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Research article

Boundedness in a quasilinear attraction–repulsion chemotaxis system with variable logistic source

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Abstract: This paper deals with a quasilinear attraction–repulsion chemotaxis system with a source term of variable logistic type $u_t = \nabla\cdot(\phi(u)\nabla u) - \nabla\cdot(\psi(u)\nabla v) + \nabla\cdot(\varphi(u)\nabla w) + g(u)$, $\tau_1 v_t = \Delta v - v + u$, $-\Delta w = -w + u$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, and endowed with nonnegative initial data and homogeneous Neumann boundary conditions. Moreover, the logistic source verifies $g(x,s) \leq \eta s^{k(x)} - \mu s^{m(x)}$, $s > 0$ with $g(x,0) \geq 0$, $x \in \Omega$, where $\eta \geq 0$, $\mu > 0$ are constants, k, m are measurable functions fulfilling $0 \leq k^- := ess\inf_{x\in\Omega} k(x) \leq k(x) \leq k^+ := ess\sup_{x\in\Omega} k(x) < +\infty$ and $1 < m^- := ess\inf_{x\in\Omega} m(x) \leq m(x) \leq m^+ := ess\sup_{x\in\Omega} m(x) < +\infty$, as well as ϕ, ψ, and φ are regular functions satisfying $c_1 s^p \leq \phi(s)$, $\psi(s) \leq c_2 s^q$, and $\underline{c}_3 s^l \leq \varphi(s) \leq c_3 s^l$ with $p, q, l \in \mathbb{R}$, $c_1, c_2, \underline{c}_3, c_3 > 0$ and $s \geq s_0 > 1$. We show that when $q = m^- - 1$ and $l \leq m^- - 1$, there exists $\mu^* > 0$ such that if $\mu > \mu^*$, then the corresponding initial-boundary value problem possesses a unique globally bounded classical solution. Moreover, the same conclusion holds true provided that $q < m^- - 1$ and $l \leq m^- - 1$ for any $\mu > 0$.

Keywords: chemotaxis; attraction–repulsion; nonlinear diffusion; variable logistic source; global boundedness
Mathematics Subject Classification: 35A01, 35K51, 35K55, 35Q92, 92C17

1. Introduction

In this paper, we study the following quasilinear attraction–repulsion chemotaxis system with a source term of variable logistic type given by

$$\begin{cases} u_t = \nabla\cdot(\phi(u)\nabla u) - \nabla\cdot(\psi(u)\nabla v) + \nabla\cdot(\varphi(u)\nabla w) + g(u), & x\in\Omega,\ t>0,\\ \tau_1 v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \tau_2 w_t = \Delta w - w + u, & x\in\Omega,\ t>0,\\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \tau_1 v(x,0)=\tau_1 v_0(x),\ \tau_2 w(x,0)=\tau_2 w_0(x), & x\in\Omega \end{cases} \tag{1.1}$$

in a bounded and smooth domain $\Omega\subset\mathbb{R}^n$ with $n\geq 1$, where $\tau_1>0$, $\tau_2=0$. Herein, u, v, and w denote the density of cells, the concentrations of the attractive and repulsive chemical signals (chemoattractant and chemorepellent), respectively. Throughout the paper, we assume that the diffusion function ϕ and the chemotactic sensitivity functions ψ and φ satisfy

$$\phi,\psi,\varphi\in C^2([0,\infty)),\qquad \phi(s)>0,\quad \text{for } s\geq 0, \tag{1.2}$$

and

$$c_1 s^p \leq \phi(s),\quad \psi(s)\leq c_2 s^q,\quad \underline{c}_3 s^l \leq \varphi(s)\leq c_3 s^l,\quad \text{for } s\geq s_0 \tag{1.3}$$

with $p,q,l\in\mathbb{R}$, $c_1,c_2,\underline{c}_3,c_3>0$ and $s_0\geq 1$, as well as g is a variable logistic function fulfilling

$$g\in C^1(\bar{\Omega}\times[0,\infty)),\quad g(x,0)\geq 0\quad \text{and}\quad g(x,s)\leq \eta s^{k(x)} - \mu s^{m(x)},\ s>0,\ \forall x\in\Omega \tag{1.4}$$

with $\eta\geq 0$ and $\mu>0$. Let $k:\Omega\to[0,\infty)$ and $m:\Omega\to[2,\infty)$ be measurable functions. Define k^-, m^-, k^+, m^+ such that

$$0\leq k^- := ess\inf_{x\in\Omega} k(x)\leq k(x)\leq k^+ := ess\sup_{x\in\Omega} k(x) < +\infty, \tag{1.5}$$

and

$$1 < m^- := ess\inf_{x\in\Omega} m(x)\leq m(x)\leq m^+ := ess\sup_{x\in\Omega} m(x) < +\infty. \tag{1.6}$$

The nonnegative initial data are taken in this way:

$$u_0\in C^\beta(\bar{\Omega}),\quad \beta\in(0,1),\quad \tau_1 v_0\in W^{1,r}(\Omega),\ r>n. \tag{1.7}$$

In the absence of the third equation, system (1.1) with $\phi(u)\equiv 1$, $\psi(u)=\chi u$, and $g\equiv 0$ with $\chi>0$ especially contains the renowned classical Keller–Segel system (see [1]). For parabolic-parabolic simplification (i.e., $\tau_1=1$) with $-c_0(u+u^m)\leq g(u)\leq k-\mu u^m$ for some $c_0,\mu>0$, $k\geq 0$, and $m>1$, it was shown in [2] that model (1.1) possesses a global very weak solution that is uniformly-in-time bounded in the three-dimensional settings for small enough initial data $\|u_0\|_{L^\gamma(\Omega)}$ and $\|\nabla v_0\|_{L^4(\Omega)}$ if the ratio $\frac{k}{\mu}$ does not exceed a certain value and $\frac{9}{5}<\gamma<m<2$. For the case when $g(u)=ku-\mu u^2$ with $\mu>0$, the global weak solutions become classical solutions after some time provided that k is not too large, and there exists an absorbing set if $k>0$ is sufficiently small (see [3]). Otherwise, it was proved that the classical solution to (1.1) with $\tau_1>0$ is global and bounded if $n\geq 1$ and $\mu>\mu_0$ for some constant $\mu_0>0$ [4]. On the other hand, when $\tau_1=0$ and $-c_0(u+u^m)\leq g(u)\leq k-\mu u^m$ with some $c_0,\mu>0$, $k\geq 0$, and $m>1$, the global existence of a very weak solution to model (1.1) was established for any nonnegative initial data $u_0\in L^1(\Omega)$ as long as $m>2-\frac{1}{n}$ (cf. [5]). Moreover, it was asserted in [6] that (1.1) with $g(u)\leq k-\mu u^2$ for some $k,\mu>0$ admits at least one global weak solution for arbitrary $\mu > 0$ and spatial dimension, and a globally bounded classical solution was also studied if $\mu > \frac{n-2}{n}\chi$ and $n \leq 2$. When $c_1u^p \leq \phi(u)$, $c_1u^q \leq \psi(u) \leq c_2u^q$ and $g(u) \leq ku - \mu u^m$ with $c_1, c_2, \mu > 0$, $p, q \in \mathbb{R}$, $k \geq 0$ and $m > 1$, Zhang and Zheng [7] considered a parabolic–parabolic system (i.e., $\tau_1 = 1$), where the globally bounded classical solution was derived and the exact way of the logistic exponent $m > 1$ affecting the behavior of solutions was presented. Furthermore, concentrating on the variable logistic sources, the global existence and boundedness of the classical solution to (1.1) with $\tau_1 = 0$ [8] or $\tau_1 > 0$ [9] were obtained. For more relevant results, we refer the reader to [10–12].

A quasilinear attraction–repulsion chemotaxis system seems to make analysis more challenging. In the absence of logistic sources, an attraction–repulsion system (1.1) with $\phi(u) \equiv 1$, $\psi(u) = \chi u$, $\varphi(u) = \xi u$ with $\chi, \xi > 0$ and $\tau_1 = \tau_2 = 1$ was studied, and the global existence of a unique classical solution in the two-dimensional spaces and the weak solution in the three-dimensional settings was established in [13] under the assumptions that $0 \leq (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ with $u_0 \not\equiv 0$ and $\xi > \chi$. In two dimensions, it was known in [14] that model (1.1) with $\tau_1 = \tau_2 = 0$ admits globally bounded solutions whenever $\chi > \xi$ (i.e., attraction prevails over repulsion) and $\int_\Omega u_0 < \frac{4\pi}{\chi-\xi}$, whereas the finite time blow-up may occur if $\int_\Omega u_0 > \frac{4\pi}{\chi-\xi}$. Moreover, Tao and Wang [15] proved that (1.1) with $\tau_1 = \tau_2 = 1$ admits a unique uniformly bounded global solution for arbitrarily large initial data u_0 if $\xi > \chi$ and $n = 2$, and this global solution converges to the non-trivial stationary solution exponentially. Furthermore, for higher-dimensional settings (i.e., $n \geq 2$), there exists a unique triple (u, v, w) of non-negative bounded functions that solves (1.1) with $\tau_1 = \tau_2 = 0$ classically if $\xi > \chi$, while the solution might blow up in finite time if $\xi < \chi$. For a further extension in the sense that $\phi(u) = D_0u^{-\theta}$ with $D_0 > 0$ and $\theta \in \mathbb{R}$, the solution's behavior is determined by a critical exponent $\theta^* = \frac{2}{n} - 1$. More precisely, Lin et al. [16] deduced a globally bounded classical solution to (1.1) with $\tau_1 = 1$ and $\tau_2 = 0$ if $\theta < \theta^*$ and $n \geq 2$. Additionally, assuming that Ω is a ball, the radially symmetric weak solution blowing up at finite time was obtained when $\theta = \theta^*$ and $n = 3$. Focusing on a broader version of (1.1) with $\phi(u) = (u + 1)^{m_1-1}$, $\psi(u) = \chi u(u + 1)^{m_2-1}$, and $\varphi(u) = \xi u(u + 1)^{m_3-1}$ and replacing the second equation by $v_t = \Delta v - f(u)v$ with $0 \leq f(u) \leq Ku^\alpha$ for some $K, \alpha > 0$, the constellations of the impacts m_1, m_2, and m_3 of the diffusion and drift terms were identified in [17]. Taking the logistic source into account, Frassu et al. [18] generalized the results of the previous articles and established other statements, and the uniformly bounded classical solution to the parabolic–parabolic–elliptic chemotaxis model (i.e., $\tau_1 = 1$, $\tau_2 = 0$) with the nonlinear production and consumption was constructed in [19] when $k(x) \equiv 1$ and $m(x) \equiv m$ with $m > 1$. For the parabolic–elliptic–elliptic type with logistic source $g(u) \leq k - \mu u^m$ for some $k \geq 0$, $\mu > 0$, and $m > 1$, Wang [20] showed that (1.1) with $\phi(u) = D_0(u + \sigma)^{\theta-1}$ with $D_0 > 0$, $\sigma \geq 0$, and $\theta \geq 1$ admits a unique global bounded classical solution provided that $\xi > \chi$, or the logistic dampening is sufficiently strong, or the diffusion is sufficiently strong, and the large-time behavior of solutions for a specific logistic source was obtained.

Turning our attention to the case of the parabolic–parabolic–elliptic systems (i.e., $\tau_1 = 1$, $\tau_2 = 0$) with logistic source (especially variable type), the relevant literature seems to at a rather fragmentary stage. Therefore, the objective of the present work is to extend the results in previous papers and to identify the effect of variable logistic sources on the qualitative behavior of the system. In fact, when the logistic source and system functions satisfy some sufficient conditions, a unique globally bounded classical solution to the considered system (1.1) is verified to exist.

2. Preliminaries

The problem of local solvability to (1.1) for sufficiently smooth initial data can be solved in an appropriate fixed point framework by means of the methods involving standard parabolic and elliptic regularity theory. As a matter of fact, one can derive a sufficient condition for extensibility of a given local-in-time solution. We first give the following lemma (see [21–23] for the proof.)

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ *be a bounded domain with a smooth boundary,* ϕ, ψ, φ*, and* g *comply with the assumptions* (1.2)–(1.4) *with nonnegative initial data* (1.7)*. Assume that* $l \leq m^- - 1$*,* $k(x) < m(x)$*,* $\forall x \in \Omega$*, and* $k^+ < m^-$*. Then, there exists* $T_{\max} \in (0, \infty]$ *such that* (1.1) *admits a unique classical solution* (u, v, w) *fulfilling*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times [0, T_{\max})), \\ v \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \cap L^\infty_{loc}([0, T_{\max}); W^{1,r}(\Omega)), \\ w \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,0}(\bar{\Omega} \times (0, T_{\max})) \cap L^\infty_{loc}([0, T_{\max}); W^{1,r}(\Omega)). \end{cases}$$

Moreover, if $T_{\max} < \infty$*, then*

$$\limsup_{t \nearrow T_{\max}} ||u(\cdot, t)||_{L^\infty(\Omega)} = \infty.$$

On account of the term $\int_\Omega u^{q+\gamma-1} \Delta v$ is unsigned, we thereby will take advantage of the following property referred to a variation of Maximal Sobolev Regularity.

Lemma 2.2. *Let* $r \in (0, \infty)$*. Consider the following evolution equation*

$$\begin{cases} \tau \zeta_t = \Delta \zeta - \zeta + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial \zeta}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ \zeta(x, 0) = \zeta_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

A_i*) (See* [22, Lemma 2.2.]*) Let* $\tau > 0$*. For each* $\zeta_0 \in W^{2,r}(\Omega)$ $(r > n)$ *with* $\frac{\partial \zeta_0}{\partial \nu} = 0$ *on* $\partial\Omega$ *and any* $u \in L^r((0, T); L^r(\Omega))$*, there exists a unique solution*

$$\zeta \in W^{1,r}((0, T); L^r(\Omega)) \cap L^r((0, T); W^{2,r}(\Omega)).$$

Moreover, there exists $C_r > 0$ *such that if* $s_0 \in [0, T)$*,* $\zeta(\cdot, s_0) \in W^{2,r}(\Omega)$ $(r > n)$ *with* $\frac{\partial \zeta(\cdot, s_0)}{\partial \nu} = 0$*, then*

$$\int_{s_0}^T \int_\Omega e^{\frac{r}{\tau}s} |\Delta \zeta|^r \leq C_r \int_{s_0}^T \int_\Omega e^{\frac{r}{\tau}s} u^r + C_r \tau e^{\frac{r}{\tau}s_0} (||\zeta(\cdot, s_0)||^r_{L^r(\Omega)} + ||\Delta \zeta(\cdot, s_0)||^r_{L^r(\Omega)}).$$

A_{ii}*) (See* [23, Lemma 3.1.]*) Let* $\tau = 0$*. For any nonnegative* $u \in C^1(\bar{\Omega})$*, the solution* $0 \leq \zeta \in C^{2,\kappa}(\bar{\Omega})$ *with* $0 < \kappa < 1$ *of system* (2.1) *has the following property: For any* $\hat{c}, \sigma > 0$ *and* $\rho > 1$*, there exists* $\tilde{c} = \tilde{c}(\sigma, \rho) > 0$ *satisfying*

$$\hat{c} \int_\Omega \zeta^{\rho+1} \leq (\sigma + \tilde{c}) \int_\Omega u^{\rho+1}.$$

3. Statement and proof of the main result

In this section, we present the statement and proof of our main result, and we begin by stating the following theorem.

Theorem 3.1. *Let* $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ *be a bounded domain with a smooth boundary,* ϕ, ψ, φ*, and* g *comply with the assumptions* (1.2)–(1.4) *with nonnegative initial data* (1.7)*. Assume that* $l \leq m^- - 1$, $k(x) < m(x)$, $\forall x \in \Omega$*, and* $k^+ < m^-$*. For each of the following cases:*

A_i) *if* $q = m^- - 1$ *with* μ *properly large, i.e.,* $\mu > \mu^*$ *for some* $\mu^* > 0$,

A_{ii}) *if* $q < m^- - 1$ *with any* $\mu > 0$,

then system (1.1) *admits a unique global and bounded classical solution* (u, v, w) *such that*

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times [0, \infty)), \\ v \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \cap L^\infty_{loc}([0, \infty); W^{1,r}(\Omega)), \\ w \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,0}(\bar{\Omega} \times (0, \infty)) \cap L^\infty_{loc}([0, \infty); W^{1,r}(\Omega)). \end{cases}$$

Remark 3.1. *From the claim* A_i) *of Theorem* 3.1*, it is seen that the result is consistent with* [7, Proposition 3.1] *when model* (1.1) *with* $\tau_1 = 1$ *and* $g(s) \leq as - \mu s^k$ *reduces to a parabolic–parabolic chemotaxis system (i.e.,* $w \equiv 0$*), and also with* i) *of* [8, Theorem 1] *when* $\tau_1 = 0$ *and* $w \equiv 0$*; thus it could be argued that our work is a generalization of the existing literature. Moreover, our result* $q < m^- - 1$ *greatly loosens restrictions* $q \in (k^+ - 1, m^- - 1)$ *in* [8] *and* $q \in (0, m^- - 1)$ *in* [9] *imposed on chemotactic sensitivity, and the condition* $m^- > 1$ *in* (1.6) *can be replaced by* $m^- \geq 0$ *when* $w \equiv 0$.

Since the regularity obtained in Lemma 2.2 requires that the initial value satisfy homogeneous Neumann boundary conditions, we shall perform a small time shift to use any positive time as the "initial time" to warrant that the boundary condition is achieved naturally. Given any $s_0 \in (0, T_{\max}) \cap (0, 1)$, by Lemma 2.1, we obtain that $(u(\cdot, s_0), v(\cdot, s_0), w(\cdot, s_0)) \in C^2(\bar{\Omega})$ with $\frac{\partial_v(\cdot, s_0)}{\partial_\nu} = \frac{\partial_w(\cdot, s_0)}{\partial_\nu} = 0$ on $\partial\Omega$, and there exists an $M > 0$ such that

$$\begin{aligned} &\sup_{0 \leq \bar{s} \leq s_0} \|u(\cdot, \bar{s})\|_{L^\infty(\Omega)} \leq M, \ \sup_{0 \leq \bar{s} \leq s_0} \|v(\cdot, \bar{s})\|_{L^\infty(\Omega)} \leq M, \\ &\sup_{0 \leq \bar{s} \leq s_0} \|w(\cdot, \bar{s})\|_{L^\infty(\Omega)} \leq M, \ \|\Delta v(\cdot, s_0)\|_{L^\infty(\Omega)} \leq M. \end{aligned} \tag{3.1}$$

First, we consider the critical case of $q = m^- - 1$ and $l \leq m^- - 1$.

Lemma 3.1. *Let the hypotheses of Theorem* 3.1 *be satisfied. For* $q = m^- - 1$ *and any* $\gamma > 1$*, if there exists* $\mu^* > 0$ *satisfying* $\mu > \mu^*$*, then*

$$\|u(\cdot, t)\|_{L^\gamma(\Omega)} \leq C \quad \text{for all } t \in (s_0, T_{\max})$$

with some $C = C(m^-, k^+, \eta, \mu, c_2, c_3, l, \gamma, u_0, v_0, |\Omega|) > 0$.

Proof. Without loss of generality, we suppose that $\gamma > \max\{1 - q, 2 - l\}$ with $q, l \in \mathbb{R}$. Testing the first equation in (1.1) by $u^{\gamma-1}$ and applying integration by parts and the Young inequality, we obtain

$$
\begin{aligned}
\frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma \le& -c_1(\gamma-1)\int_\Omega u^{p+\gamma-2}|\nabla u|^2+(\gamma-1)\int_\Omega \nabla\Big(\int_0^u \psi(s)s^{\gamma-2}\,ds\Big)\cdot\nabla v\\
&-(\gamma-1)\int_\Omega \nabla\Big(\int_0^u \varphi(s)s^{\gamma-2}\,ds\Big)\cdot\nabla w+\eta\int_\Omega u^{k(\cdot)+\gamma-1}-\mu\int_\Omega u^{m(\cdot)+\gamma-1}\\
\le& -(\gamma-1)\int_\Omega \Big(\int_0^u \psi(s)s^{\gamma-2}\,ds\Big)\Delta v+(\gamma-1)\int_\Omega \Big(\int_0^u \varphi(s)s^{\gamma-2}\,ds\Big)\Delta w\\
&+\eta\int_\Omega u^{k(\cdot)+\gamma-1}-\mu\int_\Omega u^{m(\cdot)+\gamma-1}\\
\le& \frac{c_2(\gamma-1)}{q+\gamma-1}\int_\Omega u^{q+\gamma-1}|\Delta v|+(\gamma-1)\int_\Omega \Big(\int_0^u \varphi(s)s^{\gamma-2}\,ds\Big)(w-u)\\
&+\eta\int_\Omega u^{k(\cdot)+\gamma-1}-\mu\int_\Omega u^{m(\cdot)+\gamma-1}\\
\le& \frac{c_2(\gamma-1)}{q+\gamma-1}\int_\Omega u^{q+\gamma-1}|\Delta v|+\frac{c_3(\gamma-1)}{l+\gamma-1}\int_\Omega u^{l+\gamma-1}w-\frac{\underline{c}_3(\gamma-1)}{l+\gamma-1}\int_\Omega u^{l+\gamma}\\
&+\eta\int_\Omega u^{k(\cdot)+\gamma-1}-\mu\int_\Omega u^{m(\cdot)+\gamma-1}\\
\le& \epsilon_1\int_\Omega u^{m^-+\gamma-1}+C_1\int_\Omega |\Delta v|^{m^-+\gamma-1}+C_2\int_\Omega u^{l+\gamma}+C_3\int_\Omega w^{l+\gamma}\\
&+\eta\int_\Omega u^{k(\cdot)+\gamma-1}-\mu\int_\Omega u^{m(\cdot)+\gamma-1}\quad \text{for all } t\in(s_0,T_{\max})
\end{aligned}
\tag{3.2}
$$

with some $\epsilon_1>0$, $C_1=C_1(m^-,c_2,\epsilon_1,\gamma,q)>0$, $C_2=C_2(c_3,\gamma,l)>0$, and $C_3=C_3(c_3,\gamma,l)>0$. Since $l\le m^--1$, by employing the Young inequality to the third integral on the right-hand side of (3.2), one gets

$$
C_2\int_\Omega u^{l+\gamma}\le \epsilon_2\int_\Omega u^{m^-+\gamma-1}+C_4 \tag{3.3}
$$

for some $\epsilon_2>0$ and $C_4=C_4(m^-,c_3,\epsilon_2,\gamma,l)>0$. In order to estimate the fourth integral on the right-hand side of (3.2), one can deduce from $A_{ii})$ in Lemma 2.2 with $\rho=l+\gamma-1$, $\zeta=w$, and $\hat{c}=C_3$ together with the Young inequality and the condition $l\le m^--1$ that for some $\epsilon_3>0$ and $C_5=C_5(m^-,c_3,\epsilon_3,\gamma,l)>0$,

$$
C_3\int_\Omega w^{l+\gamma}\le(\sigma+\tilde{c})\int_\Omega u^{l+\gamma}\le\epsilon_3\int_\Omega u^{m^-+\gamma-1}+C_5. \tag{3.4}
$$

As for logistic terms, by means of the conditions (1.5) and (1.6), we have

$$
\int_\Omega u^{k(\cdot)+\gamma-1}=\int\limits_{\Omega\cap\{x:u\ge1\}} u^{k(\cdot)+\gamma-1}+\int\limits_{\Omega\cap\{x:u<1\}} u^{k(\cdot)+\gamma-1}\le\int_\Omega u^{k^++\gamma-1}+|\Omega|, \tag{3.5}
$$

and

$$
\int_\Omega u^{m(\cdot)+\gamma-1}\ge\int\limits_{\Omega\cap\{x:u\ge1\}} u^{m^-+\gamma-1}=\int_\Omega u^{m^-+\gamma-1}-\int\limits_{\Omega\cap\{x:u<1\}} u^{m^-+\gamma-1}\ge\int_\Omega u^{m^-+\gamma-1}-|\Omega|. \tag{3.6}
$$

In view of the hypothesis $k^+ < m^-$ and the Young inequality, one infers that for some $\epsilon_4 > 0$ and $C_6 = C_6(m^-, k^+, \eta, \epsilon_4, \gamma) > 0$,

$$\eta \int_\Omega u^{k^+ + \gamma - 1} \le \epsilon_4 \int_\Omega u^{m^- + \gamma - 1} + C_6. \tag{3.7}$$

Substituting (3.3)–(3.7) into (3.2), we attain that for all $t \in (s_0, T_{\max})$,

$$\begin{aligned} \frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma \le & - |\Omega|^{\frac{m^- -1}{\gamma}} \int_\Omega u^{m^- + \gamma - 1} + C_1 \int_\Omega |\Delta v|^{m^- + \gamma - 1} \\ & - \left(\mu - \tilde{\epsilon}_1 - |\Omega|^{\frac{m^- -1}{\gamma}}\right) \int_\Omega u^{m^- + \gamma - 1} + C_7, \end{aligned} \tag{3.8}$$

where $\tilde{\epsilon}_1 = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 > 0$ and $C_7 = C_4 + C_5 + C_6 + (\eta + \mu)|\Omega| > 0$.

On the other hand, by invoking Hölder's and Young's inequalities, one obtains

$$|\Omega|^{\frac{m^- -1}{\gamma}} \int_\Omega u^{m^- + \gamma - 1} \ge \left(\int_\Omega u^\gamma\right)^{\frac{m^- + \gamma - 1}{\gamma}} \ge \frac{m^- + \gamma - 1}{\gamma} \int_\Omega u^\gamma - \frac{m^- - 1}{\gamma},$$

which in conjunction with (3.8), readily implies that

$$\begin{aligned} \frac{1}{\gamma}\frac{d}{dt}\int_\Omega u^\gamma \le & - \frac{m^- + \gamma - 1}{\gamma} \int_\Omega u^\gamma + C_1 \int_\Omega |\Delta v|^{m^- + \gamma - 1} \\ & - \left(\mu - \tilde{\epsilon}_1 - |\Omega|^{\frac{m^- -1}{\gamma}}\right) \int_\Omega u^{m^- + \gamma - 1} + C_8, \end{aligned} \tag{3.9}$$

where $C_8 = C_7 + \frac{m^- -1}{\gamma} > 0$. An application of the variation-of-constants formula to (3.9) yields

$$\begin{aligned} \frac{1}{\gamma}\int_\Omega u^\gamma(\cdot, t) \le & \frac{1}{\gamma} e^{-\frac{(m^- + \gamma - 1)(t - s_0)}{\tau_1}} \int_\Omega u^\gamma(\cdot, s_0) + C_1 \int_{s_0}^t e^{-\frac{(m^- + \gamma - 1)(t-s)}{\tau_1}} ds \int_\Omega |\Delta v(\cdot, s)|^{m^- + \gamma - 1} \\ & - \left(\mu - \tilde{\epsilon}_1 - |\Omega|^{\frac{m^- -1}{\gamma}}\right) \int_{s_0}^t e^{-\frac{(m^- + \gamma - 1)(t-s)}{\tau_1}} ds \int_\Omega u^{m^- + \gamma - 1}(\cdot, s) + C_8 \int_{s_0}^t e^{-\frac{(m^- + \gamma - 1)(t-s)}{\tau_1}} ds \\ \le & C_1 \int_{s_0}^t e^{-\frac{(m^- + \gamma - 1)(t-s)}{\tau_1}} ds \int_\Omega |\Delta v(\cdot, s)|^{m^- + \gamma - 1} + C_9 \\ & - \left(\mu - \tilde{\epsilon}_1 - |\Omega|^{\frac{m^- -1}{\gamma}}\right) \int_{s_0}^t e^{-\frac{(m^- + \gamma - 1)(t-s)}{\tau_1}} ds \int_\Omega u^{m^- + \gamma - 1}(\cdot, s) \quad \text{for all } t \in (s_0, T_{\max}), \end{aligned} \tag{3.10}$$

where

$$C_9 = \sup_{t \ge s_0} \frac{1}{\gamma} \int_\Omega u^\gamma(\cdot, t) + C_8 \int_{s_0}^t e^{-\frac{(m^- + \gamma - 1)(t-s)}{\tau_1}} ds$$

is independent of t. Next, we employ A_i) in Lemma 2.2 to derive that there is a $C_{m^-,\gamma} > 0$ satisfying

$$\begin{aligned} & \int_{s_0}^t e^{-\frac{(m^- + \gamma - 1)(t-s)}{\tau_1}} ds \int_\Omega |\Delta v(\cdot, s)|^{m^- + \gamma - 1} \\ & \le C_{m^-,\gamma} e^{-\frac{(m^- + \gamma - 1)t}{\tau_1}} \left(\int_{s_0}^t \int_\Omega e^{\frac{(m^- + \gamma - 1)s}{\tau_1}} u^{m^- + \gamma - 1}(\cdot, s)\, ds + \tau_1 e^{\frac{(m^- + \gamma - 1)s_0}{\tau_1}} \|v(\cdot, s_0)\|^{m^- + \gamma - 1}_{W^{2, m^- + \gamma - 1}(\Omega)} \right). \end{aligned} \tag{3.11}$$

Inserting (3.11) into (3.10), one arrives at

$$\frac{1}{\gamma}\int_\Omega u^\gamma(\cdot,t) \le C_1 C_{m^-,\gamma} e^{-(m^-+\gamma-1)(t-s_0)} \|v(\cdot,s_0)\|^{m^-+\gamma-1}_{W^{2,m^-+\gamma-1}(\Omega)} + C_9$$
$$- \left(\mu - \tilde{\epsilon}_1 - |\Omega|^{\frac{m^- -1}{\gamma}} - C_1 C_{m^-,\gamma}\right) \int_{s_0}^t e^{-\frac{(m^-+\gamma-1)(t-s)}{\tau_1}}\, ds \int_\Omega u^{m^-+\gamma-1}(\cdot,s)$$

for all $t \in (s_0, T_{\max})$. Let $\mu^* = \mu^*(m^-, c_2, \gamma, q, |\Omega|) = \inf_{\tilde{\epsilon}_1>0}(\tilde{\epsilon}_1 + |\Omega|^{\frac{m^- -1}{\gamma}} + C_1 C_{m^-,\gamma}) > 0$. It is entailed that for any $\mu > \mu^*$,

$$\frac{1}{\gamma}\int_\Omega u^\gamma(\cdot,t) \le C_{10} \quad \text{for all } t \in (s_0, T_{\max})$$

with $C_{10} = C_9 + C_1 C_{m^-,\gamma} e^{-(m^-+\gamma-1)(t-s_0)} \|v(\cdot,s_0)\|^{m^-+\gamma-1}_{W^{2,m^-+\gamma-1}(\Omega)}$. The proof is complete by recalling Lemma 2.1 and (3.1).

Next, we will establish the similar L^γ estimate for the case of $q < m^- - 1$ and $l \le m^- - 1$.

Corollary 3.1. *If $q < m^- - 1$ and $l \le m^- - 1$, then for any $\gamma > 1$ and $\mu > 0$, there exists a $C = C(m^-, k^+, \eta, \mu, c_2, c_3, q, l, \gamma, u_0, v_0, |\Omega|) > 0$ such that*

$$\|u(\cdot,t)\|_{L^\gamma(\Omega)} \le C \quad \textit{for all } t \in (s_0, T_{\max}).$$

Proof. Without loss of generality, suppose that $\gamma > \max\{1-q, 2-l\}$ with $q, l \in \mathbb{R}$. Since $q < m^- - 1$, manipulating the terms on the right-hand side of (3.2) involving $|\Delta v|$ leads to

$$\frac{c_2(\gamma-1)}{q+\gamma-1}\int_\Omega u^{q+\gamma-1}|\Delta v| \le \lambda\Big(\int_\Omega u^{m^-+\gamma-1} + \int_\Omega |\Delta v|^{m^-+\gamma-1}\Big) + C_{11}|\Omega|$$

with some $\lambda > 0$ and $C_{11} > 0$. The remaining procedures are similar to the proof of Lemma 3.1 (i.e., (3.2)–(3.11)), we can assert that

$$\frac{1}{\gamma}\int_\Omega u^\gamma(\cdot,t) \le \lambda C_{m^-,\gamma} e^{-\frac{(m^-+\gamma-1)(t-s_0)}{\tau_1}} \|v(\cdot,s_0)\|^{m^-+\gamma-1}_{W^{2,m^-+\gamma-1}(\Omega)} + C_{12}$$
$$- \left(\mu - \tilde{\epsilon}_2 - \lambda - |\Omega|^{\frac{m^- -1}{\gamma}} - \lambda C_{m^-,\gamma}\right) \int_{s_0}^t e^{-\frac{(m^-+\gamma-1)(t-s)}{\tau_1}}\, ds \int_\Omega u^{m^-+\gamma-1}(\cdot,s)$$

for all $t \in (s_0, T_{\max})$, where $\tilde{\epsilon}_2 = \epsilon_2 + \epsilon_3 + \epsilon_4$ and

$$C_{12} = \sup_{t\ge s_0} C_{11}|\Omega| \int_{s_0}^t e^{-\frac{(m^-+\gamma-1)(t-s)}{\tau_1}}\, ds + C_9$$

is independent of t. Letting $\tilde{\epsilon}_2$, λ be sufficiently small and γ be arbitrarily large, one can deduce that $\mu - \tilde{\epsilon}_2 - \lambda - |\Omega|^{\frac{m^- -1}{\gamma}} - \lambda C_{m^-,\gamma} > 0$ for any $\mu > 0$. The proof of Corollary 3.1 is thereby finished.

Finally, we invoke the standard Alikakos–Moser iteration to obtain the boundedness of (u, v, w).

Proof of Theorem 3.1. In line with Moser's iteration [24, Lemma A.1], we assert that there is $\gamma_0 > n > 0$, determined via (A.8)–(A.10) in Lemma A.1 of [24], such that if

$$\|u(\cdot,t)\|_{L^\gamma(\Omega)} \le \infty \tag{3.12}$$

for all $\gamma > \gamma_0$ and $t \in (s_0, T_{\max})$; then there exists $C_0 > 0$ fulfilling

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C_0 \quad \text{for all } t \in (s_0, T_{\max}).$$

Actually, (3.12) ensures that ∇v and ∇w are bounded. It is not difficult to verify that all assumptions of Lemma A.1 are satisfied. Noticing that (3.12) is warranted by Lemma 3.1 and Corollary 3.1, we then derive from (3.1) that u is bounded in $(0, T_{\max})$. Furthermore, the boundedness of v and w can be achieved by the standard parabolic and elliptic regularity. Consequently, the statement of Lemma 2.1 entails that (u, v, w) is globally bounded.

4. Conclusions

In this paper, we establish rigorous criteria ensuring the global existence and boundedness of the classical solution to a complex quasilinear attraction-repulsion chemotaxis system with a variable logistic source. The critical interplay between the degradation strength μ, the chemotactic sensitivities governed by the exponents q and l, and the lower bound m^- of the degradation exponent $m(x)$ is precisely characterized. From the perspective of blow-up prevention, the rigorous proof that sufficiently strong degradation (controlled by μ) and sufficiently weak attraction sensitivity (controlled by q) can enforce the boundedness of the classical solution and avoid cell aggregation provides crucial predictive power. Our results extend the theory for chemotaxis systems by successfully incorporating spatially variable exponents in the logistic source and nonlinear diffusion.

From the standpoint of practical applications, system (1.1), including nonlinear diffusion, is a generalization of the parabolic–parabolic–elliptic version, which might be used to describe the aggregation of microglia observed in Alzheimer's disease (cf. [25]) to a certain extent. Moreover, the variable logistic source realistically captures spatially heterogeneous proliferation and crowding effects, relevant in contexts like tumor invasion, disease infection, or population dynamics within non-uniform environments. Our study provides a theoretical basis for regulating biological population dynamics, and our results mean that the model solutions remain biologically bounded under the identified parameter regimes, which might offer quantitative guidance for practical issues such as maintaining ecological balance and disease treatment.

On the other hand, it is important to consider the impact of stochastic noise on the dynamics of chemotaxis systems, as environmental fluctuations are ubiquitous in biological contexts. Recent works have developed powerful methodologies for analyzing complex stochastic systems, including mean-field stochastic differential equations driven by G-Brownian motion [26] and event-triggered control for stability of stochastic delay systems with Lévy noise [27]. Extending the deterministic model (1.1) to incorporate stochastic perturbations represents a significant and challenging direction for future research (e.g., defining stochastic integrals with the nonlinear sensitivities, handling multiplicative noise, proving existence and uniqueness for stochastic differential equations with chemotactic terms, and establishing stabilization and noise-induced transition results).

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

References

1. E. F. Keller, L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.*, **26** (1970), 399–415. https://doi.org/10.1016/0022-5193(70)90092-5
2. G. Viglialoro, Boundedness properties of very weak solutions to a fully parabolic chemotaxis-system with logistic source, *Nonlinear Anal.-Real*, **34** (2017), 520–535. https://doi.org/10.1016/j.nonrwa.2016.10.001
3. J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, *J. Differ. Equations*, **258** (2015), 1158–1191. https://doi.org/10.1016/j.jde.2014.10.016
4. M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, *Commun. Part. Diff. Eq.*, **35** (2010), 1516–1537. https://doi.org/10.1080/03605300903473426
5. M. Winkler, Chemotaxis with logistic source: Very weak global solutions and their boundedness properties, *J. Math. Anal. Appl.*, **348** (2008), 708–729. https://doi.org/10.1016/j.jmaa.2008.07.071
6. J. I. Tello, M. Winkler, A chemotaxis system with logistic source, *Commun. Part. Diff. Eq.*, **32** (2007), 849–877. https://doi.org/10.1080/03605300701319003
7. Y. Zhang, S. Zheng, Global boundedness of solutions to a quasilinear parabolic-parabolic Keller–Segel system with logistic source, *Appl. Math. Lett.*, **52** (2016), 15–20. https://doi.org/10.1016/j.jmaa.2022.126482
8. R. Ayazoglu, Global boundedness of solutions to a quasilinear parabolic-parabolic Keller–Segel system with variable logistic source, *J. Math. Anal. Appl.*, **516** (2022), 126482. https://doi.org/10.1016/j.jmaa.2022.126482
9. R. Ayazoglu, E. Akkoyunlu, Boundedness of solutions to a quasilinear parabolic-parabolic chemotaxis model with variable logistic source, *Z. Angew. Math. Phys.*, **73** (2022), 212. https://doi.org/10.1007/s00033-022-01847-0
10. Y. Tao, M. Winkler, Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant, *J. Differ. Equations*, **252** (2012), 2520–2543. https://doi.org/10.1016/j.jde.2011.07.010
11. M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, *J. Differ. Equations*, **248** (2010), 2889–2905. https://doi.org/10.1016/j.jde.2010.02.008
12. M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller–Segel system, *J. Math. Pure. Appl.*, **100** (2013), 748–767. https://doi.org/10.1016/j.matpur.2013.01.020
13. H. Y. Jin, Boundedness of the attraction-repulsion Keller–Segel system, *J. Math. Anal. Appl.*, **422** (2015), 1463–1478. https://doi.org/10.1016/j.jmaa.2014.09.049
14. Q. Guo, Z. Jiang, S. Zheng, Critical mass for an attraction-repulsion chemotaxis system, *Appl. Anal.*, **97** (2018), 2349–2354. https://doi.org/10.1080/00036811.2017.1366989

15. Y. Tao, Z. A. Wang, Competing effects of attraction vs. repulsion in chemotaxis, *Math. Mod. Meth. Appl. S.*, **23** (2013), 1–36. https://doi.org/10.1142/S0218202512500443
16. K. Lin, C. Mu, Y. Gao, Boundedness and blow up in the higher-dimensional attraction-repulsion chemotaxis system with nonlinear diffusion, *J. Differ. Equations*, **261** (2016), 4524–4572. https://doi.org/10.1016/j.jde.2016.07.002
17. S. Frassu, C. V. D. Mee, G. Viglialoro, Boundedness in a nonlinear attraction-repulsion Keller–Segel system with production and consumption, *J. Math. Anal. Appl.*, **504** (2021), 125428. https://doi.org/10.1016/j.jmaa.2021.125428
18. S. Frassu, T. Li, G. Viglialoro, Improvements and generalizations of results concerning attraction-repulsion chemotaxis models, *Math. Method. Appl. Sci.*, **45** (2022), 11067–11078. https://doi.org/10.1002/mma.8437
19. T. Li, S. Frassu, G. Viglialoro, Combining effects ensuring boundedness in an attraction-repulsion chemotaxis model with production and consumption, *Z. Angew. Math. Phys.*, **74** (2023), 109. https://doi.org/10.1007/s00033-023-01976-0
20. Y. Wang, A quasilinear attraction-repulsion chemotaxis system of parabolic-elliptic type with logistic source, *J. Math. Anal. Appl.*, **441** (2016), 259–292. https://doi.org/10.1016/j.jmaa.2016.03.061
21. X. Cao, Boundedness in a quasilinear parabolic-parabolic Keller–Segel system with logistic source, *J. Math. Anal. Appl.*, **412** (2014), 181–188. https://doi.org/10.1016/j.jmaa.2013.10.061
22. C. Yang, X. Cao, Z. Jiang, S. Zheng, Boundedness in a quasilinear fully parabolic Keller–Segel system of higher dimension with logistic source, *J. Math. Anal. Appl.*, **430** (2015), 585–591. https://doi.org/10.1016/j.jmaa.2015.04.093
23. G. Viglialoro, Influence of nonlinear production on the global solvability of an attraction-repulsion chemotaxis system, *Math. Nachr.*, **294** (2021), 2441–2454. https://doi.org/10.1002/mana.201900465
24. Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller–Segel system with subcritical sensitivity, *J. Differ. Equations*, **252** (2012), 692–715. https://doi.org/10.1016/j.jde.2011.08.019
25. M. Luca, A. C. Ross, L. E. Keshet, A. Mogilner, Chemotactic signaling, microglia, and Alzheimer's disease senile plaques: Is there a connection? *B. Math. Biol.*, **65** (2003), 693–730. https://doi.org/10.1016/S0092-8240(03)00030-2
26. H. Yuan, Q. Zhu, The well-posedness and stabilities of mean-field stochastic differential equations driven by G-Brownian motion, *SIAM J. Control Optim.*, **63** (2025), 596–624. https://doi.org/10.1137/23M1593681
27. Q. Zhu, Event-triggered sampling problem for exponential stability of stochastic nonlinear delay systems driven by Lévy processes, *IEEE T. Automat. Contr.*, **70** (2025), 1176–1183. http://doi.org/10.1109/TAC.2024.3448128