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Research article

Influence of the β-time fractional derivative on soliton structures and stability in nonlinear polarization-preserving optical fibers

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Abstract: This paper investigates the influence of the β-time fractional derivative on Kudryashov's formulation of the nonlinear refractive index in polarization-preserving optical fiber systems. The β-fractional derivative framework offers a more general and memory-inclusive model of wave propagation compared to classical integer-order approaches. Using the improved modified extended tanh method, we derive several exact analytical solutions, including dark solitons, singular periodic solutions, and Jacobi elliptic function solutions. These solutions reveal the rich nonlinear dynamics introduced by the fractional temporal operator and provide insights into the modulation and stability of optical solitons in birefringent media. We demonstrate that the β-fractional derivative significantly modifies soliton behavior, especially affecting amplitude, width, and propagation speed. The originality of this work lies in introducing the β-time fractional derivative into the polarization-preserving optical fiber equation with Kudryashov-type nonlinearity-an approach not previously reported-and in obtaining new exact analytical solutions via the improved modified extended tanh method. These solutions, together with a detailed stability analysis, extend the current understanding of fractional-order soliton dynamics in nonlinear optical media.

Keywords: optical solitons; local fractional derivative; Kudryashov-type nonlinearity; birefringent media
Mathematics Subject Classification: 35R11, 35C07, 35C08

1. Introduction

The study of nonlinear wave propagation in optical fibers has attracted significant attention due to its applications in high-speed telecommunications, photonic devices, and nonlinear signal processing [1–3]. Recent advances in soliton theory continue to expand our understanding of nonlinear wave phenomena in both classical and fractional-order systems. For instance, the study in [4] reported novel soliton dynamics and stability properties in a nonlinear optical system, demonstrating the intricate interplay between nonlinearity and dispersion in determining pulse behavior.

A number of studies have explored soliton dynamics in polarization-preserving optical fibers under various forms of nonlinearity and dispersion. For instance, AlQahtani and Alngar [5] investigated perturbed nonlinear Schrödinger equation (NLSE) models with higher-order nonlinearities in birefringent media, while Bilal et al. [6] and Parasuraman et al. [7] analyzed optical solitons under different perturbation and dispersion regimes. Kudryashov's nonlinear refractive index law has been employed in several recent works to model intensity-dependent optical behaviors [8–10], including cubic-quartic and dual-form nonlinearities. Nonlinear evolution equations (NLEEs) are fundamental in describing a variety of nonlinear physical phenomena, including optical solitons, modulation instability, and wave pattern formation. Recent studies have explored their solutions under diverse nonlinear laws and dispersive effects. For example, Akinyemi et al. [11] investigated the influence of higher-order dispersion on solitary waves and modulation instability in a monomode fiber, revealing how dispersive terms modify soliton stability. In another work, Abbagari et al. [12] analyzed W-chirped solitons and modulated wave patterns in a parabolic law medium with anti-cubic nonlinearity, demonstrating the emergence of complex chirp structures. Furthermore, Abbagari et al. [13] examined rogue and solitary waves in a coupled nonlinear Schrödinger system modeling a left-handed transmission line with second-neighbor coupling, highlighting the interplay between coupling strength and wave localization. These works demonstrate the rich variety of wave phenomena that NLEEs can produce and the importance of exploring such models under different nonlinear refractive index laws and fractional-order frameworks, as undertaken in the present study. Although traditional integer-order models are effective in certain regimes, they fall short in capturing memory effects and complex temporal behaviors characteristic of many real-world optical systems [14, 15]. To overcome these limitations, fractional calculus-particularly time-fractional derivatives-has emerged as a powerful generalization of classical models [16, 17]. The β-time fractional derivative, in particular, introduces a nonlocal temporal operator that enables a more comprehensive modeling of optical pulse evolution in dispersive and dissipative media [18–20]. Recent research has increasingly focused on the application of fractional derivatives in nonlinear optical systems, especially due to their ability to model memory effects and complex dispersion phenomena. Analytical techniques for solving such systems have evolved, including studies on fractional quantum models and coupled field systems. For example, exact soliton solutions in fractional spatiotemporal quantum mechanics have been explored using analytical methods in [21], while a generalized Khater method has been employed to study soliton dynamics in fractional coupled Higgs systems [22]. These works highlight the growing role of fractional calculus in nonlinear wave theory and further support the integration of fractional operators like the β-time derivative in optical fiber models.

In this work, we investigate the effect of incorporating the β-time fractional derivative into the polarization-preserving optical fiber equation governed by Kudryashov's nonlinear refractive index profile [23, 24]. To obtain exact solutions, we apply the improved modified extended tanh method (IMETHM), a refined analytical technique known for its effectiveness in solving nonlinear fractional partial differential equations (FPDEs) [25, 26]. The method yields a range of soliton structures, including dark solitons, singular solitons, and Jacobi elliptic function solutions, thereby revealing the influence of the fractional parameter β on solution characteristics [27, 28].

The mathematical framework of fractional calculus has proven instrumental in modeling complex physical phenomena where memory effects and non-local interactions play crucial roles [29]. In the context of optical fiber communications, the β- fractional derivative provides a natural extension to classical models by incorporating temporal memory effects that arise from material dispersion, nonlinear interactions, and environmental fluctuations [30]. This approach has gained considerable traction in recent years, with applications spanning from anomalous diffusion processes to complex wave propagation in heterogeneous media [31].

Kudryashov's formulation of the nonlinear refractive index represents a significant advancement in modeling optical fiber systems, particularly in capturing the intricate balance between dispersion and nonlinearity that governs soliton formation and stability [32]. When combined with the β -fractional temporal operator, this formulation opens new avenues for understanding and controlling optical pulse propagation in advanced fiber-optic systems [33].

The polarization-preserving aspect of optical fibers introduces additional complexity through birefringence effects, which can significantly influence pulse evolution and soliton dynamics [34]. The interplay between fractional temporal effects and polarization preservation creates a rich mathematical structure that demands sophisticated analytical techniques for solution construction and stability analysis [35]. In the open literature, the dynamics of solitons in polarization-preserving optical fibers have been studied extensively for integer-order models, particularly in the context of cubic, cubic-quintic, and cubic-quartic nonlinearities. Kudryashov's nonlinear refractive index formulation has also been employed to model nonlinear effects with improved accuracy over the Kerr-only approximation. Recently, fractional-order derivatives have gained attention in nonlinear optics for capturing memory effects and temporal nonlocality that cannot be represented by integer-order formulations. Analytical techniques such as the tanh method, sine-cosine method, and Jacobi elliptic function expansions have been applied to various fractional models, but their application to polarization-preserving models with Kudryashov-type nonlinearity remains unexplored. Despite these advances, no study has combined the β-time fractional derivative with the polarization-preserving fiber model using Kudryashov's refractive index law, nor systematically obtained multiple classes of exact soliton solutions for this formulation. Furthermore, stability properties of these fractional-order solutions have not been analytically examined. The incorporation of fractional derivatives into this optical model is necessary to more accurately represent pulse evolution in realistic fiber systems, where dispersive and nonlinear effects are influenced by memory-dependent phenomena. Understanding the stability of these solutions is essential for practical optical communication applications. To the best of our knowledge, this is the first analytical investigation of the β-time fractional polarization-preserving fiber model with Kudryashov-type nonlinearity, yielding new soliton solutions and stability results not reported in the existing literature.

Governing equation and model description

The polarization-preserving fiber model considered in this study is a generalized nonlinear evolution equation incorporating a time-fractional operator of order $\beta \in (0, 1]$. Unlike prior models based on complete (integer-order) derivatives such as the one presented in [36], where cubic-quartic optical solitons were analyzed for Kudryashov's nonlinear refractive index, we propose a refined formulation using the fractional β-time derivative to account for memory effects and nonlocal temporal interactions. The equation, expressed in terms of the complex envelope of the electric field $u(x,t)$, is given by:

$$D_t^\beta u + iau_{xxx} + bu_{xxxx} + u\left(\frac{c_1}{|u|^2} + \frac{c_2}{|u|} + c_3\,|u| + c_4\,|u|^2\right) = 0, \tag{1.1}$$

is a generalized Kudryashov-type expression, capturing diverse nonlinear optical behaviors:

- $\frac{c_1}{|u|^2}$: Models inverse cubic nonlinearity, often associated with self-defocusing effects at extremely low intensities.
- $\frac{c_2}{|u|}$: Represents inverse quadratic nonlinearity, contributing to intensity-dependent refractive index modulation.
- $c_3|u|$: A standard Kerr-type nonlinearity responsible for self-phase modulation and soliton formation.
- $c_4|u|^2$: Corresponds to higher-order saturation effects in the refractive index, relevant in high-power regimes where the nonlinear response saturates.

Together, these terms allow for modeling complex refractive behaviors in nonlinear optical fibers under varying intensity levels. The fractional derivative $D_t^\beta u$ is defined as:

$$D^\beta f(t) = \lim_{\delta\to 0} \frac{f(t + \delta(t + \frac{1}{\Gamma(\beta)})^{1-\beta}) - f(t)}{\delta}, \tag{1.2}$$

where $\Gamma(\cdot)$ is the gamma function, for all $t \geq 0, \beta \in (0,1]$. Then, if the limit of the above exists, f is said to be β-differentiable. The fractional derivative D^β used in this study satisfies several important properties, which we summarize below:

$$D^\beta\left(af(t) + bg(t)\right) = aD^\beta f(t) + bD^\beta g(t), \quad \forall a, b \in \mathbb{R} \quad \text{(linearity)},$$

$$D^\beta\left(f(t)g(t)\right) = f(t)D^\beta g(t) + g(t)D^\beta f(t) \quad \text{(product rule)},$$

$$D^\beta\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)D^\beta f(t) - f(t)D^\beta g(t)}{(g(t))^2} \quad \text{(quotient rule)},$$

$$D^\beta f(t) = (t + \frac{1}{\Gamma(\beta)})^{1-\beta}\frac{df(t)}{dt} \quad \text{(fundamental definition)},$$

$$D^\beta\left(f(g(t))\right) = g'(t)^\beta D^\beta f(g(t)) \quad \text{(fractional chain rule)}.$$

The proposed model will be examined in this study using the improved modified extended tanh function technique (IMETM). This algorithm can help produce a variety of solutions. These solutions include exponential, Weierstrass elliptic, Jacobi elliptic, singular periodic, and brilliant solitons. The novelty of this work lies in the integration of the β-time fractional derivative into the polarization-preserving optical fiber equation governed by Kudryashov's nonlinear refractive index. To our knowledge, this combination has not been previously explored in the literature. By applying the improved modified extended tanh method (IMETM), we obtain new analytical solutions that capture the rich dynamics introduced by fractional temporal operators. Furthermore, the stability analysis of the dark soliton solution under the β-fractional framework reveals novel insights into the asymptotic behavior of perturbations, which practical implications for long-distance optical pulse propagation. While the above studies have advanced the understanding of soliton dynamics in fractional- and integer-order nonlinear systems, several limitations remain. Many works on fractional nonlinear Schrödinger equations focus on standard Kerr-type or cubic-quintic nonlinearities, without incorporating more complex refractive index laws such as Kudryashov's, which can capture higher-order nonlinear effects more accurately. Additionally, the majority of prior research treats either classical integer-order models or applies fractional operators without performing a systematic stability analysis. Moreover, in works where exact solutions are derived, the scope of solution types is often limited, with few studies reporting multiple classes of solutions (e.g., dark soliton, Jacobi elliptic, and singular periodic) for the same model. Finally, the physical interpretation of fractional-order effects on soliton shape, width, and stability is often lacking, making it difficult to connect theoretical results with potential experimental applications. This study is structured in the following way: Section 2 provides a brief overview of the proposed methodology. In Section 3, the method is applied to obtain precise solutions for the model under investigation. Section 4 includes graphical illustrations of some of these solutions to demonstrate the features of the propagating wave. The concluding section wraps up the work.

2. The proposed scheme

This section provides a brief discussion of the improved modified extended tanh method (IMETM), an analytical tool extensively employed to derive exact solutions of nonlinear evolution equations, including fractional and higher-order optical models. IMETM refines the classical tanh method by incorporating auxiliary parameters and transformations, allowing it to handle complex nonlinearities such as those arising in Kudryashov-type refractive index models. Its flexibility and robustness have made it a preferred method in soliton theory, where it successfully generates dark, singular, and Jacobi elliptic solitons under various nonlinear laws [37, 38].

Consider the nonlinear partial differential equation (NLPDE):

$$V(f, D_t^{\alpha} f, f_x, f_{xx}, ...) = 0. \tag{2.1}$$

The following steps need to be completed in order to apply the recommended technique to Eq (2.1).

Step (1): Equation (2.1) is converted to an ordinary differential equation (ODE) by applying the following wave transformation:

$$\begin{aligned} f(x,t) &= M(z)\, e^{i\phi}, \\ z &= hx - \frac{k\left(\frac{1}{\Gamma(\beta)} + t\right)^{\beta}}{\beta}, \\ \phi &= x - \frac{\omega\left(\frac{1}{\Gamma(\beta)} + t\right)^{\beta}}{\beta}. \end{aligned} \tag{2.2}$$

In this case, k denotes the wave's speed. Afterward, Eq (2.1) becomes:

$$H(M, M', M'', M''', M^{(4)}, ...) = 0. \tag{2.3}$$

Step (2): Write the converted ODE's solution as

$$M(z) = \sum_{j=0}^{N} a_j \Upsilon^j(z) + \sum_{j=-1}^{-N} b_{-j} \Upsilon^j(z), \tag{2.4}$$

where the following DE is satisfied by $\Upsilon(z)$:

$$\Upsilon'(z) = \sqrt{d_0 + d_1\Upsilon(z) + d_2\Upsilon^2(z) + d_3\Upsilon^3(z) + d_4\Upsilon^4(z)}. \tag{2.5}$$

Step (3): The balancing rule is used to calculate the parameter N.

Step (4): Eqs (2.5) and (2.4) are substituted into Eq (2.3) to produce a set of nonlinear algebraic equations, after which the coefficients of $\Upsilon^p(z)$ are gathered.

Step (5): The created system in step 4 is solved using Mathematica software packages to yield a_j, b_j, and the parameters h, k, ω, and ν.

Step (6): Finally, by assigning distinct numerical values to the parameters d_0, d_1, d_2, d_3, and d_4, various solutions can be derived

Case 1: $d_1 = d_3 = 0$

$$\Upsilon(z) = \sqrt{\frac{-d_2}{2d_4}} \tanh\left(\sqrt{\frac{-d_2}{2}}\, z\right) \qquad , d_2 < 0,\ d_4 > 0,\ d_0 = \frac{d_2^2}{4d_4}.$$

Case 2: $d_1 = d_3 = 0$

$$\Upsilon(z) = \sqrt{-\frac{d_2 m^2}{d_4(1+m^2)}} \mathrm{sn}(\sqrt{\frac{-d_2}{(1+m^2)}} z), \quad d_2 < 0,\ d_4 > 0,\ d_0 = \frac{d_2^2 m^2}{d_4(1+m^2)^2}.$$

Note: In the previous analytical solution in Case 2, we utilize the *Jacobi elliptic sine function*, denoted as $\mathrm{sn}(z \mid m)$. This function arises in the context of elliptic integrals and generalizes the standard trigonometric sine function. Specifically, it is defined as the inverse of the elliptic integral of the first kind:

$$z = \int_0^{\mathrm{sn}(z|m)} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}},$$

where $m \in (0, 1)$ is the modulus. The Jacobi elliptic function $\mathrm{sn}(z \mid m)$ is periodic in z and is commonly used to describe periodic wave solutions in nonlinear differential equations.

Case 3: $d_0 = d_1 = d_3 = 0$

$$\Upsilon(z) = \sqrt{-\frac{d_2}{d_4}} \sec\left(\sqrt{-d_2} z\right), \quad d_2 < 0,\ d_4 > 0.$$

Step (7): To obtain solutions for Eq (2.1), add the constants a_j, b_{-j} to Eq (2.4) in addition to the described general solutions of Eq (2.5). To elucidate the rationale for selecting the improved modified extended tanh function method (IMETM), it is instructive to compare it with other established analytical techniques, such as Hirota's bilinear method and the inverse scattering transform (IST).

Hirota's method excels in deriving multi-soliton solutions for integrable nonlinear partial differential equations (NLPDEs) by transforming them into bilinear forms, but its applicability is often limited to soliton solutions and integrable systems, which may not fully accommodate the non-local nature of β-fractional derivatives in the fractional NLSE. Similarly, IST provides a rigorous framework for integrable systems, yielding soliton solutions through scattering analysis, yet its computational intensity and complexity make it less practical for non-integrable or fractional systems. In contrast, IMETM offers distinct strengths: its versatility in generating diverse solution types, including solitons, Jacobi elliptic, Weierstrass elliptic, and exponential functions, enables a broader exploration of the fractional dynamics. Additionally, IMETM's straightforward algebraic approach, which reduces the NLPDE to a system of nonlinear equations solvable via software like Mathematica, enhances computational efficiency and accessibility. Its seamless incorporation of fractional derivatives through wave transformation further optimizes its suitability for modeling complex, non-Gaussian wave phenomena, making IMETM a robust and flexible tool for this study.

3. Applying to studied model

Assuming the following, our aim is to obtain solutions for Eq (1.1):

$$u(x,t) = Q(z)e^{-i\phi}, \text{ where } z = hx - \frac{k\left(\frac{1}{\Gamma(\beta)} + t\right)^{\beta}}{\beta}, \text{ and } \phi = x - \frac{\omega\left(\frac{1}{\Gamma(\beta)} + t\right)^{\beta}}{\beta}. \tag{3.1}$$

This transformation maps the original space-time-dependent solution $u(x,t)$ into a profile function $Q(z)$ that evolves in a traveling wave coordinate z, with a time-dependent oscillatory phase ϕ. The parameters involved are interpreted as follows:

- h : A spatial scaling parameter controlling the stretching or compression of the soliton profile in space.
- k : A velocity-related parameter associated with the temporal evolution of the wave in the coordinate frame. It governs the propagation speed in the z direction after fractional transformation.
- β : The order of the beta time-fractional derivative, with $0 < \beta \leq 1$. It encapsulates the memory effect and temporal non-locality of the medium.
- ω : The frequency-related parameter dictating the rate of phase modulation of the wave. It influences the temporal chirp or the oscillatory behavior of the wave envelope.
- ϕ : The nonlinear phase shift, which is both space and time dependent, accounting for dispersion and nonlinear effects encoded in ω.
- $Q(z)$: The wave profile in the transformed frame.

This transformation is essential for reducing the fractional partial differential equation into a more tractable form amenable to exact solution techniques such as the improved modified extended tanh method.

The substitution of Eq (3.1) into Eq (1.1) converts the fractional NLPDE to a complete derivative ODE, yielding the subsequent equation:

The imaginary part is

$$h^3(a - 4b)Q^{(3)}(z) - Q'(z)(3ah - 4bh + k) = 0 \tag{3.2}$$

and if we equate the coefficient to zero, we get the following:

$$a = 4b, \quad k = 4bh - 3ah = -8bh. \tag{3.3}$$

The real part is:

$$\begin{aligned} &3h^2(a-2b)Q''(z) - Q(z)(a-b+\omega) + bh^4 Q^{(4)}(z) \\ &+\frac{c_1}{Q(z)} + c_3 Q(z)^2 + c_4 Q(z)^3 + c_2 = 0, \end{aligned} \tag{3.4}$$

and if we multiply by $Q(z)$ on both sides, we get:

$$\begin{aligned} &3h^2(a-2b)Q(z)Q''(z) - Q(z)(Q(z)(a-b+\omega)) + bh^4 Q(z)Q^{(4)}(z) \\ &+c_2 Q(z) + c_3 Q(z)^3 + c_4 Q(z)^4 + c_1 = 0. \end{aligned} \tag{3.5}$$

To implement the suggested technique, determining the integer N is essential. By utilizing the balance rule in Eq (3.5), and by balancing $Q(z)Q^{(4)}(z)$ with $Q^4(z)$, we ascertain that $N = 2$.

The solution to the resulting ordinary differential equation (ODE) can be expressed as follows:

$$Q(z) = s_0 + s_1 \Upsilon(z) + \frac{s_2}{\Upsilon(z)} + s_3 \Upsilon^2(z) + \frac{s_4}{\Upsilon^2(z)}. \tag{3.6}$$

By substituting in Eq (3.6) along with Eq (2.5) into Eq (3.5), and making the coefficients of $\lambda(z)$ equal to zero, some nonlinear algebraic equations are generated. These resulting equations are solved using Mathematica software packages, which yield the subsequent findings:

Case (1): $d_1 = d_3 = 0, p_0 = \frac{p_2^2}{4p_4}$

$$\begin{aligned} &s_1 = 0, \ s_2 = 0, \ s_4 = 0, \\ &\omega = \frac{b\left(360 d_4^2 h^4 s_0^2 + 34 d_2^2 h^4 s_3^2 - 240 d_2 d_4 h^4 s_0 s_3 + 24 d_2 h^2 s_3^2 - 72 d_4 h^2 s_0 s_3 - 3 s_3^2\right)}{s_3^2}, \\ &c_1 = 0, \ c_2 = \frac{bh^2\left(2 d_4 s_0 - d_2 s_3\right)\left(60 d_4^2 h^2 s_0^2 + 2 d_2^2 h^2 s_3^2 - 30 d_2 d_4 h^2 s_0 s_3 + 3 d_2 s_3^2 - 18 d_4 s_0 s_3\right)}{d_4 s_3^2}, \\ &d_2 = \frac{-18\sqrt{-bc_4} + 3\sqrt{30} c_4 s_0 + \sqrt{30} c_3}{60 h^2 \sqrt{-bc_4}}, \ s_3 = \frac{2\sqrt{-30b} d_4 h^2}{\sqrt{c_4}}. \end{aligned}$$

Then a dark soliton solitary solution is obtained:

$$u(x,t) = \frac{\left(-3\sqrt{c_4}\left(5 s_0 \sqrt{-bc_4} + \sqrt{30}\, b\right) - 5\sqrt{-b} c_3\right)\tanh^2\left(\frac{\sqrt{\frac{3}{5} - \frac{3c_4 s_0 + c_3}{\sqrt{-30 b c_4}}}\left(8b\left(\frac{1}{\Gamma(\beta)} + t\right)^{\beta} + \beta x\right)}{2\beta}\right)}{10\sqrt{-b} c_4} + s_0. \tag{3.7}$$

Case (2): $d_1 = d_3 = 0, d_0 = \frac{d_2^2 m^2}{d_4(1+m^2)^2},\ 0 < m \le 1$

$$
\begin{aligned}
M &= \frac{m^2}{(m^2+1)^2},\ s_1 = 0,\ s_2 = 0,\ s_4 = 0,\\
d_2 &= -\frac{3}{10h^2},\ c_1 = 0,\\
c_2 &= \frac{6\left(2500bd_4^3h^6s_0^3 + 135bd_4h^2Ms_0s_3^2 - 120bd_4h^2s_0s_3^2 - 18bMs_3^3\right)}{125d_4h^2s_3^2},\\
c_3 &= \frac{360bd_4^2h^4s_0}{s_3^2},\\
d_4 &= \sqrt{\frac{c_4s_3^2}{120bh^4}},\ \omega = \frac{3\left(3000bd_4^2h^4s_0^2 + 54bMs_3^2 - 73bs_3^2\right)}{25s_3^2}.
\end{aligned}
\tag{3.8}
$$

then a Jacobi periodic solution is obtained:

$$
u(x,t) = \frac{3\sqrt{\frac{6}{5}}m^2\operatorname{sn}\left(\sqrt{\frac{3}{10}}\sqrt{\frac{1}{m^2+1}}\left(\frac{8b\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta}{\beta} + x\right)\middle| m\right)^2}{(m^2+1)\sqrt{\frac{c_4}{b}}} + s_0. \tag{3.9}
$$

Case 3: $d_0 = d_1 = d_3 = 0,\ d_2 < 0,\ d_4 > 0$

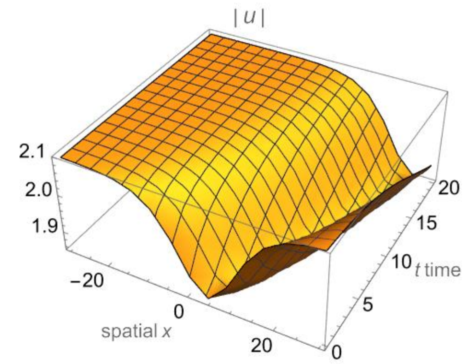
$$
\begin{aligned}
&s_1 = 0,\ s_2 = 0,\ s_4 = 0,\\
&d_2 = \frac{\sqrt{30}\sqrt{-9bc_4^3h^4s_0^2 - 6bc_3c_4^2h^4s_0 - bc_3^2c_4h^4} - 18bc_4h^2}{60bc_4h^4},\\
&d_4 = \frac{s_3\sqrt{-bc_4h^4(3c_4s_0+c_3)^2}}{2\sqrt{30}(3bc_4h^4s_0 + bc_3h^4)},\\
&c_2 = \frac{\frac{18\sqrt{30}s_0\sqrt{-bc_4h^4(3c_4s_0+c_3)^2}}{h^2} - 432bc_4s_0 + 60c_4^2s_0^3 + 15c_3c_4s_0^2 - 10c_3^2s_0}{75c_4},\\
&\omega = \frac{\frac{18\sqrt{30}\sqrt{-bc_4h^4(3c_4s_0+c_3)^2}}{h^2} - 657bc_4 + 90c_4c_3s_0 + 135c_4^2s_0^2 - 10c_3^2}{75c_4},\ c_1 = 0.
\end{aligned}
\tag{3.10}
$$

Then a singular periodic solution is obtained:

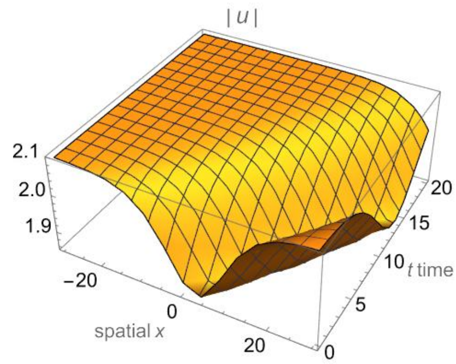
$$
u(x,t) = \frac{\left(-3\sqrt{30}\sqrt{-bc_4} - 5(3c_4s_0 + c_3)\right)\sec^2\left(\frac{\sqrt{\frac{\sqrt{\frac{2}{15}}(3c_4s_0+c_3)}{\sqrt{-bc_4}} + \frac{6}{5}}\left(8b\left(\frac{1}{\Gamma(\beta)}+t\right)^\beta + \beta x\right)}{2\beta}\right)}{5c_4} + s_0. \tag{3.11}
$$

4. Graphical visualization of some solutions

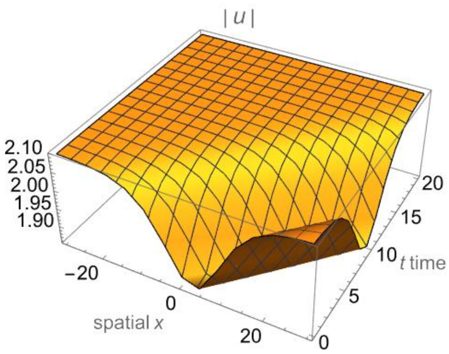
Figure 1 presents a graphical simulation of the obtained results to showcase the features of selected solutions, specifically showcasing the dark soliton identified by Eq (3.7). The depicted parameters for this solution include $b = -0.15, s_0 = 1.915, c_3 = 0.0064, c_4 = 0.0322, t = 6.5$ *seconds*. Figure 1 illustrates the dark soliton solutions of Eq (14) for various fractional orders β. The dark soliton represents a localized dip in the continuous-wave background intensity. As β decreases from 1 to lower fractional values, the soliton's width increases and its depth decreases, indicating that memory effects in the fractional model cause the wave to spread more in space and reduce its contrast. This behavior is consistent with the dispersive nature of fractional temporal operators, which tend to weaken strong localization over long propagation distances.



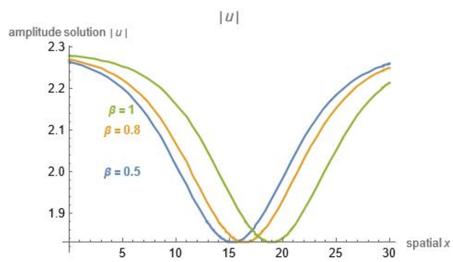
(a) $\alpha = 0.55$



(b) $\alpha = 0.7$



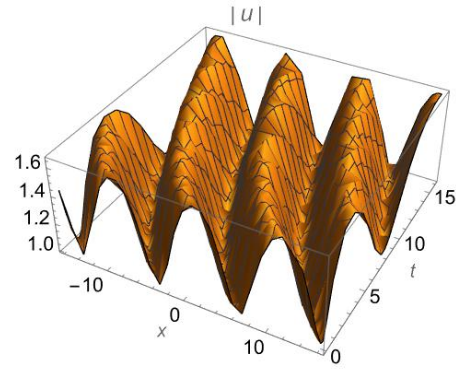
(c) $\alpha = 1$



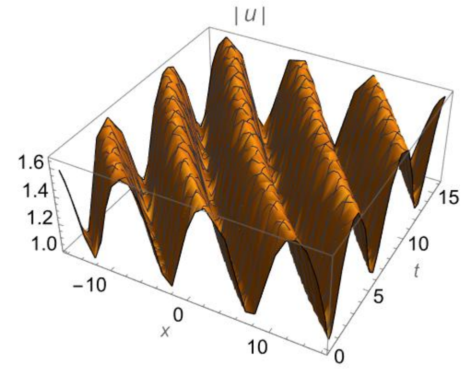
(d)

Figure 1. Graphical visualization of the dark soliton solution given by Eq (14) for different values of the fractional order β. The spatial domain is $x \in [-30, 30]$, and the wave amplitude $|u(x, t)|$ is plotted at fixed time $t = 6.5$. The parameters used are: $b = -0.15$, $s_0 = 1.915$, $c_3 = 0.0064$, $c_4 = 0.0322$. The soliton retains its shape as β varies from 0.55 to 1.0.

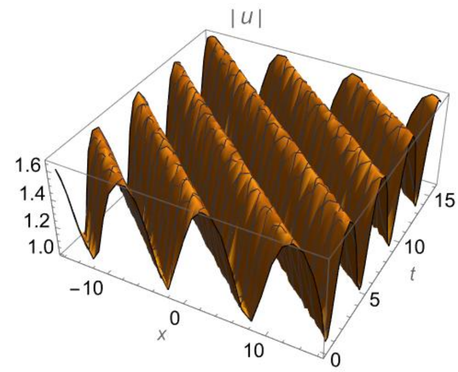
Figure 2 displays a graph of the periodic solution to Eq (3.9) with $b = 0.295$, $s_0 = 1$, $m = 0.762$, $c_4 = 0.985$, $t = 4.22$ seconds. Figure 2 presents periodic solutions expressed in terms of the Jacobi elliptic sine function $sn(z|m)$. Physically, these correspond to a periodic train of pulses in the optical fiber. Varying β modifies the wavelength and amplitude of the periodic pattern: lower β values lead to broader periods and reduced amplitude, again reflecting enhanced dispersive spreading due to memory effects. In practical terms, adjusting β could model changes in the effective dispersion profile of the fiber medium.



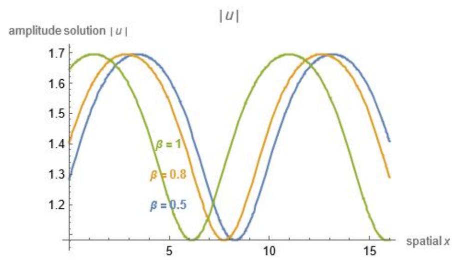
(a) $\alpha = 0.5$



(b) $\alpha = 0.8$



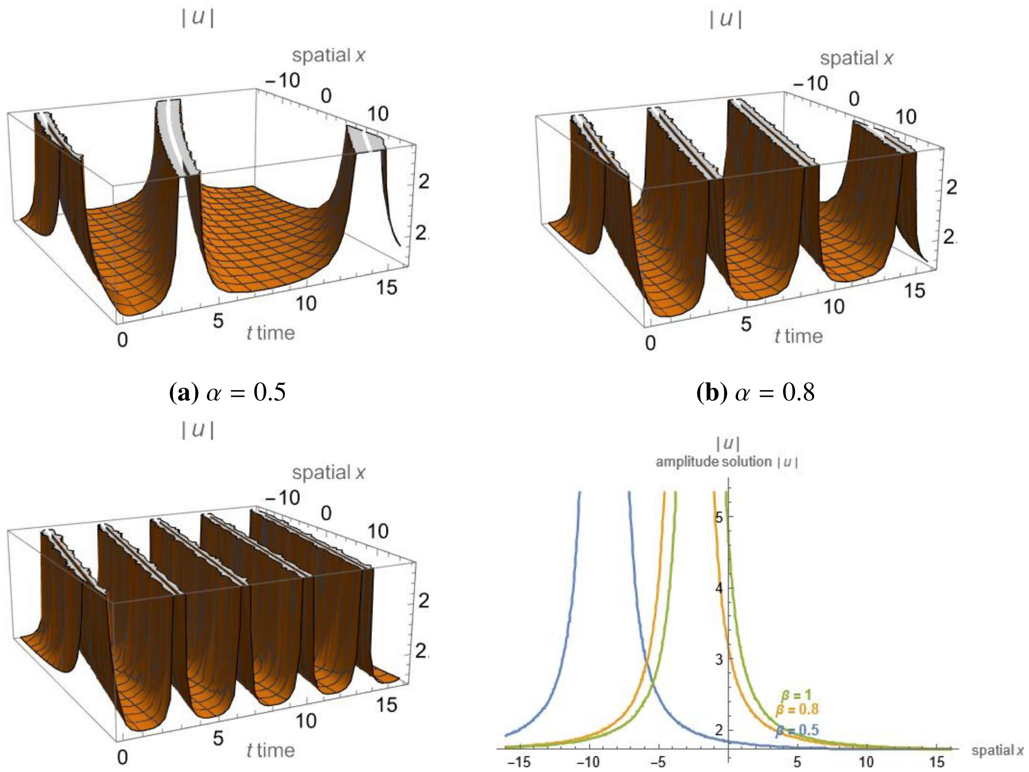
(c) $\alpha = 1$



(d)

Figure 2. Graphical simulation of the Jacobi elliptic periodic solution from Eq (16) for various values of β. The spatial domain is $x \in [-30, 30]$, and the solution $|u(x,t)|$ is plotted at fixed time $t = 4.22$. Parameter values used: $b = 0.295$, $s_0 = 1$, modulus $m = 0.762$, $c_4 = 0.985$. The figures illustrate the influence of fractional order on the periodic waveform.

Figure 3 displays a graph of singular periodic solution to Eq (3.11) with $b = 1.67, s_0 = 1.67, c_3 = 2.67, c_4 = -1.46, t = 0.9$ *seconds*. These solutions describe waveforms that maintain shape and velocity over long propagation distances due to a balance between dispersion and nonlinearity. The interaction between dispersion and nonlinearity is pivotal in the creation and conduct of solitons. While dispersion causes a pulse to widen over time, counteracting this effect is nonlinearity's ability to produce self-focus, allowing the soliton to maintain its condensed structure. A precise balance between these two factors allows for distortion-free propagation without any significant deterioration occurring within the soliton waveform itself.



(a) $\alpha = 0.5$

(b) $\alpha = 0.8$

(c) $\alpha = 1$

(d)

Figure 3. Graphical simulation of the singular periodic solution from Eq (3.11) for various values of β. The spatial domain is $x \in [-15, 15]$, and the solution $|u(x,t)|$ is plotted at fixed time $t = 0.9$. Parameter values used: $b = 1.67$, $s_0 = 1.67$, $c_4 = -1.46$, $c_3 = 2.26$. The figures illustrate the influence of fractional order on the periodic waveform.

5. Stability analysis of the dark soliton

To evaluate the dynamical stability of the derived dark soliton solution, we introduce a small complex-valued perturbation into the base solution as follows:

$$u(x,t) = \left[u_0(x,t) + \epsilon\,\eta(x,t)\right] e^{-i\phi(x,t)},$$

where $u_0(x,t)$ is the exact dark soliton given by Eq (3.7), and $\eta(x,t)$ is a small complex-valued perturbation.

By substituting Eq (5) in governing model equation (1.1), the resulting linearized perturbation equation becomes:

$$\left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{\partial \eta}{\partial t} + ia\,\eta_{xxx} + b\,\eta_{xxxx} + L[\eta] = 0,$$

where $L[\eta]$ denotes the linearized action of the nonlinear refractive index around the background soliton.

Assuming a perturbation of the form $\eta(x,t) = e^{ikx+\lambda(t)t}$, the dispersion relation is derived as:

$$\lambda(t) = \frac{-iak^3 - bk^4 - \mathcal{N}(k)}{\left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}}.$$

Since the denominator grows with time for $\beta < 1$, it follows that:

$$\lim_{t\to\infty} \lambda(t) = 0,$$

implying that any initial instability introduced by positive $\Re[\lambda(t)]$ will eventually decay. This result suggests that any initial perturbation will decay over time, establishing the asymptotic stability of the dark soliton solution under the influence of the β-time fractional derivative. Stability is particularly enhanced in regimes with strong nonlinear saturation (i.e., large c_4) and moderate dispersion. To analyze the pulse evolution, we apply the wave transformation $u(x,t) = Q(z)e^{-i\frac{\partial}{\partial t}}$ with $z = hx - \frac{k\left(\frac{1}{\Gamma(\beta)}+t\right)^{\beta}}{\beta}$. Setting $z = 0$ simplifies the transformation, allowing us to explore the relationship between the spatial coordinate x and time t under varying β values. This approach reveals the influence of the fractional derivative on the pulse peak position. The resulting plot in figure 4, generated for $\beta = 0.5, 0.6, 0.8, 1.0$, illustrates how the fractional order modulates the propagation characteristics. Figure 4 shows the evolution of the pulse peak position over time for different β values. A linear trend indicates constant-velocity propagation, while deviations from linearity suggest changes in propagation speed due to fractional dispersion effects. Lower β values produce slightly slower propagation, implying stronger temporal dispersion and memory effects in the fractional regime.

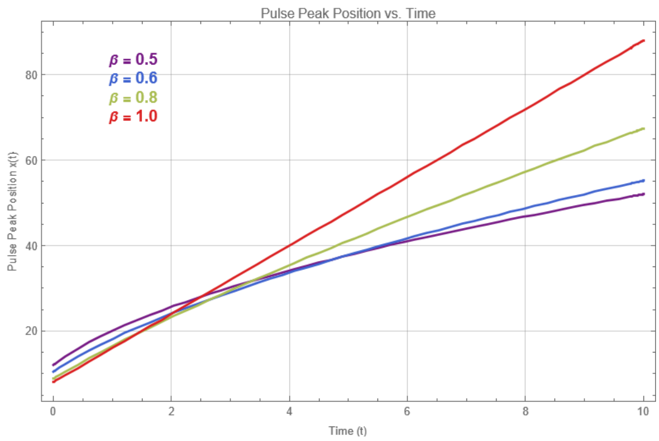


Figure 4. Pulse peak position versus time for different β values ($\beta = 0.5, 0.6, 0.8, 1.0$). The plots demonstrate the effect of the β-time fractional derivative on the spatial-temporal evolution of the pulse, with higher β values leading to increased peak positions over time.

The derivative of the pulse peak position with respect to time, representing the peak velocity, provides further insight into the stability and dynamics influenced by β. Plots of this velocity for $\beta = 0.5, 0.6, 0.8, 1.0$ show a decay in velocity over time, with the fractional order affecting the rate of decay and the initial velocity magnitude is presented in Figure 5. Figure 5 depicts the variation of pulse peak velocity over time. For $\beta = 1$ (classical case), the velocity remains nearly constant, characteristic of stable soliton motion. For fractional β values, the velocity exhibits a gradual decay, signifying that fractional memory effects cause solitons to lose speed over long propagation distances. This highlights the potential of β as a tunable parameter for controlling pulse dynamics in optical fiber systems.

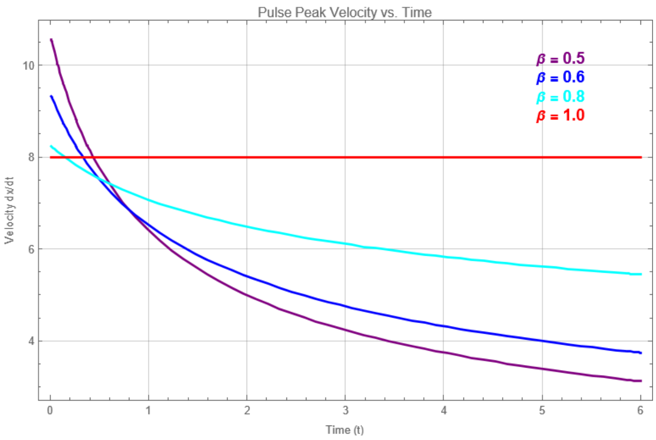


Figure 5. Pulse peak velocity versus time for different β values ($\beta = 0.5, 0.6, 0.8, 1.0$). The plots indicate a decay in velocity, with lower β values exhibiting a more pronounced initial drop, reflecting the impact of fractional derivatives on soliton stability.

6. Conclusions

In this study, we investigated the nonlinear dynamics of polarization-preserving optical fibers by introducing a β-time fractional derivative into a generalized Kudryashov-type nonlinear refractive index model. By employing the improved modified extended tanh method (IMETM), we successfully derived exact analytical solutions–including dark soliton, Jacobi elliptic, and singular periodic waveforms.

A central contribution of this work is the demonstration of how the fractional-order parameter β governs key soliton characteristics such as amplitude, width, and stability. Analytical solutions and graphical simulations reveal that memory effects introduced by the fractional operator enrich the dynamical behavior of optical solitons and enhance their controllability.

Furthermore, this study confirms the effectiveness of IMETM as a robust analytical technique for fractional nonlinear systems. Its simplicity and versatility offer significant advantages over traditional approaches like Hirota's bilinear method or the inverse scattering transform, particularly for non-integrable and memory-dependent models. From a practical perspective, these findings have direct implications for optical fiber communication systems, where stable, shape-preserving pulse propagation is essential. The results may also support the development of advanced nonlinear photonic devices–such as modulators, waveguides, and metamaterials–with tailored dispersion and nonlinear properties.

Looking forward, this framework can be extended to incorporate spatio-temporal perturbations, higher-dimensional fractional systems, and nonlocal nonlinearities. Additionally, numerical simulations and variational approaches may further augment the understanding of solution stability under realistic perturbations.

This work, therefore, not only reinforces the theoretical underpinnings of fractional soliton theory but also opens promising pathways for applied research in fiber optics, materials science, and complex systems modeling.

Author contributions

Rawan Bossly: Formal analysis, Software, Writing-original draft; Noorah Mshary: Resources, Investigation, Writing-original draft; Hamdy M. Ahmed: Conceptualization, Methodology, Writing-review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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