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Research article

Solving Volterra-Hammerstein nonlinear integral equations via fixed point theory in complex-valued suprametric spaces

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Abstract: The Volterra-Hammerstein integral equation plays a crucial role in the control of robotic manipulators, which are widely used in industrial automation, medical robotics, and space exploration. These systems present significant control challenges due to their nonlinear behaviors and memory-dependent effects. In this research, we establish novel common fixed point theorems for generalized rational contractions within the framework of complex-valued suprametric spaces. Leveraging these theoretical advancements, we apply the derived results to solve the Volterra-Hammerstein integral equation, demonstrating its significance in robotic manipulator control. To emphasize the originality and practical applicability of our findings, a comprehensive illustrative example is provided.

Keywords: fixed points; complex-valued suprametric spaces; Volterra-Hammerstein integral equations
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1. Introduction

Fixed point (FP) theory is a vast and diverse field with three principal branches: Metric, topological, and discrete FP theory. Among these, metric FP theory serves as a foundational pillar, concentrating on proving the existence and uniqueness of FPs for self-mappings defined on metric spaces (MSs). This theory is intrinsically tied to the notions of distance and convergence, which are the defining characteristics of MSs. The concept of a MS, developed by M. Fréchet [1] in 1906, describes a set equipped with a distance function, or metric, that quantifies the distance between any two points within the space. Over time, the original idea of MSs has been extended and generalized by numerous mathematicians [2–4], leading to the development of various sophisticated frameworks. Notable generalizations include partial MSs of Matthews [5], b-MSs of Czerwik [6], rectangular MSs of Branciari [7], cone MSs of Huang [8], and supra metric spaces (SMS) of Berzig [9], among others.

These extensions have significantly broadened the scope of metric FP theory, enabling the exploration of more complex mathematical structures and their applications. A novel relaxed version of the triangle inequality has been applied in the so-called SMS, first introduced by Berzig [9]. Specifically, the Banach contraction theorem has been demonstrated in SMSs. This framework has been used to investigate several matrix and nonlinear integral equations. Later on, Berzig [10, 11] extended the concept of SMS by generalizing the triangle inequality axiom, introducing two new MSs: Generalized SMSs and b-SMSs.

On the other hand, complex numbers, introduced by the Italian mathematician Gerolamo Cardano in the 16th century while solving cubic equations, have become an integral part of mathematics and its applications. Represented in the form $z = a + bi$, where a and b are real numbers and i is the imaginary unit satisfying $i^2 = -1$, complex numbers extend the concept of real numbers and enable the solution of equations that lack real solutions, such as $x^2 + 1 = 0$. The significance of complex numbers is evident across various domains. In mathematics, they are fundamental, underpinning the Fundamental Theorem of Algebra and playing a pivotal role in analysis, geometry, and number theory. Their impact extends to physics, where they are essential for describing quantum mechanics, electromagnetism, and wave phenomena. In engineering, they simplify computations in signal processing, electrical circuit analysis, and control systems. Furthermore, in computer science, they are instrumental in graphics, fractals, and algorithms for modeling and simulations. Building upon the foundation of complex numbers, the idea of complex-valued metric spaces (CVMS) was given by Azam et al. [12] to extend classical MSs by defining the distance between points as a complex number rather than a real number. A complex-valued MS consists of a set $\mathcal{G}$ and a mapping $\vee : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ satisfying properties analogous to those of real-valued MSs, including non-negativity, symmetry, and a generalized triangle inequality. These spaces have proven valuable in FP theory, enabling the exploration of more general contraction mappings and providing a framework for solving advanced mathematical problems, including nonlinear equations and integral equations. Rouzkard et al. [13] extended Azam et al.'s results [12] by including a rational expression to the contractive condition. Following this, Sintunavarat et al. [14] further extended these results using control functions of one variable. Sitthikul et al. [15] advanced the theory by employing two variables control functions under the scope of CVMSs. Quite recently, Panda et al. [16] merged the thoughts of SMS and CVMS and pioneered the notion of complex-valued suprametric spaces (CVSMSs). In this way, they subsequently proved several common FP theorems for rational contractions within this framework. They applied their results to solve complex nonlinear integral equations via contractive mappings. For a more in-depth discussion, readers may consult references [17–20].

From another perspective, FP theorems are fundamental to functional analysis, providing a robust framework for establishing the existence and uniqueness of solutions for a diverse array of mathematical equations, including integral equations. Their significance is particularly pronounced in addressing intricate problems such as Volterra-Hammerstein integral equations, where FP theorems offer a structured methodology for proving solvability and guaranteeing uniqueness under specific constraints. Volterra-Hammerstein integral equations are paramount in the field of robot manipulator control, where dynamic systems exhibit nonlinear and memory-dependent characteristics. Robot manipulators, widely employed in industrial automation, medical robotics, and space exploration, necessitate precise control strategies to manage complex dynamics, uncertainties, and external disturbances. For further information, readers are encouraged to consult references [21–23].

In this research article, we focus on developing novel CFP theorems for generalized rational contractions that incorporate control functions of a single variable, all within the framework of CVSMSs. These theorems not only generalize existing FP results but also unify various contraction conditions previously studied in the literature. As a direct consequence of our main results, we also establish CFP theorems for mappings satisfying contraction conditions involving constant parameters instead of variable control functions, thereby broadening the applicability of our theoretical framework. To demonstrate the practical relevance of these findings, we apply the developed results to solve the Volterra-Hammerstein integral equation, which has significant implications in the control of robotic manipulators. Furthermore, to illustrate the effectiveness and applicability of our approach, a detailed and comprehensive example is provided, highlighting the step-by-step implementation of the proposed theorems and their impact on solving nonlinear integral equations. This structure allows readers to understand the logical progression of the paper, from theoretical development to practical application and illustrative demonstration.

2. Preliminaries

The concept of MS was introduced by Fréchet [1] in 1906 and is defined as follows:

Definition 1. *[1] Let $\mathcal{G} \neq \emptyset$. Define a function $\curlyvee : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^+$ as follows*

(i) $0 \leq \curlyvee(\mathfrak{b}, \mathfrak{l})$ *and* $\curlyvee(\mathfrak{b}, \mathfrak{l}) = 0 \Longleftrightarrow \mathfrak{b} = \mathfrak{l}$,
(ii) $\curlyvee(\mathfrak{b}, \mathfrak{l}) = \curlyvee(\mathfrak{l}, \mathfrak{b})$,
(iii) $\curlyvee(\mathfrak{b}, \mathfrak{l}) \leq \curlyvee(\mathfrak{b}, \varpi) + \curlyvee(\varpi, \mathfrak{l})$,

for all $\mathfrak{b}, \mathfrak{l}, \varpi \in \mathcal{G}$, *then* $(\mathcal{G}, \curlyvee)$ *is said to be a MS.*

Berzig [9] introduced the concept of SMS in this way.

Definition 2. *[9] Let $\mathcal{G} \neq \emptyset$ and $\aleph \geq 0$. Define a function $\curlyvee : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^+$ that adheres to the following axioms:*

(i) $0 \leq \curlyvee(\mathfrak{b}, \mathfrak{l})$ *and* $\curlyvee(\mathfrak{b}, \mathfrak{l}) = 0 \Longleftrightarrow \mathfrak{b} = \mathfrak{l}$,
(ii) $\curlyvee(\mathfrak{b}, \mathfrak{l}) = \curlyvee(\mathfrak{l}, \mathfrak{b})$,
(iii) $\curlyvee(\mathfrak{b}, \mathfrak{l}) \leq \curlyvee(\mathfrak{b}, \varpi) + \curlyvee(\varpi, \mathfrak{l}) + \aleph \curlyvee (\mathfrak{b}, \varpi) \curlyvee (\varpi, \mathfrak{l})$,

for all $\mathfrak{b}, \mathfrak{l}, \varpi \in \mathcal{G}$, *then* $(\mathcal{G}, \curlyvee)$ *is called a SMS.*

Example 1. *Let $\mathcal{G} = \{0, 1, 2\}$. Define $\curlyvee : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^+$ as follows:*

$$\curlyvee(0, 1) = 1.5 = \curlyvee(1, 0), \quad \curlyvee(0, 2) = 1 = \curlyvee(2, 0), \quad \curlyvee(1, 2) = 2 = \curlyvee(2, 1),$$

and

$$\curlyvee(0, 0) = \curlyvee(1, 1) = \curlyvee(2, 2) = 0.$$

Let's choose $\aleph = 1.5$. Then $(\mathcal{G}, \curlyvee)$ is a SMS but not a MS because the triangle of MS is not satisfied; that is,

$$2 = \curlyvee(1, 2) > \curlyvee(1, 0) + \curlyvee(0, 2) = 0.5 + 1.$$

Let $z_1, z_2 \in \mathbb{C}$. It is well-known that

$$z_1 \precsim z_2 \;\Leftrightarrow\; \operatorname{Re}(z_1) \leqslant \operatorname{Re}(z_2),\; \operatorname{Im}(z_1) \leqslant \operatorname{Im}(z_2),$$

where $\precsim$ is a partial order on $\mathbb{C}$. Therefore,

$$z_1 \precsim z_2,$$

if one of these axioms is met:

(a) $Re(z_1) = Re(z_2),\; Im(z_1) < Im(z_2)$,
(b) $Re(z_1) < Re(z_2),\; Im(z_1) = Im(z_2)$,
(c) $Re(z_1) < Re(z_2),\; Im(z_1) < Im(z_2)$,
(d) $Re(z_1) = Re(z_2),\; Im(z_1) < Im(z_2)$.

Azam et al. [12] defined the concept of CVMS as follows:

Definition 3. *[12] Let $\mathcal{G} \neq \emptyset$. Define a function $\curlyvee : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ such that*

(i) $0 \precsim \curlyvee(\mathfrak{b}, \mathfrak{l})$ *and* $\curlyvee(\mathfrak{b}, \mathfrak{l}) = 0 \Longleftrightarrow \mathfrak{b} = \mathfrak{l}$,
(ii) $\curlyvee(\mathfrak{b}, \mathfrak{l}) = \curlyvee(\mathfrak{l}, \mathfrak{b})$,
(iii) $\curlyvee(\mathfrak{b}, \mathfrak{l}) \precsim \curlyvee(\mathfrak{b}, \varpi) + \curlyvee(\varpi, \mathfrak{l})$,

for all $\mathfrak{b}, \mathfrak{l}, \varpi \in \mathcal{G}$, then $(\mathcal{G}, \curlyvee)$ is said to be a CVMS.

Example 2. *[12] Let $\mathcal{G} = [0, 1]$ and $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$. Define $\curlyvee : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ by*

$$\curlyvee(\mathfrak{b}, \mathfrak{l}) = \begin{cases} 0, & \text{if } \mathfrak{b} = \mathfrak{l}, \\ \frac{i}{2}, & \text{if } \mathfrak{b} \neq \mathfrak{l}. \end{cases}$$

Then $(\mathcal{G}, \curlyvee)$ is a CVMS.

Panda et al. [16] defined the notion of CVSMS in this manner.

Definition 4. *[16] Let $\mathcal{G} \neq \emptyset$ and $\curlyvee : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ be a complex-valued metric function. Define a partial order $\precsim$ on $\mathcal{G}$ as follows:*

For $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$:

$$\mathfrak{b} \precsim \mathfrak{l}$$

if and only if $\curlyvee(\mathfrak{b}, \mathfrak{l})$ is a non-negative real number.

With this partial order, we can define a CVSMS as follows:

(i) *For any $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$, $\curlyvee(\mathfrak{b}, \mathfrak{l})$ is a non-negative real number if and only if $\mathfrak{b} \precsim \mathfrak{l}$. Moreover* $\curlyvee(\mathfrak{b}, \mathfrak{l}) = 0 \Longleftrightarrow \mathfrak{b} = \mathfrak{l}$.
(ii) $\curlyvee(\mathfrak{b}, \mathfrak{l}) = \overline{\curlyvee(\mathfrak{l}, \mathfrak{b})}$ *if and only if $\mathfrak{b} \precsim \mathfrak{l}$ and $\mathfrak{l} \precsim \mathfrak{b}$.*
(iii) *For any $\mathfrak{b}, \mathfrak{l}, \varpi \in \mathcal{G}$ and $\aleph \geq 0$, we have*

$$\curlyvee(\mathfrak{b}, \mathfrak{l}) \precsim \curlyvee(\mathfrak{b}, \varpi) + \curlyvee(\varpi, \mathfrak{l}) + \aleph \curlyvee(\mathfrak{b}, \varpi) \cdot \curlyvee(\varpi, \mathfrak{l}),$$

if and only if $\mathfrak{b} \precsim \varpi$ and $\varpi \precsim \mathfrak{l}$. Then, $(\mathcal{G}, \curlyvee)$ is considered as a CVSMS.

Example 3. *[16] Let $\mathcal{G} = \mathbb{C}$. Define a function $\curlyvee : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ by*

$$\curlyvee(\mathfrak{b}, \mathfrak{l}) = \frac{1}{2}\left|\mathfrak{e}^{\mathfrak{b}} - \mathfrak{e}^{\mathfrak{l}}\right|,$$

where $\mathfrak{e}^{\mathfrak{p}}$ denotes the exponential function applied to the complex number $\mathfrak{p}$. Then $(\mathcal{G}, \curlyvee)$ is a CVSMS.

Definition 5. *Let $(\mathcal{G}, \curlyvee)$ be a CVSMS and A be a subset of $\mathcal{G}$. A point $\mathfrak{b} \in \mathcal{G}$ is said to be an interior point of a set A if there exists an element $r \in \mathbb{C}$ with $0 \prec r$ such that*

$$B(\mathfrak{b}, r) = \{\mathfrak{l} \in \mathcal{G} : \curlyvee(\mathfrak{b}, \mathfrak{l}) \prec r\} \subseteq A.$$

A point $\mathfrak{b} \in \mathcal{G}$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(\mathfrak{b}, r) \cap (A \setminus \{\mathfrak{b}\}) \neq \emptyset.$$

A is called open whenever each element of A is an interior point of A. Moreover, a subset $B \subseteq \mathcal{G}$ is called closed whenever each limit point of B belongs to B. The family

$$\Phi = \{B(\mathfrak{b}, r) : \mathfrak{b} \in \mathcal{G}, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on $\mathcal{G}$.

Let $\mathfrak{b}_n$ be a sequence in $\mathcal{G}$ and $\mathfrak{b} \in \mathcal{G}$. The sequence $\{\mathfrak{b}_n\}$ is said to converge to $\mathfrak{b}$, if for every $c \in \mathbb{C}$ with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that $\curlyvee(\mathfrak{b}_n, \mathfrak{b}) \prec c$, for all $n > n_0$. In this case, $\mathfrak{b}$ is called the limit of $\{\mathfrak{b}_n\}$, and we write $\lim_{n\to\infty} \mathfrak{b}_n = \mathfrak{b}$, or equivalently $\mathfrak{b}_n \longrightarrow \mathfrak{b}$, as $n \to \infty$.

Similarly, the sequence $\{\mathfrak{b}_n\}$ is called a Cauchy sequence in $(\mathcal{G}, \curlyvee)$ if, for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $n_0 \in \mathbb{N}$ such that $\curlyvee(\mathfrak{b}_n, \mathfrak{b}_{n+m}) \prec c$, for all $n > n_0$ and for all $m \in \mathbb{N}$.

Finally, the space $(\mathcal{G}, \curlyvee)$ is said to be complete if every Cauchy sequence in $(\mathcal{G}, \curlyvee)$ is convergent.

Next, we present the following lemmas, which will be instrumental in establishing our main results.

Lemma 1. *[16] Let $(\mathcal{G}, \curlyvee)$ be a CVSMS and let $\{\mathfrak{b}_\mathfrak{p}\} \subseteq \mathcal{G}$. Then $\{\mathfrak{b}_\mathfrak{p}\}$ converges to $\mathfrak{b}$ if and only if $\left|\curlyvee(\mathfrak{b}_\mathfrak{p}, \mathfrak{b})\right| \to 0$ as $\mathfrak{p} \to \infty$.*

Lemma 2. *[16] Let $(\mathcal{G}, \curlyvee)$ be a CVSMS and let $\{\mathfrak{b}_\mathfrak{p}\} \subseteq \mathcal{G}$. Then $\{\mathfrak{b}_\mathfrak{p}\}$ is a Cauchy sequence iff $\left|\curlyvee(\mathfrak{b}_\mathfrak{p}, \mathfrak{b}_{\mathfrak{p}+m})\right| \to 0$ as $\mathfrak{p} \to \infty$, where $m \in \mathbb{N}$.*

3. Main results

We now state the main theorem, which serves as the foundation of this section.

Theorem 1. *Let $(\mathcal{G}, \curlyvee)$ be a complete CVSMS and $\mathfrak{I}, \mathfrak{K} : \mathcal{G} \to \mathcal{G}$. Assuming the existence of functions $\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3 : \mathcal{G} \to [0, 1)$ such that*

(a) $\mathbb{k}_1(\mathfrak{I}\mathfrak{b}) \leq \mathbb{k}_1(\mathfrak{b})$ *and* $\mathbb{k}_1(\mathfrak{K}\mathfrak{b}) \leq \mathbb{k}_1(\mathfrak{b})$,
$\mathbb{k}_2(\mathfrak{I}\mathfrak{b}) \leq \mathbb{k}_2(\mathfrak{b})$ *and* $\mathbb{k}_2(\mathfrak{K}\mathfrak{b}) \leq \mathbb{k}_2(\mathfrak{b})$,
$\mathbb{k}_3(\mathfrak{I}\mathfrak{b}) \leq \mathbb{k}_3(\mathfrak{b})$ *and* $\mathbb{k}_3(\mathfrak{K}\mathfrak{b}) \leq \mathbb{k}_3(\mathfrak{b})$,

(b) $\mathbb{k}_1(\mathfrak{b}) + 2\mathbb{k}_2(\mathfrak{b}) + \mathbb{k}_3(\mathfrak{b}) < 1$,

(c)

$$\curlyvee(\mathfrak{I}\mathfrak{b}, \mathfrak{K}\mathfrak{l}) \leq \mathbb{k}_1(\mathfrak{b}) \curlyvee(\mathfrak{b}, \mathfrak{l}) + \mathbb{k}_2(\mathfrak{b}) (\curlyvee(\mathfrak{b}, \mathfrak{I}\mathfrak{b}) + \curlyvee(\mathfrak{l}, \mathfrak{K}\mathfrak{l})) + \mathbb{k}_3(\mathfrak{b}) \frac{\curlyvee(\mathfrak{b}, \mathfrak{I}\mathfrak{b}) \curlyvee(\mathfrak{l}, \mathfrak{K}\mathfrak{l})}{1 + \curlyvee(\mathfrak{b}, \mathfrak{l})}, \tag{3.1}$$

for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$ with $\curlyvee(\mathfrak{b}, \mathfrak{l}) \neq -1$, then $\mathfrak{I}$ and $\mathfrak{K}$ admit a unique CFP.

Proof. Let $\mathfrak{b}_0 \in \mathcal{G}$ be given. Define the sequence $\{\mathfrak{b}_\mathfrak{p}\}$ inductively by

$$\mathfrak{b}_{2\mathfrak{p}+1} = \mathfrak{I}\mathfrak{b}_{2\mathfrak{p}} \text{ and } \mathfrak{b}_{2\mathfrak{p}+2} = \mathfrak{K}\mathfrak{b}_{2\mathfrak{p}+1}, \tag{3.2}$$

for all $\mathfrak{p} = 0, 1, 2, ..$ Now by (3.1), we have

$$\begin{aligned}
\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2}) &= \curlyvee(\mathfrak{I}\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{K}\mathfrak{b}_{2\mathfrak{p}+1}) \preceq \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}})\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1}) \\
&\quad + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})(\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{I}\mathfrak{b}_{2\mathfrak{p}}) + \curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{K}\mathfrak{b}_{2\mathfrak{p}+1})) \\
&\quad + \mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}})\frac{\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{I}\mathfrak{b}_{2\mathfrak{p}})\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{K}\mathfrak{b}_{2\mathfrak{p}+1})}{1 + \curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})} \\
&= \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}})\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1}) \\
&\quad + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})(\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1}) + \curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})) \\
&\quad + \mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}})\frac{\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})}{1 + \curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})} \\
&= \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}})\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1}) \\
&\quad + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1}) + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2}) \\
&\quad + \mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}})\frac{\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})}{1 + \curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})},
\end{aligned}$$

which implies that

$$\begin{aligned}
\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| &\leq \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| \\
&\quad + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \\
&\quad + \mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}})\frac{\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right|}{\left|1 + \curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right|}\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \\
&\leq \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| \\
&\quad + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \\
&\quad + \mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \\
&= \mathbb{k}_1(\mathfrak{K}\mathfrak{b}_{2\mathfrak{p}-1})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| \\
&\quad + \mathbb{k}_2(\mathfrak{K}\mathfrak{b}_{2\mathfrak{p}-1})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \mathbb{k}_2(\mathfrak{K}\mathfrak{b}_{2\mathfrak{p}-1})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \\
&\quad + \mathbb{k}_3(\mathfrak{K}\mathfrak{b}_{2\mathfrak{p}-1})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \\
&\leq \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}-1})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| \\
&\quad + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}-1})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}-1})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \\
&\quad + \mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}-1})\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \\
&\leq ... \leq \mathbb{k}_1(\mathfrak{b}_0)\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \mathbb{k}_2(\mathfrak{b}_0)\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| \\
&\quad + \mathbb{k}_2(\mathfrak{b}_0)\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| + \mathbb{k}_3(\mathfrak{b}_0)\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right|.
\end{aligned}$$

It follows that

$$\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \leq \frac{\mathbb{k}_1(\mathfrak{b}_0) + \mathbb{k}_2(\mathfrak{b}_0)}{1 - \mathbb{k}_2(\mathfrak{b}_0) - \mathbb{k}_3(\mathfrak{b}_0)}\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right|.$$

Let $\mu = \frac{\Bbbk_1(\mathfrak{b}_0)+\Bbbk_2(\mathfrak{b}_0)}{1-\Bbbk_2(\mathfrak{b}_0)-\Bbbk_3(\mathfrak{b}_0)} < 1$. Then

$$\left|\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\right| \leq \frac{\Bbbk_1(\mathfrak{b}_0)+\Bbbk_2(\mathfrak{b}_0)}{1-\Bbbk_2(\mathfrak{b}_0)-\Bbbk_3(\mathfrak{b}_0)}\left|\curlyvee(\mathfrak{b}_{2p},\mathfrak{b}_{2p+1})\right|. \tag{3.3}$$

Similarly, we have

$$\begin{aligned}
\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3}) &= \curlyvee(\mathfrak{K}\mathfrak{b}_{2p+1},\mathfrak{I}\mathfrak{b}_{2p+2}) = \curlyvee(\mathfrak{I}\mathfrak{b}_{2p+2},\mathfrak{K}\mathfrak{b}_{2p+1})\\
&\leq \Bbbk_1(\mathfrak{b}_{2p+2})\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})\\
&\quad+\Bbbk_2(\mathfrak{b}_{2p+2})\left(\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{I}\mathfrak{b}_{2p+2})+\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{K}\mathfrak{b}_{2p+1})\right)\\
&\quad+\Bbbk_3(\mathfrak{b}_{2p+2})\frac{\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{I}\mathfrak{b}_{2p+2})\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{K}\mathfrak{b}_{2p+1})}{1+\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})}\\
&= \Bbbk_1(\mathfrak{b}_{2p+2})\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})\\
&\quad+\Bbbk_2(\mathfrak{b}_{2p+2})\left(\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})+\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\right)\\
&\quad+\Bbbk_3(\mathfrak{b}_{2p+2})\frac{\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})}{1+\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})}\\
&= \Bbbk_1(\mathfrak{b}_{2p+2})\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})\\
&\quad+\Bbbk_2(\mathfrak{b}_{2p+2})\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})+\Bbbk_2(\mathfrak{b}_{2p+2})\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\\
&\quad+\Bbbk_3(\mathfrak{b}_{2p+2})\frac{\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})}{1+\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})},
\end{aligned}$$

which implies that

$$\begin{aligned}
\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right| &\leq \Bbbk_1(\mathfrak{b}_{2p+2})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})\right|\\
&\quad+\Bbbk_2(\mathfrak{b}_{2p+2})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|+\Bbbk_2(\mathfrak{b}_{2p+2})\left|\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\right|\\
&\quad+\Bbbk_3(\mathfrak{b}_{2p+2})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|\frac{\left|\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\right|}{\left|1+\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})\right|}\\
&\leq \Bbbk_1(\mathfrak{b}_{2p+2})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})\right|\\
&\quad+\Bbbk_2(\mathfrak{b}_{2p+2})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|+\Bbbk_2(\mathfrak{b}_{2p+2})\left|\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\right|\\
&\quad+\Bbbk_3(\mathfrak{b}_{2p+2})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|\\
&= \Bbbk_1(\mathfrak{K}\mathfrak{b}_{2p+1})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})\right|\\
&\quad+\Bbbk_2(\mathfrak{K}\mathfrak{b}_{2p+1})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|+\Bbbk_2(\mathfrak{K}\mathfrak{b}_{2p+1})\left|\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\right|\\
&\quad+\Bbbk_3(\mathfrak{K}\mathfrak{b}_{2p+1})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|\\
&\leq \Bbbk_1(\mathfrak{b}_{2p+1})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})\right|+\Bbbk_2(\mathfrak{b}_{2p+1})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|\\
&\quad+\Bbbk_2(\mathfrak{b}_{2p+1})\left|\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\right|+\Bbbk_3(\mathfrak{b}_{2p+1})\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|\\
&\leq \ldots \leq \Bbbk_1(\mathfrak{b}_0)\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+1})\right|+\Bbbk_2(\mathfrak{b}_0)\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|\\
&\quad+\Bbbk_2(\mathfrak{b}_0)\left|\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\right|+\Bbbk_3(\mathfrak{b}_0)\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right|.
\end{aligned}$$

It follows that

$$\left|\curlyvee(\mathfrak{b}_{2p+2},\mathfrak{b}_{2p+3})\right| \leq \frac{\Bbbk_1(\mathfrak{b}_0)+\Bbbk_2(\mathfrak{b}_0)}{1-\Bbbk_2(\mathfrak{b}_0)-\Bbbk_3(\mathfrak{b}_0)}\left|\curlyvee(\mathfrak{b}_{2p+1},\mathfrak{b}_{2p+2})\right|.$$

Let $\mu = \frac{\Bbbk_1(\mathfrak{b}_0)+\Bbbk_2(\mathfrak{b}_0)}{1-\Bbbk_2(\mathfrak{b}_0)-\Bbbk_3(\mathfrak{b}_0)} < 1$. Then

$$\left|\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}+2}, \mathfrak{b}_{2\mathfrak{p}+3}\right)\right| \leq \mu \left|\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2}\right)\right|. \tag{3.4}$$

Now by (3.3) and (3.4), we have

$$\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_{\mathfrak{p}+1}\right)\right| \leq \mu \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}-1}, \mathfrak{b}_{\mathfrak{p}}\right)\right|,$$

for all $\mathfrak{p} \in \mathbb{N}$. Recursively, we can generate a sequence $\{\mathfrak{b}_{\mathfrak{p}}\}$ in $\mathcal{G}$ such that

$$\begin{aligned}
\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_{\mathfrak{p}+1}\right)\right| &\leq \mu \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}-1}, \mathfrak{b}_{\mathfrak{p}}\right)\right| \\
&\leq \mu^2 \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}-2}, \mathfrak{b}_{\mathfrak{p}-1}\right)\right| \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&\leq \mu^{\mathfrak{p}} \left|\curlyvee\left(\mathfrak{b}_0, \mathfrak{b}_1\right)\right|,
\end{aligned}$$

that is,

$$\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_{\mathfrak{p}+1}\right)\right| \leq \mu^{\mathfrak{p}} \left|\curlyvee\left(\mathfrak{b}_0, \mathfrak{b}_1\right)\right|, \tag{3.5}$$

for all $\mathfrak{p} \in \mathbb{N}$. For $m > \mathfrak{p}$, we have

$$\begin{aligned}
\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_m\right)\right| &\leq \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_{\mathfrak{p}+1}\right)\right| + \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+1}, \mathfrak{b}_m\right)\right| + \aleph \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_{\mathfrak{p}+1}\right)\right| \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+1}, \mathfrak{b}_m\right)\right| \\
&= \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_{\mathfrak{p}+1}\right)\right| + \left(1 + \aleph \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_{\mathfrak{p}+1}\right)\right|\right) \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+1}, \mathfrak{b}_m\right)\right|.
\end{aligned} \tag{3.6}$$

Now, we can apply the triangle inequality again to the second term $\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+1}, \mathfrak{b}_m\right)\right|$, we have

$$\begin{aligned}
\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+1}, \mathfrak{b}_m\right)\right| &\leq \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+1}, \mathfrak{b}_{\mathfrak{p}+2}\right)\right| + \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+2}, \mathfrak{b}_m\right)\right| + \aleph \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+1}, \mathfrak{b}_{\mathfrak{p}+2}\right)\right| \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+2}, \mathfrak{b}_m\right)\right| \\
&= \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+1}, \mathfrak{b}_{\mathfrak{p}+2}\right)\right| + \left(1 + \aleph \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+1}, \mathfrak{b}_{\mathfrak{p}+2}\right)\right|\right) \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+2}, \mathfrak{b}_m\right)\right|.
\end{aligned}$$

Leveraging the triangle inequality once more on the term $\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+2}, \mathfrak{b}_m\right)\right|$, we obtain

$$\begin{aligned}
\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+2}, \mathfrak{b}_m\right)\right| &\leq \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+2}, \mathfrak{b}_{\mathfrak{p}+3}\right)\right| + \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+3}, \mathfrak{b}_m\right)\right| + \aleph \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+2}, \mathfrak{b}_{\mathfrak{p}+3}\right)\right| \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+3}, \mathfrak{b}_m\right)\right| \\
&= \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+2}, \mathfrak{b}_{\mathfrak{p}+3}\right)\right| + \left(1 + \aleph \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+2}, \mathfrak{b}_{\mathfrak{p}+3}\right)\right|\right) \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}+3}, \mathfrak{b}_m\right)\right|,
\end{aligned}$$

and so on

$$\begin{aligned}
|\curlyvee(\mathfrak{b}_{m-2}, \mathfrak{b}_m)| &\leq |\curlyvee(\mathfrak{b}_{m-2}, \mathfrak{b}_{m-1})| + |\curlyvee(\mathfrak{b}_{m-1}, \mathfrak{b}_m)| + \aleph |\curlyvee(\mathfrak{b}_{m-2}, \mathfrak{b}_{m-1})| |\curlyvee(\mathfrak{b}_{m-1}, \mathfrak{b}_m)| \\
&= |\curlyvee(\mathfrak{b}_{m-2}, \mathfrak{b}_{m-1})| + (1 + \aleph |\curlyvee(\mathfrak{b}_{m-2}, \mathfrak{b}_{m-1})|) |\curlyvee(\mathfrak{b}_{m-1}, \mathfrak{b}_m)|.
\end{aligned}$$

Recursively substituting each inequality into the preceding one (3.6) and simplifying, we have

$$\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_m\right)\right| \leq \sum_{k=\mathfrak{p}}^{m-1} \left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}}, \mathfrak{b}_{\mathfrak{p}+1}\right)\right| \prod_{j=\mathfrak{p}}^{k-1} \left(1 + \aleph \left|\curlyvee\left(\mathfrak{b}_j, \mathfrak{b}_{j+1}\right)\right|\right).$$

By the inequality (3.5), we have

$$\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}},\mathfrak{b}_{m}\right)\right|\leq\left|\curlyvee\left(\mathfrak{b}_{0},\mathfrak{b}_{1}\right)\right|\sum_{k=\mathfrak{p}}^{m-1}\mu^{\mathfrak{p}}\prod_{j=\mathfrak{p}}^{k-1}\left(1+\aleph\mu^{j}\left|\curlyvee\left(\mathfrak{b}_{0},\mathfrak{b}_{1}\right)\right|\right). \tag{3.7}$$

Now observe that $\left(1+\aleph\mu^{j}\left|\curlyvee\left(\mathfrak{b}_{0},\mathfrak{b}_{1}\right)\right|\right)\geq 1$, for all j. Therefore,

$$\prod_{j=\mathfrak{p}}^{k-1}\left(1+\aleph\mu^{j}\left|\curlyvee\left(\mathfrak{b}_{0},\mathfrak{b}_{1}\right)\right|\right)\geq 1.$$

Now with the product term bounded below by 1, we can simplify the summation

$$\left|\curlyvee\left(\mathfrak{b}_{0},\mathfrak{b}_{1}\right)\right|\sum_{k=\mathfrak{p}}^{m-1}\mu^{\mathfrak{p}}\prod_{j=\mathfrak{p}}^{k-1}\left(1+\aleph\mu^{j}\left|\curlyvee\left(\mathfrak{b}_{0},\mathfrak{b}_{1}\right)\right|\right)\geq\left|\curlyvee\left(\mathfrak{b}_{0},\mathfrak{b}_{1}\right)\right|\sum_{k=\mathfrak{p}}^{m-1}\mu^{\mathfrak{p}}. \tag{3.8}$$

Since $\sum\limits_{k=\mathfrak{p}}^{m-1}\mu^{\mathfrak{p}}$ is a finite geometric series with the first term $\mu^{\mathfrak{p}}$ and the common ratio μ. The sum of a finite geometric series is given by

$$\sum_{k=\mathfrak{p}}^{m-1}\mu^{\mathfrak{p}}=\mu^{\mathfrak{p}}\frac{(1-\mu^{m-\mathfrak{p}})}{1-\mu}.$$

As $m\to\infty$, $\mu^{m-\mathfrak{p}}\to 0$. Therefore, the sum converges to $\frac{\mu^{\mathfrak{p}}}{1-\mu}$, that is,

$$\lim_{m\to\infty}\mu^{\mathfrak{p}}\frac{(1-\mu^{m-\mathfrak{p}})}{1-\mu}=\frac{\mu^{\mathfrak{p}}}{1-\mu}.$$

Since $\mu<1$, the expression $\frac{\mu^{\mathfrak{p}}}{1-\mu}$ approaches 0 as $\mathfrak{p}\to\infty$. Letting $\mathfrak{p}\to\infty$ in inequality (3.7) and employing the established facts, we arrive at

$$\lim_{\mathfrak{p}\to\infty}\left|\curlyvee\left(\mathfrak{b}_{\mathfrak{p}},\mathfrak{b}_{m}\right)\right|=0.$$

Therefore, by lemma (2), $\{\mathfrak{b}_{\mathfrak{p}}\}$ is Cauchy. The completeness of $\mathcal{G}$ implies the existence of $\mathfrak{b}^{*}$ such that $\mathfrak{b}_{\mathfrak{p}}\to\mathfrak{b}^{*}\in\mathcal{G}$ as $\mathfrak{p}\to\infty$. Thus,

$$\lim_{\mathfrak{p}\to\infty}\mathfrak{b}_{\mathfrak{p}}=\mathfrak{b}^{*}.$$

Now, we show that $\mathfrak{b}^{*}$ is a FP of $\mathfrak{I}$. From (3.1), we have

$$\begin{aligned}
\curlyvee\left(\mathfrak{b}^{*},\mathfrak{I}\mathfrak{b}^{*}\right) &\leq \curlyvee\left(\mathfrak{b}^{*},\mathfrak{b}_{2\mathfrak{p}+2}\right)+\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}+2},\mathfrak{I}\mathfrak{b}^{*}\right)+\aleph\curlyvee\left(\mathfrak{b}^{*},\mathfrak{b}_{2\mathfrak{p}+2}\right)\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}+2},\mathfrak{I}\mathfrak{b}^{*}\right)\\
&= \curlyvee\left(\mathfrak{b}^{*},\mathfrak{b}_{2\mathfrak{p}+2}\right)+\curlyvee\left(\mathfrak{R}\mathfrak{b}_{2\mathfrak{p}+1},\mathfrak{I}\mathfrak{b}^{*}\right)+\aleph\curlyvee\left(\mathfrak{b}^{*},\mathfrak{R}\mathfrak{b}_{2\mathfrak{p}+1}\right)\curlyvee\left(\mathfrak{R}\mathfrak{b}_{2\mathfrak{p}+1},\mathfrak{I}\mathfrak{b}^{*}\right)\\
&= \curlyvee\left(\mathfrak{b}^{*},\mathfrak{R}\mathfrak{b}_{2\mathfrak{p}+1}\right)+\curlyvee\left(\mathfrak{I}\mathfrak{b}^{*},\mathfrak{R}\mathfrak{b}_{2\mathfrak{p}+1}\right)\\
&\leq \curlyvee\left(\mathfrak{b}^{*},\mathfrak{b}_{2\mathfrak{p}+2}\right)+\left(\begin{array}{c}\mathbb{k}_{1}\left(\mathfrak{b}^{*}\right)\curlyvee\left(\mathfrak{b}^{*},\mathfrak{b}_{2\mathfrak{p}+1}\right)\\ \mathbb{k}_{2}\left(\mathfrak{b}^{*}\right)\left(\curlyvee\left(\mathfrak{b}^{*},\mathfrak{I}\mathfrak{b}^{*}\right)+\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}+1},\mathfrak{R}\mathfrak{b}_{2\mathfrak{p}+1}\right)\right)\\ +\mathbb{k}_{3}\left(\mathfrak{b}^{*}\right)\frac{\curlyvee\left(\mathfrak{b}^{*},\mathfrak{I}\mathfrak{b}^{*}\right)\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}+1},\mathfrak{R}\mathfrak{b}_{2\mathfrak{p}+1}\right)}{1+\curlyvee\left(\mathfrak{b}^{*},\mathfrak{b}_{2\mathfrak{p}+1}\right)}\end{array}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\aleph \curlyvee (\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}_{2\mathfrak{p}+1}) \curlyvee (\mathfrak{K}\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{I}\mathfrak{b}^*) \\
= &\curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+2}) + \begin{pmatrix} \mathbb{k}_1(\mathfrak{b}^*) \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1}) \\ \mathbb{k}_2(\mathfrak{b}^*)(\curlyvee (\mathfrak{b}^*, \mathfrak{I}\mathfrak{b}^*) + \curlyvee (\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})) \\ +\mathbb{k}_3(\mathfrak{b}^*) \frac{\curlyvee(\mathfrak{b}^*, \mathfrak{I}\mathfrak{b}^*)\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})}{1+\curlyvee(\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1})} \end{pmatrix} \\
&+\aleph \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+2}) \curlyvee (\mathfrak{b}_{2\mathfrak{p}+2}, \mathfrak{I}\mathfrak{b}^*).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|\curlyvee (\mathfrak{b}^*, \mathfrak{I}\mathfrak{b}^*)| \leq &\left|\curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+2})\right| + \begin{pmatrix} \mathbb{k}_1(\mathfrak{b}^*) \left|\curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1})\right| \\ \mathbb{k}_2(\mathfrak{b}^*) |\curlyvee (\mathfrak{b}^*, \mathfrak{I}\mathfrak{b}^*)| + \mathbb{k}_2(\mathfrak{b}^*) \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right| \\ +\mathbb{k}_3(\mathfrak{b}^*) \frac{\left|\curlyvee(\mathfrak{b}^*, \mathfrak{I}\mathfrak{b}^*)\right|\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{b}_{2\mathfrak{p}+2})\right|}{\left|1+\curlyvee(\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1})\right|} \end{pmatrix} \\
&+\aleph \left|\curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+2})\right| \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}+2}, \mathfrak{I}\mathfrak{b}^*)\right|.
\end{aligned}
$$

Letting $\mathfrak{p} \to \infty$, we have $(1 - \mathbb{k}_2(\mathfrak{b}^*)) |\curlyvee (\mathfrak{b}^*, \mathfrak{I}\mathfrak{b}^*)| = 0$. But $(1 - \mathbb{k}_2(\mathfrak{b}^*)) \neq 0$, thus $|\curlyvee (\mathfrak{b}^*, \mathfrak{I}\mathfrak{b}^*)| = 0$ implies $\mathfrak{b}^* = \mathfrak{I}\mathfrak{b}^*$. To demonstrate that $\mathfrak{b}^*$ is a FP of $\mathfrak{K}$, we observe from (3.1) that

$$
\begin{aligned}
\curlyvee (\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*) &\leq \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1}) + \curlyvee (\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{K}\mathfrak{b}^*) + \aleph \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1}) \curlyvee (\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{K}\mathfrak{b}^*) \\
&\leq \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1}) + \curlyvee (\mathfrak{I}\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{K}\mathfrak{b}^*) + \aleph \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1}) \curlyvee (\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{K}\mathfrak{b}^*) \\
&\leq \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1}) + \begin{pmatrix} \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}}) \curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*) \\ \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})(\curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{I}\mathfrak{b}_{2\mathfrak{p}}) + \curlyvee (\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)) \\ +\mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}}) \frac{\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{I}\mathfrak{b}_{2\mathfrak{p}})\curlyvee(\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)}{1+\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*)} \end{pmatrix} \\
&\quad +\aleph \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1}) \curlyvee (\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{K}\mathfrak{b}^*) \\
&\leq \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1}) + \begin{pmatrix} \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*) \curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*) \\ \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}})(\curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1}) + \curlyvee (\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)) \\ +\mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*) \frac{\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\curlyvee(\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)}{1+\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*)} \end{pmatrix} \\
&\quad +\aleph \curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1}) \curlyvee (\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{K}\mathfrak{b}^*).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|\curlyvee (\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)| &\leq \left|\curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \begin{pmatrix} \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}}) \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*)\right| \\ \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}}) \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}}) |\curlyvee (\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)| \\ +\mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}}) \frac{\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right||\curlyvee(\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)|}{\left|1+\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*)\right|} \end{pmatrix} \\
&\quad +\aleph \left|\curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1})\right| \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{K}\mathfrak{b}^*)\right| \\
&= \left|\curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \begin{pmatrix} \mathbb{k}_1(\mathfrak{K}\mathfrak{b}_{2\mathfrak{p}-1}) \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*)\right| \\ \mathbb{k}_2(\mathfrak{K}\mathfrak{b}_{2\mathfrak{p}-1}) \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \mathbb{k}_2(\mathfrak{K}\mathfrak{b}_{2\mathfrak{p}-1}) |\curlyvee (\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)| \\ +\mathbb{k}_3(\mathfrak{K}\mathfrak{b}_{2\mathfrak{p}-1}) \frac{\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right||\curlyvee(\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)|}{\left|1+\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*)\right|} \end{pmatrix} \\
&\quad +\aleph \left|\curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1})\right| \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}+1}, \mathfrak{K}\mathfrak{b}^*)\right| \\
&\leq \left|\curlyvee (\mathfrak{b}^*, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \begin{pmatrix} \mathbb{k}_1(\mathfrak{b}_{2\mathfrak{p}-1}) \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*)\right| \\ \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}-1}) \left|\curlyvee (\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right| + \mathbb{k}_2(\mathfrak{b}_{2\mathfrak{p}-1}) |\curlyvee (\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)| \\ +\mathbb{k}_3(\mathfrak{b}_{2\mathfrak{p}-1}) \frac{\left|\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}_{2\mathfrak{p}+1})\right||\curlyvee(\mathfrak{b}^*, \mathfrak{K}\mathfrak{b}^*)|}{\left|1+\curlyvee(\mathfrak{b}_{2\mathfrak{p}}, \mathfrak{b}^*)\right|} \end{pmatrix}
\end{aligned}
$$

$$+\aleph\left|\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}_{2\mathfrak{p}+1}\right)\right|\left|\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}+1},\mathfrak{R}\mathfrak{b}^*\right)\right|$$

$$\leq\ \ldots\leq\left|\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}_{2\mathfrak{p}+1}\right)\right|+\left(\begin{array}{c}\mathbb{k}_1\left(\mathfrak{b}_0\right)\left|\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}},\mathfrak{b}^*\right)\right|\\ \mathbb{k}_2\left(\mathfrak{b}_0\right)\left|\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}},\mathfrak{b}_{2\mathfrak{p}+1}\right)\right|+\mathbb{k}_2\left(\mathfrak{b}_0\right)\left|\curlyvee\left(\mathfrak{b}^*,\mathfrak{R}\mathfrak{b}^*\right)\right|\\ +\mathbb{k}_3\left(\mathfrak{b}_0\right)\frac{\left|\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}},\mathfrak{b}_{2\mathfrak{p}+1}\right)\right|\left|\curlyvee\left(\mathfrak{b}^*,\mathfrak{R}\mathfrak{b}^*\right)\right|}{\left|1+\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}},\mathfrak{b}^*\right)\right|}\end{array}\right)$$

$$+\aleph\left|\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}_{2\mathfrak{p}+1}\right)\right|\left|\curlyvee\left(\mathfrak{b}_{2\mathfrak{p}+1},\mathfrak{R}\mathfrak{b}^*\right)\right|.$$

Letting $\mathfrak{p}\to\infty$, we have $(1-\mathbb{k}_2(\mathfrak{b}_0))\left|\curlyvee(\mathfrak{b}^*,\mathfrak{R}\mathfrak{b}^*)\right|=0$, but $(1-\mathbb{k}_2(\mathfrak{b}_0))\neq 0$; hence, $\left|\curlyvee(\mathfrak{b}^*,\mathfrak{R}\mathfrak{b}^*)\right|=0$ implies $\mathfrak{b}^*=\mathfrak{R}\mathfrak{b}^*$. Thus $\mathfrak{b}^*$ is a CFP of $\mathfrak{I}$ and $\mathfrak{R}$. To establish the uniqueness of $\mathfrak{b}^*$, suppose, to the contrary, that there exists another CFP $\mathfrak{b}^/$ of $\mathfrak{I}$ and $\mathfrak{R}$. This implies that

$$\mathfrak{b}^/=\mathfrak{I}\mathfrak{b}^/=\mathfrak{R}\mathfrak{b}^/,$$

but $\mathfrak{b}^*\neq\mathfrak{b}^/$. Now from (3.1), we have

$$\begin{aligned}\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}^/\right)&=\curlyvee\left(\mathfrak{I}\mathfrak{b}^*,\mathfrak{R}\mathfrak{b}^/\right)\\&\leq\mathbb{k}_1(\mathfrak{b}^*)\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}^/\right)+\mathbb{k}_2(\mathfrak{b}^*)\left(\curlyvee(\mathfrak{b}^*,\mathfrak{I}\mathfrak{b}^*)+\curlyvee\left(\mathfrak{b}^/,\mathfrak{R}\mathfrak{b}^/\right)\right)\\&\quad+\mathbb{k}_3(\mathfrak{b}^*)\frac{\curlyvee(\mathfrak{b}^*,\mathfrak{I}\mathfrak{b}^*)\curlyvee\left(\mathfrak{b}^/,\mathfrak{R}\mathfrak{b}^/\right)}{1+\curlyvee(\mathfrak{b}^*,\mathfrak{b}^/)}\\&=\mathbb{k}_1(\mathfrak{b}^*)\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}^/\right).\end{aligned}$$

This implies that, we have

$$\left|\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}^/\right)\right|\leq\mathbb{k}_1(\mathfrak{b}^*)\left|\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}^/\right)\right|,$$

which implies $(1-\mathbb{k}_1(\mathfrak{b}^*))\left|\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}^/\right)\right|=0$. Since $(1-\mathbb{k}_1(\mathfrak{b}^*))\neq 0$, thus, we have

$$\left|\curlyvee\left(\mathfrak{b}^*,\mathfrak{b}^/\right)\right|=0.$$

Hence $\mathfrak{b}^*=\mathfrak{b}^/$. □

By specializing Theorem 1 to the case where $\mathbb{k}_3=0$, we arrive at the upcoming corollary.

Corollary 1. *Let $(\mathcal{G},\curlyvee)$ be a complete CVSMS and $\mathfrak{I},\mathfrak{R}:\mathcal{G}\to\mathcal{G}$. Assuming the existence of functions $\mathbb{k}_1,\mathbb{k}_2:\mathcal{G}\to[0,1)$ such that*

(a) $\mathbb{k}_1(\mathfrak{I}\mathfrak{b})\leq\mathbb{k}_1(\mathfrak{b})$ *and* $\mathbb{k}_1(\mathfrak{R}\mathfrak{b})\leq\mathbb{k}_1(\mathfrak{b})$,
$\mathbb{k}_2(\mathfrak{I}\mathfrak{b})\leq\mathbb{k}_2(\mathfrak{b})$ *and* $\mathbb{k}_2(\mathfrak{R}\mathfrak{b})\leq\mathbb{k}_2(\mathfrak{b})$,

(b) $\mathbb{k}_1(\mathfrak{b})+2\mathbb{k}_2(\mathfrak{b})<1$,

(c) $\curlyvee(\mathfrak{I}\mathfrak{b},\mathfrak{R}\mathfrak{l})\leq\mathbb{k}_1(\mathfrak{b})\curlyvee(\mathfrak{b},\mathfrak{l})+\mathbb{k}_2(\mathfrak{b})(\curlyvee(\mathfrak{b},\mathfrak{I}\mathfrak{b})+\curlyvee(\mathfrak{l},\mathfrak{R}\mathfrak{l}))$,

for all $\mathfrak{b},\mathfrak{l}\in\mathcal{G}$, then $\mathfrak{I}$ and $\mathfrak{R}$ admit a unique CFP.

Setting $\mathbb{k}_2=0$ in the Theorem 1 yields the following outcome.

Corollary 2. *Let $(\mathcal{G},\curlyvee)$ be a complete CVSMS and $\mathfrak{I},\mathfrak{R}:\mathcal{G}\to\mathcal{G}$. Assuming the existence of functions $\mathbb{k}_1,\mathbb{k}_3:\mathcal{G}\to[0,1)$ such that*

(a) $\Bbbk_1(\Im\mathfrak{b}) \le \Bbbk_1(\mathfrak{b})$ *and* $\Bbbk_1(\Re\mathfrak{b}) \le \Bbbk_1(\mathfrak{b})$,
$\Bbbk_3(\Im\mathfrak{b}) \le \Bbbk_3(\mathfrak{b})$ *and* $\Bbbk_3(\Re\mathfrak{b}) \le \Bbbk_3(\mathfrak{b})$,
(b) $\Bbbk_1(\mathfrak{b}) + \Bbbk_3(\mathfrak{b}) < 1$,
(c) $\curlyvee(\Im\mathfrak{b}, \Re\mathfrak{l}) \le \Bbbk_1(\mathfrak{b})\curlyvee(\mathfrak{b},\mathfrak{l}) + \Bbbk_3(\mathfrak{b})\frac{\curlyvee(\mathfrak{b},\Im\mathfrak{b})\curlyvee(\mathfrak{l},\Re\mathfrak{l})}{1+\curlyvee(\mathfrak{b},\mathfrak{l})}$,
for all $\mathfrak{b},\mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b},\mathfrak{l}) \neq -1$, *then* $\Im$ *and* $\Re$ *admit a unique CFP.*

By restricting Theorem 1 to the case $\Bbbk_2 = \Bbbk_3 = 0$, we arrive at this result.

Corollary 3. *Let* $(\mathcal{G},\curlyvee)$ *be a complete CVSMS and* $\Im,\Re : \mathcal{G} \to \mathcal{G}$. *Assuming the existence of function* $\Bbbk_1 : \mathcal{G} \to [0,1)$ *such that*
(a) $\Bbbk_1(\Im\mathfrak{b}) \le \Bbbk_1(\mathfrak{b})$ *and* $\Bbbk_1(\Re\mathfrak{b}) \le \Bbbk_1(\mathfrak{b})$,
(b) $\Bbbk_1(\mathfrak{b}) < 1$,
(c) $\curlyvee(\Im\mathfrak{b}, \Re\mathfrak{l}) \le \Bbbk_1(\mathfrak{b})\curlyvee(\mathfrak{b},\mathfrak{l})$,
for all $\mathfrak{b},\mathfrak{l} \in \mathcal{G}$, *then* $\Im$ *and* $\Re$ *admit a unique CFP.*

Zooming in on the special case $\Im = \Re$ within Theorem 1, we discern the following result.

Corollary 4. *Let* $(\mathcal{G},\curlyvee)$ *be a complete CVSMS and* $\Im : \mathcal{G} \to \mathcal{G}$. *Assuming the existence of functions* $\Bbbk_1, \Bbbk_2, \Bbbk_3 : \mathcal{G} \to [0,1)$ *such that*
(a) $\Bbbk_1(\Im\mathfrak{b}) \le \Bbbk_1(\mathfrak{b})$,
$\Bbbk_2(\Im\mathfrak{b}) \le \Bbbk_2(\mathfrak{b})$,
$\Bbbk_3(\Im\mathfrak{b}) \le \Bbbk_3(\mathfrak{b})$,
(b) $\Bbbk_1(\mathfrak{b}) + 2\Bbbk_2(\mathfrak{b}) + \Bbbk_3(\mathfrak{b}) < 1$,
(c) $\curlyvee(\Im\mathfrak{b}, \Im\mathfrak{l}) \le \Bbbk_1(\mathfrak{b})\curlyvee(\mathfrak{b},\mathfrak{l}) + \Bbbk_2(\mathfrak{b})(\curlyvee(\mathfrak{b},\Im\mathfrak{b}) + \curlyvee(\mathfrak{l},\Im\mathfrak{l})) + \Bbbk_3(\mathfrak{b})\frac{\curlyvee(\mathfrak{b},\Im\mathfrak{b})\curlyvee(\mathfrak{l},\Im\mathfrak{l})}{1+\curlyvee(\mathfrak{b},\mathfrak{l})}$
for all $\mathfrak{b},\mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b},\mathfrak{l}) \neq -1$, *then* $\Im$ *admits a unique FP.*

Example 4. *Let* $\mathcal{G} = \mathbb{R}$ *and define a function* $\curlyvee : \mathcal{G}\times\mathcal{G} \to \mathbb{C}$ *by*

$$\curlyvee(\mathfrak{b},\mathfrak{l}) = \frac{1}{2}\left|e^{\mathfrak{b}} - e^{\mathfrak{l}}\right|$$

for all $\mathfrak{b},\mathfrak{l} \in \mathcal{G}$. *Then* $(\mathcal{G},\curlyvee)$ *forms a CVSMS. Consider the self-mapping* $\Im : \mathcal{G} \to \mathcal{G}$ *defined by*

$$\Im\mathfrak{b} = \ln\left(\frac{1}{2}e^{\mathfrak{b}} + 1\right).$$

Let the contractive functions $\Bbbk_1, \Bbbk_2, \Bbbk_3 : \mathcal{G} \to [0,1)$ *be given as*

$$\Bbbk_1(\mathfrak{b}) = 0.6,\ \Bbbk_2(\mathfrak{b}) = 0.1 \text{ and } \Bbbk_3(\mathfrak{b}) = 0.05.$$

Then $\Bbbk_1(\mathfrak{b}) + 2\Bbbk_2(\mathfrak{b}) + \Bbbk_3(\mathfrak{b}) = 0.85 < 1$. *Hence, all the hypotheses of Corollary 4 are satisfied and the mapping* $\Im$ *has a unique fixed point* $\mathfrak{b}^* = \ln 2$.

Corollary 5. *Let* $(\mathcal{G},\curlyvee)$ *be a complete CVSMS and* $\Im : \mathcal{G} \to \mathcal{G}$. *Assuming the existence of functions* $\Bbbk_1, \Bbbk_2, \Bbbk_3 : \mathcal{G} \to [0,1)$ *such that*
(a) $\Bbbk_1(\Im\mathfrak{b}) \le \Bbbk_1(\mathfrak{b})$,
$\Bbbk_2(\Im\mathfrak{b}) \le \Bbbk_2(\mathfrak{b})$,
$\Bbbk_3(\Im\mathfrak{b}) \le \Bbbk_3(\mathfrak{b})$,
(b) $\Bbbk_1(\mathfrak{b}) + 2\Bbbk_2(\mathfrak{b}) + \Bbbk_3(\mathfrak{b}) < 1$,

(c)

$$\curlyvee(\Im^n\mathfrak{b},\Im^n\mathfrak{l}) \leq \Bbbk_1(\mathfrak{b})\curlyvee(\mathfrak{b},\mathfrak{l}) + \Bbbk_2(\mathfrak{b})(\curlyvee(\mathfrak{b},\Im^n\mathfrak{b}) + \curlyvee(\mathfrak{l},\Im^n\mathfrak{l})) + \Bbbk_3(\mathfrak{b})\frac{\curlyvee(\mathfrak{b},\Im^n\mathfrak{b})\curlyvee(\mathfrak{l},\Im^n\mathfrak{l})}{1+\curlyvee(\mathfrak{b},\mathfrak{l})}, \tag{3.9}$$

for all $\mathfrak{b},\mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b},\mathfrak{l}) \neq -1$, *then there exists a unique point* $\mathfrak{b}^* \in \mathcal{G}$ *such that* $\Im\mathfrak{b}^* = \mathfrak{b}^*$.

Proof. By Corollary (4), there exists $\mathfrak{b} \in \mathcal{G}$ such that $\Im^n\mathfrak{b} = \mathfrak{b}$. Now, from

$$\begin{aligned}
\curlyvee(\Im\mathfrak{b},\mathfrak{b}) &= \curlyvee(\Im\Im^n\mathfrak{b},\Im^n\mathfrak{b}) \\
&= \curlyvee(\Im^n\Im\mathfrak{b},\Im^n\mathfrak{b}) \leq \Bbbk_1(\Im\mathfrak{b})\curlyvee(\Im\mathfrak{b},\mathfrak{b}) \\
&\quad +\Bbbk_2(\Im\mathfrak{b})(\curlyvee(\Im\mathfrak{b},\Im^n\Im\mathfrak{b}) + \curlyvee(\mathfrak{b},\Im^n\mathfrak{b})) \\
&\quad +\Bbbk_3(\Im\mathfrak{b})\frac{\curlyvee(\Im\mathfrak{b},\Im^n\Im\mathfrak{b})\curlyvee(\mathfrak{b},\Im^n\mathfrak{b})}{1+\curlyvee(\Im\mathfrak{b},\mathfrak{b})} \\
&\leq \Bbbk_1(\Im\mathfrak{b})\curlyvee(\Im\mathfrak{b},\mathfrak{b}) + \Bbbk_2(\Im\mathfrak{b})(\curlyvee(\Im\mathfrak{b},\Im\mathfrak{b}) + \curlyvee(\mathfrak{b},\mathfrak{b})) \\
&\quad +\Bbbk_3(\Im\mathfrak{b},\mathfrak{b})\frac{\curlyvee(\Im\mathfrak{b},\Im\mathfrak{b})\curlyvee(\mathfrak{b},\mathfrak{b})}{1+\curlyvee(\Im\mathfrak{b},\mathfrak{b})} \\
&= \Bbbk_1(\Im\mathfrak{b},\mathfrak{b})\curlyvee(\Im\mathfrak{b},\mathfrak{b}),
\end{aligned}$$

which implies that

$$|\curlyvee(\Im\mathfrak{b},\mathfrak{b})| \leq \Bbbk_1(\Im\mathfrak{b})|\curlyvee(\Im\mathfrak{b},\mathfrak{b})|,$$

which is possible only whenever $(1-\Bbbk_1(\Im\mathfrak{b}))|\curlyvee(\Im\mathfrak{b},\mathfrak{b})| = 0$, but $(1-\Bbbk_1(\Im\mathfrak{b})) \neq 0$, thus $|\curlyvee(\Im\mathfrak{b},\mathfrak{b})| = 0$ implies $\Im\mathfrak{b} = \mathfrak{b}$. □

Corollary 6. *Let* $(\mathcal{G},\curlyvee)$ *be a complete CVSMS and* $\Im,\Re : \mathcal{G} \to \mathcal{G}$. *Suppose there is* $\Bbbk_1,\Bbbk_2,\Bbbk_3 \in [0,1)$ *with* $\Bbbk_1 + 2\Bbbk_2 + \Bbbk_3 < 1$ *such that*

$$\curlyvee(\Im\mathfrak{b},\Re\mathfrak{l}) \leq \Bbbk_1\curlyvee(\mathfrak{b},\mathfrak{l}) + \Bbbk_2(\curlyvee(\mathfrak{b},\Im\mathfrak{b}) + \curlyvee(\mathfrak{l},\Re\mathfrak{l})) + \Bbbk_3\frac{\curlyvee(\mathfrak{b},\Im\mathfrak{b})\curlyvee(\mathfrak{l},\Re\mathfrak{l})}{1+\curlyvee(\mathfrak{b},\mathfrak{l})},$$

for all $\mathfrak{b},\mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b},\mathfrak{l}) \neq -1$, *then* $\Im$ *and* $\Re$ *possess a unique CFP.*

Proof. Define $\Bbbk_1,\Bbbk_2,\Bbbk_3 : \mathcal{G} \to [0,1)$ by $\Bbbk_1(\cdot) = \Bbbk_1$, $\Bbbk_2(\cdot) = \Bbbk_2$ and $\Bbbk_3(\cdot) = \Bbbk_3$ in the Theorem 1. □

Corollary 7. *Let* $(\mathcal{G},\curlyvee)$ *be a complete CVSMS and let* $\Im : \mathcal{G} \to \mathcal{G}$. *Suppose there is* $\Bbbk_1,\Bbbk_2,\Bbbk_3 \in [0,1)$ *with* $\Bbbk_1 + 2\Bbbk_2 + \Bbbk_3 < 1$ *such that*

$$\curlyvee(\Im\mathfrak{b},\Im\mathfrak{l}) \leq \Bbbk_1\curlyvee(\mathfrak{b},\mathfrak{l}) + \Bbbk_2(\curlyvee(\mathfrak{b},\Im\mathfrak{b}) + \curlyvee(\mathfrak{l},\Im\mathfrak{l})) + \Bbbk_3\frac{\curlyvee(\mathfrak{b},\Im\mathfrak{b})\curlyvee(\mathfrak{l},\Im\mathfrak{l})}{1+\curlyvee(\mathfrak{b},\mathfrak{l})},$$

for all $\mathfrak{b},\mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b},\mathfrak{l}) \neq -1$, *then* $\Im$ *has a unique FP.*

Proof. Letting $\Im$ equal $\Re$ in the previous result. □

Corollary 8. *Let* $(\mathcal{G},\curlyvee)$ *be a complete CVSMS and* $\Im,\Re : \mathcal{G} \to \mathcal{G}$. *Suppose there is* $\Bbbk_1,\Bbbk_3 \in [0,1)$ *with* $\Bbbk_1 + \Bbbk_3 < 1$ *such that*

$$\curlyvee(\Im\mathfrak{b},\Re\mathfrak{l}) \leq \Bbbk_1\curlyvee(\mathfrak{b},\mathfrak{l}) + \Bbbk_3\frac{\curlyvee(\mathfrak{b},\Im\mathfrak{b})\curlyvee(\mathfrak{l},\Re\mathfrak{l})}{1+\curlyvee(\mathfrak{b},\mathfrak{l})},$$

for all $\mathfrak{b},\mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b},\mathfrak{l}) \neq -1$, *then* $\Im$ *and* $\Re$ *have a unique CFP.*

Proof. By setting $\Bbbk_2$ to zero in Corollary 6. □

Corollary 9. *Let* $(\mathcal{G}, \curlyvee)$ *be a complete CVSMS and* $\Im : \mathcal{G} \to \mathcal{G}$. *Suppose there is* $\Bbbk_1, \Bbbk_3 \in [0, 1)$ *with* $\Bbbk_1 + \Bbbk_3 < 1$ *such that*

$$\curlyvee(\Im\mathfrak{b}, \Im\mathfrak{l}) \preceq \Bbbk_1 \curlyvee(\mathfrak{b}, \mathfrak{l}) + \Bbbk_3 \frac{\curlyvee(\mathfrak{b}, \Im\mathfrak{b}) \curlyvee(\mathfrak{l}, \Im\mathfrak{l})}{1 + \curlyvee(\mathfrak{b}, \mathfrak{l})},$$

for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b}, \mathfrak{l}) \neq -1$, *then* $\Im$ *has a unique FP.*

Proof. Letting $\Im$ equal $\Re$ in the previous result. □

Corollary 10. *Let* $(\mathcal{G}, \curlyvee)$ *be a complete CVSMS and* $\Im, \Re : \mathcal{G} \to \mathcal{G}$. *Suppose there is* $\Bbbk_1, \Bbbk_2 \in [0, 1)$ *with* $\Bbbk_1 + 2\Bbbk_2 < 1$ *such that*

$$\curlyvee(\Im\mathfrak{b}, \Re\mathfrak{l}) \preceq \Bbbk_1 \curlyvee(\mathfrak{b}, \mathfrak{l}) + \Bbbk_2 \left(\curlyvee(\mathfrak{b}, \Im\mathfrak{b}) + \curlyvee(\mathfrak{l}, \Re\mathfrak{l})\right),$$

for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$, *then* $\Im$ *and* $\Re$ *possess a unique CFP.*

Proof. By setting $\Bbbk_3$ to zero in corollary 6. □

Corollary 11. *Let* $(\mathcal{G}, \curlyvee)$ *be a complete CVSMS and* $\Im : \mathcal{G} \to \mathcal{G}$. *Suppose there is* $\Bbbk_1, \Bbbk_2 \in [0, 1)$ *with* $\Bbbk_1 + 2\Bbbk_2 < 1$ *such that*

$$\curlyvee(\Im\mathfrak{b}, \Im\mathfrak{l}) \preceq \Bbbk_1 \curlyvee(\mathfrak{b}, \mathfrak{l}) + \Bbbk_2 \left(\curlyvee(\mathfrak{b}, \Im\mathfrak{b}) + \curlyvee(\mathfrak{l}, \Im\mathfrak{l})\right),$$

for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$, *then* $\Im$ *possesses a unique FP.*

Proof. Take $\Im = \Re$ in above corollary. □

Corollary 12. *Let* $(\mathcal{G}, \curlyvee)$ *be a complete CVSMS and* $\Im, \Re : \mathcal{G} \to \mathcal{G}$. *Suppose there is* $\Bbbk_1 \in [0, 1)$ *such that*

$$\curlyvee(\Im\mathfrak{b}, \Re\mathfrak{l}) \preceq \Bbbk_1 \curlyvee(\mathfrak{b}, \mathfrak{l}),$$

for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$, *then* $\Im$ *and* $\Re$ *possess a unique CFP.*

Proof. By setting $\Bbbk_2 = \Bbbk_3 = 0$ in Corollary 6. □

Corollary 13. *Let* $(\mathcal{G}, \curlyvee)$ *be a complete CVSMS and* $\Im : \mathcal{G} \to \mathcal{G}$. *Suppose there is* $\Bbbk_1 \in [0, 1)$ *such that*

$$\curlyvee(\Im\mathfrak{b}, \Im\mathfrak{l}) \preceq \Bbbk_1 \curlyvee(\mathfrak{b}, \mathfrak{l}),$$

for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$, *then* $\Im$ *possesses a unique FP.*

Proof. Take $\Im = \Re$ in above corollary. □

4. Key findings in complex-valued metric spaces

In this way, we derive some fixed point results in the framework of CVMSs by taking $\aleph = 0$ in the Definition 4 and applying the Theorem 1.

Corollary 14. *Let* $(\mathcal{G}, \curlyvee)$ *be a complete CVMS and* $\Im, \Re : \mathcal{G} \to \mathcal{G}$. *Assuming the existence of functions* $\Bbbk_1, \Bbbk_2, \Bbbk_3 : \mathcal{G} \to [0, 1)$ *such that*

(a) $\Bbbk_1(\Im\mathfrak{b}) \leq \Bbbk_1(\mathfrak{b})$ *and* $\Bbbk_1(\Re\mathfrak{b}) \leq \Bbbk_1(\mathfrak{b})$,
$\Bbbk_2(\Im\mathfrak{b}) \leq \Bbbk_2(\mathfrak{b})$ *and* $\Bbbk_2(\Re\mathfrak{b}) \leq \Bbbk_2(\mathfrak{b})$,
$\Bbbk_3(\Im\mathfrak{b}) \leq \Bbbk_3(\mathfrak{b})$ *and* $\Bbbk_3(\Re\mathfrak{b}) \leq \Bbbk_3(\mathfrak{b})$,
(b) $\Bbbk_1(\mathfrak{b}) + 2\Bbbk_2(\mathfrak{b}) + \Bbbk_3(\mathfrak{b}) < 1$,
(c) $\curlyvee(\Im\mathfrak{b}, \Re\mathfrak{l}) \leq \Bbbk_1(\mathfrak{b}) \curlyvee(\mathfrak{b}, \mathfrak{l}) + \Bbbk_2(\mathfrak{b}) \left(\curlyvee(\mathfrak{b}, \Im\mathfrak{b}) + \curlyvee(\mathfrak{l}, \Re\mathfrak{l})\right) + \Bbbk_2(\mathfrak{b}) \frac{\curlyvee(\mathfrak{b},\Im\mathfrak{b})\curlyvee(\mathfrak{l},\Re\mathfrak{l})}{1+\curlyvee(\mathfrak{b},\mathfrak{l})}$,
for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b}, \mathfrak{l}) \neq -1$, *then* $\Im$ *and* $\Re$ *have a unique CFP.*

In this way, we derive the main result of Sintunavarat et al. [14] directly from the above result.

Corollary 15. *[14] Let* $(\mathcal{G}, \curlyvee)$ *be a complete CVMS and* $\Im, \Re : \mathcal{G} \to \mathcal{G}$. *Assuming the existence of functions* $\Bbbk_1, \Bbbk_3 : \mathcal{G} \to [0, 1)$ *such that*

(a) $\Bbbk_1(\Im\mathfrak{b}) \leq \Bbbk_1(\mathfrak{b})$ *and* $\Bbbk_1(\Re\mathfrak{b}) \leq \Bbbk_1(\mathfrak{b})$,
$\Bbbk_3(\Im\mathfrak{b}) \leq \Bbbk_3(\mathfrak{b})$ *and* $\Bbbk_3(\Re\mathfrak{b}) \leq \Bbbk_3(\mathfrak{b})$,
(b) $\Bbbk_1(\mathfrak{b}) + \Bbbk_3(\mathfrak{b}) < 1$,
(c) $\curlyvee(\Im\mathfrak{b}, \Re\mathfrak{l}) \leq \Bbbk_1(\mathfrak{b}) \curlyvee(\mathfrak{b}, \mathfrak{l}) + \Bbbk_3(\mathfrak{b}) \frac{\curlyvee(\mathfrak{b},\Im\mathfrak{b})\curlyvee(\mathfrak{l},\Re\mathfrak{l})}{1+\curlyvee(\mathfrak{b},\mathfrak{l})}$,
for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b}, \mathfrak{l}) \neq -1$, *then* $\Im$ *and* $\Re$ *have a unique CFP.*

Proof. Define $\Bbbk_2 : \mathcal{G} \to [0, 1)$ by $\Bbbk_2(\mathfrak{b}) = 0$ in the Corollary 14. □

The principal result obtained by Azam et al. [12] can be obtained in this way by the Corollary 14.

Corollary 16. *[12] Let* $(\mathcal{G}, \curlyvee)$ *be a complete CVMS and* $\Im, \Re : \mathcal{G} \to \mathcal{G}$. *If there exist the nonnegative real numbers* $\Bbbk_1$ *and* $\Bbbk_2$ *with* $\Bbbk_1 + \Bbbk_2 < 1$ *such that*

$$\curlyvee(\Im\mathfrak{b}, \Re\mathfrak{l}) \leq \Bbbk_1 \curlyvee(\mathfrak{b}, \mathfrak{l}) + \Bbbk_3 \frac{\curlyvee(\mathfrak{b}, \Im\mathfrak{b}) \curlyvee(\mathfrak{l}, \Re\mathfrak{l})}{1 + \curlyvee(\mathfrak{b}, \mathfrak{l})},$$

for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b}, \mathfrak{l}) \neq -1$, *then* $\Im$ *and* $\Re$ *have a unique CFP.*

Proof. Define $\Bbbk_1, \Bbbk_2, \Bbbk_3 : \mathcal{G} \to [0, 1)$ by $\Bbbk_1(\mathfrak{b}) = \Bbbk_1, \Bbbk_3(\mathfrak{b}) = \Bbbk_3$ and $\Bbbk_2(\mathfrak{b}) = 0$ in the Corollary 14. □

Corollary 17. *[12] Let* $(\mathcal{G}, \curlyvee)$ *be a complete CVMS and* $\Im : \mathcal{G} \to \mathcal{G}$. *If there exist the nonnegative real numbers* $\Bbbk_1$ *and* $\Bbbk_2$ *with* $\Bbbk_1 + \Bbbk_2 < 1$ *such that*

$$\curlyvee(\Im\mathfrak{b}, \Im\mathfrak{l}) \leq \Bbbk_1 \curlyvee(\mathfrak{b}, \mathfrak{l}) + \Bbbk_3 \frac{\curlyvee(\mathfrak{b}, \Im\mathfrak{b}) \curlyvee(\mathfrak{l}, \Im\mathfrak{l})}{1 + \curlyvee(\mathfrak{b}, \mathfrak{l})},$$

for all $\mathfrak{b}, \mathfrak{l} \in \mathcal{G}$ *with* $\curlyvee(\mathfrak{b}, \mathfrak{l}) \neq -1$, *then* $\Im$ *has a unique FP.*

Proof. Take $\Im = \Re$ in the preceding corollary. □

5. Applications

The Volterra-Hammerstein integral equation, given as

$$\mathfrak{b}(t) = f(t) + \int_a^t \varphi(t, s) H(s, \mathfrak{b}(s))\, ds, \tag{5.1}$$

plays a significant role in the field of robot manipulator control, where dynamic systems involve nonlinear and memory-dependent behaviors. Robot manipulators, which are extensively used in industrial automation, medical robotics, and space exploration, require precise control strategies to handle complex dynamics, uncertainties, and external disturbances. Related works on efficient algorithms and spectral methods for Volterra-type equations are reported in [24–26].

The motion of a robotic manipulator is governed by the nonlinear differential equation:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau,$$

where,

- q represents the joint positions (generalized coordinates),
- $\dot{q}$ and $\ddot{q}$ are joint velocities and accelerations, respectively,
- $M(q)$ is the inertia matrix,
- $C(q, \dot{q})$ represents Coriolis and centrifugal forces,
- $G(q)$ is the gravitational force,
- τ is the control input (torques/forces).

When external disturbances and nonlinearities are considered, the system dynamics can be rewritten in an integral form, which closely resembles the Volterra-Hammerstein equation:

$$q(t) = q_0 + \int_0^t \varphi(t, s) H(s, q(s))\, ds,$$

where,

- q_0 models initial conditions and known system inputs,
- $\varphi(t, s)$ represents the system memory and delay effects,
- $H(s, q(s))$ introduces nonlinear actuator or friction forces.

In this section, we apply the fixed point theorem developed in this research to solve the Volterra-Hammerstein equation. For this, we consider $(\mathcal{G}, \curlyvee)$ as a CVSMS, where $\mathcal{G}$ is the space of continuous functions defined on $[a, b]$ and $d : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ is defined as follows

$$d(\mathfrak{b}, \mathfrak{l}) = \sup_{t \in [a,b]} |\mathfrak{b}(t) - \mathfrak{l}(t)|\, e^{i\frac{\pi}{4}}.$$

Theorem 2. *Let $(\mathcal{G}, \curlyvee)$ be a complete CVSMS, and let $\Im : \mathcal{G} \to \mathcal{G}$ be a mapping defined by*

$$\Im\mathfrak{b}(t) = f(t) + \int_a^t \varphi(t, s) H(s, \mathfrak{b}(s))\, ds.$$

If the following conditions hold:

(i) Lipschitz continuity: $H(s, \mathfrak{b}(s))$ satisfies a Lipschitz condition with constant $L > 0$, i.e.,

$$|H(s, \mathfrak{b}(s)) - H(s, \mathfrak{l}(s))| \leq L |\mathfrak{b}(s) - \mathfrak{l}(s)|,$$

for all $s \in [a, b]$.

(ii) Kernel condition: The kernel $\varphi(t, s)$ satisfies:

$$K = \sup_{t\in[a,b]} \int_a^t \varphi(t, s) ds < 1.$$

(iii) There exists $\Bbbk_1 = LK$ such that $\Bbbk_1 \in [0, 1)$. Then $\Im$ admits a unique fixed point $\mathfrak{b}^*(t)$, i.e.,

$$\Im(\mathfrak{b}^*)(t) = \mathfrak{b}^*(t),$$

which solves the Volterra-Hammerstein integral equation:

$$\mathfrak{b}(t) = f(t) + \int_a^t \varphi(t, s) H(s, \mathfrak{b}(s)) ds.$$

Proof. Let $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{G}$. Then

$$\Im(\mathfrak{b})(t) = f(t) + \int_a^t \varphi(t, s) H(s, \mathfrak{b}(s)) ds.$$

Now for every $t \in [a, b]$, we have

$$\begin{aligned} |\Im(\mathfrak{b})(t) - \Im(\mathfrak{l})(t)| &= \left|\int_a^t \varphi(t, s)(H(s, \mathfrak{b}(s)) - H(s, \mathfrak{l}(s))) ds\right| \\ &\leq \int_a^t |\varphi(t, s)| |H(s, \mathfrak{b}(s)) - H(s, \mathfrak{l}(s))| ds \\ &\leq \int_a^t |\varphi(t, s)| \cdot L |\mathfrak{b}(s) - \mathfrak{l}(s)| ds. \end{aligned}$$

Taking the supremum over $t \in [a, b]$:

$$\begin{aligned} \curlyvee(\Im(\mathfrak{b}), \Im(\mathfrak{l})) &= \sup_{t\in[a,b]} |\Im(\mathfrak{b})(t) - \Im(\mathfrak{l})(t)| e^{i\frac{\pi}{4}} \\ &\leq L \sup_{t\in[a,b]} \int_a^t |\varphi(t, s)| \cdot d(\mathfrak{b}, \mathfrak{l}) \\ &= LKd(\mathfrak{b}, \mathfrak{l}) \\ &= \Bbbk_1 d(\mathfrak{b}, \mathfrak{l}). \end{aligned}$$

Hence all the conditions of Corollary 13 are satisfied. Therefore, the Volterra-Hammerstein integral equation admits a unique solution. □

Example 5. *Let us turn our attention to the subsequent Volterra-Hammerstein integral equation*

$$\mathfrak{b}(t) = \frac{t}{e^{t^2}} + \int_0^t 2ts\mathrm{e}^{-\mathfrak{b}^2(s)} ds,$$

which admits a unique solution, namely, $\mathfrak{b}(t) = t$.

Example 6. *The ensuing analysis focuses on the subsequent Volterra-Hammerstein integral equation*

$$\mathfrak{b}(t) = t - \frac{e^t}{2} + \left(\frac{\sin t + \cos t}{2}\right) + \int_0^t e^{t-s} \sin(\mathfrak{b}(s))\, ds.$$

It can be shown that the integral equation possesses a unique solution given by $\mathfrak{b}(t) = t$.

6. Conclusions

In this study, we investigated the idea of complex-valued suprametric spaces and established novel common fixed point theorems for generalized rational contractions involving control functions of a single variable. These results not only extend and generalize the earlier contributions of Berzig [9], Azam et al. [12], and Sintunavarat et al. [14], but also unify different contraction conditions previously studied in the literature. As a further outcome, we derived common fixed point theorems for mappings governed by constant parameters rather than variable control functions, which broadens the applicability of our theoretical framework.

To emphasize the practical relevance of our findings, we applied the developed results to analyze the solution of the Volterra-Hammerstein integral equation, which is directly connected to the dynamics of robotic manipulator control. This demonstrates that the theoretical advances are not only mathematically significant but also carry real-world implications. Moreover, a comprehensive illustrative example was provided to highlight the effectiveness of our approach and to show step-by-step how the proposed theorems can be implemented in solving nonlinear integral equations.

Overall, the logical progression of the work, moving from theoretical development to practical application and finally to illustrative demonstration, highlights both the originality and applicability of the results and provides a new pathway for further research in fixed point theory and its applications to nonlinear analysis and control problems.

Future research will focus on extending the CFP theorems to encompass multi-valued, fuzzy, and L-fuzzy mappings within the foundation of CVSMSs. Additionally, the study of differential and integral inclusions in this context will be explored. It is expected that the outcomes of this research will encourage further investigations and improvements, potentially broadening the range of applications for these findings.

Author contributions

Amnah Essa Shammaky: Conceptualization, writing–review and editing, funding acquisition; Ali H. Hakami: Writing–original draft, methodology, formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of interest

The authors declare no conflicts of interest.

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