Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(9): 22445–22470.
DOI: 10.3934/math.20251000
Received: 11 July 2025
Revised: 02 September 2025
Accepted: 15 September 2025
Published: 28 September 2025

Research article

Classes of Ma-Minda type analytic functions associated with a kidney-shaped domain

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Abstract: Ma-Minda-type analytic functions have been a central topic in geometric function theory due to their importance in studying properties such as univalence, convexity, and starlikeness. However, the class of Ma-Minda functions associated with kidney-shaped domains has not been systematically investigated before, leaving a clear research gap. In this paper, we introduced and studied the geometric properties of class $\mathcal{S}_k^*$. Our approach combined rigorous geometric analysis with Python programming (version 3.13.2) and graphical illustrations, providing both theoretical depth and computational support. Several inclusion relations and radius problems for $\mathcal{S}_k^*$ were established in comparison with other known classes. Furthermore, we examined various problems related to the initial coefficient bounds, with particular emphasis on their sharpness.

Keywords: starlike functions; subordination; radius problem; coefficient estimates; Hankel determinants
Mathematics Subject Classification: 05A30, 11B65, 30C45, 47B38

1. Introduction and motivation

Let $\mathbb{A}$ be the family of analytic functions in $\mathbb{U} = \{z \in \mathbb{C}\colon |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathbb{U}.$$

We define

$$\mathcal{S}_k^* = \left\{ f \in \mathbb{A} :\ zf'(z)/f(z) \prec 2 - \sqrt[3]{4/(1+e^{2z})^2},\ z \in \mathbb{U} \right\}.$$

Also, let $\mathcal{S}$ be the set of all functions $f \in \mathbb{A}$ that are univalent in $\mathbb{U}$. For $\alpha \in [0,1)$, the subclasses $\mathcal{S}_\alpha^*$ and C_α are defined as the collections of functions in $\mathbb{A}$ that are starlike and convex of order α, respectively. These subclasses are characterized analytically as follows:

$$\mathcal{S}_\alpha^* = \left\{ f \in \mathbb{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\},$$

$$C_\alpha = \left\{ f \in \mathbb{A} : Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha \right\}.$$

When $\alpha = 0$, we recover the standards of $\mathcal{S}^*$ and C, respectively. The relationship between these two classes is described by the well-known Alexander theorem, which states $f(z) \in C_\alpha$ if and only if $zf'(z) \in \mathcal{S}_\alpha^*$. For $\delta \in (0,1]$, the class $\mathcal{SS}^*(\delta)$ ($\mathcal{SC}(\delta)$) represents strongly starlike (strongly convex) functions of order δ, if it satisfies the inequality $|arg(zf'(z)/f(z))| < \delta\pi/2$ ($|arg((zf''(z)/f'(z)) + 1)| < \delta\pi/2$). Notably, when $\delta = 1$, $\mathcal{SS}^*(1) = \mathcal{S}^*$.

For analytic functions f and h, we say that f is subordinate to h, denoted $f \prec h$, if there exists an analytic function $u : \mathbb{U} \longrightarrow \mathbb{U}$ with $u(0) = 0$ (Schwarz function) that satisfies $f(z) = h(u(z))$. In this case, it follows that $f(0) = h(0)$ and $f(\mathbb{U}) \subset h(\mathbb{U})$. Furthermore, if $h(z)$ is univalent, then $f \prec h$ if and only if $f(0) = h(0)$ and $f(\mathbb{U}) \subset h(\mathbb{U})$.

In 1992, Ma and Minda [1] introduced a creative approach to constructing subclasses of starlike and convex functions by leveraging the concept of subordination. Their generalization is based on an analytic function $\varphi : \mathbb{U} \longrightarrow \mathbb{C}$ that satisfies specific conditions, detailed as follows:

(1) Univalence and the real part is positive.

(2) The image $\varphi(\mathbb{U})$ is symmetric with respect to the real axis.

(3) The starlikeness of $\varphi(\mathbb{U})$ is satisfied about the point 1.

(4) The derivative at the origin satisfies $\varphi'(0) > 0$.

In our work, whenever the function φ is mentioned, it is assumed to satisfy the aforementioned properties. Based on this foundation, Ma and Minda defined the following function classes:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathbb{A} :\ \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

$$C(\varphi) = \left\{ f \in \mathbb{A} :\ \frac{zf''(z)}{f'(z)} + 1 \prec \varphi(z) \right\}.$$

These definitions establish subclasses of analytic functions that are starlike and convex with respect to the chosen $\varphi(z)$. Table 1 below demonstrates that certain well-known classes of functions are obtained by selecting specific values of φ.

Table 1. Special Ma-Minda classes.

Starlike class	Convex class	$\varphi(z)$	References
$\mathcal{S}^*$	C	$(1+z)/(1-z)$	[1]
$\mathcal{S}^*_\alpha$	C_α	$(1+(1-2\alpha)z)/(1-z)$, $\alpha \in [0,1)$	[1]
$\mathcal{SS}^*(\alpha)$	$\mathcal{SC}(\alpha)$	$((1+z)/(1-z))^\alpha$, $\alpha \in (0,1]$	[1]
$\mathcal{S}^*[A,B]$	$C[A,B]$	$(1+Az)/(1+Bz)$, $-1 \le B < A \le 1$	[2]

The development of various specialized function classes is based on defining specific functions $\varphi(z)$ within $\mathbb{U}$. For instance, the parabolic starlike class $\mathcal{S}^*_p$, defined and studied by Rønning [3], and the uniformly convex class UCV, proposed by Goodman [4], are constructed using the function

$$\varphi(z) = 1 + \frac{2}{\pi^2}\left(log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \ z \in \mathbb{U}.$$

Additionally, Kanas and Wisniowska [5] defined the classes $k-\mathcal{ST}$ and $k-UCV$ for $k \in [0,\infty)$. These classes are derived by selecting $\varphi(z)$ as

$$\varphi_k(z) = \frac{1}{1-k^2}cosh\left(\frac{2cos^{-1}k}{\pi}log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right) - \frac{k^2}{1-k^2}.$$

Moreover, Sokół [6] established the starlike class $\mathcal{S}^*_{q_c}$, where $q_c(z) = \sqrt{1+cz}$ $(0 < c \le 1)$. The function $q_c(z)$ maps $\mathbb{U}$ onto the interior of the right loop of the Cassinian ovals, represented by the equation

$$(u^2+v^2)^2 - 2\left(u^2-v^2\right) = c^2 - 1.$$

In a special case, if $c = 1$, we have $\mathcal{S}^*_{q_c} = \mathcal{S}^*_L$, which was introduced by Sokól and Stankiewicz [7]. To clearly position our work within the existing literature and highlight its novelty, we provide a critical review of relevant studies in Table 2. This review covers previous work from earlier years up to the most recent in 2025. By doing so, we identify the research gaps that motivated the current study and explicitly illustrate how our approach, focusing on kidney-shaped domains and combining geometric analysis with Python programming, extends and advances the state-of-the-art methods. Table 2 below highlights the recent introduction of Ma-Minda-type classes.

Table 2. Classes of starlike functions based on the Ma-Minda framework.

$\mathcal{S}^*(\varphi)$	$\varphi(z)$	References
$\mathcal{S}^*_{RL}$	$\varphi_{RL}(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}$	[8]
$\mathcal{S}^*_{☾}$	$\varphi_{☾}(z) = z + \sqrt{1+z^2}$	[9–11]
$\mathcal{S}^*_{e}$	$\varphi_e(z) = e^z$	[12]
$\mathcal{S}^*_{C}$	$\varphi_C(z) = 1 + \frac{4z}{3} + \frac{2z^2}{3}$	[13]
$\mathcal{S}^*_{R}$	$\varphi_R(z) = 1 + \frac{z}{k}\left(\frac{k+z}{k-z}\right),\ k = 1 + \sqrt{2}$	[14]
$\mathcal{S}^*_{lim}$	$\varphi_{lim}(z) = 1 + \sqrt{2}z + \frac{z^2}{2}$	[15]
$\mathcal{BS}(\vartheta)$	$\varphi(z) = 1 + \frac{z}{1-\vartheta z^2},\ \vartheta \in [0, 1)$	[16]
$\mathcal{S}^*_{S}$	$\varphi_S(z) = 1 + sinz$	[17]
$\mathcal{S}^*_{\vartheta,L}$	$\varphi_{\vartheta,L}(z) = \vartheta + (1 - \vartheta)\sqrt{1+z},\ \vartheta \in [0, 1)$	[18]
$\mathcal{S}^*_{\vartheta,e}$	$\varphi_{\vartheta,e}(z) = \vartheta + (1 - \vartheta) e^z,\ \vartheta \in [0, 1)$	[18]
$\mathcal{S}^*_{SG}$	$\varphi_{SG}(z) = \frac{2}{1+e^{-z}}$	[19]
$\mathcal{S}^*_{n\mathcal{L}}$	$\varphi_{n\mathcal{L}}(z) = 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n$	[20]
$\mathcal{S}^*_{\varrho}$	$\varphi_{\varrho}(z) = cosh\sqrt{z}$	[21]
$\mathcal{S}^*_{\rho}$	$\varphi_{\rho}(z) = 1 + sinh^{-1}z$	[22]
$\mathcal{S}^*_{H}$	$\varphi_H(z) = 1 + z + \frac{1}{3}z^2 - \frac{1}{9}z^3$	[23]

Motivated by the foundational studies outlined earlier, this paper introduces two new classes $\mathcal{S}^*_k = \left\{f \in \mathbb{A} : \frac{zf'(z)}{f(z)} \prec \varphi_k(z),\ z \in \mathbb{U}\right\}$ and $\mathcal{C}_k = \left\{f \in \mathbb{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi_k(z),\ z \in \mathbb{U}\right\}$. These classes are defined in terms of the analytic function $\varphi_k : \mathbb{U} \longrightarrow \mathbb{C}$, such that

$$\varphi_k(z) = 2 - \sqrt[3]{4/(1+e^{2z})^2} = 1 + \frac{2z}{3} + \frac{z^2}{9} - \frac{14z^3}{81} - \frac{11z^4}{243} + \frac{214z^5}{3645} + \frac{572z^6}{32805} + \ldots.$$

Observe that $\varphi_k(\mathbb{U})$ maps $\mathbb{U}$ onto the region

$$\Omega_k := \left\{\omega \in \mathbb{C} : \left|log\left(\frac{2}{\sqrt{(2-\omega)^3}} - 1\right)\right| < 2\right\}.$$

The function $\varphi_k(z) = 2 - \sqrt[3]{4/(1+e^{2z})^2}$ is analytic and satisfies the following Ma-Minda conditions (see [1]):

(1) $\varphi_k(0) = 1,\ \varphi_k'(0) > 0$,

(2) $Re(\varphi_k(z)) > 0$ (see Figure 1),

(3) $\varphi_k(z)$ is univalent since $Re\left(\varphi_k'(z)\right) > 0$ (see Figure 2(a)),

(4) $\varphi_k(z)$ is starlike with respect to 1 since $Re\left(\Phi(z) := z\varphi_k'(z) / (\varphi_k(z) - 1)\right) > 0$ (see Figure 2(b)),

(5) $\varphi_k(z)$ is symmetric about the real axis (clearly evident, also see Figure 1).

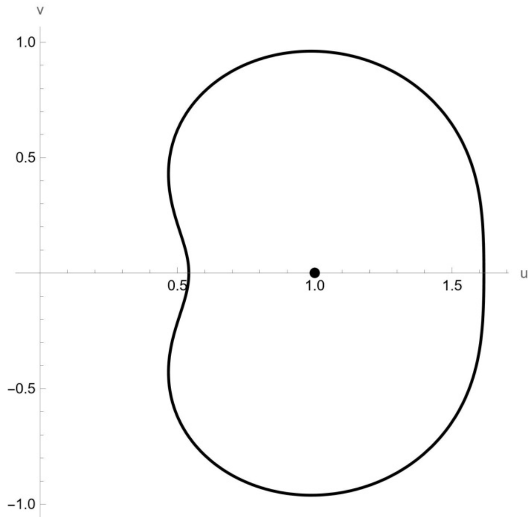

Figure 1. Plot boundary of $\varphi_k(\mathbb{U})$.

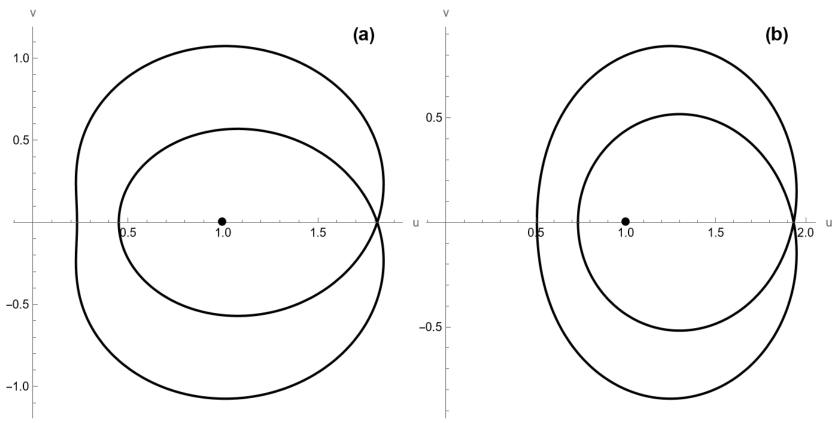

Figure 2. Plot boundary of (a) $\varphi_k'(\mathbb{U})$ and (b) $\Phi(\mathbb{U})$.

In this paper, we focus on a kidney-shaped domain due to its asymmetric and non-convex nature. Its varying boundary curvature allows us to study subtle behaviors of starlike functions that are not observable in classical domains such as circles or ellipses. This motivates the investigation of coefficient bounds, inclusion relations, and radius problems in such a geometrically rich setting. Our work is divided into three parts. In the first part, we provide several examples and discuss the geometric properties of the classes $\mathcal{S}_k^*$ and C_k. The second part presents results related to specific inclusion relationships and radius problems. Finally, motivated by recent contributions in coefficient estimation (for example, see [24–27]), the last part offers significant results, including sharp upper bounds for the initial Maclaurin coefficients, the precise Fekete-Szegö inequality, and the sharp bound for the second Hankel determinant. Moreover, we investigate the sharpness of logarithmic coefficients and present related special inequalities (see [28–31]).

2. About $\mathcal{S}_k^*$

The following two results give the representation formulas:

Theorem 2.1. *An analytic function f belongs in $\mathcal{S}_k^*$ if and only if*

$$f(z) = zexp\left(\int_0^z \frac{q(\tau)-1}{\tau}\, d\tau\right), \ (z \in \mathbb{U}), \tag{2.1}$$

for some analytic function $q(z)$ subordinate to φ_k.

Proof. Let $f \in \mathcal{S}_k^*$, $q(\tau) := \tau f'(\tau)/f(\tau)$, and $k(\tau) := f(\tau)/\tau$. Then $q(\tau) \prec \varphi_k$ and $k(\tau) \neq 0$. By logarithmic differentiation of $k(\tau)$, we have

$$\frac{k'(\tau)}{k(\tau)} = \frac{q(\tau)-1}{\tau}, \ (\tau \in \mathbb{U}). \tag{2.2}$$

Thus (2.1) follows from integrating (2.2) from 0 to z. □

Corollary 2.1. *An analytic function g belongs in C_k if and only if*

$$g(z) = \int_0^z exp\left(\int_0^w \frac{q(\tau)-1}{\tau}\, d\tau\right) dw \ \ (z \in \mathbb{U}), \tag{2.3}$$

for some analytic function $q(z)$ subordinate to φ_k.

Proof. This result follows immediately from applying Alexander's relation between $\mathcal{S}_k^*$ and C_k. □

Theorem 2.1 and Corollary 2.1 provide some non-trivial examples bellow:

Example 2.1. *Consider the following analytic functions:*

$$\begin{aligned} q_1(z) &= 1 + \frac{zsinz}{3}, \\ q_2(z) &= 1 + \frac{2ze^z}{9}, \\ q_3(z) &= 1 + \frac{3z - z^3}{6}, \\ q_4(z) &= 1 + \frac{4}{9}z - \frac{1}{11}z^5, \\ q_5(z) &= \frac{7+4z}{7+3z}. \end{aligned}$$

It is straightforward to conclude that $q_i(0) = 1$ and $q_i(\mathbb{U}) \subseteq \varphi_k(\mathbb{U})$ for $i = 1, 2, \ldots, 5$. Since $\varphi_k \in \mathcal{S}$, then $q_i \prec \varphi_k$ for $i = 1, 2, \ldots, 5$ (see Figure 3). Therefore, Theorem 2.1 and Corollary 2.1 imply that the functions $f_i(z)$ (or $g_i(z)$) $(i = 1, 2, \ldots, 5)$ listed below belong to $\mathcal{S}_k^*$ (or C_k).

$$\begin{aligned} f_1(z) &= zexp\left(\frac{1-cosz}{3}\right) = z + \frac{z^3}{6} - \frac{7z^7}{6480} + \frac{z^9}{217728} + \frac{z^{11}}{194400} + \ldots, \\ f_2(z) &= zexp\left(\frac{2}{9}(expz - 1)\right) = z + \frac{2z^2}{9} + \frac{11z^3}{81} + \frac{139z^4}{2187} + \frac{2087z^5}{78732} + \frac{37267z^6}{3542940} + \ldots, \\ f_3(z) &= zexp\left(\frac{9z - z^3}{18}\right) = z + \frac{z^2}{2} + \frac{z^3}{8} - \frac{5z^4}{144} - \frac{29z^5}{1152} - \frac{77z^6}{11520} + \ldots, \end{aligned}$$

$$f_4(z) = z\exp\left(\frac{4}{9}z - \frac{1}{55}z^5\right) = z + \frac{4z^2}{9} + \frac{8z^3}{81} + \frac{32z^4}{2187} + \frac{32z^5}{19683} - \frac{175739z^6}{9743085} + \ldots,$$

$$f_5(z) = z\sqrt[3]{(7+3z)/7} = z + \frac{z^2}{7} - \frac{z^3}{49} + \frac{5z^4}{1029} - \frac{10z^5}{7203} + \frac{22z^6}{50421} + \ldots.$$

Furthermore,

$$g_1(z) = z + \frac{z^3}{18} - \frac{z^7}{6480} + \frac{z^9}{1959552} + \ldots,$$
$$g_2(z) = z + \frac{z^2}{9} + \frac{11z^3}{243} + \frac{139z^4}{8748} + \frac{2087z^5}{393660} + \ldots,$$
$$g_3(z) = z + \frac{z^2}{42} + \frac{z^3}{24} - \frac{5z^4}{720} - \frac{29z^5}{5760} + \ldots,$$
$$g_4(z) = z + \frac{2z^2}{9} + \frac{8z^3}{243} + \frac{8z^4}{2187} + \frac{32z^5}{98415} + \ldots,$$
$$g_5(z) = z + \frac{z^2}{14} - \frac{z^3}{147} + \frac{5z^4}{4116} - \frac{2z^5}{7203} + \ldots.$$

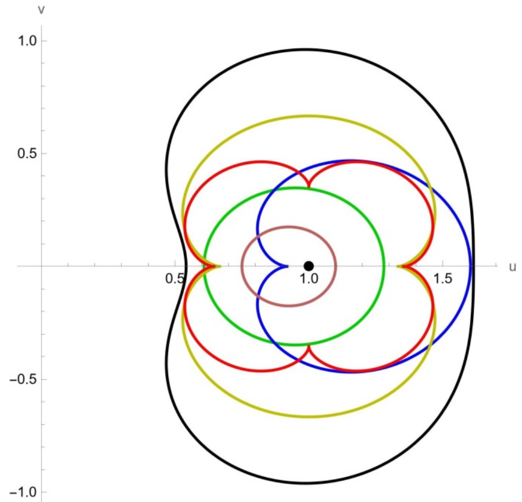

Figure 3. Plot boundaries of $q_1(\mathbb{U})$, $q_2(\mathbb{U})$, $q_3(\mathbb{U})$, $q_4(\mathbb{U})$, and $q_5(\mathbb{U})$, in colors green, blue, yellow, red, and pink, respectively.

Remark 2.1. If we specifically select $q(z) = \varphi_k(z) = 2 - \sqrt[3]{4/(1+e^{2z})^2}$ in Theorem 2.1, we obtain that

$$f_k(z) = z + \frac{2z^2}{3} + \frac{5z^3}{18} + \frac{7z^4}{243} - \frac{161z^5}{5832} - \frac{1517z^6}{218700} + \frac{123341z^7}{23619600} + \ldots \tag{2.4}$$

can be classified as a member of class $\mathcal{S}_k^*$ (see Figure 4). Subsequently, it will be demonstrated that the function f_k described by (2.4) is used as an extremal function for several extremal problems with class $\mathcal{S}_k^*$. Additionally, the corresponding function that belong to C_k can be established using (2.3) as follows:

$$g_k(z) = z + \frac{z^2}{3} + \frac{5z^3}{54} + \frac{7z^4}{972} - \frac{161z^5}{29160} - \frac{1517z^6}{1312200} + \frac{123341z^7}{165337200} + \ldots, \tag{2.5}$$

which belongs to C_k and fulfills the role of the Koebe function within this class.

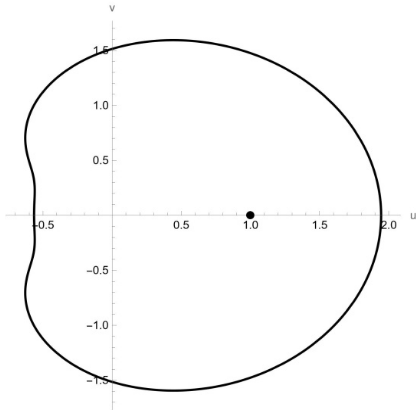

Figure 4. Boundary plot of $f_k(\mathbb{U})$.

In [1], Ma and Minda established some results, which imply that if $f \in \mathcal{S}_k^*$ ($g \in C_k$), then the following result is easy to obtain.

Theorem 2.2. *Let $f \in \mathcal{S}_k^*$, $g \in C_k$, and $|z| = r < 1$. Then*

(i)

$$\frac{zf'(z)}{f(z)} \prec \frac{zf_k'(z)}{f_k(z)} \text{ and } \frac{f(z)}{z} \prec \frac{f_k(z)}{z};$$

(ii)

$$\frac{zg''(z)}{g'(z)} \prec \frac{zg_k''(z)}{g_k'(z)} \text{ and } g'(z) \prec g_k'(z)\ .$$

(iii) Growth theorems give

$$-f_k(-r) \le |f(z)| \le f_k(r) \text{ and } -g_k(-r) \le |g(z)| \le g_k(r).$$

The equality holds if and only if f (or g) is a rotation of f_k (or g_k).

(iv) Rotation theorems give

$$\left|arg\left(\frac{f(z)}{z}\right)\right| \le \max_{|z|=r} arg\left(\frac{f_k(z)}{z}\right) \text{ and } \left|arg\left(g'(z)\right)\right| \le \max_{|z|=r} arg\left(g_k'(z)\right).$$

The equality holds if and only if f (or g) is a rotation of f_k (or g_k).

The following result provides the convexity radius of $\varphi_k(z)$:

Theorem 2.3. *$\varphi_k(z)$ is convex for $|z| < r_0$ whenever $r_0 \approx 0.7293401$.*

Proof. Let $z = re^{i\theta}$. Thus, after several computations and simplifications, we obtain

$$Re\left(1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right) = \frac{3 + e^{2z}(3 - 4z) + 6z}{3(1 + e^{2z})} = \frac{\Upsilon(r,\theta)}{\Xi(r,\theta)} =: C(r,\theta),$$

where the numerator

$$\Upsilon(r,\theta) = 1 + 2rcos\theta + e^{4rcos\theta}\left(1 - \frac{4}{3}rcos\theta\right)$$
$$+ e^{2rcos\theta}\left(2cos(2rsin\theta) + r\left(2cos(\theta - 2rsin\theta) - \frac{4}{3}cos(\theta + 2rsin\theta)\right)\right),$$

and the denominator

$$\Xi(r,\theta) = 1 + e^{2rcos\theta}\, cos^2(2rsin\theta) + e^{4rcos\theta}\, sin^2(2rsin\theta).$$

Since φ_k is symmetric about the real axis, it is sufficient to consider $0 \le \theta \le \pi$. Now, we need to find the smallest value r_0 that ensures $C(r,\theta) > 0$ for all $r \le r_0$. This process is known as determining the radius of $\varphi_k(z)$. Clearly, $\Xi(r,\theta) > 0$ for all $0 < r < 1$. Therefore, we must now find the smallest r_0 that ensures $\Upsilon(r,\theta) > 0$. Specifically, for an arbitrary fixed r, the minimum of $\Upsilon(r,\theta)$ occurs at $\theta = \pi$ (see Figure 5). Therefore,

$$\Upsilon(r,\pi) = 1 - 2r + e^{-4r}\left(1 + \frac{4}{3}r\right) + e^{-2r}\left(2 - \frac{2}{3}r\right)$$
$$= 1 + 2e^{-2r} + e^{-4r} - \left(2 + \frac{2}{3}e^{-2r} - \frac{4}{3}e^{-4r}\right)r.$$

Thus, solving the equation $\Upsilon(r,\pi) = 0$ gives $r_0 \approx 0.7293401$, completing the proof. □

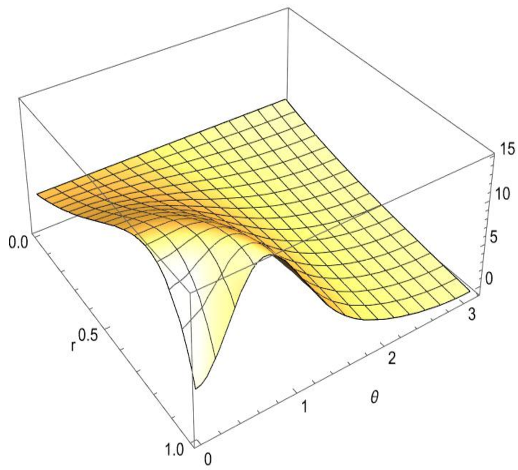

Figure 5. Plot of the function $\Upsilon(r,\pi)$.

Remark 2.2. To determine the largest disk centered at $z = 1$ that lies entirely within Ω_k, we first need to describe the parametric equations derived from the real and imaginary parts of the expression $\varphi_k(e^{i\theta})$. Next, we formulate the expression for the squared distance from $z = 1$ to an arbitrary point on the curve $\varphi_k(e^{i\theta})$. By finding the local maxima and minima, as well as the regions of increase and decrease, we can identify the minimum value of the distance-squared function, which yields the desired radius. However, implementing these steps is highly challenging due to the complexity of describing the parametric equations and the difficulties in handling them. Therefore, we provide an alternative approach below, using Python software code (version 13.3.2) to compute this radius. Similarly, we find the smallest disk centered at $z = 1$ that contains Ω_k. The code below will produce the optimal radiuses along with the corresponding value of θ.

```
Computational code 1:
import numpy as np
from scipy.optimize import minimize, minimize_scalar
def phi_k(z):
return 2 - (4 / (1 + np.exp(2 * z))**2)**(1/3)
def polar_form(theta):
z = np.exp(1j * theta)  # z = e^(i theta)
return phi_k(z)
def u(theta):
return np.real(polar_form(theta))
def v(theta):
return np.imag(polar_form(theta))
fixed_point = (1, 0)
def distance(theta):
u_val = u(theta)
v_val = v(theta)
return np.sqrt((u_val - fixed_point[0])**2 + v_val**2)
min_result = minimize_scalar(distance, bounds=(0, 2 * np.pi), method='bounded')
min_distance = min_result.fun
min_theta = min_result.x
max_result = minimize_scalar(lambda t: -distance(t), bounds=(0, 2 * np.pi), method='bounded')
max_distance = -max_result.fun
max_theta = max_result.x
print(f"Minimum Distance: {min_distance}")
print(f"Theta for Minimum Distance: {min_theta}")
print(f"Maximum Distance: {max_distance}")
print(f"Theta for Maximum Distance: {max_theta}")
Execution:
r=0.458603,   0=3.14159
R=0.961087,  0=4.79737
```

Therefore, we have the following inequality:

$$\mathbb{D}_r = \{\omega \in \mathbb{C}\colon\ |\omega - 1| < r \approx 0.458603\} \subseteq \Omega_k \subseteq \mathbb{D}_R = \{\omega \in \mathbb{C}\colon\ |\omega - 1| < R \approx 0.961087\}. \qquad (2.6)$$

Figure 6 provides an illustrative depiction of the largest disk that can be entirely contained within the studied domain, as well as the smallest disk that completely encloses the domain, thereby offering a clear visual representation of the domain's boundaries, as well as its internal containment and expansion limits.

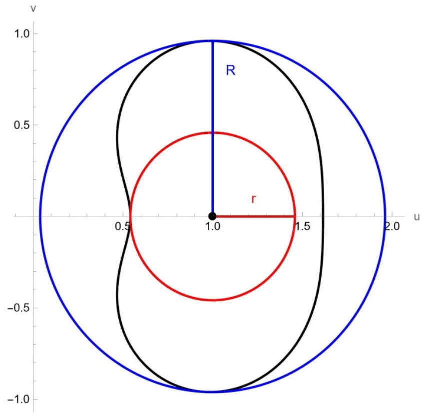

Figure 6. Plot of Ω_k, $\mathbb{D}_r$, and $\mathbb{D}_R$ in colors, black, red, and blue, respectively.

3. Inclusion results and radius problems

Before providing the main results, we will find the bounds of the real part of $\varphi_k(z)$ over $|z| = 1$. The following Python software code (version 13.3.2) contains the maximum and minimum values, as shown in Figure 7.

$$\left.\begin{array}{l} \max_{|z|=1} \operatorname{Re} \varphi_k(z) = 2 - \sqrt[3]{4/\left(1+e^2\right)^2} \approx 1.6155, \\ \min_{|z|=1} \operatorname{Re} \varphi_k(z) = 2 - \operatorname{Re}\left(\sqrt[3]{4/\left(1+e^{2e^{i\frac{13\pi}{20}}}\right)^2} \right) \approx 0.46732 \end{array}\right\}. \tag{3.1}$$

Computational code 2:

```
\noindent import numpy as np
\noindent def phi\_k(z):
\noindent     return 2 - (4 / (1 + np.exp(2 * z))**2)**(1/3)
\noindent def Lambda(theta):
\noindent     z = np.exp(1j * theta)  \# z = e${\wedge}$(i$\theta$)
\noindent     return np.real(phi\_k(z))
\noindent sample\_points = np.linspace(0, np.pi, 101)
\noindent sample\_values = np.array([Lambda(theta) for theta in sample\
    _points])
\noindent max\_value = np.max(sample\_values)
\noindent min\_value = np.min(sample\_values)
\noindent max\_index = np.argmax(sample\_values)
\noindent min\_index = np.argmin(sample\_values)
\noindent max\_point = sample\_points[max\_index]
\noindent min\_point = sample\_points[min\_index]
\noindent max\_value\_approx = round(max\_value, 5)
\noindent min\_value\_approx = round(min\_value, 5)
\noindent max\_point\_approx = round(max\_point, 3)
\noindent min\_point\_approx = round(min\_point, 3)
```

```
\noindent print(f"Maximum Value: ${\{}$max\_value\_approx${\}}$, at Theta: $
    {\{}$max\_point\_approx${\}}$")
\noindent print(f"Minimum Value: ${\{}$min\_value\_approx${\}}$, at Theta: $
    {\{}$min\_point\_approx${\}}$").
```

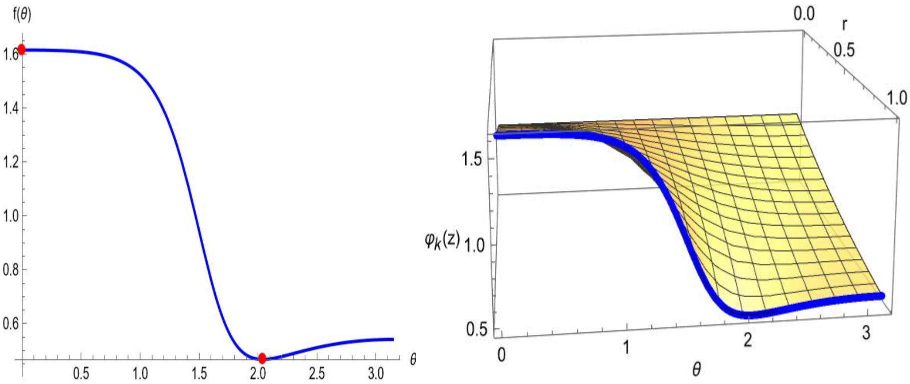

Figure 7. Graphs showing the maximum and minimum values of $Re\ \varphi_k(z)$ as red points.

Recall that the class $\mathcal{M}(\gamma)\,(\gamma > 1)$ is defined by $Re\,(zf'(z)/f(z)) < \gamma$ (see [32]). The following result provides inclusion relations between the class $\mathcal{S}_k^*$ and certain classes that were previously mentioned in Table 2.

Theorem 3.1. *The class $\mathcal{S}_k^*$ satisfies the following inclusion relations:*

(a) $\mathcal{S}_k^* \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^*$ *whenever* $0 \le \alpha \le 0.46732$.

(b) $\mathcal{S}_k^* \subset \mathcal{M}(\gamma)$ *whenever* $\gamma \ge 1.6155$.

(c) $\mathcal{S}_k^* \subset \mathcal{SS}^*(\upsilon)$ *for* $0.5869227 \approx \upsilon_0 \le \upsilon < 1$.

(d) $k - \mathcal{ST} \subset \mathcal{S}_k^*$ *whenever* $k > 2.624653$.

(e) $\mathcal{S}_{qc}^* \subset \mathcal{S}_k^*$ *whenever* $0 < c \le 0.70689$.

(f) $\mathcal{S}_k^* \subset \mathcal{S}_e^*$.

(g) $\mathcal{S}_{\vartheta,L}^* \subset \mathcal{S}_k^*$ *whenever* $\vartheta \ge 0.53268$.

Proof. (a) and (b):

For $f \in \mathcal{S}_k^*$ and by (3.1), it can be readily concluded that

$$0.46732 = \min_{|z|=1} Re\ \varphi_k(z) \quad < Re\frac{zf'(z)}{f(z)} < \max_{|z|=1} Re\varphi_k(z) = 1.6155.$$

Hence,

$$f \in \mathcal{S}^*(0.46732) \cap \mathcal{M}(1.6155)\,.$$

(c) If $f \in \mathcal{S}_k^*$, then

$$\left| arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \max_{|z|=1} arg\,(\varphi_k(z))$$

$$= \max_{-\pi<\theta\le\pi} arg\left(\varphi_k\left(e^{i\theta}\right)\right).$$

We will find the maximum value of the right side by writing and implementing the following Python software code (version 13.3.2):

Computational code 3:

```
\noindent import numpy as
\noindent from scipy.optimize import minimize\_scalar
\noindent def f(z):
\noindent     return 2 - (4 / (1 + np.exp(2 * z))**2)**(1/3)
\noindent def z\_polar(theta):
\noindent     return np.exp(1j * theta)
\noindent def argF(theta):
\noindent     z = z\_polar(theta)
\noindent     return np.angle(f(z))
\noindent result = minimize\_scalar(lambda theta: -argF(theta), bounds=(-np.
    pi, np.pi), method='bounded')
\noindent max\_arg = -result.fun
\noindent theta\_at\_max = result.x
\noindent v = 2 * max\_arg / np.pi
\noindent print(f"Maximum Arg Value: ${\{}$max\_arg${\}}$")
\noindent print(f"Theta at Maximum Arg: ${\{}$theta\_at\_max${\}}$")
\noindent print(f"v Value: ${\{}$v${\}}$")
\noindent \textit{Execution: }
$${\mathop{{max}}_{-\pi <\theta \le \pi } {{arg} \left({\varphi }_k\left(e^{i
    \theta }\right)\right) }}=0.921936{, \; }{\theta }{=}1.74425, \; {\upsilon
     }_0=0.5869227.$$
```

(d) Let $f \in k-\mathcal{ST}$. Then $zf'(z)/f(z) \prec \varphi_k(z)$ (see [5]). It is shown that the function $\varphi_k(z)$ maps $\mathbb{U}$ onto the region $\{\omega \in \mathbb{C} : Re(\omega) > k\,|\omega - 1|\}$, which is enclosed by the ellipse

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1,$$

where $(x_0, y_0) = \left(k^2/k^2-1, 0\right)$, $a = k/(k^2-1)$, and $b = 1/\sqrt{k^2-1}$.

As $a > b$, so this is a horizontal ellipse. Therefore, for the elliptic region above be included in $\varphi_k(\mathbb{U})$, we must have

$$0.541397 \le \frac{k_1^2}{k_1^2-1} - \frac{k_1}{\left(k_1^2-1\right)},\ and\ \frac{k_2^2}{k_2^2-1} + \frac{k_2}{\left(k_2^2-1\right)} \le 1.615516.$$

So this leads to $k > max\{k_1, k_2\} = \{1.180535,\ 2.624653\} = 2.624653$.

(e) It was established [6] that the Carathéodory function $q_c(z) = \sqrt{1+cz}$ ($c \in (0,1]$) maps $\mathbb{U}$ onto the interior of the curve $\left|w^2 - 1\right| = c$. Straightforward analysis reveals that $q_c(z)$ is contained in $\varphi_k(\mathbb{U})$ if and only if $\sqrt{1-c} \ge 1 - r = 0.541397$, which is equivalent to $c \le 0.70689$.

(f) Use (2.6) and [12, Lemma 2.2].

(g) Based on [18, Lemma 2.3], since the function $\varphi_{\vartheta,L}(z) = \vartheta + (1-\vartheta)\sqrt{1+z}$ maps $\mathbb{U}$ onto $\left\{w : \left|\left(\frac{w-\vartheta}{1-\vartheta}\right)^2 - 1\right| < 1,\ Re\,(w) > 0\right\}$, a straightforward computation reveals that this region is contained within $\varphi_k(\mathbb{U})$ given that $1-\vartheta < 0.46732$. Subsequent simplification yields $\vartheta \geq 0.53268$.

Therefore, the proof is complete, with all the inclusions that have been established illustrated precisely in Figure 8. □

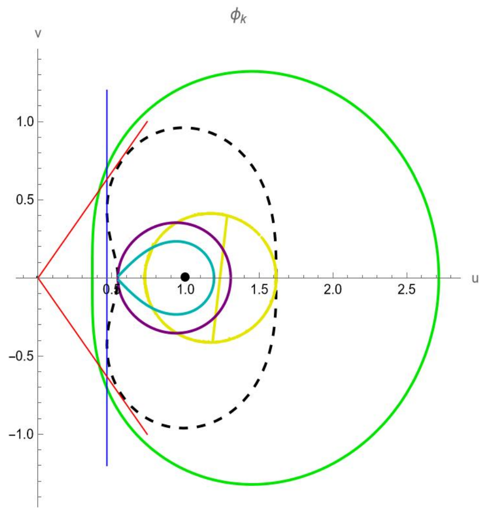

Figure 8. A plot showing the inclusions (a)–(g) in Theorem 3.1 as colors (a) blue, (c) red, (d) yellow, (e) purple, (f) green, and (g) cyan, respectively.

Let us recall that for any arbitrary subclasses M_1 and M_2 of $\mathbb{A}$, the M_1-radius for the class M_2 represents the largest value ρ, where $0 < \delta < 1$, ensuring that $\rho^{-1} f(\rho z) \in M_1$ holds true whenever $f \in M_2$ and $0 < \rho \leq \delta$. This radius, which is denoted by $R_{M_1}(M_2)$, highlights the following results that explore specific radius problems for $\mathcal{S}_k^*$:

Example 3.1. *The radius connection between $\mathcal{S}^*_{n\mathcal{L}}$ and $\mathcal{S}^*_k$ can be described as follows: Let $f \in \mathcal{S}^*_{n\mathcal{L}}$. Then*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n \quad (n \geq 2).$$

Hence, for $|z| = r$, we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r}\left|\frac{n}{n+1}z + \frac{1}{n+1}z^n\right| \leq \frac{n}{n+1}r + \frac{1}{n+1}r^n. \tag{3.2}$$

The disk (3.2) *lies entirely inside Ω_k if it satisfies* (2.6) $(n \geq 3)$. *Individually, we also obtain $\rho_2 =$* 0.68500. *Therefore, $r \leq \rho_n$ $(n \geq 2)$ (see Table 3).*

Table 3. The values of $R_{\mathcal{S}_k^*}(\mathcal{S}_{n\mathcal{L}}^*)$ at $n = 2, 3, \ldots, 31$.

n	ρ_n	n	ρ_n	n	ρ_n
2	0.68500	12	0.49680	22	0.47944
3	0.55460	13	0.49387	23	0.47854
4	0.55032	14	0.49135	24	0.47771
5	0.54105	15	0.48917	25	0.47694
6	0.53128	16	0.48726	26	0.47624
7	0.52259	17	0.48557	27	0.47558
8	0.51530	18	0.48408	28	0.47498
9	0.50930	19	0.48274	29	0.47441
10	0.50435	20	0.48153	30	0.47389
11	0.50025	21	0.48044	31	0.47339

Theorem 3.2. *The subsequent results for the radius are valid:*

(1) $R_{\mathcal{S}_k^*}\left(\mathcal{S}_{\vartheta,e}^*\right) = \rho_\vartheta := \log\left((1.458603 - \vartheta)/(1 - \vartheta)\right),\ 0 \le \vartheta \le 0.733103.$

(2) $R_{\mathcal{S}_k^*}\left(\mathcal{S}_{\vartheta,L}^*\right) = \rho_\vartheta := (0.70639 - 0.917206\vartheta)/(1 - \vartheta)^2,\ 0 \le \vartheta \le 0.57728.$

(3) $R_{\mathcal{S}_k^*}\left(\mathcal{BS}^*(\vartheta)\right) = \rho_\vartheta := \left(-1 + \sqrt{1 + 0.841266\vartheta}\right)/0.917206\vartheta,\ 0 \le \vartheta < 1.$

(4) $R_{\mathcal{S}_k^*}\left(\mathcal{S}_C^*\right) = \rho_C \approx 0.299193.$

(5) $R_{\mathcal{S}_k^*}\left(\mathcal{S}_S^*\right) = \rho_S \approx 0.443882.$

(6) $R_{\mathcal{S}_k^*}\left(\mathcal{S}_R^*\right) = \rho_R \approx 0.642009.$

(7) $R_{\mathcal{S}_k^*}\left(\mathcal{S}_{\leftmoon}^*\right) = \rho_{\leftmoon} \approx 0.382541.$

Proof. (1) If $f \in \mathcal{S}_{\vartheta,e}^*$, then $zf'(z)/f(z) \prec \vartheta + (1 - \vartheta)\, e^z$. Hence, for $|z| = r$, we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \max_{|z|=r} |\vartheta + (1 - \vartheta)\, e^z - 1| \le (1 - \vartheta)(e^r - 1) \le 0.458603.$$

Thus, $r \le \log\left((1.458603 - \vartheta)/(1 - \vartheta)\right) = \rho_\vartheta$.

(2) If $f \in \mathcal{S}_{\vartheta,L}^*$, then $zf'(z)/f(z) \prec \vartheta + (1 - \vartheta)\sqrt{1 + z}$. Hence, for $|z| = r$, we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \max_{|z|=r} \left|\vartheta + (1 - \vartheta)\sqrt{1 + z} - 1\right| \le (1 - \vartheta)(1 - \sqrt{1 - r}) \le 0.458603.$$

Thus, $r \le (0.70639 - 0.917206\vartheta)/(1 - \vartheta)^2 = \rho_\vartheta$.

(3) If $f \in \mathcal{BS}^*$, then $zf'(z)/f(z) \prec 1 + z/(1 - \vartheta z^2)$. Hence, for $|z| = r$, we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \max_{|z|=r} \left|z/(1 - \vartheta z^2)\right| \le r/(1 - \vartheta r^2) \le 0.458603.$$

Thus, $0.458603\vartheta r^2 + r - 0.458603 \le 0$, and this gives $r \le \frac{-1+\sqrt{1+0.841266\vartheta}}{0.917206\vartheta} = \rho_\vartheta$.

(4) If $f \in \mathcal{S}_C^*$, then $zf'(z)/f(z) \prec 1 + 4z/3 + 2z^2/3$. Hence, for $|z| = r$, we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r} \left|4z/3 + 2z^2/3\right| \leq 4r/3 + 2r^2/3 \leq 0.458603.$$

Thus, $r \leq 0.299193 = \rho_C$.

(5) If $f \in \mathcal{S}_S^*$, then $zf'(z)/f(z) \prec 1 + sinz$. Hence, for $|z| = r$, we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r} |sinz| \leq sinhr \leq 0.458603.$$

Thus, $r \leq sinh^{-1}0.458603 \approx 0.443882 = \rho_S$.

(6) If $f \in \mathcal{S}_R^*$, then $zf'(z)/f(z) \prec 1 + z\left(\sqrt{2} + 1 + z\right)/\left(\sqrt{2} + 1\right)\left(\sqrt{2} + 1 - z\right)$. Hence, for $|z| = r$, we get

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - 1\right| &\leq \max_{|z|=r} \left|z\left(\sqrt{2} + 1 + z\right)/\left(\sqrt{2} + 1\right)\left(\sqrt{2} + 1 - z\right)\right| \\ &\leq r\left(\sqrt{2} + 1 + r\right)/\left(\sqrt{2} + 1\right)\left(\sqrt{2} + 1 - r\right) \leq 0.458603.\end{aligned}$$

Thus, $r \leq 0.642009 = \rho_R$.

(7) If $f \in \mathcal{S}_{☾}^*$, then $zf'(z)/f(z) \prec z + \sqrt{1 + z^2}$. Hence, for $|z| = r$, we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \max_{|z|=r} \left|z + \sqrt{1 + z^2} - 1\right| \leq 1 + r - \sqrt{1 - r^2} \leq 0.458603.$$

Thus, $r \leq 0.382541 = \rho_{☾}$.

□

4. Coefficient bounds

In this section, we derive estimates for the Maclaurin coefficients and determine the upper bound for the 2nd Hankel determinant for every function in $\mathcal{S}_k^*$. Furthermore, we present new results on the sharp bounds for the logarithmic coefficients associated with all functions in this class. However, we list below several concepts and lemmas that serve as fundamental tools for the results in this section. Carefully chosen for their importance and relevance, each lemma plays a distinct role in advancing the coefficient problems explored in this study.

(1) The q^{th} Hankel determinant of any function f is expressed as

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad (z \in \mathbb{U}),$$

and defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (q \geq 1).$$

Specifically, we have

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \text{ and } H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

(2) Denote by $\mathbb{P}$ the class of analytic functions p with $Re(p(z)) > 0$ $(z \in \mathbb{U})$ by the following series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} \sigma_n z^n, \ z \in \mathbb{U}. \tag{4.1}$$

Lemma 4.1. *[13, 33] Let $p \in \mathbb{P}$, and then*

$$|\sigma_n| \leq 2, \ n \geq 1, \tag{4.2}$$

$$|\sigma_{r+n} - \beta\sigma_r\sigma_n| < 2, \ 0 < \beta \leq 1, \tag{4.3}$$

$$\left|\sigma_2 - \delta\sigma_1^2\right| < 2max\{1, \ |2\delta - 1|\}, \ \delta \in \mathbb{C}. \tag{4.4}$$

Lemma 4.2. *[34] Let $0 \leq A \leq 1$ with $A(2A-1) \leq B \leq A$, and let $p \in \mathbb{P}$ be of the form (4.1). Then*

$$\left|B\sigma_1^3 - 2A\sigma_1\sigma_2 + \sigma_3\right| \leq 2. \tag{4.5}$$

Lemma 4.3. *[35, 36] Let $p \in \mathbb{P}$ be of the form (4.1). Then,*

$$2\sigma_2 = \sigma_1^2 + \gamma\left(4 - \sigma_1^2\right), \tag{4.6}$$

$$4\sigma_3 = \sigma_1^3 + 2\sigma_1\left(4 - \sigma_1^2\right)\gamma - \sigma_1\left(4 - \sigma_1^2\right)\gamma^2 + 2\left(4 - \sigma_1^2\right)\left(1 - |\gamma|^2\right)\eta, \tag{4.7}$$

for some $\gamma, \eta \in \mathbb{C}$ with $max\{|\gamma|, |\eta|\} \leq 1$.

Theorem 4.1. *Let $f \in \mathcal{S}_k^*$. Then,*

$$|a_2| \leq \frac{2}{3}, \ |a_3| \leq \frac{1}{3}, \ and \ |a_4| \leq \frac{2}{9}.$$

These bounds are sharp.

Proof. Let $f \in \mathcal{S}_k^*$, and then there exists a Schwarz function $u : \mathbb{U} \longrightarrow \mathbb{U}$ that satisfies $u(0) = 0$, such that

$$\frac{zf'(z)}{f(z)} = 2 - \sqrt[3]{4/(1 + e^{2u(z)})^2}, \ z \in \mathbb{U}. \tag{4.8}$$

Define $p \in \mathbb{P}$ by

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \sigma_1 z + \sigma_2 z^2 + \sigma_3 z^3 + \ldots, \ z \in \mathbb{U}. \tag{4.9}$$

Therefore,

$$\begin{aligned} 2 - \sqrt[3]{4/(1 + e^{2u(z)})^2} = 1 &+ \frac{\sigma_1 z}{3} + \left(\frac{1}{3}\sigma_2 - \frac{5}{36}\sigma_1^2\right)z^2 + \left(\frac{11}{324}\sigma_1^3 - \frac{45}{162}\sigma_1\sigma_2 + \frac{27}{81}\sigma_3\right)z^3 \\ &+ \left(\frac{17}{1944}\sigma_1^4 + \frac{99}{972}\sigma_1^2\sigma_2 - \frac{135}{972}\sigma_2^2 - \frac{135}{486}\sigma_1\sigma_3 + \frac{81}{243}\sigma_4\right)z^4 + \ldots, \end{aligned} \tag{4.10}$$

and

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right) z^2 + \left(3a_4 - 3a_2 a_3 + a_2^3\right) z^3 \tag{4.11}$$

$$+ \left(4a_5 - 4a_2 a_4 - 2a_3^2 + 4a_3 a_2^2 - a_2^4\right) z^4 + \cdots . \tag{4.12}$$

Comparing the first four coefficients in (4.10) and (4.11), we obtain

$$a_2 = \frac{1}{3}\sigma_1, \tag{4.13}$$

$$a_3 = \frac{1}{6}\sigma_2 - \frac{1}{72}\sigma_1^2, \tag{4.14}$$

$$a_4 = -\frac{11}{1944}\sigma_1^3 - \frac{1}{27}\sigma_1\sigma_2 + \frac{1}{9}\sigma_3. \tag{4.15}$$

From (4.2) and (4.13), we have

$$|a_2| \leq \frac{2}{3}.$$

Using (4.3) for $\beta = 1/12$, thus (4.14) gives

$$|a_3| = \frac{1}{6}\left|\sigma_2 - \frac{1}{12}\sigma_1^2\right| \leq \frac{1}{3}.$$

Put $A = 9/54$, $B = -99/1944$, then $A(2A-1) = -0.1111 \leq B = -0.0509 \leq A = 0.1666$. By applying (4.5) to (4.15), we obtain

$$|a_4| = \frac{1}{9}\left|-\frac{99}{1944}\sigma_1^3 - \frac{9}{27}\sigma_1\sigma_2 + \sigma_3\right| \leq \frac{2}{9}.$$

For sharpness, Theorem 2.1 gives that

$$k_n(z) = z exp\left(\int_0^z \frac{1 - \sqrt[3]{4/(1+e^{2n\tau})^2}}{\tau}\, d\tau\right) \qquad (z \in \mathbb{U},\ n = 1,2,3) \tag{4.16}$$

belongs to $\mathcal{S}_k^*$, and

$$k_1(z) = z + \frac{2z^2}{3} + \frac{5z^3}{18} + \frac{7z^4}{243} - \frac{161z^5}{5832} - \frac{1517z^6}{218700} + \ldots, \tag{4.17}$$

$$k_2(z) = z + \frac{z^3}{3} + \frac{z^5}{12} - \frac{13z^7}{972} + \ldots, \tag{4.18}$$

$$k_3(z) = z + \frac{2z^4}{9} + \frac{7z^7}{162} - \frac{29z^{10}}{2187} + \cdots . \tag{4.19}$$

The proof is complete. □

Theorem 4.2. *Let $f \in \mathcal{S}_k^*$. Then for $\delta \in \mathbb{C}$, we have*

$$\left|a_3 - \delta a_2^2\right| \leq \frac{1}{3} max\left\{1,\ \frac{|8\delta - 5|}{6}\right\}.$$

The sharpness is given by (4.17) *and* (4.18).

Proof. This follows from (4.13) and (4.14), as well as applying (4.4). □

Theorem 4.3. *Let $f \in \mathcal{S}_k^*$. Then*

$$|a_2a_3 - a_4| \leq \frac{2}{9} .$$

The sharpness is given by (4.19).

Proof. It follows from (4.13), (4.14), and (4.15) that

$$|a_2a_3 - a_4| = \left|\frac{\sigma_1^3}{972} + \frac{5\sigma_1\sigma_2}{54} - \frac{\sigma_3}{9}\right| = \frac{1}{9}\left|-\frac{9\sigma_1^3}{972} - \frac{45\sigma_1\sigma_2}{54} + \sigma_3\right|.$$

Putting $A = 45/108$ and $B = -9/972$, we have $A(2A-1) = -0.069444 \leq B = -0.009259 \leq A = 0.416666$. Therefore, Lemma 4.2 leads to

$$|a_2a_3 - a_4| \leq \frac{2}{9}.$$

Thus, the desired result is achieved, and the proof is complete. □

Theorem 4.4. *Let $f \in \mathcal{S}_k^*$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{9} .$$

The sharpness is given by (4.18).

Proof. It follows from (4.13), (4.14), and (4.15) that

$$\left|a_2a_4 - a_3^2\right| = \left|-\frac{97\sigma_1^4}{46656} - \frac{5\sigma_1^2\sigma_2}{648} + \frac{\sigma_1\sigma_3}{27} - \frac{\sigma_2^2}{36}\right|.$$

Taking $\sigma_1 = \sigma$ and $\tau = |\gamma|$ in Lemma 4.3,

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| = &|-\frac{169}{46656}\sigma_1^4 - \frac{1}{108}\sigma_1^2\left(4-\sigma_1^2\right)\gamma^2 + \frac{1}{1296}\sigma_1^2\left(4-\sigma_1^2\right)\gamma \\ &+ \frac{1}{54}\sigma_1\left(4-\sigma_1^2\right)\left(1-|\gamma|^2\right)\eta - \frac{1}{144}\gamma^2\left(4-\sigma_1^2\right)^2| \\ \leq &\frac{169}{46656}\sigma^4 + \frac{1}{108}\sigma^2\left(4-\sigma^2\right)\tau^2 \\ &+ \frac{1}{1296}\sigma^2\left(4-\sigma^2\right)\tau + \frac{1}{54}\sigma\left(4-\sigma^2\right)\left(1-\tau^2\right) + \frac{1}{144}\tau^2\left(4-\sigma^2\right)^2 \\ := &\Upsilon(\tau,\sigma).\end{aligned}$$

Now, we need to maximize $\Upsilon(\tau,\sigma)$, so

$$\begin{aligned}\frac{\partial\Upsilon(\tau,\sigma)}{\partial\tau} &= \frac{1}{54}\sigma^2\left(4-\sigma^2\right)\tau + \frac{1}{1296}\sigma^2\left(4-\sigma^2\right) - \frac{1}{27}\sigma\left(4-\sigma^2\right)\tau + \frac{1}{72}\tau\left(4-\sigma^2\right)^2 \\ &= \frac{1}{1296}\left(4-\sigma^2\right)\left(72\tau + 6\sigma^2\tau - 48\sigma\tau + \sigma^2\right).\end{aligned}$$

Clearly, it follows that $\frac{\partial \Upsilon(\tau,\sigma)}{\partial \tau} \geq 0$ in $[0, 1]$. Thus, the maximum value of $\Upsilon(\tau, \sigma)$ is attained when $\tau = 1$, yielding:

$$\Upsilon(\sigma) := \Upsilon(1, \sigma) = \frac{169}{46656}\sigma^4 + \frac{13}{1296}\sigma^2\left(4 - \sigma^2\right) + \frac{1}{144}\left(4 - \sigma^2\right)^2.$$

Now, we want to find the value of σ that maximizes $\Upsilon(\sigma)$. Therefore,

$$\Upsilon'(\sigma) = \frac{5}{11664}\sigma\left(-72 + 5\sigma^2\right).$$

After setting it to 0, we obtain the three roots: $\sigma = 0$, $\sigma = -6\sqrt{2/5}$, and $\sigma = 6\sqrt{2/5}$. Since the only root within $[0, 2]$ is $\sigma = 0$, and $\Upsilon''(0) < 0$, therefore, the maximum is attained at $\sigma = 0$, and this gives

$$\left|a_2a_4 - a_3^2\right| \leq \Upsilon(1, 0) = \frac{1}{9}.$$

Thus, the proof is established, affirming the validity of the result. □

The logarithmic coefficients for any function $f \in \mathcal{S}$ are defined as follows:

$$log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty}\xi_n z^n\ ,\ z \in \mathbb{U}. \tag{4.20}$$

The reason for this is Theorem 3.1(a) ($\mathcal{S}_k^* \subset \mathcal{S}$), as the logarithmic coefficients can be defined for $f \in \mathcal{S}_k^*$. In the following results, we will present some bounds on the coefficients for functions that lie within $\mathcal{S}_k^*$.

Theorem 4.5. *Let $f \in \mathcal{S}_k^*$. Then,*

$$|\xi_n| \leq \frac{1}{3n}\ (n = 1, 2, 3).$$

These bounds are sharp.

Proof. Let $f \in \mathcal{S}_k^*$, then

$$log\frac{f(z)}{z} = a_2z + \frac{1}{2}\left(-a_2^2 + 2a_3\right)z^2 + \frac{1}{3}\left(a_2^3 - 3a_2a_3 + 3a_4\right)z^3 + \ldots. \tag{4.21}$$

The equality of (4.20) and (4.21) gives

$$\xi_1 = \frac{a_2}{2},\ \xi_2 = \frac{1}{4}\left(-a_2^2 + 2a_3\right),\ and\ \xi_3 = \frac{1}{6}\left(a_2^3 - 3a_2a_3 + 3a_4\right). \tag{4.22}$$

Substituting (4.13), (4.14), and (4.15) into (4.22), we obtain

$$\xi_1 = \frac{\sigma_1}{6}, \tag{4.23}$$

$$\xi_2 = \frac{\sigma_2}{12} - \frac{5\sigma_1^2}{144}, \tag{4.24}$$

$$\xi_3 = \frac{11\sigma_1^3}{1944} - \frac{5\sigma_2\sigma_1}{108} + \frac{\sigma_3}{18}. \tag{4.25}$$

By (4.2) and (4.23), we have

$$|\xi_1| \leq \frac{1}{3}.$$

Using (4.3) for $\beta = 5/12$, thus (4.24) gives

$$|\xi_2| = \frac{1}{12}\left|\sigma_2 - \frac{5\sigma_1^2}{12}\right| \leq \frac{1}{6}\ .$$

Put $A = 90/216$, $B = 198/1944$, then $A(2A-1) = -0.069444 \leq B = 0.101851 \leq A = 0.416666$. By applying (4.5) to (4.25), we obtain

$$|\xi_3| = \frac{1}{18}\left|\frac{198\sigma_1^3}{1944} - \frac{90\sigma_2\sigma_1}{108} + \sigma_3\right| \leq \frac{1}{9}\ .$$

Since

$$\frac{1}{2}log\frac{k_1(z)}{z} = \frac{z}{3} + \frac{z^2}{36} - \frac{7z^3}{243} - \frac{11z^4}{1944} + \frac{107z^5}{18225} + \ldots, \tag{4.26}$$

$$\frac{1}{2}log\frac{k_2(z)}{z} = \frac{z^2}{6} + \frac{z^4}{72} - \frac{7z^6}{486} + \frac{167z^8}{46656} - \frac{7z^{10}}{43740} + \ldots, \tag{4.27}$$

$$\frac{1}{2}log\frac{k_3(z)}{z} = \frac{z^3}{9} + \frac{z^6}{108} - \frac{7z^9}{729} + \ldots. \tag{4.28}$$

Thus (4.17), (4.18), and (4.19) give sharp bounds. □

Theorem 4.6. *Let $f \in \mathcal{S}_k^*$. Then for $\delta \in \mathbb{C}$, we have*

$$\left|\xi_2 - \delta\xi_1^2\right| \leq \frac{1}{6}max\left\{1,\ \frac{|4\delta - 1|}{6}\right\}.$$

The sharpness is given by (4.17) *and* (4.18).

Proof. This follows from (4.23), (4.24), and (4.4). □

Theorem 4.7. *Let $f \in \mathcal{S}_k^*$. Then*

$$|\xi_1\xi_2 - \xi_3| \leq \frac{1}{9}\ .$$

The sharpness is given by (4.19) *and* (4.28).

Proof. It follows from (4.23), (4.24), and (4.15) that

$$|\xi_1\xi_2 - \xi_3| = \left|-\frac{89\sigma_1^3}{7776} + \frac{13\sigma_1\sigma_2}{216} - \frac{\sigma_3}{18}\right| = \frac{1}{18}\left|\frac{1602\sigma_1^3}{7776} - \frac{234\sigma_1\sigma_2}{216} + \sigma_3\right|.$$

Putting $A = 234/432$ and $B = 1602/7776$, we have $A(2A-1) = 0.045138 \leq B = 0.206018 \leq A = 0.541666$. Thus, Lemma 4.2 gives

$$|\xi_1\xi_2 - \xi_3| \leq \frac{1}{9}.$$

Thus, the proof is finalized. □

Theorem 4.8. *Let $f \in \mathcal{S}_k^*$. Then*

$$\left|\xi_1\xi_3 - \xi_2^2\right| \leq \frac{1}{36}.$$

The sharpness is given by (4.18) *and* (4.27).

Proof. It follows from (4.23), (4.24), and (4.25) that

$$\left|\xi_1\xi_3 - \xi_2^2\right| = \left|-\frac{49\sigma_1^4}{186624} - \frac{5\sigma_1^2\sigma_2}{2592} + \frac{\sigma_1\sigma_3}{108} - \frac{\sigma_2^2}{144}\right|.$$

Taking $\sigma_1 = \sigma$ and $\tau = |\gamma|$ in Lemma 4.3,

$$\begin{aligned}
\left|\xi_1\xi_3 - \xi_2^2\right| &= |-\frac{21}{186624}\sigma_1^4 - \frac{1}{432}\sigma_1^2\left(4-\sigma_1^2\right)\gamma^2 + \frac{1}{5184}\sigma_1^2\left(4-\sigma_1^2\right)\gamma \\
&\quad + \frac{2}{432}\sigma_1\left(4-\sigma_1^2\right)\left(1-|\gamma|^2\right)\eta - \frac{1}{576}\gamma^2\left(4-\sigma_1^2\right)^2| \\
&\leq \frac{21}{186624}\sigma^4 + \frac{1}{432}\sigma^2\left(4-\sigma^2\right)\tau^2 \\
&\quad + \frac{1}{5184}\sigma^2\left(4-\sigma^2\right)\tau + \frac{2}{432}\sigma\left(4-\sigma^2\right)\left(1-\tau^2\right) + \frac{1}{576}\tau^2\left(4-\sigma^2\right)^2 \\
&:= \Upsilon(\tau,\sigma).
\end{aligned}$$

Now, we need to maximize $\Upsilon(\tau,\sigma)$, so

$$\begin{aligned}
\frac{\partial\Upsilon(\tau,\sigma)}{\partial\tau} &= \frac{1}{216}\sigma^2\left(4-\sigma^2\right)\tau + \frac{1}{5184}\sigma^2\left(4-\sigma^2\right) - \frac{1}{108}\sigma\left(4-\sigma^2\right)\tau + \frac{1}{288}\tau\left(4-\sigma^2\right)^2 \\
&= \frac{1}{5184}\left(4-\sigma^2\right)(72\tau + 6\sigma^2\tau - 48\sigma\tau + \sigma^2).
\end{aligned}$$

Clearly, it follows that $\frac{\partial\Upsilon(\tau,\sigma)}{\partial\tau} \geq 0$ in $[0,1]$. Thus, the maximum value of $\Upsilon(\tau,\sigma)$ is attained when $\tau = 1$, yielding:

$$\Upsilon(\sigma) := \Upsilon(1,\sigma) = \frac{21}{186624}\sigma^4 + \frac{13}{5184}\sigma^2\left(4-\sigma^2\right) + \frac{1}{576}\left(4-\sigma^2\right)^2.$$

Now, we want to find the value of σ that maximizes $\Upsilon(\sigma)$. Therefore,

$$\Upsilon'(\sigma) = \frac{-1}{15552}\sigma(120 + 41\sigma^2).$$

After setting it to 0, we obtain the three roots: $\sigma = 0$, $\sigma = -2i\sqrt{30/41}$, and $\sigma = 2i\sqrt{30/41}$. Since the only root within $[0,2]$ is $\sigma = 0$, and $\Upsilon''(0) < 0$, therefore, the maximum is attained at $\sigma = 0$, and this gives

$$\left|\xi_1\xi_3 - \xi_2^2\right| \leq \Upsilon(1,0) = \frac{1}{36}.$$

Thus, the proof is established, affirming the validity of the result. □

5. Conclusions

In this paper, we have provided a new geometric domain characterized by a univalent property with a kidney-like shape, defined using a subordination notion on starlike and convex functions. We have addressed the main constraints on the coefficients a_n, along with Hankel determinants and logarithms, and carefully examined the sharpness of the obtained results. The importance of this study lies in its multiple practical applications. The developed mathematical framework can be used to characterize heat diffusion problems and model fluid motion, in addition to its role in image processing and geometric shape analysis in computer vision. These results can also be applied to problems of electromagnetism and electrical potential distribution, as well as to the study of elastic deformations in elasticity theory. Furthermore, this work opens up avenues for research in complex dynamics and the study of systems with fractal behavior, giving the results a dual value that combines theoretical foundations with practical applications.

Author contributions

Adel Salim Tayyah: Conceptualization, methodology, software, validation, formal analysis, investigation, resources, writing—original draft preparation, writing—review and editing, visualization, supervision; Sarem H. Hadi: Conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing—original draft preparation, writing—review and editing, visualization, supervision, project administration; Zhi-Gang Wang: Methodology, software, validation, formal analysis, investigation, writing—original draft preparation, visualization; Alina Alb Lupaş: Validation, formal analysis, investigation, writing—original draft preparation, visualization, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The publication of this paper was supported by University of Oradea, Romania.

The authors would like to express their sincere gratitude to the editorial team and dedicated reviewers for their valuable contributions to their work.

Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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