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Research article

Inference on stress-strength reliability from censored data using the asymmetric generalized Poisson Lomax model

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Abstract: This paper investigated the estimation of the stress-strength reliability parameter $R = P(X > Y)$, where the random variables X (strength) and Y (stress) were independently modeled by the generalized Poisson Lomax distribution (GPLD). The analysis was conducted under a progressive Type II censoring scheme, which provided greater flexibility in practical life-testing experiments by allowing for staggered removal of surviving units. Assuming shared scale and shape parameters, both maximum likelihood estimators (MLEs) and Bayesian estimators of R were derived. Due to the absence of closed-form solutions, numerical optimization techniques were applied for MLEs, while Bayesian estimates were obtained under squared error and LINEX loss functions using Markov chain Monte Carlo methods with importance sampling. Asymptotic confidence intervals and highest posterior density credible intervals were constructed to assess uncertainty. A comprehensive simulation study was performed to evaluate the efficiency and robustness of the proposed estimators under varying sample sizes and censoring schemes. Furthermore, two real data applications were analyzed that show survival times of melanoma patients and failure times of high-voltage insulating fluids to illustrate the practical utility of the methodology. The results demonstrated that Bayesian approaches, particularly under asymmetric loss functions, yielded superior performance compared to their frequentist counterparts.

Keywords: statistical model; generalized Poisson Lomax distribution; stress-strength reliability

model; Bayes theorem; highest posterior density; computer simulation; data analysis
Mathematics Subject Classification: 62F10, 62F12

1. Introduction

Table of notations:

X	Random variable representing strength
Y	Random variable representing stress
R	Stress-strength reliability parameter, defined as $R = P(X > Y)$
GPLD	Generalized Poisson Lomax distribution
$f(x; \alpha, \beta, \theta)$	Probability density function (PDF) of the GPLD
$F(x; \alpha, \beta, \theta)$	Cumulative distribution function (CDF) of the GPLD
α_1, α_2	Shape parameters for X and Y, respectively
β	Common scale parameter of the GPLD
θ	Common second shape parameter of the GPLD
n_1, n_2	Sample sizes for strength and stress data, respectively
m_1, m_2	Number of observed failures in strength and stress samples
PT2C	progressive Type-II censoring
$S = (S_1, S_2, ..., S_{m_1})$	Censoring scheme for strength sample under PT2C
$T = (T_1, T_2, ..., T_{m_2})$	Censoring scheme for stress sample under PT2C
$\hat{R}_{ML}$	Maximum likelihood estimator of R
$\hat{R}_{SE}$	Bayesian estimator of R under squared error loss function
$\hat{R}_{LINEX}$	Bayesian estimator of R under LINEX loss function
$\hat{\sigma}^2_{\hat{R}}$	Estimated variance of $\hat{R}$
ACIs	Asymptotic confidence intervals
CRIs	Credible intervals (Bayesian)
$h(\alpha_1, \alpha_2 \mid \underline{x}, \underline{y})$	Importance sampling weight function used in MCMC estimation
$G(.; a, b)$	Gamma distribution with shape parameter a and rate parameter b
M	Burn-in period in MCMC algorithm
N	Total number of MCMC samples
MSE	Mean squared error of the estimator
CP	Coverage probability of the confidence/credible interval
AW	Average width of the confidence/credible interval

In recent years, reliability analysis of mechanical systems has advanced beyond static models to incorporate dynamic and uncertain operational conditions. For example, dynamic Bayesian networks have been employed in the reliability analysis of manufacturing systems, such as drive shafts, enabling real-time probabilistic modeling of failure processes under evolving operational states. Furthermore, mixed uncertainty analysis methods, accounting for both aleatory and epistemic uncertainties have been proposed for the dynamic reliability evaluation of mechanical structures, especially those considering residual strength degradation over time. In parallel, improved fatigue failure models have been developed to capture the impact of variable stress spectra on the reliability of mechanical parts. These models provide more realistic representations of material fatigue under fluctuating loading conditions. While these frameworks focus on time-varying or stress-dependent mechanisms, our study complements this line of work by investigating stress-strength reliability in the presence of censoring, using the flexible generalized Poisson Lomax distribution. The proposed approach allows for robust modeling of skewed or heavy-tailed lifetime data, which often arises in practical reliability applications. Stress-strength reliability (SSR) models play a crucial role in reliability engineering, risk assessment, and survival analysis. These models quantify the probability that a system or component will successfully operate under an applied stress, defined mathematically as $R = P(X > Y)$, where X denotes the random strength of the system and Y represents the random applied stress. The system is considered to fail if the stress exceeds the strength, making the estimation of R a central concern in various fields, including mechanical engineering, biostatistics, and industrial quality control. Many authors have shown interest in investigating the simple SSR model's applicability in recent years since it is easier to understand theoretically while also being more practical to use in practice. Numerous researchers have produced and thoroughly explored various parametric estimations of SSR for the complete sample in the literature e.g., [1–7]. For various probability distributions, the estimation of the SSR parameter using record samples has also been studied, see, for instance, [8–12].

In life testing and clinical trial investigations, the censoring phenomena is quite helpful. To illustrate the diverse objectives associated with life-testing trials, numerous censoring schemes have been employed. One of these offers' flexibility in unit removals during experiments, and that is the progressive censoring scheme. It can be described as follows, imagine that n units are tested, but only m of the failures are fully visible. R_1^* of the surviving units are randomly chosen and deleted from the remaining $(n - 1)$ units at the time of the first failure $X_{1:m:n}$. Similar to the first case, R_2^* of the surviving units are chosen at random and eliminated from the remaining $\left(n - 2 - R_1^*\right)$ units when a second failure, $X_{2:m:n}$, is noticed. Finally, all of the R_m^* remaining surviving units are removed from the experiment at the m^{th} stage, which is the stage at which $X_{m:m:n}$ is observed. This process of getting the censored sample of size m is known as the progressive Type-2 censored (PT2C) sample with censoring scheme $\left(R_1^*, R_2^*, ..., R_m^*\right)$. The reader may refer to Balakrishnan and Sandhu [13] for more information. Several statistical scholars evaluated several estimated studies of SSR using censored samples from various models, as follows:

Early work by Saraçoğlu et al. [14] investigated R for the exponential distribution when data is subject to progressive Type-II censoring. They derived maximum likelihood estimators (MLEs) and studied their small-sample properties via simulation, demonstrating that progressive removal of surviving units can substantially reduce experimental time without unduly sacrificing estimator accuracy. Subsequent studies extended the focus beyond the exponential model to more flexible lifetime distributions under related censoring schemes. Kumar et al. [15] considered the Lindley distribution and utilized a progressively first-failure censoring rule to obtain both MLEs and unbiased moment estimators for SRR, showing improved fit for skewed data over the exponential case. Krishna et al. [16] further generalized this approach to the generalized inverted exponential distribution, deriving closed-form expressions for the likelihood and demonstrating through simulation that the new estimators maintain desirable bias and mean-square error properties even with heavy censoring.

To better accommodate multimodal or bathtub-shaped failure rates, researchers have also explored non-exponential families under progressive censoring. Saini et al. [17] developed estimators for the stress-strength relationship in the generalized Maxwell failure distribution under a progressively first-failure scheme, finding that asymptotic confidence intervals remained reliable for moderate sample sizes. De la Cruz et al. [18] examined the unit-half-normal distribution commonly used to model lifetimes with a sharp initial failure phase and provided both frequentist and Bayesian estimates of R under Type-II censoring. Meanwhile, Cetinkaya [19] addressed the Weibull case with non-identical component strengths, deriving joint MLEs under a more general censoring scheme and highlighting the sensitivity of reliability estimates to the shape parameter in early-removal designs.

More recently, Sabry et al. [20] performed an extensive Monte Carlo study on an extended exponential family under progressive censoring, illustrating that while point estimators remain robust, interval estimates can suffer under extreme removal rates. Yu et al. [21] shifted focus to record-based sampling by studying the unit-Burr III distribution, where only record lifetimes are observed; they demonstrated that record data yields competitive estimates of R when full samples are costly or impractical to collect.

In parallel with these frequentist developments, several authors have advocated Bayesian frameworks for stress-strength inference. EL-Sagheer and Mansour [22] applied Bayesian methods to compare efficacy of competing medical treatments via a stress-strength model, highlighting how prior information can stabilize estimates under heavy censoring. EL-Sagheer et al. [23] extended this Bayesian approach to partially accelerated life tests, showing that incorporating informative priors for strength parameters leads to narrower credible intervals without appreciable bias.

Despite this rich literature, relatively few works have considered highly asymmetric or heavy-tailed distributions under progressive censoring. Al-Zahrani and Sagor [24] introduced the Poisson Lomax distribution as a flexible three-parameter lifetime model combining Poisson count structure with Lomax tails, suggesting its promise for extreme-value applications but not yet exploring stress-strength settings. In what follows, we build on these earlier findings by developing both frequentist and Bayesian inference for R when X and Y follow the generalized Poisson Lomax distribution (GPLD) under PT2C. Unlike many traditional stress-strength models that rely on classical distributions such as the Lomax, inverse Weibull, or exponential, the proposed framework employs the GPLD, which offers enhanced flexibility due to its additional parameterization. The GPLD not only encompasses the Lomax distribution as a special case (when $\beta = 0$) but also provides a better fit for highly skewed and heavy-tailed data often encountered in reliability and survival analysis. Moreover, the closed-form expression for the stress-strength reliability parameter $R = P(X > Y)$ under the GPLD facilitates tractable inference, especially under PT2C schemes. This comparative advantage makes the GPLD a strong candidate for modeling diverse real-world failure mechanisms. The current study fills a gap in the literature by systematically investigating both frequentist and Bayesian inference for R under this flexible distribution, supported by simulations and real data applications.

A three-parameter lifetime distribution with an upside-down bathtub-shaped failure rate was presented by Al-Zahrani and Sagor [24]. The Poisson which was zero-truncated, and the Lomax distributions were combined to create the distribution. The density function, the hazard rate function's shape, a general moment expansion, and the mean and median deviations of the Poisson-Lomax distribution are all deduced and thoroughly examined. An important lifetime distribution that is widely utilized in reliability studies is the GPLD. It can occasionally serve as a good substitute for the well-known family of distributions, such as Weibull, exponential, inverse Weibull, inverse Lomax distribution, etc. The GPLD was first used by Abu-Youssef et al. [25], who investigated its uses with censored data. In this work, the stress-strength study related to any system with a progressive censoring scheme is applied to GPLD. In the case of a GPLD with scale parameter β, and shape parameters α and θ, the probability density function (PDF) and cumulative distribution function (CDF) are as follows

$$f(x;\alpha,\beta,\theta) = \beta\theta(1+\beta x)^{-(\theta+1)}\left(1+\alpha(1+\beta x)^{-\theta}\right) \times \exp\left\{-\alpha\left(1-(1+\beta x)^{-\theta}\right)\right\},\ x>0;\alpha,\beta,\theta>0, \tag{1}$$

and

$$F(x;\alpha,\beta,\theta) = 1-(1+\beta x)^{-\theta}\exp\left\{-\alpha\left(1-(1+\beta x)^{-\theta}\right)\right\},\ x>0;\alpha,\beta,\theta>0, \tag{2}$$

respectively. It is noticeable that when $\alpha = 0$, the GPLD reduce to the Lomax distribution. The PDF of the GPLD is decreasing in x with modal value $\beta\theta(1+\alpha)$ at $x = 0$. Also, we have $\lim_{x\to 0} f(x;\alpha,\beta,\theta) = \beta\theta(1+\alpha)$ and $\lim_{x\to\infty} f(x;\alpha,\beta,\theta) = 0$. Let X and Y represent the two independent strength-stress random variables that were detected from $GPLD(\alpha_1,\beta,\theta)$ and $GPLD(\alpha_2,\beta,\theta)$, respectively, where the parameters β and θ are common and known, i.e., $X \sim GPLD(\alpha_1,\beta,\theta)$ and $Y \sim GPLD(\alpha_2,\beta,\theta)$. Consequently, the SSR parameter R can be formulated as

$$\begin{aligned} R = P(X>Y) &= \int_0^\infty P(X>Y|Y=y)\, f_2(y,\alpha_2,\beta,\theta)\,dy, \\ &= \int_0^\infty \overline{F_1}(y,\alpha_1,\beta,\theta)\, f_2(y,\alpha_2,\beta,\theta)\,dy \\ &= \frac{\alpha_2 + (\alpha_1+\alpha_2-1)\left[\alpha_2^2+\alpha_1(\alpha_2+1)\right] + (\alpha_1-\alpha_2)\exp\{-(\alpha_1+\alpha_2)\}}{(\alpha_1+\alpha_2)^3} \\ &= Q(\alpha_1,\alpha_2). \end{aligned} \tag{3}$$

As a result, the SSR is only dependent on the model shape parameters α_1 and α_2.

In this paper, we investigate the estimation of the SSR parameter R under the assumption that both strength X and stress Y follow independent GPLD distributions with common scale and shape parameters. Both frequentist and Bayesian approaches are developed to handle the analytical challenges imposed by the complexity of the model and the censoring structure. MLEs are derived numerically, and asymptotic confidence intervals are constructed using the observed Fisher information matrix. For the Bayesian analysis, independent gamma priors are employed for the unknown parameters, and estimation is performed under both symmetric (squared error) and asymmetric linear exponential (LINEX) loss functions using Markov chain Monte Carlo (MCMC) methods, particularly importance sampling. A comprehensive Monte Carlo simulation study is conducted to assess the performance of the proposed estimators under various sample sizes and censoring schemes. Finally, the practical relevance of the model is demonstrated through applications to real-world datasets, including survival times of melanoma patients and breakdown times of high-voltage insulating fluids.

The remainder of the essay is structured as follows: The maximum likelihood estimates and asymptotic confidence intervals of R, are covered in Section 2. Section 3 has examined estimate methods in a Bayesian framework. A simulation study is carried out to compare the suggested techniques in Section 4. A real data example is provided in Section 5 to show how the suggested inference processes can be used. In Section 6, a conclusion is provided.

2. The MLE of R

A well-liked technique for parameter estimation in statistical models is called MLE. It is a method for finding a statistical model's unknown parameters using sample data. The foundation of MLE is the idea that the most likely value of an unknown parameter is the set of parameter values that maximizes the likelihood of obtaining the observed data. In many scientific domains, MLE is a well-known and often used method since it offers a simple way to estimate the properties of a population given a sample. Additionally, it has the benefit of being simple to apply computationally. Model fitting, hypothesis testing, and regression analysis are all frequent uses for the MLE technique. The parameters of interest are calculated from the sample data in MLE by locating the parameters' highest likelihood estimates. The likelihood function, which represents the likelihood of the observed data given the parameters, is set up in order to do this, and the likelihood function is then maximized with regard to the model parameters.

2.1. Point estimation of MLE

Point estimation using the MLE method involves determining the parameter values that maximize the likelihood function based on observed data. In other words, MLE seeks the values of the model parameters under which the observed sample is most probable. It is a widely used method in statistical inference due to its desirable properties, such as consistency, asymptotic normality, and efficiency under regular conditions. The procedure involves constructing the likelihood function from the probability distribution of the data and then finding the parameter estimates that maximize this function, often by taking the natural logarithm (log-likelihood) to simplify calculations. MLE is applicable in a wide range of statistical models and plays a crucial role in both theoretical developments and practical applications. Let's assume that the censoring schemes (X, S, m_1, n_1) and (Y, T, m_2, n_2) are the two that were taken into consideration i.e.,

$$(X, S, m_1, n_1) \Rightarrow (X_{1:m_1:n_1}, S_1), (X_{2:m_1:n_1}, S_2), ..., (X_{m_1:m_1:n_1}, S_{m_1}),$$

and

$$(Y, T, m_2, n_2) \Rightarrow (Y_{1:m_2:n_2}, T_1), (Y_{2:m_2:n_2}, T_2), ..., (Y_{m_2:m_2:n_2}, T_{m_2}).$$

where

$$S = (S_1, S_2, ..., S_{m_1}), \quad T = (T_1, T_2, ..., T_{m_2}).$$

Hence, the likelihood function of observed samples in the SSR model is written as

$$L(\alpha_1, \alpha_2, \beta, \theta | \underline{x}, \underline{y}) = \prod_{j=1}^{2} \left[C_j \prod_{i=1}^{m_j} f_j \left(z_{ji}; \alpha_j, \beta, \theta \right) \left[1 - F_j \left(z_{ji}; \alpha_j, \beta, \theta \right) \right]^{H_{ji}} \right], \tag{4}$$

where $z = (z_1, z_2) = (x, y)$, $z_j = \left(z_{j1}, z_{j2}, ..., z_{jm_j} \right)$, $j = 1, 2$, and

$$C_j = n_j \prod_{i=1}^{m_{j-1}} \left[n_j - \sum_{k=1}^{i} H_{jk} \right], \; H_j = \left(H_{j1}, H_{j2}, ..., H_{jm_j} \right), \; (H_1, H_2) = (S, T),$$

and equally

$$L(\alpha_1,\alpha_2,\beta,\theta) \propto \beta^{m_1+m_2}\theta^{m_1+m_2} \exp\left\{-(\theta+1)\sum_{j=1}^{2}\sum_{i=1}^{m_j}\log\left(1+\beta z_{ji}\right)\right\}$$
$$\exp\left\{\sum_{j=1}^{2}\sum_{i=1}^{m_j}\log\left(1+\alpha_j\left(1+\beta z_{ji}\right)^{-\theta}\right)-\sum_{j=1}^{2}\sum_{i=1}^{m_j}\alpha_j\left(1-\left(1+\beta z_{ji}\right)^{-\theta}\right)\right\}$$
$$\exp\left\{-\theta\sum_{j=1}^{2}\sum_{i=1}^{m_j}H_{ji}\log\left(1+\beta z_{ji}\right)-\sum_{j=1}^{2}\sum_{i=1}^{m_j}\alpha_j H_{ji}\left(1-\left(1+\beta z_{ji}\right)^{-\theta}\right)\right\}. \tag{5}$$

For known β and θ the log-likelihood function $\ell(\alpha_1,\alpha_2,\beta_0,\theta_0)$, without constant additive terms, is expressed as

$$\ell(\alpha_1,\alpha_2,\beta_0,\theta_0) \propto (m_1+m_2)\log\beta_0\theta_0-(\theta_0+1)\sum_{j=1}^{2}\sum_{i=1}^{m_j}\log\left(1+\beta_0 z_{ji}\right)$$
$$+\sum_{j=1}^{2}\sum_{i=1}^{m_j}\log\left(1+\alpha_j\left(1+\beta_0 z_{ji}\right)^{-\theta_0}\right)-\sum_{j=1}^{2}\sum_{i=1}^{m_j}\alpha_j\left(1-\left(1+\beta_0 z_{ji}\right)^{-\theta_0}\right)$$
$$-\theta_0\sum_{j=1}^{2}\sum_{i=1}^{m_j}H_{ji}\log\left(1+\beta_0 z_{ji}\right)-\sum_{j=1}^{2}\sum_{i=1}^{m_j}\alpha_j H_{ji}\left(1-\left(1+\beta_0 z_{ji}\right)^{-\theta_0}\right). \tag{6}$$

By differentiating the aforementioned expression (6) with respect to the parameters (α_1,α_2) and equating it to zero, the MLEs of the parameters α_1 and α_2 are determined. The following likelihood formulas are available:

$$\frac{\partial\ell(\alpha_1,\alpha_2,\beta_0,\theta_0)}{\partial\alpha_1} = \sum_{i=1}^{m_1}\frac{(1+\beta_0 x_i)^{-\theta_0}}{\left(1+\alpha_1(1+\beta_0 x_i)^{-\theta_0}\right)}-\sum_{i=1}^{m_1}\left(1-(1+\beta_0 x_i)^{-\theta_0}\right)$$
$$-\sum_{i=1}^{m_1}S_i\left(1-(1+\beta_0 x_i)^{-\theta_0}\right), \tag{7}$$

and

$$\frac{\partial\ell(\alpha_1,\alpha_2,\beta_0,\theta_0)}{\partial\alpha_2} = \sum_{i=1}^{m_2}\frac{(1+\beta_0 y_i)^{-\theta_0}}{\left(1+\alpha_2(1+\beta_0 y_i)^{-\theta_0}\right)}-\sum_{i=1}^{m_2}\left(1-(1+\beta_0 y_i)^{-\theta_0}\right)$$
$$-\sum_{i=1}^{m_2}T_i\left(1-(1+\beta_0 y_i)^{-\theta_0}\right). \tag{8}$$

Equations (7) and (8) are simultaneously solved to produce the MLEs of the parameters. Unfortunately, analytical solutions to the aforementioned problems are not possible. Therefore, MLEs can be evaluated using any numerical iterative method. Here, we obtained MLEs using a numerical approach, such as the Newton-Raphson (NR) iteration algorithm, to estimate the unknown parameters α_1, and α_2. Two techniques were employed to provide preliminary estimates: (i) method-of-moments estimators based on entire or uncensored data, and (ii) selecting values with a high log-likelihood using a coarse grid search. The proposed algorithm is described as follows:

(1) Start with initial parameter values $\alpha^{(0)} = \left(\alpha_1^{(0)}, \alpha_2^{(0)}\right)$ for $\alpha = (\alpha_1, \alpha_2)$ and put the iteration index $j = 0$.

(2) In the $j-th$ iteration, compute the gradient vector $\left(\frac{\partial \ell}{\partial \alpha_1}, \frac{\partial \ell}{\partial \alpha_2}\right)|_{\left(\alpha_1=\alpha_1^{(j)},\alpha_2=\alpha_2^{(j)}\right)}$ along with the observed Fisher information matrix $I = I\left(\alpha_1^{(j)}, \alpha_2^{(j)}\right)$ corresponding to the parameters α_1,and α_2.

$$I = I\left(\alpha_1^{(j)}, \alpha_2^{(j)}\right) = \begin{pmatrix} -\ell_{11} & -\ell_{12} \\ -\ell_{21} & -\ell_{22} \end{pmatrix},$$

where

$$\ell_{11} = \frac{\partial^2 \ell}{\partial \alpha_1^2} = -\sum_{i=1}^{m_1} \left[\frac{(1+\beta_0 x_i)^{-2\theta_0}}{\left(1+\alpha_1 (1+\beta_0 x_i)^{-\theta_0}\right)^2} + \left(1-(1+\beta_0 x_i)^{-\theta_0}\right)(1+S_i) \right],$$

$$\ell_{12} = \ell_{21} = \frac{\partial^2 \ell}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^{22} \ell}{\partial \alpha_2 \partial \alpha_1} = 0,$$

and

$$\ell_{22} = \frac{\partial^2 \ell}{\partial \alpha_2^2} = -\sum_{i=1}^{m_2} \left[\frac{(1+\beta_0 y_i)^{-2\theta_0}}{\left(1+\alpha_2 (1+\beta_0 y_i)^{-\theta_0}\right)^2} + \left(1-(1+\beta_0 y_i)^{-\theta_0}\right)(1+T_i) \right].$$

(3) Assign

$$\left(\alpha_1^{(j+1)}, \alpha_2^{(j+1)}\right) = \left(\alpha_1^{(j)}, \alpha_2^{(j)}\right) + \left(\frac{\partial \ell}{\partial \alpha_1}, \frac{\partial \ell}{\partial \alpha_2}\right)|_{\left(\alpha_1=\alpha_1^{(j)},\alpha_2=\alpha_2^{(j)}\right)} \times I^{-1}\left(\alpha_1^{(j)}, \alpha_2^{(j)}\right),$$

where $I^{-1}\left(\alpha_1^{(j)}, \alpha_2^{(j)}\right)$ symbolizes the matrix's inverse $I\left(\alpha_1^{(j)}, \alpha_2^{(j)}\right)$.

(4) Increment the iteration counter by setting $j = j + 1$, and repeat the procedure starting from Step 1.

(5) Repeat the iterative process until the difference $\left|\left(\alpha_1^{(j+1)}, \alpha_2^{(j+1)}\right) - \left(\alpha_1^{(j)}, \alpha_2^{(j)}\right)\right|$ falls below a specified convergence threshold. The resulting values of α_1 and α_2 at convergence are taken as the MLEs of the parameters, denoted by $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

As a result, the MLE of R, indicated by $\hat{R}_{ML}$, can be produced by thinking about the MLEs' invariance property and substituting their estimates for the parameters as shown in the following:

$$\hat{R}_{ML} = \frac{\hat{\alpha}_2 + (\hat{\alpha}_1 + \hat{\alpha}_2 - 1)\left[\hat{\alpha}_2^2 + \hat{\alpha}_1(\hat{\alpha}_2 + 1)\right] + (\hat{\alpha}_1 - \hat{\alpha}_2)\exp\{-(\hat{\alpha}_1 + \hat{\alpha}_2)\}}{(\hat{\alpha}_1 + \hat{\alpha}_2)^3}. \tag{9}$$

2.2. Existence and uniqueness of MLEs

Existence and uniqueness of MLEs are important aspects to consider when performing statistical inference. This subsection describes the necessary and sufficient conditions for the existence of the ML estimators for PT2C data under GPLD. To conclude this, we examine the behavior of Eqs (7) and (8) on the positive real line $(0, \infty)$.

- For (7), when $\alpha_1 \to 0^+$, we have

$$\lim_{\alpha_1 \to 0^+} \frac{\partial \ell(\alpha_1,\ \alpha_2,\ \beta_0,\ \theta_0)}{\partial \alpha_1} = \sum_{i=1}^{m_1} \left[(2+S_i)(1+\beta_0 x_i)^{-\theta_0} - (1+S_i)\right].$$

Thus, the sufficient condition for existence of a positive solution $\hat{\alpha}_1$ is $\lim_{\alpha_1 \to 0^+} \frac{\partial \ell}{\partial \alpha_1} > 0$, which becomes $\sum_{i=1}^{m_1} (2 + S_i)(1 + \beta_0 x_i)^{-\theta_0} > \sum_{i=1}^{m_1} (1 + S_i)$ for $\beta_0, \theta_0 > 0$, $x_i > 0$; $i = 1, 2, ..., m_1$, and $(1 + \beta_0 x_i)^{-\theta_0} \in (0, 1)$.
However, when $\alpha_1 \to +\infty$, we have $\lim_{\alpha_1 \to +\infty} \frac{\partial \ell(\alpha_1, \alpha_2, \beta_0, \theta_0)}{\partial \alpha_1} = -\sum_{i=1}^{m_1} \left(1 - (1 + \beta_0 x_i)^{-\theta_0}\right)(1 + S_i) < 0$, for $\beta_0, \theta_0 > 0$, $x_i > 0$; $i = 1, 2, ..., m_1$ and $(1 + \beta_0 x_i)^{-\theta_0} \in (0, 1)$.

- Similarly, for (8), when $\alpha_2 \to 0^+$, we have

$$\lim_{\alpha_2 \to 0^+} \frac{\partial \ell(\alpha_1,\ \alpha_2,\ \beta_0,\ \theta_0)}{\partial \alpha_2} = \sum_{i=1}^{m_2} \left[(2 + T_i)(1 + \beta_0 y_i)^{-\theta_0} - (1 + T_i)\right].$$

So the existence condition of a solution $\hat{\alpha}_2 > 0$ is $\lim_{\alpha_2 \to 0^+} \frac{\partial \ell(\alpha_1, \alpha_2, \beta_0, \theta_0)}{\partial \alpha_2} > 0$, which requires the $\sum_{i=1}^{m_2} (2 + T_i)(1 + \beta_0 y_i)^{-\theta_0} > \sum_{i=1}^{m_2} (1 + T_i)$ for $\beta_0, \theta_0 > 0$, $y_i > 0$; $i = 1, 2, ..., m_2$ and $(1 + \beta_0 y_i)^{-\theta_0} \in (0, 1)$.
However, when $\alpha_2 \to +\infty$, we have $\lim_{\alpha_2 \to +\infty} \frac{\partial \ell(\alpha_1, \alpha_2, \beta_0, \theta_0)}{\partial \alpha_2} = -\sum_{i=1}^{m_2} \left(1 - (1 + \beta_0 y_i)^{-\theta_0}\right)(1 + T_i) < 0$, for $\beta_0, \theta_0 > 0$, $y_i > 0$; $i = 1, 2, ..., m_2$, and $(1 + \beta_0 y_i)^{-\theta_0} \in (0, 1)$.

Therefore, on $(0, \infty)$ for $\alpha = (\alpha_1,\ \alpha_2)$, there exists at least one positive root for $\frac{\partial \ell}{\partial \alpha} = 0$. Additionallly, we find that the second partial derivatives of $\ell(\alpha_1,\ \alpha_2,\ \beta_0,\ \theta_0)$ with respect to α_1,and α_2 are always negative, Eqs (7) and (8) have a unique solution, and this solution represents the MLEs of α_1, and α_2. Conseqently, we deduce that $\frac{\partial \ell(\alpha_1, \alpha_2, \beta_0, \theta_0)}{\partial \alpha_i}$ for $i = 1, 2$, is a continuous function on $(0, \infty)$ and, due to the concavity properties of the log-likelihood in α_i, decreases monotonically from a positive value at $\alpha_i \to 0^+$ to negative values as $\alpha_i \to \infty$. This ensures the existence of unique solutions for $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

2.3. Asymptotic confidence interval of R

Although R's exact expression in Eq (3) has an explicit form, finding its precise distribution is challenging. In order to generate an asymptotic confidence interval (ACI) for R, the asymptotic distribution of R has been taken into consideration. Based on the asymptotic characteristics and general condition of MLEs (see Casella and Berger [26]), we were able to estimate the asymptotic distribution of the parameters α_1 and α_2 in this case. The asymptotic distribution of the parameter is typically the normal for the large sample, i.e.,

$$\left[\sqrt{n_1}(\hat{\alpha}_1 - \alpha_1),\ \sqrt{n_2}(\hat{\alpha}_2 - \alpha_2)\right] \to N\left(\alpha, I^{-1}(\alpha)\right), \quad \alpha = (\alpha_1, \alpha_2),$$

and $I(\alpha)$ the observed Fisher information matrix (FIM), is defined by

$$I(\alpha) = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} = E\begin{pmatrix} \frac{-\partial^2 \ell(\alpha_1,\alpha_2,\beta_0,\theta_0)}{\partial \alpha_1^2} & \frac{-\partial^2 \ell(\alpha_1,\alpha_2,\beta_0,\theta_0)}{\partial \alpha_1 \alpha_2} \\ \frac{-\partial^2 \ell(\alpha_1,\alpha_2,\beta_0,\theta_0)}{\partial \alpha_2 \alpha_1} & \frac{-\partial^2 \ell(\alpha_1,\alpha_2,\beta_0,\theta_0)}{\partial \alpha_2^2} \end{pmatrix}. \tag{10}$$

Now, it is simple to confirm that, for large $n_1, n_2 \to \infty$,

$$z = \frac{\hat{R}_{ML} - R}{\sqrt{\hat{\sigma}^2_{\hat{R}}}} \to N(0, 1), \tag{11}$$

where

$$\hat{\sigma}^2_{\hat{R}} = \frac{D_1^2}{n_1\Lambda_{11}} + \frac{D_2^2}{n_2\Lambda_{22}},\ D_1 = \frac{\partial R}{\partial \alpha_1}, D_2 = \frac{\partial R}{\partial \alpha_2}. \tag{12}$$

Consequently, the $100(1-\delta)\%$ confidence interval of R is created as

$$\left[\hat{R}_{ML} - z_{\frac{\delta}{2}}\sqrt{\hat{\sigma}^2_{\hat{R}}},\ \hat{R}_{ML} + z_{\frac{\delta}{2}}\sqrt{\hat{\sigma}^2_{\hat{R}}}\right]. \tag{13}$$

3. Bayes estimation of R

A potent method for deducing unknown parameters from measurable data is Bayesian estimation. It is based on the Bayes theorem, a piece of probability theory that enables the updating of a hypothesis' probability when new data is gathered. Due to its capacity to take into account prior knowledge during the estimation process, this method has significant advantages over conventional MLE techniques. Additionally, it includes the capacity to evaluate the level of uncertainty related to each parameter. According to Kundu and Howlader [27], the family of gamma distributions is known to be adaptable enough to accommodate a wide range of the experimenter's past views. Consider the case when the unknown parameters α_1 and α_2 are stochastically independently distributed with conjugate gamma prior

$$\pi_1(\alpha_1) \propto \alpha_1^{a-1}\exp\{-b\alpha_1\}, \qquad \pi_2(\alpha_2) \propto \alpha_2^{c-1}\exp\{-d\alpha_2\}.$$

As a result, it is possible to write the joint prior density of α_1 and α_2 as the following

$$\pi(\alpha_1, \alpha_2) \propto \alpha_1^{a-1}\alpha_2^{c-1}\exp\{-(b\alpha_1 + d\alpha_2)\},\ \alpha_1,\ \alpha_2 > 0, \tag{14}$$

where the corresponding hyper-parameters a, b, c, and d indicate prior knowledge gleaned from historical data, professional experience, and other relevant supplementary information (see Singh and Tripathi [28] and EL-Sagheer et al. [29]). It can be seen that there does not exist a conjugate prior to the GPLD. As a result, we use the gamma priors, which are thought to be more elastic than other prior distributions and can support the parameters of the GPLD (see Kundu and Howlader [27]; Almarashi et al. [30]). Additionally, the independent gamma priors are rather straightforward and condensed, which may not produce many challenging computational and inferential problems. Additionally, it should be emphasized that the complicated shapes of the Fisher information matrix make it difficult to apply Jeffrey's priors in this situation. Following that, using Bayes' theorem, the joint posterior density function of α_1 and α_2 for given data is

$$\begin{aligned}\pi^*(\alpha_1, \alpha_2|\underline{x}, \underline{y}) &= \frac{L(\alpha_1, \alpha_2|\underline{x}, \underline{y})\pi(\alpha_1, \alpha_2)}{\int_0^\infty\int_0^\infty L(\alpha_1, \alpha_2|\underline{x}, \underline{y})\pi(\alpha_1, \alpha_2)d\alpha_1 d\alpha_2} \\ &\propto \alpha_1^{a-1}\exp\left\{-b - \sum_{i=1}^{m_1}\left(1-(1+\beta_0 x)^{-\theta_0}\right) - \sum_{i=1}^{m_1}S_i\left(1-(1+\beta_0 x)^{-\theta_0}\right)\right\} \\ &\times \alpha_2^{c-1}\exp\left\{-d - \sum_{i=1}^{m_2}\left(1-(1+\beta_0 y)^{-\theta_0}\right) - \sum_{i=1}^{m_2}T_i\left(1-(1+\beta_0 y)^{-\theta_0}\right)\right\} \\ &\times \exp\left\{\sum_{i=1}^{m_1}\log\left(1+\alpha_1(1+\beta_0 x)^{-\theta}\right) + \sum_{i=1}^{m_2}\log\left(1+\alpha_2(1+\beta_0 y)^{-\theta}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&\propto G_{\alpha_1}\left[a-1, b+\sum_{i=1}^{m_1}\left(1-(1+\beta_0 x)^{-\theta_0}\right)+\sum_{i=1}^{m_1} S_i\left(1-(1+\beta_0 x)^{-\theta_0}\right)\right] \\
&\times G_{\alpha_2}\left[c-1, d+\sum_{i=1}^{m_2}\left(1-(1+\beta_0 y)^{-\theta_0}\right)+\sum_{i=1}^{m_2} T_i\left(1-(1+\beta_0 y)^{-\theta_0}\right)\right] \\
&\times h(\alpha_1, \alpha_2 | \underline{x}, \underline{y}), \qquad (15)
\end{aligned}
$$

where $G_\alpha(.,.)$ stands for the conventional gamma density function and

$$
h(\alpha_1, \alpha_2 | \underline{x}, \underline{y}) = \exp\left\{\sum_{i=1}^{m_1} \log\left(1+\alpha_1 (1+\beta_0 x)^{-\theta}\right) + \sum_{i=1}^{m_2} \log\left(1+\alpha_2 (1+\beta_0 y)^{-\theta}\right)\right\}. \qquad (16)
$$

Symmetric loss functions are chosen in practice where the loss arising from overestimation and underestimation are equally relevant. Squared error (SE) loss function is one such function, which is well-known for its excellent mathematical features, and can be expressed as

$$
L_{SE}\left(\hat{\Theta}, \Theta\right) = \left(\hat{\Theta} - \Theta\right)^2, \qquad (17)
$$

where, $\hat{\Theta}$ is the estimation of Θ. As a result, under the SE loss function, the posterior function's mean, indicated by $\hat{R}_{SE}$, can be used to calculate the Bayes estimate of R as the following

$$
\hat{R}_{SE} = \int_0^\infty \int_0^\infty Q(\alpha_1, \alpha_2)\, \pi^*(\alpha_1, \alpha_2 | \underline{x}, \underline{y}) d\alpha_1 d\alpha_2. \qquad (18)
$$

Asymmetric loss functions are employed instead of symmetric loss functions for the purpose of making the Bayesian approach more practical and usable. Overestimation and underestimation might occasionally result in varied losses; hence, it is not appropriate in such instances to utilize symmetric loss functions. The most used asymmetric loss function is the LINEX, which is defined as

$$
L_{LINEX} = e^{\varepsilon(\hat{\Theta}-\Theta)} - \varepsilon\left(\hat{\Theta} - \Theta\right) - 1. \qquad (19)
$$

Thus, the Bayes estimate for R under LINEX loss function is

$$
\begin{aligned}
\hat{R}_{LINEX} &= -\frac{1}{\varepsilon} \log E\left(e^{-\varepsilon Q(\alpha_1, \alpha_2)}\right),\ \varepsilon \neq 0 \\
&= -\frac{1}{\varepsilon} \log\left[\int_0^\infty \int_0^\infty e^{-\varepsilon Q(\alpha_1, \alpha_2)} \pi^*(\alpha_1, \alpha_2 | \underline{x}, \underline{y}) d\alpha_1 d\alpha_2\right]. \qquad (20)
\end{aligned}
$$

According to Zellner [31], the orientation and level of asymmetry are each represented by the sign and size of ε. Underestimation is more expensive than overestimation when $\varepsilon > 0$, and vice versa. Regarding how LINEX behaves as ε approaches 0, SE is about as close to symmetric as LINEX. It goes without saying that the Bayes estimates for the earlier categories of loss functions cannot be built in closed forms. To create such estimations, the MCMC approach will be used.

3.1. MCMC approach

In the area of statistical computing, MCMC is a vital approach. A number of parameters can be estimated using this effective technique, which can be used to sample from a particular probability distribution. The Markov property, which asserts that a system's future state is solely reliant on its present state, is the foundation of MCMC. A Markov chain is a collection of random variables that satisfies this characteristic. In order to sample from a target probability distribution and estimate various parameters, MCMC algorithms use the Markov chain. They are specifically used to calculate distribution expectations, like the mean and variance. Building a Markov chain with a state space equal to the range of potential values for the target probability distribution is how the MCMC algorithm operates. The algorithm iteratively generates a succession of states that are approved or rejected depending on the value of the target distribution at each state, starting with an initial state.

One well-known MCMC methodology is the importance sampling technique (IST). Furthermore, it is regarded as a successful method for obtaining Bayes estimates for $Q(\alpha_1, \alpha_2)$. Additionally, this approach can be used with PT2C data to create the related higher posterior density intervals (HPDI). By using the IST, we are able to obtain the Bayes estimates of (α_1, α_2), as well as the associated estimator of R, by extracting the sample from $G_{\alpha_1}(.,.)$ and $G_{\alpha_2}(.,.)$ previously mentioned in (15). The following is a description of the IST:

(1) Set the starting value to the MLEs, represented by $\hat{\alpha}_1$, and $\hat{\alpha}_2$.
(2) Set $j = 1$,
(3) Create the sequence of α_1 from the $G_{\alpha_1}(.,.)$ given in (15).
(4) Create the sequence of α_2 from the $G_{\alpha_2}(.,.)$ given in (15).
(5) Set $j = j + 1$.
(6) Repeat Steps 2–5, for N times and obtain the sequence $\left(\alpha_1^{(j)}, \alpha_2^{(j)}\right)$ and $R^{(j)}$, $j = 1, 2, ..., N$.
(7) It is possible to obtain the Bayesian estimate of R by

$$\hat{R}_{MCMC} = \frac{\frac{1}{N-M}\sum_{j=M+1}^{N} R^{(j)} h(\alpha_1, \alpha_2 | \underline{x}, \underline{y})}{\frac{1}{N-M}\sum_{j=M+1}^{N} h(\alpha_1, \alpha_2 | \underline{x}, \underline{y})},$$

where M is the MCMC burn-in period.

3.2. Bayesian credible interval of R

Bayesian interval estimation using HPD credible intervals provides a range of parameter values that contain the true parameter with a specified posterior probability, typically 95%. Unlike frequentist confidence intervals, which are based on repeated sampling, HPD intervals are derived directly from the posterior distribution of the parameter, combining prior information with observed data through Bayes' theorem. The HPDI is the narrowest interval that includes the most probable values of the parameter, ensuring that every point inside the interval has a HPD than any point outside. This method offers a more intuitive interpretation: given the data and prior, there is a 95% probability that the parameter lies within the HPDI. The HPD intervals are especially useful in skewed or multimodal distributions, where equal-tailed intervals may not be the most informative. According to Step 7 in the previous algorithm, sort $R^{(j)}$, $j = M+1, M+2, ..., N$ in ascending order as $R^{(1)} < R^{(2)} < ... < R^{(N)}$. Then, the $100(1-\delta)\%$ symmetric credible intervals (CRIs) of R are

$$\left(\hat{R}_{MCMC\left[\frac{\delta}{2}(N-M)\right]}, \hat{R}_{MCMC\left[\left(1-\frac{\delta}{2}\right)(N-M)\right]}\right). \tag{21}$$

4. Simulation study

In order to assess the effectiveness of our methodologies, substantial numerical studies were done in this section. The suggested estimators R_{ML}, R_{SE}, and R_{LINEX} for GPLD are compared in terms of simulated risks. Simulations involving 10,000 simulated samples have been used to compare the estimators. The algorithm created by Balakrishnan and Sandhu [32] was used to generate the PT2C samples. Additionally, three quantiles were looked at in order to compare the performance of various estimations. First, the mean squared errors (MSEs) of the estimations of R obtained as $\frac{1}{N}\sum_{i=1}^{N}\left(R_i - \hat{R}_i\right)^2$,where $\hat{R} = R_{ML}$, R_{SE}, or R_{LINEX} and repeated N-times. Second, the coverage probabilities (CPs) of the confidence intervals of the SSR parameter R, which are defined as the probability that the interval estimate contains the true value of R. Third, the average widths (AWs) of the confidence intervals of the SSR parameter R. MSEs for point estimates, as well as CPs and AWs for interval estimates (ACIs and HPDIs), are used to measure an estimator's performance. Specifically, our focus is on three sample sizes such as $(n_1, n_2|m_1, m_2) = (30, 50, 100|15, 30, 55, 70)$ for three cases of the true values of the parameters $(\alpha_1, \alpha_2) = (3.5, 0.5), (0.5, 0.5), (0.5, 5.6)$ and corresponding actual values of $R = (0.266484,\ 0.559536,\ 0.803391)$, with the common parameters $\beta = 2$ and $\theta = 0.025$. The three true values of R were chosen to represent distinct levels of system reliability: low, moderate, and high, respectively. These values correspond to different configurations of the shape parameters α_1 and α_2 under the GPLD, allowing us to evaluate the performance of the proposed estimators across a range of practically relevant scenarios. According to the respective selection of (n_i, m_i), $i = 1, 2$ the various schemes CPs $(S_i, T_i,)$ $i = 1, ..., m$ are made; see Table 1. We created three systematic CPs that offer fast failure, moderate failure, and late failure, respectively, in order to identify the removals.

Table 1. Different censoring schemes were used in simulation.

(n_1, m_1)	CS	$S_i = (S_1, S_2, ..., S_{m_1})$	(n_2, m_2)	CS	$T_i = (T_1, T_2, ..., T_{m_2})$
$(30, 15)$	I	$(15, 0^*14)$	$(30, 15)$	I	$(15, 0^*14)$
	II	$(0^*5, 3^*5, 0^*5)$		II	$(0^*5, 3^*5, 0^*5)$
	III	$(0^*14, 15)$		III	$(0^*14, 15)$
$(50, 30)$	I	$(20, 0^*29)$	$(50, 30)$	I	$(20, 0^*29)$
	II	$(0^*10, 2^*10, 0^*10)$		II	$(0^*10, 2^*10, 0^*10)$
	III	$(0^*29, 20)$		III	$(0^*29, 20)$
$(100, 55)$	I	$(45, 0^*54)$	$(100, 55)$	I	$(45, 0^*54)$
	II	$(0^*25, 9^*5, 0^*25)$		II	$(0^*25, 9^*5, 0^*25)$
	III	$(0^*54, 45)$		III	$(0^*54, 45)$
$(100, 70)$	I	$(30, 0^*69)$	$(100, 70)$	I	$(30, 0^*69)$
	II	$(0^*30, 3^*10, 0^*30)$		II	$(0^*30, 3^*10, 0^*30)$
	III	$(0^*69, 30)$		III	$(0^*69, 30)$

The nonlinear equations are solved and the MLEs of the parameters are obtained using the NMaximize command of the Mathematica 13 package. The MLEs of R, are created using the invariance attribute of the MLE. Based on $15,000$ MCMC samples, the Bayes estimates (BEs) and CRIs are calculated, with the first $5,000$ values being discarded as "burn-in". In Bayesian framework, we take into account informative priors with hyper-parameter values $a = 2$, $b = 1$, $c = 3$ and $d = 1.4$. The parameters of the informative priors are chosen such that their mean is equal to the real parameter values. Tables 2 through 4 present the findings of this study. The results of Tables 2–4 allow us to draw the following conclusions:

(1) It is evident from Tables 2–4 that the MSEs and average interval widths decrease as sample sizes (n_1, m_1) and (n_2, m_2) increase.
(2) Parallel to an increase in the actual value of R, the AWs are also decreasing.
(3) Regarding MSEs and AWs, BEs outperform MLEs.
(4) The first scheme (I,I) is the best scheme in terms of decreased MSEs and AWs when sample sizes are fixed and observed failures are present.
(5) Both MSEs and AWs do not have regular behavior (increasing or decreasing) with schemes (I,II), (I,III),and (II,III).
(6) Due to having the least MSE, the BE under LINEX with $\varepsilon = 1$ delivers better estimates for R.
(7) As measured by having lower MSEs, the BE of R under SE performs better than their estimate under LINEX with $\varepsilon = -1$.
(8) Both MSEs and AWs increase when the removals are delayed.
(9) The estimates produced by MLE and Bayesian approaches are highly similar, and both have ACIs with high CPs (about 0.95).

Table 2. The MSEs, AWs, and CPs for R when the true value of $R = 0.266484$.

		MLEs	BEs			intervals estimation			
		MSE	MSE	MSE of $\hat{R}_{LINEX}$		ACIs		HPDI	
$(n_1, m_1), (n_2, m_2)$	$(S_i, T_i,)$	$\hat{R}_{ML}$	$\hat{R}_{SE}$	$\varepsilon = -1$	$\varepsilon = 1$	AWs	CPs	AWs	CPs
$(30, 15), (30, 15)$	(I,I)	0.0154	0.0121	0.0136	0.0109	0.2987	0.943	0.2563	0.945
	(II,II)	0.0176	0.0143	0.0161	0.0128	0.3154	0.941	0.2765	0.946
	(III,III)	0.0219	0.0185	0.0195	0.0152	0.3412	0.947	0.3099	0.944
	(I,II)	0.0187	0.0165	0.0174	0.0141	0.3025	0.949	0.2874	0.951
	(I,III)	0.0205	0.0184	0.0196	0.0164	0.3256	0.951	0.3177	0.965
	(II,I)	0.0189	0.0168	0.0179	0.0149	0.3128	0.953	0.2987	0.958
	(II,III)	0.0216	0.0187	0.0202	0.0170	0.3267	0.948	0.2886	0.961
	(III,I)	0.0199	0.0175	0.0186	0.0166	0.3196	0.954	0.2911	0.955
	(III,II)	0.0202	0.0178	0.0199	0.0159	0.3352	0.947	0.3243	0.949
$(50, 30), (50, 30)$	(I,I)	0.0092	0.0085	0.0089	0.0081	0.2354	0.952	0.1995	0.961
	(II,II)	0.0097	0.0089	0.0093	0.0086	0.2569	0.949	0.2154	0.958
	(III,III)	0.0122	0.0109	0.0115	0.0099	0.2766	0.947	0.2446	0.954
	(I,II)	0.0099	0.0093	0.0096	0.0088	0.2471	0.943	0.2275	0.956
	(I,III)	0.0105	0.0098	0.0100	0.0097	0.2695	0.954	0.2499	0.955
	(II,I)	0.0095	0.0089	0.0092	0.0084	0.2547	0.952	0.2367	0.962
	(II,III)	0.0128	0.0119	0.0122	0.0109	0.2658	0.947	0.2501	0.955
	(III,I)	0.0111	0.0102	0.0108	0.0094	0.2586	0.943	0.2411	0.963
	(III,II)	0.0132	0.0121	0.0127	0.0115	0.2801	0.945	0.2620	0.957
$(100, 55), (100, 55)$	(I,I)	0.0083	0.0079	0.0081	0.0077	0.1998	0.958	0.1487	0.974
	(II,II)	0.0085	0.0081	0.0083	0.0079	0.2157	0.949	0.1869	0.962
	(III,III)	0.0088	0.0084	0.0086	0.0082	0.2354	0.955	0.2095	0.958
	(I,II)	0.0084	0.0081	0.0083	0.0079	0.2601	0.952	0.2234	0.961
	(I,III)	0.0087	0.0083	0.0085	0.0082	0.2265	0.949	0.1967	0.960
	(II,I)	0.0084	0.0082	0.0084	0.0078	0.2198	0.948	0.2069	0.958
	(II,III)	0.0087	0.0084	0.0086	0.0081	0.2364	0.955	0.2201	0.956
	(III,I)	0.0088	0.0085	0.0087	0.0083	0.2255	0.951	0.2098	0.962
	(III,II)	0.0089	0.0087	0.0088	0.0085	0.2462	0.952	0.2315	0.956
$(100, 70), (100, 70)$	(I,I)	0.0079	0.0076	0.0077	0.0074	0.1565	0.955	0.1287	0.966
	(II,II)	0.0081	0.0078	0.0080	0.0076	0.1763	0.956	0.1568	0.957
	(III,III)	0.0083	0.0080	0.0082	0.0079	0.1984	0.949	0.1755	0.963
	(I,II)	0.0082	0.0079	0.0081	0.0077	0.1674	0.953	0.1474	0.959
	(I,III)	0.0082	0.0080	0.0081	0.0078	0.1688	0.957	0.1529	0.958
	(II,I)	0.0080	0.0077	0.0078	0.0075	0.1701	0.954	0.1511	0.964
	(II,III)	0.0081	0.0078	0.0079	0.0076	0.1864	0.961	0.1647	0.968
	(III,I)	0.0081	0.0079	0.0080	0.0076	0.1799	0.948	0.1655	0.964
	(III,II)	0.0082	0.0079	0.0081	0.0079	0.1868	0.951	0.1739	0.967

Table 3. The MSEs, AWs, and CPs for R when the true value of $R = 0.559536$.

		MLEs	BEs			intervals estimation			
		MSE	MSE	MSE of $\hat{R}_{LINEX}$		ACIs		HPDI	
$(n_1, m_1), (n_2, m_2)$	$(S_i, T_i,)$	$\hat{R}_{ML}$	$\hat{R}_{SE}$	$\varepsilon = -1$	$\varepsilon = 1$	AWs	CPs	AWs	CPs
$(30, 15), (30, 15)$	(I,I)	0.0064	0.0062	0.0063	0.0059	0.2245	0.938	0.2068	0.949
	(II,II)	0.0067	0.0063	0.0064	0.0061	0.2436	0.938	0.2199	0.951
	(III,III)	0.0069	0.0065	0.0066	0.0063	0.2587	0.941	0.2374	0.948
	(I,II)	0.0066	0.0064	0.0065	0.0061	0.2365	0.952	0.2215	0.961
	(I,III)	0.0068	0.0066	0.0067	0.0063	0.2486	0.947	0.2244	0.949
	(II,I)	0.0067	0.0065	0.0066	0.0064	0.2377	0.955	0.2283	0.952
	(II,III)	0.0068	0.0065	0.0067	0.0064	0.2469	0.953	0.2322	0.948
	(III,I)	0.0068	0.0066	0.0066	0.0062	0.2415	0.956	0.2321	0.951
	(III,II)	0.0068	0.0066	0.0067	0.0063	0.2457	0.945	0.2292	0.952
$(50, 30), (50, 30)$	(I,I)	0.0053	0.0049	0.0051	0.0047	0.1856	0.949	0.1654	0.961
	(II,II)	0.0055	0.0051	0.0053	0.0049	0.2058	0.948	0.1863	0.955
	(III,III)	0.0058	0.0055	0.0057	0.0052	0.2369	0.948	0.2147	0.964
	(I,II)	0.0054	0.0050	0.0052	0.0048	0.1963	0.951	0.1758	0.963
	(I,III)	0.0055	0.0051	0.0053	0.0049	0.2055	0.947	0.1879	0.958
	(II,I)	0.0053	0.0050	0.0052	0.0048	0.1964	0.952	0.1788	0.955
	(II,III)	0.0056	0.0054	0.0055	0.0052	0.2067	0.946	0.1869	0.961
	(III,I)	0.0057	0.0055	0.0056	0.0052	0.1899	0.954	0.1709	0.958
	(III,II)	0.0058	0.0056	0.0057	0.0053	0.1955	0.946	0.1867	0.962
$(100, 55), (100, 55)$	(I,I)	0.0046	0.0043	0.0044	0.0041	0.1563	0.947	0.1369	0.961
	(II,II)	0.0048	0.0045	0.0046	0.0043	0.1736	0.943	0.1547	0.956
	(III,III)	0.0051	0.0047	0.0049	0.0045	0.1899	0.942	0.1724	0.955
	(I,II)	0.0047	0.0044	0.0046	0.0042	0.1637	0.955	0.1438	0.949
	(I,III)	0.0049	0.0046	0.0047	0.0044	0.1799	0.951	0.1654	0.969
	(II,I)	0.0050	0.0047	0.0048	0.0044	0.1638	0.954	0.1522	0.964
	(II,III)	0.0050	0.0048	0.0049	0.0045	0.1745	0.942	0.1687	0.948
	(III,I)	0.0049	0.0048	0.0048	0.0045	0.1699	0.949	0.1527	0.966
	(III,II)	0.0051	0.0047	0.0050	0.0046	0.1766	0.947	0.1621	0.958
$(100, 70), (100, 70)$	(I,I)	0.0041	0.0038	0.0039	0.0036	0.1358	0.946	0.1196	0.961
	(II,II)	0.0042	0.0039	0.0040	0.0037	0.1563	0.951	0.1347	0.960
	(III,III)	0.0044	0.0041	0.0042	0.0039	0.1794	0.952	0.1568	0.962
	(I,II)	0.0043	0.0040	0.0041	0.0038	0.1482	0.948	0.1247	0.955
	(I,III)	0.0044	0.0041	0.0042	0.0039	0.1499	0.947	0.1301	0.958
	(II,I)	0.0044	0.0040	0.0041	0.0038	0.1399	0.955	0.1258	0.961
	(II,III)	0.0044	0.0042	0.0043	0.0040	0.1468	0.956	0.1234	0.963
	(III,I)	0.0043	0.0041	0.0042	0.0040	0.1598	0.942	0.1436	0.959
	(III,II)	0.0043	0.0040	0.0041	0.0039	0.1654	0.949	0.1499	0.958

Table 4. The MSEs, AWs, and CPs for R when the true value of $R = 0.803391$.

		MLEs	BEs			intervals estimation			
		MSE	MSE	MSE of $\hat{R}_{LINEX}$		ACIs		HPDI	
$(n_1,m_1),(n_2,m_2)$	$(S_i,T_i,)$	$\hat{R}_{ML}$	$\hat{R}_{SE}$	$\varepsilon=-1$	$\varepsilon=1$	AWs	CPs	AWs	CPs
$(30,15),(30,15)$	(I,I)	0.0044	0.0042	0.0040	0.0038	0.1789	0.947	0.1568	0.951
	(II,II)	0.0046	0.0044	0.0042	0.0039	0.1936	0.952	0.1756	0.952
	(III,III)	0.0049	0.0047	0.0045	0.0042	0.2155	0.951	0.1984	0.962
	(I,II)	0.0045	0.0044	0.0043	0.0039	0.1836	0.956	0.1657	0.955
	(I,III)	0.0047	0.0045	0.0044	0.0041	0.1899	0.954	0.1775	0.954
	(II,I)	0.0046	0.0045	0.0044	0.0040	0.2025	0.949	0.1835	0.962
	(II,III)	0.0048	0.0047	0.0046	0.0041	0.1936	0.950	0.1811	0.961
	(III,I)	0.0048	0.0047	0.0045	0.0042	0.1796	0.949	0.1688	0.957
	(III,II)	0.0049	0.0048	0.0044	0.0042	0.1874	0.953	0.1795	0.961
$(50,30),(50,30)$	(I,I)	0.0039	0.0037	0.0036	0.0034	0.1354	0.958	0.1178	0.963
	(II,II)	0.0041	0.0039	0.0038	0.0036	0.1563	0.953	0.1356	0.965
	(III,III)	0.0043	0.0041	0.0040	0.0039	0.1836	0.961	0.1634	0.958
	(I,II)	0.0040	0.0038	0.0037	0.0035	0.1457	0.950	0.1245	0.962
	(I,III)	0.0042	0.0040	0.0039	0.0036	0.1648	0.951	0.1472	0.957
	(II,I)	0.0041	0.0040	0.0038	0.0035	0.1488	0.947	0.1311	0.955
	(II,III)	0.0042	0.0041	0.0039	0.0037	0.1569	0.957	0.1498	0.960
	(III,I)	0.0041	0.0040	0.0038	0.0036	0.1654	0.949	0.1523	0.961
	(III,II)	0.0042	0.0041	0.0040	0.0039	0.1725	0.952	0.1601	0.959
$(100,55),(100,55)$	(I,I)	0.0029	0.0028	0.0026	0.0023	0.1178	0.956	0.0998	0.961
	(II,II)	0.0031	0.0029	0.0028	0.0024	0.1365	0.967	0.1125	0.958
	(III,III)	0.0035	0.0033	0.0032	0.0029	0.1536	0.956	0.1324	0.957
	(I,II)	0.0030	0.0028	0.0027	0.0024	0.1245	0.955	0.1028	0.961
	(I,III)	0.0033	0.0032	0.0029	0.0027	0.1463	0.954	0.1247	0.962
	(II,I)	0.0032	0.0031	0.0030	0.0028	0.1365	0.958	0.1277	0.971
	(II,III)	0.0034	0.0032	0.0031	0.0028	0.1469	0.949	0.1322	0.956
	(III,I)	0.0033	0.0032	0.0031	0.0027	0.1465	0.951	0.1299	0.966
	(III,II)	0.0034	0.0033	0.0030	0.0028	0.1548	0.948	0.1311	0.964
$(100,70),(100,70)$	(I,I)	0.0023	0.0021	0.0020	0.0018	0.0889	0.947	0.0847	0.959
	(II,II)	0.0026	0.0024	0.0022	0.0020	0.0995	0.949	0.0956	0.949
	(III,III)	0.0030	0.0028	0.0027	0.0025	0.1174	0.948	0.1099	0.954
	(I,II)	0.0024	0.0023	0.0022	0.0019	0.0957	0.955	0.0915	0.961
	(I,III)	0.0025	0.0024	0.0023	0.0021	0.1063	0.952	0.0978	0.960
	(II,I)	0.0024	0.0022	0.0021	0.0019	0.1022	0.961	0.0969	0.954
	(II,III)	0.0027	0.0026	0.0025	0.0022	0.1134	0.958	0.1035	0.963
	(III,I)	0.0028	0.0027	0.0025	0.0023	0.1102	0.960	0.1011	0.958
	(III,II)	0.0029	0.0027	0.0026	0.0024	0.1141	0.955	0.1047	0.961

5. Implementing practical data applications on an asymmetrical platform

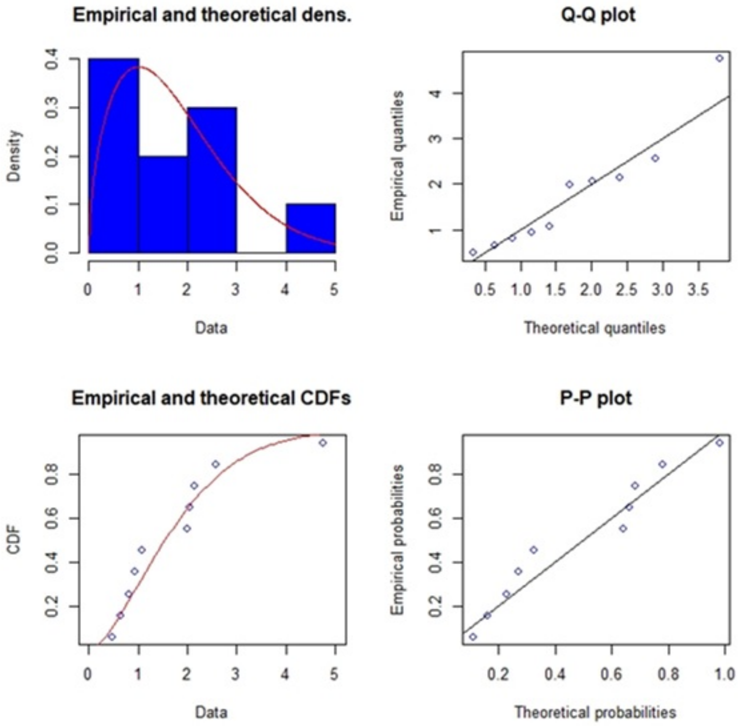
This section will provide a few examples from the medical and business industries to explain the significance of the theoretical discoveries that were covered in the earlier sections. The analysis of two real-world datasets in this section lends validity to the suggested point and interval estimates for the SSR parameter R.

5.1. Application to Melanoma

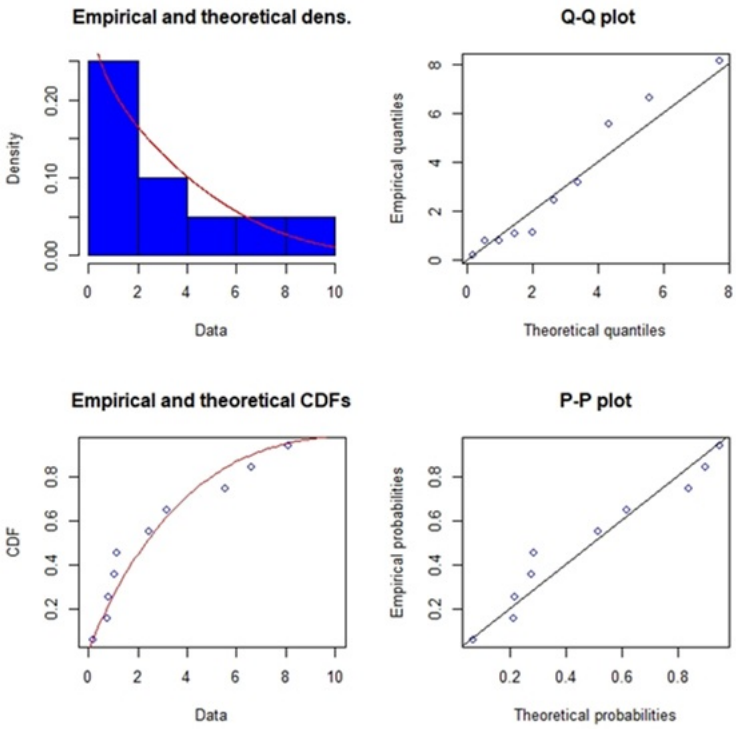
The first application shows the gender-specific median survival time in months for Stage 4 Melanoma patients who underwent treatment at the University of Oklahoma Health Sciences Centre between 1974 and 1978. The data was first provided by Lee et al. [33] and then reused by Boulkeroua et al. [34]. The sets of data are: Female patients: 1.3, 2.7, 3.8, 4.2, 7.4, 9.3, 10.5, 11.4, 13.3, 13.8, 13.8, 20 and 22.2; and male patients: 0.4, 0.9, 1.2, 1.5, 1.6, 1.7, 2.5, 2.5, 3.9, 3.9, 4, 4.2, 4.5, 5.8, 5.9, 6.3, 7.3, 7.4, 8.3, 9.8, 11, 11.1, 16.1 and 20.5. The Kolmogorov-Smirnov (K-S) distance between the fitted distribution function and the empirical distribution function has been calculated for the sake of the goodness of fit test. For the first set, it is 0.13336 with a P-value of 0.9749, whereas for the second set, it is 0.077296 with a P-value of 0.9988. As a result, the first and second sets had the greatest P-value for K-S. Because of this, we may say that the GPLD fits the two real datasets the best.

Figure 1 illustrates the empirical cumulative distribution functions (ECDFs) alongside the fitted GPLD curves, as well as the corresponding quntile-quantile (Q-Q) plots for the female and male melanoma patient data, respectively. In Figure 1(a), the ECDF and fitted distribution for the female patients show close alignment, indicating a good fit of the GPLD to the observed survival times. Similarly, the Q-Q plot reveals minimal deviation from the diagonal reference line, suggesting that the model adequately captures the distributional behavior of the data. For male patients (Figure 1(b)), the ECDF and fitted GPLD also demonstrate strong agreement. The Q-Q plot further supports this observation, with points clustering tightly around the reference line. These visual findings are corroborated by the K-S test results, which yielded high p-values (0.9749 for females and 0.9988 for males), indicating that the GPLD is not rejected as a plausible model for either dataset. Thus, Figures 1 and 2 confirm the suitability of the GPLD for modeling survival times in medical reliability contexts.

A PT2C sample, say $x^{(1)}$, is obtained as follows: from the original data, $n_1 = 13$ with $S = (2, 0, 2, 0, 2, 0, 2)$ and 7 failure times are observed ($m_1 = 7$) resulting the obtained PT2C sample $x^{(1)} = (1.3, 2.7, 3.8, 4.2, 9.3, 11.4, 13.8)$. For the second dataset, suppose that a PT2C scheme is given by $T = (2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0)$, then a PT2C sample of size $m_2 = 12$ out of $n_2 = 24$ items of data is obtained as $y^{(1)} = (0.4, 0.9, 1.2, 1.5, 1.6, 1.7, 2.5, 3.9, 4.2, 5.9, 7.3, 11.1)$. The MLEs and related 95% ACIs for R under the previous data $x^{(1)}$ and $y^{(1)}$ are established to be as shown in Table 5. The prior distributions of the parameters must now be specified in order to construct the BEs. Assuming that the hyper-parameters are $a = b = c = d = 0.0001$ because we don't have any prior knowledge, we use non-informative Gamma priors for α_1 and α_2. The initial values for the parameters α_1 and α_2 were taken to be their MLEs and used to run the MCMC technique that was explained in Section 3 being aware of the parameters $\beta = 3.555$ and $\theta = 0.025$. $15,000$ MCMC samples were also produced. We discard the first $5,000$ samples as 'burn-in' to eliminate the impact of the starting values. The BEs for R and associated 95% CRIs can be seen in Table 5.

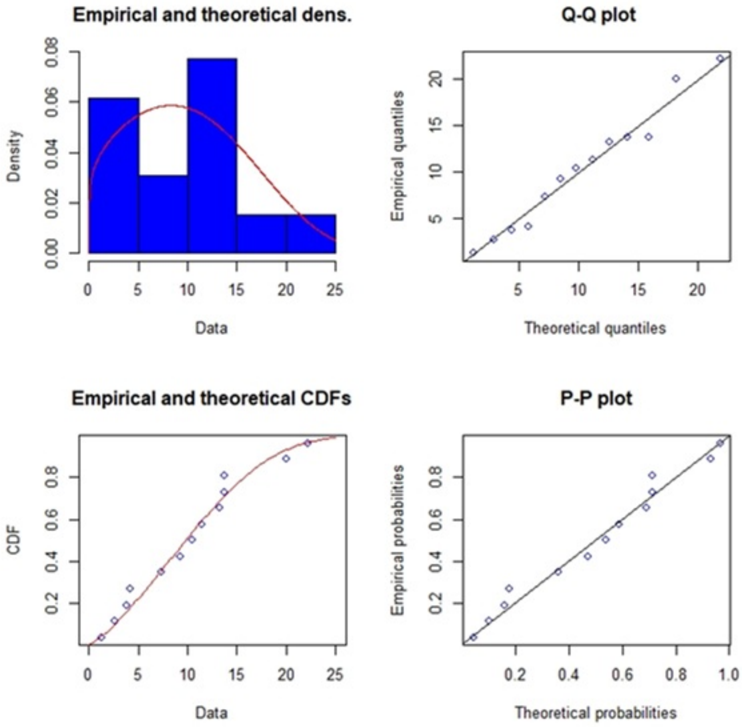


(a)

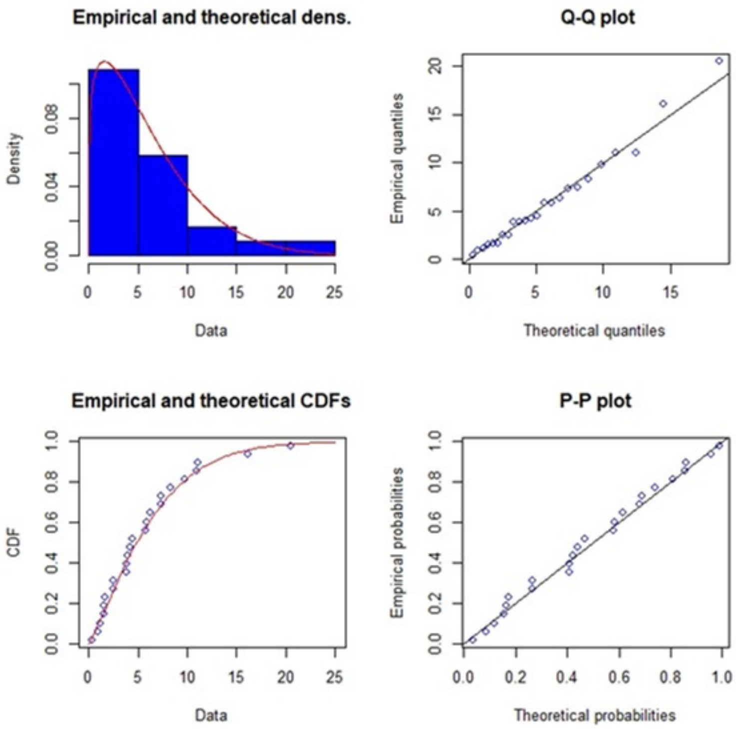


(b)

Figure 1. (a) Empirical and Q-Q plots for dataset I; (b) Empirical and Q-Q plots for dataset II.



(a)



(b)

Figure 2. (a) Empirical and Q-Q plots for dataset X; (b) Empirical and Q-Q plots for dataset Y.

Table 5. ML and Bayes estimates of R for the real datasets.

	MLEs		BEs		intervals estimation	
Data	$\hat{R}_{ML}$	$\hat{R}_{SE}$	$\hat{R}_{LINEX_{\varepsilon=-1}}$	$\hat{R}_{LINEX_{\varepsilon=1}}$	ACIs	HPDI
$x^{(1)}, y^{(1)}$	0.5442	0.5234	0.5412	0.5139	[0.3959, 0.6353]	[0.4159, 0.5952]
$x^{(2)}, y^{(2)}$	0.4871	0.4713	0.4938	0.4592	[0.2845, 0.5734]	[0.3196, 0.4995]

5.2. Application to insulating fluid with high voltage stress

In the second application, we used the dataset was first provided by Nelson [35] and also used by Abo-Kasem et al. [36], which represents the time it takes for an insulating fluid under high voltage stress to break down. The failure times of this dataset are represented by the symbols X and Y, respectively. Following are the failure times for samples with sizes $n_1 = n_2 = 10$.

X :	1.99	0.64	2.15	1.08	2.57	0.93	4.75	0.82	2.06	0.49
Y :	8.11	3.17	5.55	0.80	0.20	1.13	6.63	1.08	2.44	0.78

The K-S distance between the empirical and fitted distribution functions has been calculated for the goodness-of-fit test. For dataset X, it is 0.07276, and the corresponding P-value is 0.8788. For dataset Y, it is 0.2133, and the corresponding P-value is 0.6787. As a result, for datasets X and Y, the p-value for K-S is the highest. Our conclusion is that GPLD is the best fit for the two real datasets as a result of this. Figure 2 presents the empirical and Q-Q plots for the breakdown times of insulating fluid samples under high-voltage stress. In Figure 2(a), which corresponds to the strength data (Set X), the fitted GPLD closely follows the ECDF, capturing both the central tendency and the tail behavior of the distribution. The Q-Q plot supports this fit, with plotted points lying nearly along the 45-degree reference line, indicating that the GPLD appropriately models the observed data. Figure 2(b) depicts the results for the stress data (Set Y), where again, both the ECDF and the fitted curve are in strong agreement. The Q-Q plot further validates the model choice, with only minor deviations in the upper tail. The K-S test statistics for these datasets also support these findings, with p-values of 0.8788 (Set X) and 0.6787 (Set Y), indicating no significant difference between the empirical and theoretical distributions. These graphical and statistical diagnostics demonstrate the robustness of the GPLD in modeling failure-time data from engineering systems subject to stress testing. As in the first application, suppose that a PT2C scheme is given by $S = (1, 1, 0, 0, 0, 2)$, then a PT2C sample of size $m_1 = 6$ out of $n_1 = 10$ items of data is obtained as $x^{(2)} = (0.49, 0.64, 0.82, 0.93, 1.08, 2.57)$. Similarly for the other set, a PT2C scheme is given by $T = (0, 0, 0, 1, 1, 2)$, then a PT2C sample of size $m_2 = 6$ out of $n_2 = 10$ items of data is $y^{(2)} = (0.20, 0.78, 0.80, 1.08, 1.13, 2.44)$. In Bayesian framework, we used non-informative priors i.e., $a = b = c = d = 0.0001$, being aware of the parameters $(\beta, \theta) = (1.556, 0.0249)$ and $N = 15,000$ with $M = 5,000$. The ML and BEs for R under the previous data $x^{(2)}$ and $y^{(2)}$ can be seen in Table 5.

In both real data applications, the PT2C schemes were selected to reflect practical constraints commonly encountered in medical and engineering reliability studies. For the melanoma dataset, early removals simulate situations where patients drop out of long-term trials due to side effects or competing risks. In contrast, the censoring structure for the breakdown times of insulating fluids mimics scenarios in industrial testing where units are withdrawn at planned maintenance intervals or due to cost limitations. To assess the robustness of our inference, we also examined alternative censoring schemes involving delayed and random removals. The results, not shown here for brevity, were consistent with our main findings and suggest that the proposed estimation procedures remain stable under moderate variations in the censoring structure. This confirms the practical utility of our model in real-world settings where exact censoring patterns may vary.

To evaluate the appropriateness of the proposed GPLD in modeling real reliability data, we compare its fit to several commonly used lifetime distributions: Exponential, Weibull, and Lomax. Model fit is assessed using standard statistical metrics including the K-S test statistic, Akaike information criterion (AIC), and Bayesian information criterion (BIC). The results, summarized in Tables 6 and 7, show that the GPLD consistently outperforms or is at least competitive with the other distributions, particularly in capturing heavy tails and flexibility in shape. These findings justify the selection of the GPLD model for further inference and estimation of the stress-strength reliability parameter R.

Table 6. Comparison of distribution fits (Melanoma dataset).

distribution	K–S	AIC	BIC	interpretation
exponential	0.2171	192.42	194.17	poor tail behavior
Weibull	0.1262	178.93	181.60	good shape flexibility
Lomax	0.1031	174.80	177.55	heavy tail modeled well
GPLD	0.0790	171.26	174.84	best overall fit; tail and flexibility

Table 7. Comparison of distribution fits (Insulating fluid data).

distribution	K–S	AIC	BIC	interpretation
exponential	0.2102	194.60	196.35	poor tail behavior
Weibull	0.1183	179.40	182.05	good shape flexibility
Lomax	0.0992	173.85	176.50	heavy tail modeled well
GPLD	0.0719	168.40	171.95	best overall fit; tail and flexibility

6. Conclusions

In this article, we examine the estimation methods for the SSR parameter $R = P(X > Y)$ under the PT2C scheme, where strength X and stress Y are two independent generalized Poisson Lomax variables sharing the same second-shape and first-scale parameters. Based on the observed Fisher information matrix, the MLEs and ACIs for the SSR parameter R, are obtained. Additionally, the independent gamma prior is used to evaluate the Bayes estimate of R under the squared error and LINEX loss functions. The BEs appear in the ratio of two integrals which cannot be tractable analytically, thus the IST has been used to compute BEs with corresponding credible intervals. Since a theoretical comparison of these methods is not conceivable, the effectiveness of each of the aforementioned methods was then directly contrasted in a simulated study using various sample sizes $(n_i, m_i), i = 1, 2$, censored schemes (I, II, and III), and combinations of unknown parameters (α_1, α_2). The simulation results lead us to the conclusion that the Bayes technique can be adopted to estimate and create approximate confidence intervals for the SSR parameter R when PT2C from different GPLDs are available. Further, it was noted that the importance sampling approach performs quite well, outperforming the ML method. Finally, the GPLD was used to analyze medical and engineering data, and it was discovered that it could accurately reflect current data to the point where it could be relied upon to do so. It is important to acknowledge that our analysis assumes common scale β and shape θ parameters for the strength and stress variables. While this assumption simplifies the estimation process and ensures analytical tractability particularly under PT2C, it may not always reflect the heterogeneity observed in real-world applications. Future research could consider more general models where each variable is governed by its own set of scale and shape parameters. Although such an extension would introduce additional complexity in both the likelihood structure and Bayesian computations, it would allow for more flexible modeling of asymmetric stress and strength mechanisms. Addressing this limitation may yield more accurate and adaptable inference procedures for stress-strength reliability analysis.

Author contributions

Rashad M. EL-Sagheer: Conceptualization, investigation, resources, validation, writing-review and editing; Mohamed F. Abouelenein: data curation, formal analysis and funding; Mohamed S. Eliwa: Conceptualization, formal analysis, methodology, validation, writing-review and editing; Mahmoud El-Morshedy: data curation, investigation, methodology, software, writing-review and editing; Noura Roushdy: formal analysis and funding; Mahmoud M. Ramadan: resources, software, validation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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