0.5892	1.0007	0.0007	0.0007	0.0521
	WLS	1.5969	0.0969	0.0646	0.4577	1.0103	0.0103	0.0103	0.0456
	MPS	1.3795	0.1205	0.0803	0.2050	0.9332	0.0668	0.0668	0.0354
50	ML	1.5963	0.0963	0.0642	0.1306	1.0302	0.0302	0.0302	0.0184
	AD	1.5483	0.0483	0.0322	0.1308	1.0102	0.0102	0.0102	0.0189
	CVM	1.6153	0.1153	0.0769	0.2157	1.0305	0.0305	0.0305	0.0261
	OLS	1.5247	0.0247	0.0165	0.1711	0.9968	0.0032	0.0032	0.0234
	WLS	1.5409	0.0409	0.0273	0.1456	1.0061	0.0061	0.0061	0.0203
	MPS	1.4081	0.0919	0.0613	0.0964	0.9554	0.0446	0.0446	0.0173
100	ML	1.5491	0.0491	0.0328	0.0546	1.0161	0.0161	0.0161	0.0085
	AD	1.5285	0.0285	0.0190	0.0583	1.0073	0.0073	0.0073	0.0092
	CVM	1.5611	0.0611	0.0407	0.0831	1.0181	0.0181	0.0181	0.0121
	OLS	1.5176	0.0176	0.0117	0.0735	1.0014	0.0014	0.0014	0.0114
	WLS	1.5279	0.0279	0.0186	0.0614	1.0068	0.0068	0.0068	0.0096
	MPS	1.4404	0.0596	0.0397	0.0469	0.9725	0.0275	0.0275	0.0084
200	ML	1.5226	0.0226	0.0151	0.0240	1.0074	0.0074	0.0074	0.0040
	AD	1.5124	0.0124	0.0083	0.0268	1.0030	0.0030	0.0030	0.0045
	CVM	1.5270	0.0270	0.0180	0.0362	1.0079	0.0079	0.0079	0.0057
	OLS	1.5059	0.0059	0.0039	0.0342	0.9996	0.0004	0.0004	0.0056
	WLS	1.5136	0.0136	0.0090	0.0276	1.0034	0.0034	0.0034	0.0046
	MPS	1.4606	0.0394	0.0263	0.0227	0.9824	0.0176	0.0176	0.0041
300	ML	1.5179	0.0179	0.0119	0.0166	1.0059	0.0059	0.0059	0.0028
	AD	1.5114	0.0114	0.0076	0.0189	1.0031	0.0031	0.0031	0.0031
	CVM	1.5222	0.0222	0.0148	0.0250	1.0069	0.0069	0.0069	0.0040
	OLS	1.5082	0.0082	0.0054	0.0239	1.0014	0.0014	0.0014	0.0039
	WLS	1.5128	0.0128	0.0085	0.0193	1.0037	0.0037	0.0037	0.0032
	MPS	1.4735	0.0265	0.0177	0.0158	0.9880	0.0120	0.0120	0.0028

Table 11. Estimation of the EHaq distribution for $\delta = 1.5$ and $\tau = 1.5$.

n	Method	$\hat{\delta}$				$\hat{\tau}$			
		Estimate	Bias	MRE	MSE	Estimate	Bias	MRE	MSE
25	ML	1.7137	0.2137	0.1425	0.3588	1.6040	0.1040	0.0694	0.1112
	AD	1.6079	0.1079	0.0719	0.3055	1.5378	0.0378	0.0252	0.1028
	CVM	1.7741	0.2741	0.1828	0.7776	1.6096	0.1096	0.0731	0.1654
	OLS	1.5696	0.0696	0.0464	0.4636	1.4980	0.0020	0.0014	0.1306
	WLS	1.5851	0.0851	0.0567	0.3885	1.5155	0.0155	0.0103	0.1153
	MPS	1.3843	0.1157	0.0771	0.1941	1.4006	0.0994	0.0663	0.0886
50	ML	1.5935	0.0935	0.0623	0.1211	1.5497	0.0497	0.0332	0.0465
	AD	1.5464	0.0464	0.0309	0.1220	1.5183	0.0183	0.0122	0.0477
	CVM	1.6135	0.1135	0.0757	0.2023	1.5526	0.0526	0.0351	0.0678
	OLS	1.5237	0.0237	0.0158	0.1602	1.4985	0.0015	0.0010	0.0601
	WLS	1.5385	0.0385	0.0257	0.1325	1.5121	0.0121	0.0081	0.0512
	MPS	1.4069	0.0931	0.0620	0.0904	1.4300	0.0700	0.0467	0.0430
100	ML	1.5473	0.0473	0.0315	0.0538	1.5243	0.0243	0.0162	0.0219
	AD	1.5248	0.0248	0.0166	0.0567	1.5088	0.0088	0.0058	0.0236
	CVM	1.5561	0.0561	0.0374	0.0800	1.5256	0.0256	0.0171	0.0311
	OLS	1.5132	0.0132	0.0088	0.0711	1.4990	0.0010	0.0007	0.0293
	WLS	1.5240	0.0240	0.0160	0.0593	1.5080	0.0080	0.0053	0.0244
	MPS	1.4397	0.0603	0.0402	0.0467	1.4548	0.0452	0.0301	0.0216
200	ML	1.5232	0.0232	0.0155	0.0247	1.5128	0.0128	0.0086	0.0104
	AD	1.5131	0.0131	0.0087	0.0272	1.5058	0.0058	0.0039	0.0114
	CVM	1.5291	0.0291	0.0194	0.0368	1.5147	0.0147	0.0098	0.0148
	OLS	1.5081	0.0081	0.0054	0.0346	1.5014	0.0014	0.0010	0.0143
	WLS	1.5147	0.0147	0.0098	0.0280	1.5068	0.0068	0.0046	0.0117
	MPS	1.4616	0.0384	0.0256	0.0233	1.4729	0.0271	0.0181	0.0104
300	ML	1.5169	0.0169	0.0113	0.0159	1.5081	0.0081	0.0054	0.0069
	AD	1.5087	0.0087	0.0058	0.0177	1.5028	0.0028	0.0019	0.0076
	CVM	1.5175	0.0175	0.0117	0.0232	1.5077	0.0077	0.0052	0.0097
	OLS	1.5036	0.0036	0.0024	0.0224	1.4990	0.0010	0.0007	0.0095
	WLS	1.5100	0.0100	0.0067	0.0180	1.5037	0.0037	0.0024	0.0078
	MPS	1.4728	0.0272	0.0181	0.0152	1.4796	0.0204	0.0136	0.0070

From Tables 3–11, it is interesting to note that:

- The bias and MSE consistently decrease as n increases, regardless of the estimation approach or parameter values.
- For the higher sample sizes (n = 200, 300), all the estimation approaches converge to more accurate and stable estimates, but OLS, AD, and WLS consistently outperform others.
- The OLS estimation approach is the most efficient estimator overall, with excellent performance in terms of accuracy and consistency for both parameters τ and δ.
- The methods AD and WLS are strong alternatives, especially if computational simplicity or robustness to the small parameter τ is desired.

- The MLE approach is only reliable for high sample sizes, and CVM should be avoided unless essential.
- The MPS estimation technique shows potential for higher samples but should be utilized cautiously due to bias in low sample values.

5. Application

This section highlights the practical value of the EHaq distribution by applying it to two datasets, one related to radiation and another to COVID-19, and by comparing its performance with several alternative distributions. The EHaq distribution is analyzed against other alternative models in order to test the adaptiveness and efficiency in data modeling. These distributions of the probability are the Burr III (BIII) distribution, inverted Kumaraswamy (IKw) distribution [31], Haq distribution [24], ZLindley distribution [32], XLindley distribution [33], Lindley distribution, and Xgamma distribution [34].

The parameters of all considered distributions were estimated using the MLE approach. The performance of considered probability distributions is assessed by considering five well-established criteria: log-likelihood (Λ_1), Akaike information criterion (Λ_2), Bayesian information criterion (Λ_3), Kolmogorov-Smirnov statistic (Λ_4), and p-value (Λ_5). The optimal model is identified by minimizing information criteria and test statistics $(\Lambda_2, \Lambda_3, \Lambda_4)$ and maximizing the log-likelihood and p-value (Λ_1, Λ_5).

5.1. The irradiated packets dataset

The first dataset is about irradiated packets. The dataset represents the logarithmic values of F1 adult counts, as reported by [35]. They investigated the effects of rearing drugstore beetles on peppermint stored in variously treated packets. These treatments included unirradiated controls, gamma irradiation at doses of 6, 8, and 10 KGy, and microwave exposure for durations of 1, 2, and 3 minutes, under a “non-choice non-packet test” setup. The observed data are: 2.17026, 2.16137, 2.18184, 1.80618, 1.80618, 1.80618, 1.71600, 1.77085, 1.75587, 1.51851, 1.55630, 1.49136, 1.93952, 1.92942, 1.93952, 1.83885, 1.81291, 1.82607, 1.68124, 1.62325, and 1.66276. We also explore the shape of the datasets and present these graphs such as the quantile-quantile (Q_Q), total time on test (TTT), box, and violin plots in Figure 4. Details regarding the fitted distributions and their respective goodness-of-fit measures are presented in Table 12. Figure 5 presents the graphs of observed and expected PDF, CDF, probability–probability (P-P) plot, profile log-likelihood plots, and contour plot for the first dataset.

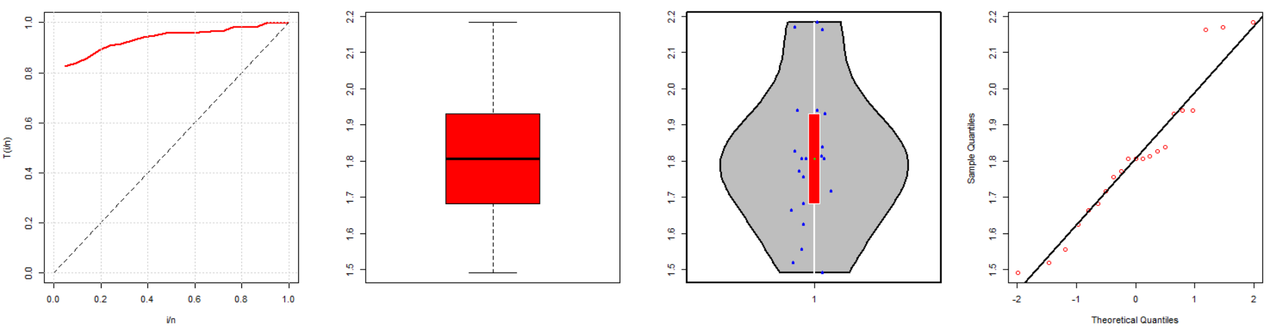

Figure 4. TTT, box, violin, and Q-Q plots for the radiation dataset.

Table 12. Estimated values, standard errors, and fitting measure values for all models applied to the first dataset.

Models	Parameter Estimates		Standard Errors		Fitting Measure Values				
	$\hat{\tau}$	$\hat{\delta}$	$\hat{\tau}$	$\hat{\delta}$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
EHaq	6.59966	36172.4	1.14494	1.28485	5.0004	-6.0009	-3.91184	0.12769	0.8833
BIII	261.469	10.3852	223.220	1.68726	4.5322	-5.0645	-2.97544	0.14098	0.7981
IKw	16.3934	12661780	0.22432	33556.8	4.7163	-5.4327	-3.34365	0.13674	0.8272
Haq	0.88394	-	0.14183	-	-34.5619	71.1238	72.1684	0.56750	0.0000
ZLindley	0.72600	-	0.13574	-	-32.4970	66.9940	68.0385	0.55510	0.0000
XLindley	0.72840	-	0.12297	-	-32.2254	66.4509	67.4954	0.53983	0.0000
Lindley	0.85138	-	0.13892	-	-30.3972	62.7944	63.8389	0.52644	0.0000
Xgamma	1.16828	-	0.17436	-	-31.7106	65.4212	66.4657	0.56160	0.0000

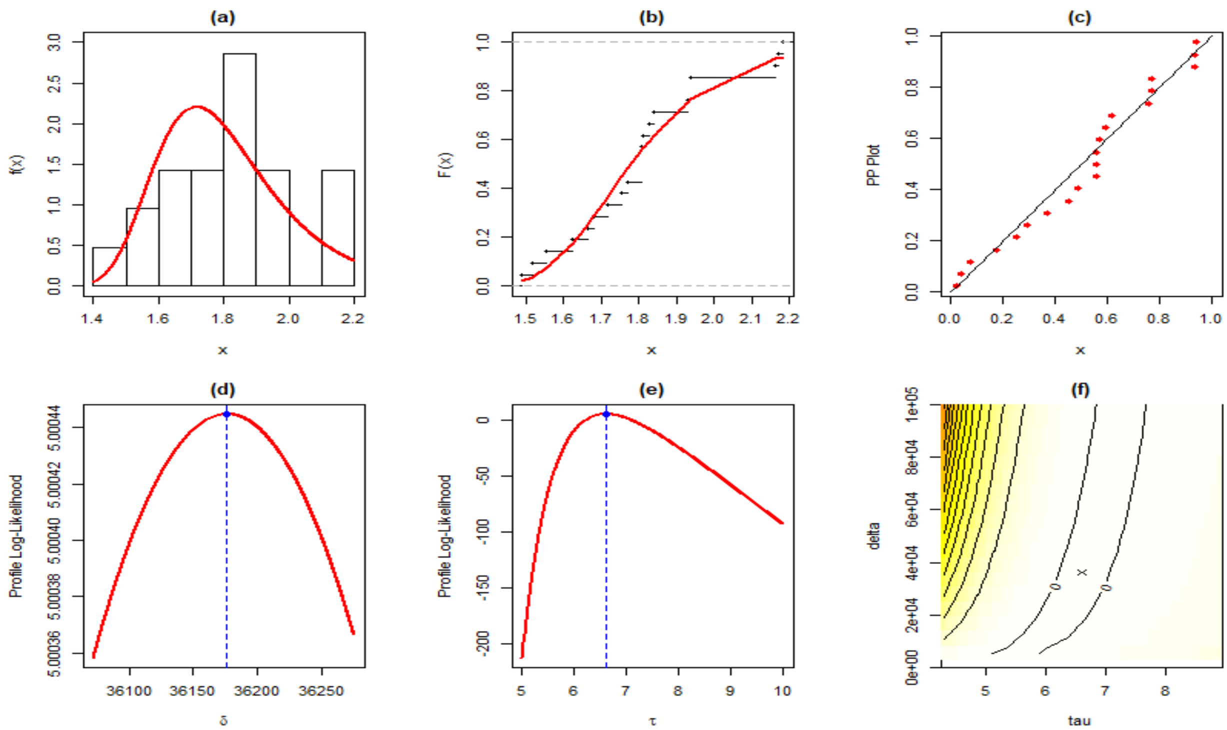

Figure 5. PDF, CDF, P-P, profile log-likelihood, and contour plots for the radiation dataset.

The descriptive characteristics of the first dataset indicate that the numerical values are roughly symmetric, with an average value of 1.80 and a median of 1.81. The standard deviation (SD) is around 0.20, indicating a moderate dispersion of the data around the central value. The coefficient of skewness is 0.03, which is close to zero, confirming the data distribution is nearly symmetry. The coefficient of kurtosis is somewhat less than 3 (≈ 2.6), indicating a platykurtic distribution with lighter tails than the normal distribution. Overall, the dataset's distribution is very stable and symmetric, with no severe outliers or heavy-tailed behavior. Similar behavior is observed in Figure 4. Further, the TTT shows increasing failure rate behavior.

Table 12 compares the fitting performance of the considered statistical models based on ML estimates, SEs, and a set of five goodness-of-fit criteria labeled Λ_1 through Λ_5. Based on these criteria, the EHaq distribution outperforms other probability distributions.

The visual diagnostics jointly illustrate that the proposed distribution provides an adequate fit to the first dataset. The density function overlay on the histogram (panel (a)) and the theoretical and empirical CDF (panel (b)) both display close alignment. The PP plot (panel (c)) demonstrates that the distribution captures the distributional structure without systematic deviations. The profile log-likelihood plots for both parameters show stable and precise ML estimates. The contour plot highlights a clear likelihood maximum with elliptical contours. On the whole, Figure 5 indicates that the EHaq distribution is a flexible and highly adequate model for the analysis of this dataset.

5.2. The COVID-19 dataset

The second dataset pertains to the COVID-19 mortality rates recorded in the Netherlands over 30 days, from March 31 to April 30, 2020 [36]. The data observations are 14.918, 10.656, 12.274, 10.289, 10.832, 7.099, 5.928, 13.211, 7.968, 7.584, 5.555, 6.027, 4.097, 3.611, 4.960, 7.498, 6.940, 5.307, 5.048, 2.857, 2.254, 5.431, 4.462, 3.883, 3.461, 3.647, 1.974, 1.273, 1.416, and 4.235. We also explore the shape of the datasets and present these graphs in Figure 6. Details regarding the fitted distributions and their respective goodness-of-fit measures are presented in Table 13. Figure 7 presents the graphs of observed and expected PDF, CDF, Q-Q plot, profile log-likelihood plots, and contour plot for the second dataset.

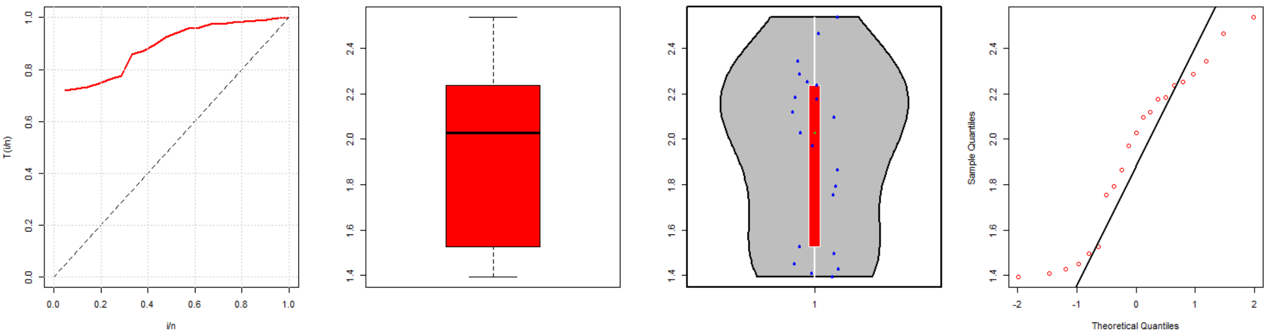

Figure 6. TTT, box, violin, and Q-Q plots for the COVID-19 dataset.

Table 13. Estimated values, standard errors, and fitting measure values for all models applied to the second dataset.

Models	Parameter Estimates		Standard Errors		Fitting Measure Values				
	$\hat{\tau}$	$\hat{\delta}$	$\hat{\tau}$	$\hat{\delta}$	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
EHaq	0.53163	2.92630	0.07317	0.91179	-76.804	157.608	160.411	0.0850	0.9690
BIII	9.97437	1.66364	2.62244	0.19896	-80.360	164.720	167.523	0.1374	0.5752
IKw	1.99924	23.5564	0.28489	10.3932	-79.643	163.286	166.088	0.1297	0.6461
Haq	0.35270	-	0.04171	-	-82.336	166.671	168.073	0.2213	0.0900
ZLindley	0.23412	-	0.03650	-	-82.959	167.918	169.319	0.2426	0.0486
XLindley	0.26313	-	0.03471	-	-81.261	164.522	165.923	0.2014	0.1522
Lindley	0.28854	-	0.03772	-	-79.964	161.928	163.329	0.1794	0.2565
Xgamma	0.40658	-	0.04620	-	-79.991	161.981	163.382	0.1712	0.3065

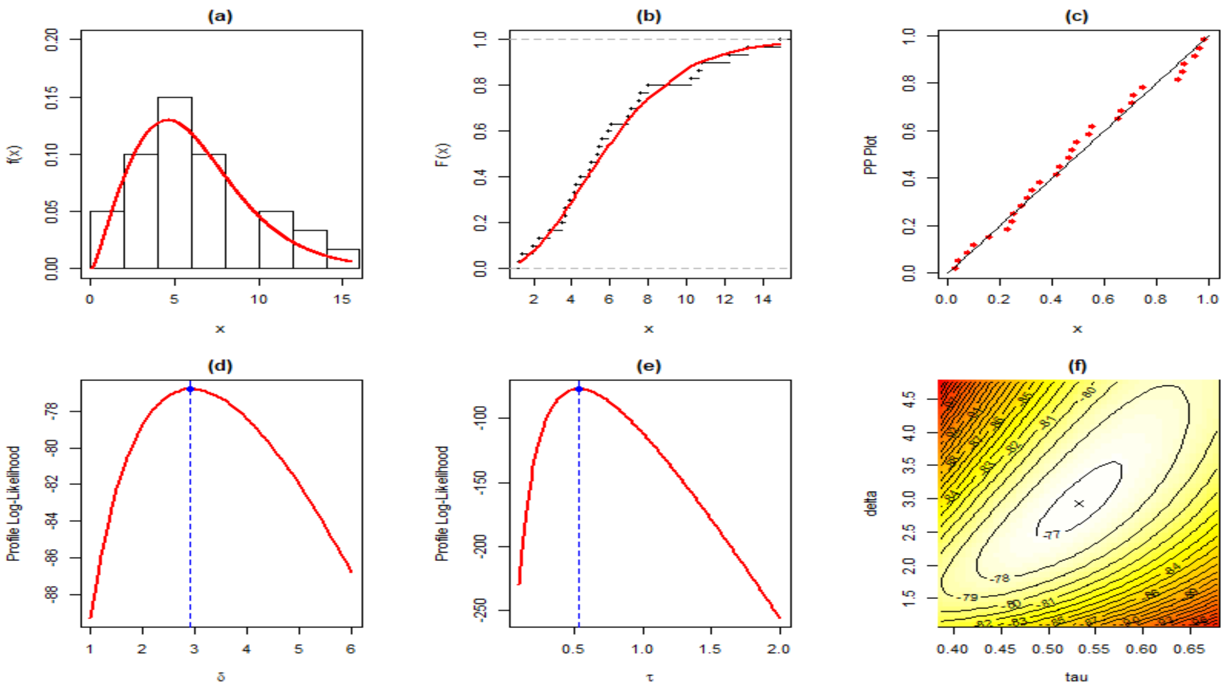

Figure 7. PDF, CDF, P-P, profile log-likelihood, and contour plots for the COVID-19 dataset.

The second dataset shows the daily COVID-19 death rates in the Netherlands for 30 days. The average mortality rate is around 6.16, which is the mean daily death rate throughout this period. The median value is around 5.43, which is less than average, depicting a right-skewed data distribution. The standard deviation is around 3.26, indicating a moderate amount of fluctuation in the daily death rate during this period. The coefficient of skewness values are positive, indicating positive skewness, suggesting a right-skewed data distribution in which the bulk of the observations are clustered at the lower end and a few higher values extend the right tail. The kurtosis exceeds 3, suggesting a leptokurtic distribution with a sharper peak and heavier tails than a normal distribution. This means that extreme values, particularly higher death rates, are more prevalent than in a normally distributed sample. Figure 6 shows a similar pattern in terms of the TTT (total test time), with an increasing failure rate.

Table 13 presents the comparison of selected statistical distributions for the COVID-19 dataset. Among all the probability distributions, the EHaq distribution achieves the highest log-likelihood value and lowest AIC and BIC values, indicating the best fit.

The visual diagnostics show that the suggested distribution is an appropriate match to the COVID-19 dataset. The density function overlay over the histogram (panel (a)) and the theoretical and empirical CDF (panel (b)) are closely aligned. The PP plot (panel (c)) shows that the distribution accurately represents the distributional structure with no consistent deviations. The profile log-likelihood graphs for both parameters reveal consistent and exact ML estimates. The contour plot shows a distinct probability maximum for elliptical contours. Figure 7 demonstrates that the EHaq distribution is a versatile and very appropriate model for analyzing the COVID-19 dataset.

6. Conclusions

In this study, a new two-parameter EHaq distribution is introduced and explored. The proposed distribution significantly enhances the flexibility of the baseline Haq distribution, enabling it to effectively model datasets with characteristics such as skewness, heavy tails, and high variability. We derived its various mathematical characteristics, including its quantile function, moments, entropy, order statistics, stress-strength reliability, and mean residual life function. For parameter estimation, six classical estimation techniques were implemented, supported by a comprehensive simulation study to assess their performance. To show its practical value, the EHaq distribution was applied to two real-world datasets: one for radiation exposure measurements and another for COVID-19 death rates in the Netherlands.

Finally, the EHaq distribution has been found to be an effective and adaptive model performing better than old models on complex real-world data. The EHaq distribution is theoretically well-endowed, with stable parameter estimation, and excellent empirical behavior in both radiation and COVID-19 data, resulting in a powerful and flexible modelling tool that can be applied in statistical applications in many varied areas of science.

7. Delimitations and limitations

The new distribution is suitable in cases when the conditions and assumptions of the distribution are satisfied. In contrast to the baseline Haq distribution, the EHaq distribution is broad scoped and can be implemented on heterogeneous forms of data sets. The EHaq distribution is especially limited to positive data sets. As the proposed distribution is more flexible than the considered competitive distributions, it may not be suitable for datasets that deviate significantly from the distribution assumptions.

8. Future work

In this study, we introduced a flexible two-parameter probability distribution and derived its properties, estimated its parameters, and applied it to real-world datasets. However, various possible directions remain open, which can enrich the theoretical understanding of the model and broaden its practical applications. The key avenues for future studies include:

- Order statistics and record values: Study the probability distribution in the context of record values and order statistics, which would provide insight for extreme value theory, statistical inference, and characterize the model based on order statistics and record values.
- Neutrosophic framework: Neutrosophic extension based on the proposed distribution can be introduced to analyze vague or imprecise datasets.
- Entropic analysis: A detailed analysis of entropy measures, such as Shannon entropy and Rényi entropy, can be done to gain a deeper understanding of information theory and uncertainty of the probability distribution.
- Censored data analysis: Extend the EHaq distribution to accommodate censored data, as predominantly right, left, and interval censoring often appear in reliability and survival studies.
- Regression modeling: Extend the proposed distribution to regression frameworks as [37] did for analyzing relationships between variables, with an emphasis on robust and non-robust estimation methods to address outliers and multicollinearity, thereby enhancing accuracy and predictive performance.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares having no competing interest.

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