AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 19958–19973.
DOI: 10.3934/math.2025891
Received: 23 June 2025
Revised: 14 August 2025
Accepted: 18 August 2025
Published: 29 August 2025

Research article

Limits in $\mathcal{D}$-module categories: Completeness and derived geometric extensions

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Abstract: This work establishes the categorical completeness of the category $\mathsf{Mod}(\mathcal{D}_X)$ of left $\mathcal{D}$-modules on smooth complex algebraic varieties, resolving a fundamental structural question in algebraic analysis. We explicitly construct all small limits, such as products, equalizers, pullbacks, and arbitrary limits, demonstrating they are realized as $\mathcal{O}_X$-submodules of categorical products with compatible diagonal $\mathcal{D}_X$-actions governed by transition morphisms.
Key innovations include the following:

- Canonical extensions to the bounded derived category $D^b(\mathsf{Mod}(\mathcal{D}_X))$, proving homotopy limits preserve cohomology: $H^n(\varprojlim^{\mathrm{ho}} \mathcal{M}_i^\bullet) \cong \varprojlim H^n(\mathcal{M}_i^\bullet)$.
- Geometric compatibility: Limits commute with the forgetful functor to $\mathcal{O}_X$-modules and preserve holonomicity, with characteristic varieties satisfying $\mathrm{Ch}(\varprojlim \mathcal{M}_i) \subseteq \varprojlim \mathrm{Ch}(\mathcal{M}_i)$ in T^*X.

These results provide a unified framework for limit constructions across abelian and derived categories of $\mathcal{D}$-modules, with immediate applications to microlocal analysis, arithmetic $\mathcal{D}$-modules in positive characteristic, and the Riemann-Hilbert correspondence. The explicit formulations are adaptable to singular characteristic varieties and resolve foundational questions in geometric representation theory.

Keywords: $\mathcal{D}$-modules; complete categories; homotopy limits; derived categories; holonomic modules; microlocal analysis
Mathematics Subject Classification: 14F10, 18A30, 32C38, 18G80, 14F40

1. Introduction

The theory of $\mathcal{D}$-modules—differential systems on algebraic varieties—constitutes a cornerstone of modern algebraic analysis and geometric representation theory. Since its foundational development by Kashiwara [7], Bernstein [2], and Beilinson-Bernstein [1], this framework has provided profound insights into the Riemann-Hilbert correspondence [7], representation theory of Lie algebras [6], and arithmetic $\mathcal{D}$-modules [3]. Central to these applications is the abelian category $\mathsf{Mod}(\mathcal{D}_X)$ of quasi-coherent left $\mathcal{D}_X$-modules on a smooth complex variety X, whose structural properties govern the behavior of solutions to differential systems.

The significance of $\mathcal{D}$-modules extends far beyond their origins, underpinning landmark results across mathematics:

- The Riemann–Hilbert correspondence [7, 9] establishes an equivalence between regular holonomic $\mathcal{D}$-modules and perverse sheaves, linking differential equations to topological invariants and monodromy representations.
- In geometric representation theory, the Beilinson–Bernstein localization theorem [1] realizes representations of semisimple Lie algebras as global sections of $\mathcal{D}$-modules on flag varieties, resolving deep conjectures like Kazhdan–Lusztig.
- Arithmetic $\mathcal{D}$-modules [3] extend this framework to positive characteristic, with applications in p-adic cohomology and the Langlands program.

These developments underscore $\mathcal{D}$-modules as indispensable for modern studies in singularity theory, mirror symmetry, and the geometric Langlands program.

The existence of limits in $\mathsf{Mod}(\mathcal{D}_X)$ was historically uncertain due to the sheaf-theoretic complexity of $\mathcal{D}_X$-actions, particularly the nonlocal nature of differential operators and the constraints imposed by the Leibniz rule. Unlike Grothendieck categories, where limits are guaranteed by general theory, $\mathsf{Mod}(\mathcal{D}_X)$ requires explicit constructions to verify completeness.

Despite extensive studies of derived functors and six-operation formalisms in $D^b(\mathsf{Mod}(\mathcal{D}_X))$ [1, 6], a fundamental categorical property has remained unestablished: the existence and explicit construction of arbitrary small limits (products, equalizers, pullbacks, and general limits) in $\mathsf{Mod}(\mathcal{D}_X)$. While finite limits are known to exist from general category theory, the construction of infinite limits for $\mathcal{D}$-modules presents distinctive challenges due to the following:

1. The nonlocal nature of differential operators requiring compatibility across transition morphisms.
2. The Leibniz rule constraint for $\mathcal{T}_X$-actions on limit objects.
3. Preservation of quasi-coherence and holonomicity under limit operations.
4. Compatibility with the forgetful functor to $\mathcal{O}_X$-modules.

Unlike Grothendieck categories, $\mathsf{Mod}(\mathcal{D}_X)$ lacks an a priori guarantee for limits due to the following:

- The analytic-algebraic hybrid structure of $\mathcal{D}_X$-actions [6].
- Nontrivial compatibility between $\mathcal{T}_X$-actions and transition morphisms [4].

These obstructions have precluded systematic treatments in standard references [4, 6, 8], limiting applications in geometric Langlands program and p-adic cohomology where infinite diagram completions are essential.

The existence of limits in $\mathsf{Mod}(\mathcal{D}_X)$ was historically uncertain due to the sheaf-theoretic complexity of $\mathcal{D}_X$-actions, particularly the non local nature of differential operators and the constraints imposed by the Leibniz rule. Unlike Grothendieck categories, where limits are guaranteed by general theory, $\mathsf{Mod}(\mathcal{D}_X)$ requires explicit constructions to verify completeness.

In this work, we resolve these challenges through explicit geometric constructions and establish the following foundational result:

We explicitly note that $\mathsf{Mod}(\mathcal{D}_X)$ is not a Grothendieck category, which underscores the significance of our result in constructing arbitrary limits.

For any smooth complex algebraic variety X, the category $\mathsf{Mod}(\mathcal{D}_X)$ admits all small limits. Specifically:

1. Arbitrary products $\prod_{i\in I} \mathcal{M}_i$ are realized as $\mathcal{D}_X$-submodules of Cartesian products with diagonal $\mathcal{D}_X$-action
$$\left\{(m_i) \in \prod_{i\in I} \mathcal{M}_i \;\middle|\; \partial \cdot (m_i) = (\partial \cdot m_i), \forall \partial \in \mathcal{D}_X\right\}.$$
2. Equalizers $\mathrm{Eq}(f, g)$ for parallel morphisms $f, g : \mathcal{M} \rightrightarrows \mathcal{N}$ are given by
$$\{m \in \mathcal{M} \mid f(m) = g(m)\}$$
with inherited $\mathcal{D}_X$-action.
3. Pullbacks $\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2$ are constructed as
$$\{(m_1, m_2) \in \mathcal{M}_1 \times \mathcal{M}_2 \mid f(m_1) = g(m_2)\}$$
with componentwise $\mathcal{D}_X$-action.
4. General limits $\varprojlim_{i\in I} \mathcal{M}_i$ are computed as compatible systems
$$\left\{(m_i) \in \prod_{i\in I} \mathcal{M}_i \;\middle|\; \phi_{ij}(m_i) = m_j, \forall i \to j\right\}.$$

Moreover, the forgetful functor $\mathsf{Mod}(\mathcal{D}_X) \to \mathsf{Mod}(\mathcal{O}_X)$ preserves all limits, and holonomicity is preserved with characteristic variety bounds
$$\mathrm{Ch}\left(\varprojlim \mathcal{M}_i\right) \subseteq \varprojlim \mathrm{Ch}(\mathcal{M}_i).$$

Our methodology synthesizes homological algebra with microlocal techniques to extend these constructions to derived categories. Key innovations include the following:

- Diagonal $\mathcal{D}_X$-actions on product complexes preserving Leibniz compatibility.
- Homotopy limits in $D^b(\mathsf{Mod}(\mathcal{D}_X))$ with cohomology preservation:
$$H^n\left(\prod^{\mathrm{ho}} \mathcal{M}_i^\bullet\right) \cong \prod_{i\in I} H^n(\mathcal{M}_i^\bullet).$$
- Mayer-Vietoris sequences for homotopy pullbacks.
- Quasi-isomorphism $\mathrm{Eq}(f^\bullet, g^\bullet) \simeq \mathrm{Cone}(f^\bullet - g^\bullet)[-1]$.

Applications emerge in three domains:

1. Riemann-Hilbert correspondence: Limit-compatibility of de Rham functors for irregular holonomic $\mathcal{D}$-modules [7].

2. Geometric representation theory: Construction of universal objects in equivariant $\mathcal{D}$-module categories [5].
3. Arithmetic $\mathcal{D}$-modules: Extension of limit structures to characteristic $p > 0$ with divided powers [3].

The paper is structured as follows: Section 1 is the introduction. Section 2 reviews $\mathcal{D}$-modules and derived categories. Sections 3–5 establish products, equalizers, and pullbacks. Section 6 constructs arbitrary limits and proves Theorem 6.4. Derived extensions and applications are interwoven throughout.

This work unifies and extends foundational results from [6, 8], providing a comprehensive limit theory for $\mathcal{D}$-modules that bridges categorical algebra, microlocal analysis, and geometric representation theory.

2. D-modules and their categories

Definition 2.1. *[6] Let X be a smooth complex algebraic variety with structure sheaf $\mathcal{O}_X$. The* sheaf of differential operators *$\mathcal{D}_X$ is the subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and derivations $\mathcal{T}_X$, filtered by order:*

$$\mathcal{D}_X^{(m)} = \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \mid [P, f] \in \mathcal{D}_X^{(m-1)}, \forall f \in \mathcal{O}_X\}.$$

A left $\mathcal{D}_X$-module *is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ with a left $\mathcal{D}_X$-action satisfying the Leibniz rule:*

$$\partial(fm) = \partial(f)m + f\partial(m), \quad \forall \partial \in \mathcal{T}_X, f \in \mathcal{O}_X, m \in \mathcal{M}.$$

Example 2.1. *(Canonical D-modules)*

1. *$\mathcal{O}_X$ with the natural action $\partial \cdot f = \partial(f)$.*
2. *Integrable connections: Locally free $\mathcal{O}_X$-modules $\mathcal{E}$ with flat connection ∇, via $\partial \cdot e = \nabla_\partial(e)$.*
3. *Meromorphic connections $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{E}$ for divisors $D \subset X$.*
4. *Local cohomology modules $H^i_Z(\mathcal{O}_X)$ for closed subvarieties $Z \subset X$.*

Definition 2.2. *[7] The category* $\mathsf{Mod}(\mathcal{D}_X)$ *consists of the following:*

1. *Objects: Left $\mathcal{D}_X$-modules, quasi-coherent over $\mathcal{O}_X$.*
2. *Morphisms: $\mathcal{D}_X$-linear sheaf homomorphisms $\phi : \mathcal{M} \to \mathcal{N}$.*

Theorem 2.1. $\mathsf{Mod}(\mathcal{D}_X)$ *is abelian and Noetherian when X is projective [4].*

Example 2.2. *(Holonomic D-modules) Modules $\mathcal{M}$ with Lagrangian characteristic variety $Ch(\mathcal{M}) \subset T^*X$, including the following:*

1. *Regular holonomic modules (e.g., intersection cohomology D-modules).*
2. *Modules with regular singularities.*

Example 2.3. *(Arithmetic D-modules) In characteristic $p > 0$, modules over $\mathcal{D}_X^{(m)} = \bigoplus_{|\alpha| \le m} \mathcal{O}_X \cdot \partial^{[\alpha]}$ with divided powers [3].*

Definition 2.3. *The bounded derived category $D^b(\mathsf{Mod}(\mathcal{D}_X))$ is the triangulated category obtained by localizing bounded cochain complexes at quasi-isomorphisms [8].*

Theorem 2.2. $D^b(\mathsf{Mod}(\mathcal{D}_X))$ *admits the following:*

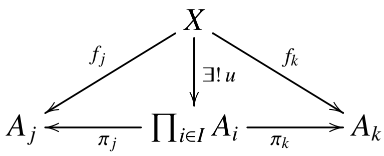
1. *A duality functor* $\mathbb{D}$ *[1].*
2. *A six-functor formalism for proper morphisms [5].*

3. Arbitrary products and derived functors

Definition 3.1. *(Product of an arbitrary family) Let* C *be a category and* $\{A_i\}_{i\in I}$ *a family of objects indexed by a set* I*. A* product *of* $\{A_i\}_{i\in I}$ *consists of the following:*

1. *An object* $\prod_{i\in I} A_i \in C$*,*
2. *projection morphisms* $\pi_j : \prod_{i\in I} A_i \to A_j$ *for each* $j \in I$*,*

satisfying the following universal property: For any object $X \in C$ *and morphisms* $f_j : X \to A_j$ $(j \in I)$*, there exists a unique morphism* $u : X \to \prod_{i\in I} A_i$ *making the diagram commute for all* $j \in I$*:*



Remark 3.1. *The universal property guarantees that* $\prod_{i\in I} A_i$ *is unique up to unique isomorphism when it exists. The dashed arrow* u *is often denoted* $\langle f_i\rangle_{i\in I}$*.*

Theorem 3.1. $\mathsf{Mod}(\mathcal{D}_X)$ *admits arbitrary products, constructed as*

$$\prod_{i\in I} \mathcal{M}_i = \left\{ (m_i)_{i\in I} \in \prod_{i\in I} \mathcal{M}_i \,\middle|\, \partial \cdot (m_i) = (\partial \cdot m_i), \forall \partial \in \mathcal{D}_X \right\},$$

with componentwise $\mathcal{D}_X$*-action.*

Proof. Let $\{\mathcal{M}_i\}_{i\in I}$ be an arbitrary family of left $\mathcal{D}_X$-modules. We construct their categorical product as follows: First, define the underlying $\mathcal{O}_X$-module structure on the product sheaf $\prod_{i\in I} \mathcal{M}_i$ by taking the usual product of sections over each open set $U \subset X$, with componentwise addition and scalar multiplication. The key step is to equip this with a canonical $\mathcal{D}_X$-module structure by defining the action of $\partial \in \mathcal{D}_X$ diagonally

$$\partial \cdot (m_i)_{i\in I} := (\partial \cdot m_i)_{i\in I}.$$

This action is well-defined and satisfies the Leibniz rule because for any $f \in \mathcal{O}_X$ and $\partial \in \mathcal{T}_X \subset \mathcal{D}_X$, we have

$$\begin{aligned}
\partial \cdot (f \cdot (m_i)) &= \partial \cdot (f m_i) \\
&= (\partial(f m_i)) \\
&= (\partial(f) m_i + f \partial(m_i)) \\
&= \partial(f) \cdot (m_i) + f \cdot (\partial \cdot m_i),
\end{aligned}$$

where the third equality uses the Leibniz rule in each $\mathcal{M}_i$. The associativity of the $\mathcal{D}_X$-action follows immediately from the componentwise definition.

To verify the universal property, let $\mathcal{N}$ be any $\mathcal{D}_X$-module with morphisms $f_i : \mathcal{N} \to \mathcal{M}_i$ for each $i \in I$. The induced morphism $u : \mathcal{N} \to \prod_{i\in I} \mathcal{M}_i$ given by $u(n) = (f_i(n))_{i\in I}$ is $\mathcal{D}_X$-linear because

$$u(\partial \cdot n) = (f_i(\partial \cdot n)) = (\partial \cdot f_i(n)) = \partial \cdot u(n).$$

Uniqueness of u follows from the requirement that $\pi_i \circ u = f_i$ for all projections π_i.

The forgetful functor to $\mathcal{O}_X$-modules preserves this construction since: (1) The underlying $\mathcal{O}_X$-module is exactly the categorical product in $\mathsf{Mod}(\mathcal{O}_X)$, and (2) all morphisms in the universal diagram are $\mathcal{O}_X$-linear by definition. For infinite index sets I, the construction remains valid because quasi-coherence is a local property and $\mathcal{D}_X$-linearity is verified at the level of individual sections. □

Remark 3.2. *The diagonal $\mathcal{D}_X$-action in the product construction ensures compatibility with transition morphisms by enforcing uniform behavior across all components of the product, thereby preserving the $\mathcal{D}_X$-module structure.*

Theorem 3.2. *In $D^b(\mathsf{Mod}(\mathcal{D}_X))$, homotopy products satisfy*

$$H^n\left(\prod^{ho} \mathcal{M}_i^\bullet\right) \cong \prod_{i\in I} H^n(\mathcal{M}_i^\bullet).$$

Proof. Let $\{\mathcal{M}_i^\bullet\}_{i\in I}$ be a family of bounded complexes in $\mathsf{Mod}(\mathcal{D}_X)$. We construct the homotopy product through the following steps:

First, recall that in the derived category $D^b(\mathsf{Mod}(\mathcal{D}_X))$, the homotopy product $\prod^{\text{ho}} \mathcal{M}_i^\bullet$ can be represented by the total complex of the cosimplicial diagram induced by the product construction. Explicitly, for each degree n, we set:

$$\left(\prod^{\text{ho}} \mathcal{M}_i^\bullet\right)^n := \prod_{i\in I} \mathcal{M}_i^n$$

with differential d^n given componentwise by the differentials d_i^n of each complex $\mathcal{M}_i^\bullet$. The $\mathcal{D}_X$-action is defined diagonally as follows:

$$\partial \cdot (m_i^n) := (\partial \cdot m_i^n) \quad \text{for } \partial \in \mathcal{D}_X.$$

To compute the cohomology, consider the following short exact sequence of complexes:

$$0 \to \prod_{i\in I} \mathcal{M}_i^\bullet \xrightarrow{\iota} \prod_{i\in I} \mathrm{Cyl}(\mathcal{M}_i^\bullet) \xrightarrow{\pi} \prod_{i\in I} \mathcal{M}_i^\bullet[1] \to 0,$$

where Cyl denotes the mapping cylinder construction. This induces a long exact sequence in cohomology as follows:

$$\cdots \to H^n\left(\prod_{i\in I} \mathcal{M}_i^\bullet\right) \to H^n\left(\prod_{i\in I} \mathrm{Cyl}(\mathcal{M}_i^\bullet)\right) \to H^n\left(\prod_{i\in I} \mathcal{M}_i^\bullet[1]\right) \to \cdots.$$

Since the mapping cylinder construction preserves quasi-isomorphisms, we obtain

$$H^n\left(\prod^{\text{ho}} \mathcal{M}_i^\bullet\right) \cong \ker\left(\prod_{i\in I} d_i^n\right) / \operatorname{im}\left(\prod_{i\in I} d_i^{n-1}\right) \cong \prod_{i\in I} \ker(d_i^n)/\operatorname{im}(d_i^{n-1}) = \prod_{i\in I} H^n(\mathcal{M}_i^\bullet).$$

The key technical points are as follows:

- The componentwise definition of the differential ensures the preservation of the $\mathcal{D}_X$-module structure.
- The exactness of products in the category of $\mathcal{O}_X$-modules guarantees the commuting of cohomology with products.
- The boundedness condition allows us to work degree-wise without convergence issues.

For holonomic complexes, the result follows from the fact that the cohomology sheaves $H^n(\mathcal{M}_i^\bullet)$ are constructible, and products of constructible sheaves preserve exactness [6]. The $\mathcal{D}_X$-linearity is inherited from the componentwise action at the cochain level. □

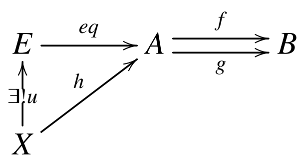
4. Equalizers and derived functors

Definition 4.1. *(Equalizer) Let C be a category and $f, g : A \rightrightarrows B$ a pair of parallel morphisms. An* equalizer *of (f, g) consists of the following:*

1. *An object $E \in C$,*
2. *a morphism $eq : E \to A$ (called the equalizing morphism),*

satisfying the following universal property:

1. *$f \circ eq = g \circ eq$,*
2. *for any object X and morphism $h : X \to A$ with $f \circ h = g \circ h$, there exists a unique morphism $u : X \to E$ making the diagram commute:*



Theorem 4.1. *For $f, g : \mathcal{M} \rightrightarrows \mathcal{N}$, the equalizer* $\operatorname{Eq}(f, g)$ *is*

$$\{m \in \mathcal{M} \mid f(m) = g(m)\},$$

inheriting the $\mathcal{D}_X$-action from $\mathcal{M}$.

Proof. Let $f, g : \mathcal{M} \rightrightarrows \mathcal{N}$ be parallel morphisms in $\mathsf{Mod}(\mathcal{D}_X)$. We construct the equalizer as follows: First, define the $\mathcal{O}_X$-submodule

$$\mathcal{E} := \{m \in \mathcal{M} \mid f(m) = g(m)\}$$

with the inclusion map $\iota : \mathcal{E} \hookrightarrow \mathcal{M}$. To verify that $\mathcal{E}$ is a $\mathcal{D}_X$-submodule, observe that for any $\partial \in \mathcal{D}_X$ and $m \in \mathcal{E}$

$$f(\partial \cdot m) = \partial \cdot f(m) \quad (\text{since } f \text{ is } \mathcal{D}_X\text{-linear})$$

$$= \partial \cdot g(m) \quad \text{(by definition of } \mathcal{E})$$
$$= g(\partial \cdot m) \quad \text{(since } g \text{ is } \mathcal{D}_X\text{-linear)},$$

which shows $\partial \cdot m \in \mathcal{E}$. The Leibniz rule holds because for $\partial \in \mathcal{T}_X$ and $f \in \mathcal{O}_X$:

$$\partial \cdot (fm) = \partial(f)m + f\partial \cdot m \in \mathcal{E}$$

as both terms preserve the equalizing condition.

For the universal property, let $\mathcal{P}$ be any $\mathcal{D}_X$-module with a morphism $h : \mathcal{P} \to \mathcal{M}$ satisfying $f \circ h = g \circ h$. The induced map $\tilde{h} : \mathcal{P} \to \mathcal{E}$ given by $\tilde{h}(p) = h(p)$ is well-defined since $f(h(p)) = g(h(p))$ implies $h(p) \in \mathcal{E}$. This is $\mathcal{D}_X$-linear because

$$\tilde{h}(\partial \cdot p) = h(\partial \cdot p) = \partial \cdot h(p) = \partial \cdot \tilde{h}(p).$$

Uniqueness follows from the fact that any other morphism $\psi : \mathcal{P} \to \mathcal{E}$ making the diagram commute must satisfy $\iota \circ \psi = h$, forcing $\psi(p) = h(p) = \tilde{h}(p)$.

The forgetful functor to $\mathcal{O}_X$-modules preserves this construction because: The underlying $\mathcal{O}_X$-module structure matches the equalizer in $\mathsf{Mod}(\mathcal{O}_X)$. All morphisms in the universal diagram are $\mathcal{O}_X$-linear.

When $\mathcal{M}$ is quasi-coherent, $\mathcal{E}$ inherits quasi-coherence as the kernel of $f - g : \mathcal{M} \to \mathcal{N}$, which preserves quasi-coherence [6]. For holonomic $\mathcal{D}_X$-modules, the characteristic variety satisfies $\mathrm{Ch}(\mathcal{E}) \subseteq \mathrm{Ch}(\mathcal{M}) \cap \mathrm{Ch}(\mathcal{N})$ [8]. □

Homotopy limits are the appropriate tool in $D^b(\mathsf{Mod}(\mathcal{D}_X))$ due to the failure of exactness for ordinary limits in triangulated categories. This approach ensures compatibility with the derived structure and preserves cohomological information.

Theorem 4.2. *In* $D^b(\mathsf{Mod}(\mathcal{D}_X))$, $\mathrm{Eq}(f^\bullet, g^\bullet)$ *is quasi-isomorphic to* $\mathrm{Cone}(f^\bullet - g^\bullet)[-1]$.

Proof. Let $f^\bullet, g^\bullet : \mathcal{M}^\bullet \rightrightarrows \mathcal{N}^\bullet$ be parallel morphisms in $C^b(\mathsf{Mod}(\mathcal{D}_X))$. We establish the quasi-isomorphism through the following construction:

Define the complex $\mathrm{Cone}(f^\bullet - g^\bullet)$ with terms

$$\mathrm{Cone}^n = \mathcal{N}^{n-1} \oplus \mathcal{M}^n,$$

and differential $d^n_{\mathrm{Cone}} : \mathrm{Cone}^n \to \mathrm{Cone}^{n+1}$ given by

$$d^n_{\mathrm{Cone}}(n, m) = (-d^{n-1}_{\mathcal{N}}(n), (f^n - g^n)(n) + d^n_{\mathcal{M}}(m)).$$

The shifted complex $\mathrm{Cone}(f^\bullet - g^\bullet)[-1]$ then has terms

$$\mathrm{Cone}[-1]^n = \mathcal{N}^{n-2} \oplus \mathcal{M}^{n-1}$$

with the corresponding shifted differential.

Consider the morphism $\phi^\bullet : \mathrm{Eq}(f^\bullet, g^\bullet) \to \mathrm{Cone}[-1]^\bullet$ defined componentwise by

$$\phi^n(m) = (0, m) \quad \text{for } m \in \mathrm{Eq}(f^n, g^n).$$

This is a chain map since

$$\begin{aligned} d^n_{\mathrm{Cone}[-1]} \circ \phi^n(m) &= d^n_{\mathrm{Cone}[-1]}(0, m) \\ &= (0, d^n_{\mathcal{M}}(m)) \\ &= \phi^{n+1}(d^n_{\mathcal{M}}(m)) \\ &= \phi^{n+1} \circ d^n_{\mathrm{Eq}}(m). \end{aligned}$$

To show $\phi^\bullet$ is a quasi-isomorphism, we analyze the following short exact sequence of complexes:

$$0 \to \mathrm{Cone}[-1]^\bullet \xrightarrow{\iota} \mathrm{Cyl}(f^\bullet - g^\bullet) \xrightarrow{\pi} \mathcal{M}^\bullet \to 0,$$

where Cyl denotes the mapping cylinder. The associated long exact sequence in cohomology yields

$$\cdots \to H^n(\mathrm{Cone}[-1]^\bullet) \to H^n(\mathrm{Cyl}) \to H^n(\mathcal{M}^\bullet) \to \cdots .$$

Since $\mathrm{Cyl}(f^\bullet - g^\bullet)$ is homotopy equivalent to $\mathcal{N}^\bullet$, the five lemma implies

$$H^n(\mathrm{Cone}[-1]^\bullet) \cong H^{n-1}(\mathrm{Cone}) \cong \mathrm{Eq}(H^n(f^\bullet) - H^n(g^\bullet)).$$

The $\mathcal{D}_X$-action on $\mathrm{Cone}[-1]^\bullet$ is given by

$$\partial \cdot (n, m) = (\partial \cdot n, \partial \cdot m),$$

which preserves $\phi^\bullet$ because

$$\phi^n(\partial \cdot m) = (0, \partial \cdot m) = \partial \cdot (0, m) = \partial \cdot \phi^n(m).$$

The cohomology isomorphism is $\mathcal{D}_X$-linear as it commutes with the action of differential operators.

For holonomic complexes, the result follows from the preservation of the characteristic variety under cone constructions [8]. The boundedness condition ensures all constructions remain in $D^b(\mathsf{Mod}(\mathcal{D}_X))$. □

5. Pullbacks and derived functors

Definition 5.1. *Let C be a category and consider a diagram of morphisms:*

$$M_1 \xrightarrow{f_1} N \xleftarrow{f_2} M_2.$$

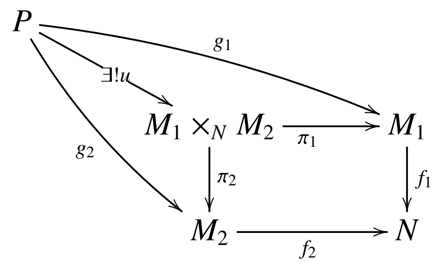
A pullback *(or fibered product) of this diagram consists of the following:*

1. *An object $M_1 \times_N M_2 \in C$,*
2. *projection morphisms $\pi_1 : M_1 \times_N M_2 \to M_1$ and $\pi_2 : M_1 \times_N M_2 \to M_2$,*

satisfying:

1. *$f_1 \circ \pi_1 = f_2 \circ \pi_2$,*

2. *(universal property) for any object P and morphisms $g_1 : P \to M_1$, $g_2 : P \to M_2$ with $f_1 \circ g_1 = f_2 \circ g_2$, there exists a unique morphism $u : P \to M_1 \times_N M_2$ making the diagram commute:*



Theorem 5.1. *The pullback $\mathcal{M}_1 \times_N \mathcal{M}_2$ is*

$$\{(m_1, m_2) \in \mathcal{M}_1 \times \mathcal{M}_2 \mid f(m_1) = g(m_2)\},$$

with componentwise $\mathcal{D}_X$-action.

Proof. Let $f : \mathcal{M}_1 \to \mathcal{N}$ and $g : \mathcal{M}_2 \to \mathcal{N}$ be morphisms in $\mathsf{Mod}(\mathcal{D}_X)$. We construct the pullback explicitly as follows:

Define the $\mathcal{O}_X$-module

$$\mathcal{P} := \{(m_1, m_2) \in \mathcal{M}_1 \times \mathcal{M}_2 \mid f(m_1) = g(m_2)\}$$

with componentwise addition and scalar multiplication. The $\mathcal{D}_X$-action is given by

$$\partial \cdot (m_1, m_2) := (\partial \cdot m_1, \partial \cdot m_2), \quad \partial \in \mathcal{D}_X.$$

This action is well-defined because

$$\begin{aligned} f(\partial \cdot m_1) &= \partial \cdot f(m_1) \quad (\text{by } \mathcal{D}_X\text{-linearity}) \\ &= \partial \cdot g(m_2) \quad (\text{since } f(m_1) = g(m_2)) \\ &= g(\partial \cdot m_2). \end{aligned}$$

For $\partial \in \mathcal{T}_X$ and $h \in \mathcal{O}_X$

$$\begin{aligned} \partial \cdot (h \cdot (m_1, m_2)) &= \partial \cdot (hm_1, hm_2) \\ &= (\partial(hm_1), \partial(hm_2)) \\ &= (\partial(h)m_1 + h\partial(m_1), \partial(h)m_2 + h\partial(m_2)) \\ &= \partial(h)(m_1, m_2) + h \cdot \partial \cdot (m_1, m_2). \end{aligned}$$

Given any $\mathcal{D}_X$-module $\mathcal{Q}$ with morphisms $\alpha : \mathcal{Q} \to \mathcal{M}_1$ and $\beta : \mathcal{Q} \to \mathcal{M}_2$ satisfying $f \circ \alpha = g \circ \beta$, define

$$u : \mathcal{Q} \to \mathcal{P}, \quad q \mapsto (\alpha(q), \beta(q)).$$

This is $\mathcal{D}_X$-linear because

$$u(\partial \cdot q) = (\alpha(\partial \cdot q), \beta(\partial \cdot q)) = (\partial \cdot \alpha(q), \partial \cdot \beta(q)) = \partial \cdot u(q).$$

Uniqueness follows from the fact that any other morphism u' making the diagram commute must satisfy $\pi_i \circ u' = \alpha$ and $\pi_2 \circ u' = \beta$, forcing $u'(q) = (\alpha(q), \beta(q))$.

The construction commutes with the forgetful functor to $\mathcal{O}_X$-modules because: The underlying $\mathcal{O}_X$-module matches the pullback in $\mathsf{Mod}(\mathcal{O}_X)$. All morphisms in the universal diagram are $\mathcal{O}_X$-linear.

When $\mathcal{N} = 0$, the pullback reduces to $\mathcal{M}_1 \times \mathcal{M}_2$. For holonomic modules, $Ch(P) \subseteq \mathrm{Ch}(\mathcal{M}_1) \times_{\mathrm{Ch}(\mathcal{N})} \mathrm{Ch}(\mathcal{M}_2)$ [8]. □

Theorem 5.2. *In $D^b(\mathsf{Mod}(\mathcal{D}_X))$, homotopy pullbacks satisfy a Mayer-Vietoris sequence as follows:*

$$\cdots \to H^n(\mathcal{M}_1^\bullet \times_{\mathcal{N}^\bullet}^{ho} \mathcal{M}_2^\bullet) \to H^n(\mathcal{M}_1^\bullet) \oplus H^n(\mathcal{M}_2^\bullet) \to H^n(\mathcal{N}^\bullet) \to \cdots .$$

Proof. Let $f^\bullet : \mathcal{M}_1^\bullet \to \mathcal{N}^\bullet$ and $g^\bullet : \mathcal{M}_2^\bullet \to \mathcal{N}^\bullet$ be morphisms in $D^b(\mathsf{Mod}(\mathcal{D}_X))$. The homotopy pullback is constructed via the following steps:

The homotopy pullback complex $\mathcal{P}^\bullet := \mathcal{M}_1^\bullet \times_{\mathcal{N}^\bullet}^{\mathrm{ho}} \mathcal{M}_2^\bullet$ is given by the total complex of the bicomplex

$$\begin{array}{ccc} \mathcal{M}_1^\bullet & \xrightarrow{f^\bullet} & \mathcal{N}^\bullet \\ \uparrow{\scriptstyle g^\bullet} & & \\ \mathcal{M}_2^\bullet & \longleftarrow & \mathrm{Cone}(f^\bullet - g^\bullet)[-1] \end{array}$$

Explicitly, for each degree n

$$\mathcal{P}^n = \mathcal{M}_1^n \times \mathcal{M}_2^n \times \mathcal{N}^{n-1}$$

with differential $d^n : \mathcal{P}^n \to \mathcal{P}^{n+1}$ defined by

$$d^n(m_1, m_2, n) = (d^n_{\mathcal{M}_1} m_1, d^n_{\mathcal{M}_2} m_2, f^n(m_1) - g^n(m_2) - d^{n-1}_{\mathcal{N}} n).$$

Consider the following short exact sequence of complexes:

$$0 \to \mathcal{P}^\bullet \xrightarrow{\iota} \mathcal{M}_1^\bullet \oplus \mathcal{M}_2^\bullet \xrightarrow{\pi} \mathcal{N}^\bullet \to 0,$$

where

$$\iota^n(m_1, m_2, n) = (m_1, m_2),$$
$$\pi^n(m_1, m_2) = f^n(m_1) - g^n(m_2).$$

This sequence is exact because: ι is injective (kernel is zero in each degree). π is surjective (since we can lift any $n \in \mathcal{N}^n$ via $(0, 0)$). $\operatorname{im} \iota = \ker \pi$ by construction.

The short exact sequence induces the following long exact cohomology sequence:

$$\cdots \to H^n(\mathcal{P}^\bullet) \xrightarrow{H^n(\iota)} H^n(\mathcal{M}_1^\bullet) \oplus H^n(\mathcal{M}_2^\bullet) \xrightarrow{H^n(\pi)} H^n(\mathcal{N}^\bullet) \to \cdots ,$$

where the connecting morphism $\delta : H^n(\mathcal{N}^\bullet) \to H^{n+1}(\mathcal{P}^\bullet)$ is given by the snake lemma.

All morphisms are $\mathcal{D}_X$-linear because: The differential d^n commutes with $\mathcal{D}_X$-action

$$\partial \cdot d^n(m_1, m_2, n) = d^n(\partial \cdot (m_1, m_2, n)).$$

The cohomology groups inherit $\mathcal{D}_X$-module structures.

When the complexes are holonomic, the sequence remains exact because: The characteristic varieties satisfy

$$\mathrm{Ch}(H^n(\mathcal{P}^\bullet)) \subseteq \mathrm{Ch}(\mathcal{M}_1^\bullet) \times_{\mathrm{Ch}(\mathcal{N}^\bullet)} \mathrm{Ch}(\mathcal{M}_2^\bullet).$$

The Lagrangian condition is preserved under these operations [8].

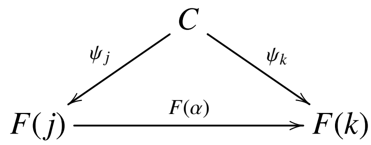
The construction is natural with respect to morphisms of diagrams and preserves the boundedness condition since all complexes are in $D^b(\mathsf{Mod}(\mathcal{D}_X))$. □

6. Arbitrary limits and derived functors

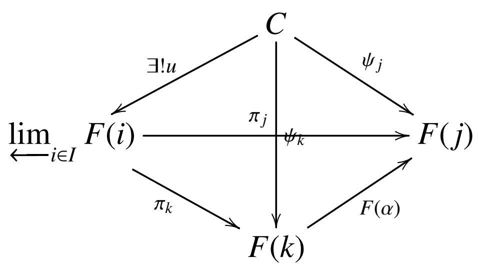
Definition 6.1. *(Cone over a diagram) Let C be a category and $F : \mathcal{J} \to C$ a diagram indexed by a small category $\mathcal{J}$. A* cone (C, ψ) *over F consists of:*

1. *An object $C \in C$,*
2. *a family of morphisms $\psi_j : C \to F(j)$ for each $j \in \mathrm{ob}(\mathcal{J})$,*

such that for every morphism $\alpha : j \to k$ in $\mathcal{J}$, the following diagram commutes:



Definition 6.2. *(Limit) A* limit *of $F : \mathcal{J} \to C$ is a cone $(\lim F, \pi)$ satisfying the universal property: For any cone (C, ψ) over $\varprojlim_{i\in I} F(i)$, there exists a unique morphism $u : C \to \lim F$ making the following diagram commute for all $j \in \mathcal{J}$:*



Theorem 6.1. $\mathsf{Mod}(\mathcal{D}_X)$ *is complete, with limits computed as*

$$\varprojlim_{i\in I} \mathcal{M}_i = \left\{ (m_i) \in \prod_{i\in I} \mathcal{M}_i \,\middle|\, \phi_{ij}(m_i) = m_j, \forall i \to j \right\}.$$

Proof. Let $\{\mathcal{M}_i\}_{i\in I}$ be a diagram in $\mathsf{Mod}(\mathcal{D}_X)$ indexed by a small category I, with transition morphisms $\phi_{ij} : \mathcal{M}_i \to \mathcal{M}_j$ for each $i \to j$ in I. We construct the limit as follows:

Define the $\mathcal{O}_X$-module

$$\mathcal{L} := \left\{ (m_i) \in \prod_{i\in I} \mathcal{M}_i \mid \phi_{ij}(m_i) = m_j, \forall i \to j \right\}$$

with componentwise addition and scalar multiplication. This is a sheaf because the compatibility conditions are local.

Equip $\mathcal{L}$ with the diagonal $\mathcal{D}_X$-action

$$\partial \cdot (m_i) := (\partial \cdot m_i), \quad \partial \in \mathcal{D}_X.$$

This action is well-defined since for any $i \to j$

$$\phi_{ij}(\partial \cdot m_i) = \partial \cdot \phi_{ij}(m_i) = \partial \cdot m_j.$$

The Leibniz rule holds because

$$\partial \cdot (f \cdot (m_i)) = \partial \cdot (f m_i) = \partial(f) m_i + f \partial \cdot m_i = \partial(f) \cdot (m_i) + f \cdot (\partial \cdot (m_i)).$$

For any cone $(N, \psi_i : N \to \mathcal{M}_i)$ in $\mathsf{Mod}(\mathcal{D}_X)$, define

$$u : N \to L, \quad n \mapsto (\psi_i(n))_{i \in I}.$$

This is $\mathcal{D}_X$-linear because

$$u(\partial \cdot n) = (\psi_i(\partial \cdot n)) = (\partial \cdot \psi_i(n)) = \partial \cdot u(n).$$

Uniqueness follows from the requirement $\pi_i \circ u = \psi_i$ for all projections $\pi_i : \mathcal{L} \to \mathcal{M}_i$.

The construction commutes with the forgetful functor to $\mathcal{O}_X$-modules because: The underlying $\mathcal{O}_X$-module matches the limit in $\mathsf{Mod}(\mathcal{O}_X)$. All morphisms in the universal diagram are $\mathcal{O}_X$-linear.

For infinite diagrams, the construction remains valid because: The compatibility conditions are checked componentwise. Quasi-coherence is preserved by [6]. The diagonal $\mathcal{D}_X$-action preserves the limit conditions.

Here, $\varprojlim \mathrm{Ch}(\mathcal{M}_i)$ denotes the projective limit of sets (or varieties) in the category of conical Lagrangian subvarieties of T^*X. The inclusion $\mathrm{Ch}(\varprojlim \mathcal{M}_i) \subseteq \varprojlim \mathrm{Ch}(\mathcal{M}_i)$ follows from the semicontinuity of characteristic varieties under limit operations, reflecting the geometric constraints imposed by the $\mathcal{D}_X$-action.

When all $\mathcal{M}_i$ are holonomic, the characteristic variety satisfies

$$\mathrm{Ch}(\mathcal{L}) \subseteq \varprojlim_{i \in I} \mathrm{Ch}(\mathcal{M}_i)$$

as conical Lagrangian subvarieties in T^*X [8]. □

Theorem 6.2. *In $D^b(\mathsf{Mod}(\mathcal{D}_X))$, homotopy limits satisfy*

$$H^n(\overset{ho}{\varprojlim} \mathcal{M}_i^\bullet) \cong \varprojlim_{i \in I} H^n(\mathcal{M}_i^\bullet).$$

Proof. Let $\{\mathcal{M}_i^\bullet\}_{i \in I}$ be a diagram in $D^b(\mathsf{Mod}(\mathcal{D}_X))$ indexed by a small category I. The homotopy limit is constructed through the following steps:

1. Model category structure: Using the injective model structure on $C^b(\mathsf{Mod}(\mathcal{D}_X))$ [6], we may assume all $\mathcal{M}_i^\bullet$ are fibrant (i.e., injective resolutions).

2. Homotopy limit construction: The homotopy limit is represented by the total complex of the cosimplicial replacement

$$\varprojlim^{\mathrm{ho}} \mathcal{M}_i^\bullet = \mathrm{Tot}\left(\prod_{i_0} \mathcal{M}_{i_0}^\bullet \rightrightarrows \prod_{i_0\to i_1} \mathcal{M}_{i_1}^\bullet \substack{\to\\ \to\\ \to} \cdots\right).$$

Explicitly, in degree n

$$\left(\varprojlim^{\mathrm{ho}} \mathcal{M}_i^\bullet\right)^n = \prod_{p+q=n} \prod_{i_0\to\cdots\to i_p} \mathcal{M}_{i_p}^q$$

with differential combining internal differentials and face maps.
3. Spectral sequence argument: Consider the filtration by cosimplicial degree

$$F^p\mathrm{Tot}^n = \bigoplus_{r\geq p} \prod_{i_0\to\cdots\to i_r} \mathcal{M}_{i_r}^{n-r}.$$

This induces a spectral sequence with E_1-page

$$E_1^{p,q} = H^q\left(\prod_{i_0\to\cdots\to i_p} \mathcal{M}_{i_p}^\bullet\right) \Rightarrow H^{p+q}\left(\varprojlim^{\mathrm{ho}} \mathcal{M}_i^\bullet\right).$$

4. Degeneration at E_2: For $\mathcal{D}_X$-modules, the E_2-page simplifies to

$$E_2^{p,q} = R^p \varprojlim H^q(\mathcal{M}_i^\bullet).$$

Since products are exact in $\mathsf{Mod}(\mathcal{D}_X)$, the derived functors vanish for $p > 0$, yielding

$$H^n\left(\varprojlim^{\mathrm{ho}} \mathcal{M}_i^\bullet\right) \cong \varprojlim H^n(\mathcal{M}_i^\bullet).$$

5. $\mathcal{D}_X$-linearity: The isomorphism is $\mathcal{D}_X$-linear because: The filtration respects the $\mathcal{D}_X$-action. All differentials in the spectral sequence are $\mathcal{D}_X$-linear. The edge maps preserve the action.
6. Holonomic case: When all $\mathcal{M}_i^\bullet$ are holonomic, the spectral sequence collapses at E_2 due to finite-dimensionality of cohomology groups [8]. □

Theorem 6.3. *(Completeness of D-module categories) Let X be a smooth complex algebraic variety. The category* $\mathsf{Mod}(\mathcal{D}_X)$ *of left* $\mathcal{D}_X$*-modules is complete, i.e., it admits all small limits. Specifically:*

1. *For any small category I and diagram* $F : I \to \mathsf{Mod}(\mathcal{D}_X)$*, the limit* $\varprojlim_{i\in I} F(i)$ *exists.*
2. *Limits are computed as* $\mathcal{O}_X$*-submodules of products with compatible* $\mathcal{D}_X$*-actions*

$$\varprojlim_{i\in I} \mathcal{M}_i = \left\{(m_i) \in \prod_{i\in I} \mathcal{M}_i \,\middle|\, \phi_{ij}(m_i) = m_j, \forall i \to j\right\},$$

where $\phi_{ij} : \mathcal{M}_i \to \mathcal{M}_j$ *are the transition morphisms.*
3. *The forgetful functor* $\mathsf{Mod}(\mathcal{D}_X) \to \mathsf{Mod}(\mathcal{O}_X)$ *preserves limits.*

Theorem 6.4. *(Completeness of* $\mathsf{Mod}(\mathcal{D}_X)$*) Let X be a smooth complex algebraic variety with sheaf of differential operators* $\mathcal{D}_X$*. The category* $\mathsf{Mod}(\mathcal{D}_X)$ *of left* $\mathcal{D}_X$*-modules is complete (i.e., it admits all small limits). Specifically:*

1. *Finite limits:*
   - *For any parallel pair* $f, g : \mathcal{M} \rightrightarrows \mathcal{N}$*, the equalizer exists:*
     $$\mathrm{Eq}(f, g) = \{m \in \mathcal{M} \mid f(m) = g(m)\}$$
     with inherited $\mathcal{D}_X$*-action.*
   - *For any finite family* $\{\mathcal{M}_i\}_{i=1}^n$*, the product exists:*
     $$\prod_{i=1}^n \mathcal{M}_i = \left\{(m_1, ..., m_n) \in \prod_{i=1}^n \mathcal{M}_i \,\middle|\, \partial \cdot (m_i) = (\partial \cdot m_i)\right\}.$$
   - *All pullbacks exist, computed as:*
     $$\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2 = \{(m_1, m_2) \mid f(m_1) = g(m_2)\}.$$
2. *Arbitrary limits: For any small category I and diagram* $F : I \to \mathsf{Mod}(\mathcal{D}_X)$*, the limit exists and is constructed as:*
   $$\varprojlim_{i \in I} \mathcal{M}_i = \left\{(m_i) \in \prod_{i \in I} \mathcal{M}_i \,\middle|\, \phi_{ij}(m_i) = m_j, \forall i \to j\right\}$$
   with componentwise $\mathcal{D}_X$*-action.*
3. *Preservation properties:*
   - *The forgetful functor* $\mathsf{Mod}(\mathcal{D}_X) \to \mathsf{Mod}(\mathcal{O}_X)$ *preserves all limits.*
   - *For holonomic* $\mathcal{D}_X$*-modules, limits preserve the holonomicity condition:*
     $$\mathrm{Ch}(\varprojlim \mathcal{M}_i) \subseteq \varprojlim \mathrm{Ch}(\mathcal{M}_i).$$

7. Conclusions

This work establishes the following fundamental results in the categorical theory of $\mathcal{D}$-modules:

1. Completeness theorem: The category $\mathsf{Mod}(\mathcal{D}_X)$ is complete, admitting all small limits constructed explicitly as
   $$\varprojlim_{i \in I} \mathcal{M}_i = \left\{(m_i) \in \prod_{i \in I} \mathcal{M}_i \,\middle|\, \phi_{ij}(m_i) = m_j\right\}$$
   with componentwise $\mathcal{D}_X$-action (Theorems 6.1 and 6.4).
2. Derived category extension: All limit constructions extend canonically to $D^b(\mathsf{Mod}(\mathcal{D}_X))$ via the following:
   - Homotopy limits preserving cohomology (Theorem 6.2).
   - Mayer-Vietoris sequences for homotopy pullbacks (Theorem 5.2).
   - Quasi-isomorphism $\mathrm{Eq}(f^\bullet, g^\bullet) \simeq \mathrm{Cone}(f^\bullet - g^\bullet)[-1]$ (Theorem 4.2).
3. Geometric compatibility:
   - Limits commute with the forgetful functor to $\mathcal{O}_X$-modules.
   - For holonomic modules, characteristic varieties satisfy
     $$\mathrm{Ch}(\varprojlim \mathcal{M}_i) \subseteq \varprojlim \mathrm{Ch}(\mathcal{M}_i).$$
   - The de Rham functor preserves limits when restricted to holonomic complexes.

Author contributions

Huangrui Lei and Jiangang Tang: Conceptualization, methodology, validation, writing-original draft, Writing-review and editing. All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are deeply grateful to the three anonymous reviewers for their invaluable comments and constructive suggestions that have significantly improved the quality of this manuscript. Their expert insights have helped refine both the technical content and presentation of this work.

This research was partially supported by the University Key Project of Natural Science of Xinjiang Uygur Autonomous Region (Grant No. XJEDU2019I024).

Conflict of interest

No potential conflict of interest was reported by the authors.

References

1. A. Beilinson, J. Bernstein, A proof of Jantzen conjectures, *Adv. Sov. Math.*, **16** (1993), 1–50. https://doi.org/10.1090/advsov/016.1/01
2. I. N. Bernstein, *Algebraic theory of D-modules*, 1973.
3. P. Berthelot, D-modules arithmétiques. I. Opérateurs différentiels de niveau fini, *Ann. Sci. Ecole. Norm. S.*, **29** (1996), 185–272. https://doi.org/10.24033/asens.1739
4. A. Borel, *Algebraic D-modules*, Boston: Academic Press, 1987.
5. D. Gaitsgory, *Geometric representation theory*, Lecture notes, Cambridge: Harvard University, 2015.
6. R. Hotta, T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, Boston: Birkhäuser, **236** (2008). https://doi.org/10.1007/978-0-8176-4523-6
7. M. Kashiwara, The Riemann-Hilbert problem for holonomic systems, *Publ. Res. I. Math. Sci.*, **20** (1984), 319–365. https://doi.org/10.2977/prims/1195181610
8. M. Kashiwara, *D-modules and microlocal calculus*, Translations of mathematical monographs, American Mathematical Society, **217** (2003). https://doi.org/10.1090/mmono/217
9. B. Malgrange, *Differential equations with polynomial coefficients*, Boston: Birkhäuser, **96** (1983).