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Research article

A new investigation of impulsive fractional stochastic delayed systems in the framework of (δ, ψ)-Hilfer derivative and Lévy processes

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Abstract: In this paper, the averaging result for impulsive (δ, ψ)-Hilfer fractional stochastic delayed differential equations (FSDDEs) caused by the Lévy process was derived. In the sense of mean square, the relationship between the equivalent solutions of the original equations and the averaged equation solutions was demonstrated. Our findings allowed us to shift our attention from the original, more complicated system to the averaged system. Additionally, to demonstrate the relevance and practicality of our findings, an example was given.

Keywords: stochastic system; generalized Hilfer fractional derivative; averaging principle; mild solutions
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1. Introduction

In recent years, the mathematical modeling of dynamical systems has been significantly enhanced by the incorporation of fractional-order derivatives. These operators allow for more accurate modeling of systems with memory and hereditary properties, offering advantages over classical integer-order models in a wide variety of scientific and engineering applications [1–3]. Fractional differential operators (FDOs), often defined via their associated fractional integral operators (FIOs), have become fundamental tools in the development of fractional calculus (FC). Common types of FDOs include the Riemann–Liouville, Caputo, Hadamard, Katugampola, and Hilfer derivatives, each exhibiting distinctive nonlocal behaviors governed by their respective kernel structures [4–6]. The flexibility and effectiveness of fractional models have stimulated ongoing research into generalized operators and their applications in emerging areas such as control theory, viscoelasticity, and anomalous diffusion [7–9]. Continued exploration of these operators is crucial for advancing theoretical understanding and enhancing real-world modeling capabilities [10].

Among recent advancements, Sousa and Oliveira [11] introduced a generalized derivative—the ψ-Hilfer $\mathcal{FDO}$—defined with respect to another function ψ, thus unifying several well-known operators. This idea was further extended by Kucche and Mali [12], who investigated properties of the (k, ψ)-Hilfer operator. These developments underpin the growing interest in generalized fractional dynamics, where system behavior is modulated not just by order but by structure-defining kernel functions.

The intersection of stochastic analysis and fractional calculus has given rise to an advanced analytical framework known as fractional stochastic differential equations (FSDEs). This emerging theory synthesizes the memory-preserving nature of fractional-order derivatives with the probabilistic modeling of random dynamical systems [13, 14]. Within this hybrid setting, fractional derivatives are employed to effectively model hereditary effects and nonlocal temporal structures, which are especially relevant in contexts such as anomalous diffusion and viscoelastic materials [15]. On the stochastic side, random fluctuations are typically represented through Gaussian noise processes, most notably standard Brownian motion and its generalization—the fractional Brownian motion (fBm)—which captures long-range dependence through its Hurst parameter [16–18].

The interplay between fractional derivatives and stochastic perturbations has catalyzed substantial progress in the study of nonlinear FSDEs. Notably, this includes models featuring delay effects, impulsive behavior, non-instantaneous dynamics, and stochastic forcing driven by both Wiener processes and Poisson jump noise [19–21]. This growing body of work has further expanded to encompass systems governed by multivalued dynamics and hemivariational inclusions, particularly within hybrid stochastic environments [22–24]. Moreover, attention has been directed toward Sobolev-type systems and their associated controllability properties, highlighting the versatility of this analytical framework [25–27]. Moreover, the increasing analytical intractability of such stochastic-fractional systems has motivated the invocation of stochastic averaging principles—an asymptotic reduction technique designed to preserve the essential statistical and dynamical signatures of the original system while simplifying its structural complexity [28, 29]. These averaging methodologies, when applied judiciously, offer profound insights into the effective behavior of FSDEs under small stochastic perturbations.

However, real-world systems often manifest abrupt jumps and heavy-tailed behaviors that are poorly captured by Gaussian noise. In such cases, non-Gaussian Lévy noise ($\mathbb{LN}$) offers a more comprehensive framework. $\mathbb{LN}$ accounts for both continuous fluctuations and discontinuous jumps, due to its infinite activity and heavy-tail properties, thereby providing a richer class of stochastic perturbations. This has motivated significant efforts in analyzing stochastic differential equations (SDEs) driven by $\mathbb{LN}$, especially in the context of stability, controllability, and averaging [30, 31].

For example, Balasubramaniam [30] investigated existence results for Hilfer-type FSDEs under $\mathbb{LN}$, while Pei et al. [31] utilized successive approximations to validate the solvability of $\mathbb{LN}$-driven models. Recently, the Ulam–Hyers–Rassias stability of Hilfer-type impulsive stochastic systems under time-changed Brownian motion was examined in [32], with applications to currency option pricing models. Ahmed and Zhu [33] explored delayed Hilfer FSDEs perturbed by $\mathbb{LN}$ in finite-dimensional spaces, yet their formulation excluded impulsive dynamics or evolution systems. Similarly, Luo et al. [34] analyzed averaging for non-Lipschitz Hilfer FSDEs but without incorporating Lévy noise, whereas Shen et al. [35] considered averaging under $\mathbb{LN}$ but did not adopt the Hilfer fractional derivative framework.

Notwithstanding recent advancements, substantial lacunae persist in the rigorous analytical treatment of impulsive fractional stochastic delay differential equations (IFSDDEs) governed by generalized fractional calculus and perturbed by Lévy-type stochasticities. Related work on dynamical behavior in impulsive fractional neutral stochastic delay systems was reported in [27], which investigated solution behavior and stability in a boundary value setting. In particular, the intricate interplay between impulsive effects and nonlocal memory kernels under the influence of Lévy noise $\mathbb{LN}$ remains inadequately charted in the literature. A pivotal aspect that continues to elude comprehensive investigation is the role of structural modulating functions—such as the generalized time-scaling map ψ—especially in stochastic hybrid frameworks where impulsive dynamics and discontinuous jump perturbations coalesce. Furthermore, emergent methodologies, including event-triggered control paradigms and sampling mechanisms tailored for Lévy-driven nonlinear stochastic delay systems, underscore the intensifying relevance and analytical intricacy of such models [36]. Motivated by the conversations recounted above, this paper aspires to investigate the averaging result for impulsive (δ, ψ)-HFSDDEs with $\mathbb{LN}$ in the following manner:

$$\begin{cases} {}^{\delta,\mathcal{H}}\mathbb{D}_{\iota+}^{\mathfrak{A},\mathfrak{B};\psi}\Big(\varpi(t) - \zeta(t, \varpi_t)\Big) = \xi(t, \varpi(t), \varpi_t) + \varrho(t, \varpi(t), \varpi_t)\frac{d\mathfrak{L}(t)}{dt}, \quad t \in \mathfrak{J} := (\iota, \mu], \quad t \neq t_k, \\ \Delta {}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_\delta;\psi}\varpi\mid_{t=t_k} = \mathbb{C}_k(\varpi(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, \mathcal{T}, \\ \varpi(t) = \chi(t), \quad -\theta \leq t \leq \iota, \\ {}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_\delta;\psi}\varpi(\iota) = \beta, \qquad A_\delta = \mathfrak{A} + \mathfrak{B}(\delta - \mathfrak{A}), \end{cases} \tag{1.1}$$

where ${}^{\delta,\mathcal{H}}\mathbb{D}_{\iota+}^{\mathfrak{A},\mathfrak{B};\psi}(\cdot)$ is the (δ, ψ)-Hilfer fractional derivative ($\mathcal{FD}$) of order $0 < \mathfrak{A} < \delta/2$, $\delta > 0$, and type $0 \leq \mathfrak{B} \leq 1$; and ${}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_\delta;\psi}(\cdot)$ is the (δ, ψ)-Riemann–Liouville fractional integral ($\mathcal{FI}$) of order $\delta - A_\delta$. The function $\varpi(\cdot)$ takes values in $\mathscr{A}$, the separable Hilbert space (sHs) with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let the Lévy process $\mathfrak{L}$ be defined on another sHs $\mathscr{B}$. The impulsive effect is given by $\Delta {}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_\delta;\psi}\varpi\Big|_{t=t_k} = \varpi(t_k^+) - \varpi(t_k^-)$, where $\varpi(t_k^+)$ and $\varpi(t_k^-)$ denote the right and left limits of ϖ at $t = t_k$. The notation $\varpi_t = \varpi(t + \epsilon)$ for $-\theta \leq \epsilon \leq \iota$ is used to indicate history dependence. The underlying probability space is $(\Omega, \mathcal{D}, \mathbb{P})$, where $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathcal{D})$, and $\{\mathcal{D}_t\}_{t\geq 0}$ is a filtration satisfying the usual conditions. Assume that there exists an ε-finite measurable space $(\bigvee, V, \nu(d\mathfrak{X}))$, and define $\aleph(t, d\mathfrak{X}) := \tilde{G}(t, d\mathfrak{X}) - t\nu(d\mathfrak{X})$. Let $\chi = \{\chi(t),\ -\theta \leq t \leq \iota\}$ satisfy $\mathbb{E}\left\{\sup_{-\theta\leq t\leq\iota} \|\chi(t)\|^2\right\} < \infty$.

This paper seeks to address the following key challenges:

- The derivation of an averaging principle for impulsive (δ, ψ)-Hilfer fractional stochastic delay differential equations (HFSDDEs) perturbed by Lévy noise, a class that combines jump discontinuities, nonlocal memory, generalized fractional structure, and stochastic irregularities.

- Establishing the existence of mild solutions within an appropriate evolution triple under assumptions of mean-square continuity and non-Lipschitz drift.
- Formulating and proving a stochastic averaging theorem that captures the asymptotic behavior of solutions in the presence of $\mathbb{LN}$ and impulsive perturbations.

These contributions are novel in several respects: (i) They extend previous work on Hilfer FSDEs by explicitly incorporating impulsive effects and delay terms; (ii) they refine the analytical framework by using (δ, ψ)-Hilfer derivatives, which generalize earlier operators; and (iii) they broaden the scope of existing stochastic averaging results to accommodate non-Gaussian dynamics.

The paper is organized as follows. Section 2 presents the necessary preliminaries, function spaces, and properties of the (δ, ψ)-Hilfer operator. Section 3 formulates the problem and enumerates the hypotheses, constructs the mild solution and establishes existence results, and presents the main stochastic averaging result and its proof. Section 4 illustrates the applicability of the theoretical findings via a concrete example. Finally, Section 5 concludes the paper with future directions.

2. Preliminaries

The investigation of the proposed problem demands the following definitions and lemmas.

(1) Let $\mathfrak{S} = [\iota, \mu]$ constitute a certain period and $\psi \in \mathbf{C}^1(\mathfrak{S}, \mathcal{A})$ be a growing function with $\psi'(t) \neq 0$, $\forall\ t \in \mathfrak{S}$. Let $0 < \mathfrak{A} < \delta$ ($\mathfrak{A}, \delta \in \mathbb{R}$), $\mathfrak{B} \in [0, 1]$, and $0 \leq 1 - \frac{A_\delta}{\delta} \leq 1$. Consider the space $\mathfrak{C}_{1-\frac{A_\delta}{\delta};\psi}(\mathfrak{S}, \mathcal{A})$ of scaled functions $\jmath$ specified on $\mathfrak{S}$ supplied by

$$\begin{aligned}\mathbb{H} \ := \ \mathfrak{C}_{1-\frac{A_\delta}{\delta};\psi}(\mathfrak{S}, \mathcal{L}^2(\Omega, \mathcal{A})) := \Big\{ \jmath : (\iota, \mu] \to \mathcal{L}^2(\Omega, \mathcal{A}) : \jmath(\iota+) \\ exists\ and\ (\psi(\cdot) - \psi(\iota))^{1-\frac{A_\delta}{\delta}} \jmath(\cdot) \in \mathbf{C}(\mathfrak{S}, \mathcal{L}^2(\Omega, \mathcal{A})) \Big\}.\end{aligned}$$

(2) Consider the weighted space

$$\mathfrak{C}^{A_\delta}_{1-\frac{A_\delta}{\delta};\psi}(\mathfrak{S}, \mathcal{L}^2(\Omega, \mathcal{A})) := \Big\{ \jmath \in \mathbb{H} : {}^{\delta,RL}\mathbb{D}^{A_\delta;\psi}_{\iota+} \jmath \in \mathbb{H} \Big\}.$$

(3) We take into account the indexed space for $n \in \mathbb{N}$,

$$\mathfrak{C}^n_{1-\frac{A_\delta}{\delta};\psi}(\mathfrak{S}, \mathcal{L}^2(\Omega, \mathcal{A})) := \Big\{ \jmath : \Big(\frac{\delta}{\psi'(t)}\frac{d}{dt}\Big)^{n-1} \jmath(t) \in \mathbf{C}(\mathfrak{S}, \mathcal{L}^2(\Omega, \mathcal{A}))\ and\ \Big(\frac{\delta}{\psi'(t)}\frac{d}{dt}\Big)^{n} \jmath(t) \in \mathbb{H} \Big\}.$$

(4) The Banach space $\mathbb{S} = PC_{1-\frac{A_\delta}{\delta};\psi}(\mathfrak{J}, \mathcal{L}^2(\Omega, \mathcal{A}))$ is defined by

$$\begin{aligned}\mathbb{S} \ := \ \Big\{ \jmath : \mathfrak{J} \to \mathcal{L}^2(\Omega, \mathcal{A}) : \jmath \in \mathfrak{C}_{1-\frac{A_\delta}{\delta};\psi}((t_k, t_{k+1}], \mathcal{A}), k = 1, 2, \cdots, \mathcal{T}, {}^{\delta}\mathbb{I}^{1-\frac{A_\delta}{\delta};\psi}_{t_k^+} \jmath(t_k^+) \\ and\ {}^{\delta}\mathbb{I}^{1-\frac{A_\delta}{\delta};\psi}_{t_k^+} \jmath(t_k^-) = {}^{\delta}\mathbb{I}^{1-\frac{A_\delta}{\delta};\psi}_{t_k} \jmath(t_k)\ exists\ for\ k = 1, 2, \cdots, \mathcal{T} \Big\},\end{aligned}$$

with the norm

$$\|\jmath\|_{\mathbb{S}} = \left(\sup_{t \in \mathfrak{J}} \mathbb{E} \left| (\psi(t) - \psi(\iota))^{1-\frac{A_\delta}{\delta}} \jmath(t) \right|^2 \right)^{1/2}.$$

Definition 2.1. *([12]) Let $\jmath$ be an integrable function defined on $[\iota,\mu]$ and $\delta>0$. Then, the (δ,ψ)-RL $\mathcal{FIO}$ of order $\lambda>0(\lambda\in\mathbb{R})$ of $\jmath$ is supplied by*

$$ {}^{\delta}\mathbb{I}_{\iota+}^{\lambda;\psi}\jmath(t)=\frac{1}{\delta\Gamma_{\delta}(\lambda)}\int_{\iota}^{t}\psi^{'}(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\lambda}{\delta}-1}\jmath(\mathfrak{G})d\mathfrak{G}, \tag{2.1}$$

$$\Gamma_{\delta}(z)=\int_{0}^{\infty}t^{z-1}e^{-\frac{t^{\delta}}{\delta}}dt,\ z\in\mathbb{C},Rez>0,\delta>0.$$

For more details and properties on $\Gamma_{\delta}(\cdot)$, see [12].

Definition 2.2. *([12]) Let $\jmath\in\mathbf{C}^{n}(\mathfrak{S},\mathcal{L}^{2}(\Omega,\mathcal{A}))$, $\psi\in\mathbf{C}^{n}(\mathfrak{S},\mathcal{L}^{2}(\Omega,\mathcal{A}))$, $\psi^{'}(t)\neq0,\ \forall t\in[0,\mu]$, $\mathfrak{A},\delta\in\mathbb{R}^{+}$ and $\mathfrak{B}\in[0,1]$. The (δ,ψ)-Hilfer $\mathcal{FDO}$ of $\jmath$ of order $\mathfrak{A}$ and type $\mathfrak{B}$ is provided by*

$$ {}^{\delta,\mathcal{H}}\mathbb{D}_{\iota+}^{\mathfrak{A},\mathfrak{B};\psi}\jmath(t)={}^{\delta}\mathbb{I}_{\iota+}^{\mathfrak{B}(\delta n-\mathfrak{A});\psi}\rho_{\psi}^{n}\,{}^{\delta}\mathbb{I}_{\iota+}^{(1-\mathfrak{B})(\delta n-\mathfrak{A});\psi}\jmath(t), \tag{2.2}$$

where $\rho_{\psi}^{n}=\left(\frac{\delta}{\psi^{'}(t)}\frac{d}{dt}\right)^{n}$ and $n=\lceil\frac{\mathfrak{A}}{\delta}\rceil$.

$\{\mathfrak{W}(t)\}_{t\geq0}$ represents a $\mathcal{B}$-valued Θ-Brownian motion (Bm) process mentioned on $(\Omega,\mathcal{D},\mathcal{F})$. Let $\breve{\mathcal{B}}=\Theta^{1/2}\mathcal{B}$, $\mathcal{L}_{\Theta}^{2}(\breve{\mathcal{B}},\mathcal{A})$ be a sHs comprising all Hilbert–Schmidt operators from $\breve{\mathcal{B}}$ to $\mathcal{A}$. Nevertheless, the initial function $\chi(t)$ is a $\mathcal{D}_{0}$-measurable PC-valued random variable (r.v.) fulfilling $\mathbb{E}\|\chi\|^{2}<\infty$. Additionally, $\mathcal{L}^{2}(\Omega,\mathcal{A})$ is a Banach space furnished with $\|\varpi(t)\|_{\mathcal{L}^{2}(\Omega,\mathcal{A})}=\left(\mathbb{E}\|\varpi(t)\|^{2}\right)^{1/2}$.

Lemma 2.1. *([37]) The expression*

$$\mathfrak{V}(t)=\mathfrak{G}t+\mathfrak{W}(t)+\int_{\|\mathfrak{X}\|<s}\mathfrak{X}\,\tilde{\aleph}(dt,d\mathfrak{X})+\int_{\|\mathfrak{X}\|\geq s}\mathfrak{X}\,\aleph(dt,d\mathfrak{X})$$

is identified as the Lévy-Itô decomposition, where $\mathfrak{G}\in\mathcal{B}$, $s>0$ is a constant, and $\mathfrak{W}$ is a $\mathcal{B}$-valued Brownian motion with covariance operator Θ. The term $\tilde{\aleph}$ denotes the compensated Poisson random measure associated with small jumps, while $\aleph$ is the Poisson random measure capturing large jumps. Both are defined on $\mathbb{R}^{+}\times(\mathcal{B}\setminus\{0\})$, and the jump measure ν satisfies the integrability condition

$$\int\left(\|\mathfrak{X}\|_{\mathcal{B}}^{2}\wedge1\right)\nu(d\mathfrak{X})<\infty.$$

In light of the earlier discussion, system (1.1) might be reorganized into the following, more thorough format:

$$\begin{aligned}
{}^{\delta,\mathcal{H}}\mathbb{D}_{\iota+}^{\mathfrak{A},\mathfrak{B};\psi}\Big(\varpi(t)-\zeta(t,\varpi_{t})\Big) &= \tilde{\xi}(t,\varpi(t),\varpi_{t})+\tilde{\varrho}(t,\varpi(t),\varpi_{t})\frac{d\mathfrak{W}(t)}{dt}\\
&\quad+\frac{1}{dt}\int_{\|\mathfrak{X}\|<s}\mathfrak{R}(t,\varpi(t),\varpi_{t},\mathfrak{X})\tilde{\aleph}(dt,d\mathfrak{X})\\
&\quad+\frac{1}{dt}\int_{\|\mathfrak{X}\|\geq s}\mathfrak{T}(t,\varpi(t),\varpi_{t},\mathfrak{X})\aleph(dt,d\mathfrak{X}),\quad t\in\mathfrak{J}:=(\iota,\mu],\quad t\neq t_{k},\\
\Delta\,{}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_{\delta};\psi}\varpi\,|_{t=t_{k}} &= \mathbb{C}_{k}(\varpi(t_{k}^{-})),\quad t=t_{k},\quad k=1,2,\cdots,\mathcal{T},\\
\varpi(t) &= \chi(t),\quad -\theta\leq t\leq\iota,\\
{}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_{\delta};\psi}\varpi(\iota) &= \beta,
\end{aligned}\tag{2.3}$$

where $\tilde{\xi}$, $\tilde{\varrho}$, $\Re$, and $\mathfrak{T}$ are measurable. Considering that we have to focus on the tiny-jump stochastic differential system, we get

$$\begin{aligned}
{}^{\delta,\mathcal{H}}\mathbb{D}_{\iota+}^{\mathfrak{A},\mathfrak{B};\psi}\Big(\varpi(t)-\zeta(t,\varpi_t)\Big) &= \tilde{\xi}(t,\varpi(t),\varpi_t)+\tilde{\varrho}(t,\varpi(t),\varpi_t)\frac{d\mathfrak{W}(t)}{dt} \\
&\quad +\frac{1}{dt}\int_{\|\mathfrak{X}\|<s}\Re(t,\varpi(t),\varpi_t,\mathfrak{X})\tilde{\aleph}(dt,d\mathfrak{X}), \quad t\in\mathfrak{J}:=(\iota,\mu], \quad t\neq t_k, \\
\Delta{}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_\delta;\psi}\varpi\,|_{t=t_k} &= \mathbb{C}_k(\varpi(t_k^-)), \quad t=t_k, \quad k=1,2,\cdots,\mathcal{T}, \\
\varpi(t) &= \chi(t), \quad -\theta\leq t\leq\iota, \\
{}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_\delta;\psi}\varpi(\iota) &= \beta.
\end{aligned} \tag{2.4}$$

Definition 2.3. *In the context of the interval $[-\theta,\mu]$, an $\mathcal{A}$-valued stochastic process $\varpi(t)$ is regarded as a moderate solution of (2.4) if*

- $\forall\, t\in[-\theta,\iota]$, $\varpi(t)=\chi(t)$;
- $\varpi(t)\in\mathcal{L}^2(\Omega,\mathcal{A})$ *is* $\mathcal{D}_\mu$*-adapted, and has càdlàg paths on* $\mathfrak{J}$ *almost surely;*
- $$\varpi(t)=\begin{cases}\Psi_1(t), & t\in(\iota,t_1]\\ \Psi_2(t), & t\in(t_k,t_{k+1}],\ k=1,2,\cdots,\mathcal{T},\end{cases}$$

  where

$$\begin{aligned}
\Psi_1(t) &= \frac{(\psi(t)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)}\Big(\beta-\zeta(\iota,\chi(\iota))\Big)+\zeta(t,\varpi_t) \\
&+\frac{1}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\tilde{\xi}(\mathfrak{G},\varpi(\mathfrak{G}),\varpi_{\mathfrak{G}})d\mathfrak{G} \\
&+\frac{1}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\tilde{\varrho}(\mathfrak{G},\varpi(\mathfrak{G}),\varpi_{\mathfrak{G}})d\mathfrak{W}(\mathfrak{G}) \\
&+\frac{1}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big\{\int_{\|\mathfrak{X}\|<s}\Re(\mathfrak{G},\varpi(\mathfrak{G}),\varpi_{\mathfrak{G}},\mathfrak{X})\Big\}\tilde{\aleph}(d\mathfrak{G},d\mathfrak{X}), \\
\Psi_2(t) &= \frac{(\psi(t)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)}\Big(\beta-\zeta(\iota,\chi(\iota))+\sum_{i=1}^{k}\mathbb{C}_i(\varpi_{t_i^-})\Big)+\zeta(t,\varpi_t) \\
&+\frac{1}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\tilde{\xi}(\mathfrak{G},\varpi(\mathfrak{G}),\varpi_{\mathfrak{G}})d\mathfrak{G} \\
&+\frac{1}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\tilde{\varrho}(\mathfrak{G},\varpi(\mathfrak{G}),\varpi_{\mathfrak{G}})d\mathfrak{W}(\mathfrak{G}) \\
&+\frac{1}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\int_{\|\mathfrak{X}\|<s}\Re(\mathfrak{G},\varpi(\mathfrak{G}),\varpi_{\mathfrak{G}},\mathfrak{X})\tilde{\aleph}(d\mathfrak{G},d\mathfrak{X}).
\end{aligned}$$

Remark 2.1. *The càdlàg requirement on the process $\varpi(t)$ is essential due to the presence of impulsive dynamics and Poisson jump terms in Eq (2.4). These elements introduce discontinuities in the trajectories, and càdlàg paths ensure the solution resides in the Skorokhod space $\mathcal{D}([-\theta,\mu];\mathcal{A})$, which provides the appropriate topological and measurability structure for well-posedness.*

3. Principal findings

In this section, we initially show that (2.4) has a mild solution, which is unique. For us to be able to establish the desired result, the functions ζ, $\tilde{\xi}$, $\tilde{\varrho}$, and $\Re$ require certain assumptions.

$(P1)$ For all $\eta_1, \eta_2, \eta_3 \in \mathbb{S}$, there exist constants $\gamma_1, \gamma_2 \in (0,1)$ such that:

(I) $\|\zeta(t,\eta_1) - \zeta(t,\eta_2)\|^2 \le \gamma_1 \|\eta_1 - \eta_2\|^2$,

(II) $\|\zeta(t,\eta_3)\|^2 \le \gamma_2(1 + \|\eta_3\|^2)$.

$(P2)$ For all $\upsilon_1, \upsilon_2 \in \mathcal{A}$, $\eta_1, \eta_2 \in \mathbb{S}$, and $\mathfrak{Z} \in \mathfrak{J}$, there exist constants $\gamma_1, \gamma_2 > 0$ such that:

(1) $\mathbb{E}\left\|\tilde{\xi}(t,\upsilon_1,\eta_1)\right\|^2 \le \gamma_1(1 + \|\upsilon_1\|^2 + \|\eta_1\|^2)$,

(II) $\mathbb{E}\left\|\tilde{\xi}(t,\upsilon_1,\eta_1) - \tilde{\xi}(t,\upsilon_2,\eta_2)\right\|^2 \le \gamma_2(\|\upsilon_1 - \upsilon_2\|^2 + \|\eta_1 - \eta_2\|^2)$.

$(P3)$ $\forall\, \upsilon_1, \upsilon_2 \in \mathcal{A}$, $\eta_1, \eta_2 \in \mathbb{S}$ and $\mathfrak{Z} \in \mathfrak{J}$, $\tilde{\varrho}$ and $\Re$ fulfill that

(I)

$$\begin{aligned}\mathbb{E}\,\|\tilde{\varrho}(t,\upsilon_1,\eta_1)\|^2 \vee \mathbb{E}\Big(\int_{\|\mathfrak{X}\|<s} \|\Re(t,\upsilon_1,\eta_1,\mathfrak{X})\|^2\,\nu(d\mathfrak{X})\Big) \\ \le G_1(t)\Big\{1 + \|\upsilon_1\|^2 + \|\eta_1\|^2\Big\},\end{aligned}$$

(II)

$$\begin{aligned}\mathbb{E}\,\|\tilde{\varrho}(t,\upsilon_1,\eta_1) - \tilde{\varrho}(t,\upsilon_2,\eta_2)\|^2 \vee \mathbb{E}\Big(\int_{\|\mathfrak{X}\|<s} \|\Re(t,\upsilon_1,\eta_1,\mathfrak{X}) - \Re(t,\upsilon_2,\eta_2,\mathfrak{X})\|^2\,\nu(d\mathfrak{X})\Big) \\ \le G_2(t)\Big\{\|\upsilon_1 - \upsilon_2\|^2 + \|\eta_1 - \eta_2\|^2\Big\},\end{aligned}$$

where for the functions $G_1(t) \in \mathcal{L}^{\frac{1}{2\mathfrak{A}_1-\delta}}(\mathfrak{J})$ and $G_2(t) \in \mathcal{L}^{\frac{1}{2\mathfrak{A}_1-\delta}}(\mathfrak{J})$, $\mathfrak{A}_1 \in (\delta/2, \mathfrak{A} + \delta/2)$.

$(P4)$ (I) The operators $\mathbb{C}_k : \mathcal{A} \to \mathcal{A}$ are completely continuous, and there exist constants $d_k > 0$, for $k = 1, 2, \ldots, \mathcal{T}$ such that:

$$\mathbb{E}\left\|\mathbb{C}_k(\varpi(t_i^-))\right\|^2 \le d_k, \quad \forall \varpi \in \mathbb{S}.$$

(II) There exists $\check{\mathfrak{P}} > 0$ such that:

$$\mathbb{E}\left\|\mathbb{C}_k(\sigma_1(t_k^-)) - \mathbb{C}_k(\sigma_2(t_k^-))\right\|^2 \le \check{\mathfrak{P}}\mathbb{E}\left\|\sigma_1(t_k^-) - \sigma_2(t_k^-)\right\|^2,$$

for all $\sigma_1, \sigma_2 \in \mathbb{S}$.

$(P5)$ $\exists\ \mathbb{G}(t,\sigma) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ which is locally integrable with regard to t, and non-decreasing, continuous, and concave with regard to σ $\forall\ t \in [\iota, \mu]$, $\int_{0^+} \frac{1}{\mathbb{G}(t,\sigma)} d\sigma = \infty$. For any $\upsilon_1, \upsilon_2 \in \mathcal{A}$, $\eta_1, \eta_2 \in \mathbb{S}$, and $\mathfrak{Z} \in \mathfrak{J}$, this inequality holds:

$$\begin{aligned}\left\|\tilde{\xi}(t,\upsilon_1,\eta_1) - \tilde{\xi}(t,\upsilon_2,\eta_2)\right\|^2 \vee \|\tilde{\varrho}(t,\upsilon_1,\eta_1) - \tilde{\varrho}(t,\upsilon_2,\eta_2)\|^2 \\ \vee \int_{\|\mathfrak{X}\|<s} \|\Re(t,\upsilon_1,\eta_1,\mathfrak{X}) - \Re(t,\upsilon_2,\eta_2,\mathfrak{X})\|^2\,\nu(d\mathfrak{X}) \\ \le \mathbb{G}\Big(t, \|\upsilon_1 - \upsilon_2\|^2 + \|\eta_1 - \eta_2\|^2\Big).\end{aligned}$$

Remark 3.1. *The conditions imposed on $\mathbb{G}(t,\sigma)$—specifically, its monotonicity and concavity with respect to σ—are classical in the analysis of integral inequalities, particularly in generalized Bihari-type frameworks. These properties ensure the integrability of $\frac{1}{\mathbb{G}(t,\sigma)}$ near zero, which is crucial for deriving uniqueness and stability results. Moreover, this setting includes many functional forms of practical relevance, such as $\mathbb{G}(t,\sigma) = a(t)\sigma^{\kappa}$ for $\kappa \in (0,1)$ and $a(t)$ locally integrable, or $\mathbb{G}(t,\sigma) = a(t)\log(1+\sigma)$, both of which are widely encountered in sublinear and logarithmic growth scenarios.*

($P6$) There exist measurable mappings: $\mathfrak{f} : \mathcal{A} \times \mathbb{S} \to \mathcal{A}$, $\mathfrak{H} : \mathcal{A} \times \mathbb{S} \to \mathcal{L}(\mathcal{B}, \mathcal{A})$, $\mathfrak{Y} : \mathcal{A} \times \mathbb{S} \times \mathfrak{U} \to \mathcal{A}$, and $\mathfrak{I} : \mathcal{A} \to \mathcal{A}$ satisfying ($P5$) such that for any $\tilde{\rho} \in \mathfrak{J}$, $\upsilon \in \mathcal{A}$, and $\hbar \in \mathbb{S}$, there exist bounded functions $\mathbb{P}_j(\tilde{\rho}) > 0$, $j = 1, 2, 3, 4$, for which:

$$\begin{aligned}
\frac{1}{\tilde{\rho}} \int_0^{\tilde{\rho}} \left\| \tilde{\xi}(\mathfrak{G}, \upsilon, \hbar) - \mathfrak{f}(\upsilon, \hbar) \right\|^2 d\mathfrak{G} &\leq \mathbb{P}_1(\tilde{\rho})(1 + \|\upsilon\|^2 + \|\hbar\|^2), \\
\frac{1}{\tilde{\rho}} \int_0^{\tilde{\rho}} \|\tilde{\varrho}(\mathfrak{G}, \upsilon, \hbar) - \mathfrak{H}(\upsilon, \hbar)\|^2 d\mathfrak{G} &\leq \mathbb{P}_2(\tilde{\rho})(1 + \|\upsilon\|^2 + \|\hbar\|^2), \\
\frac{1}{\tilde{\rho}} \int_0^{\tilde{\rho}} \Big(\int_{\|\mathfrak{X}\|<s} \|\mathfrak{R}(\mathfrak{G}, \upsilon, \hbar, \mathfrak{X}) - \mathfrak{Y}(\upsilon, \hbar, \mathfrak{X})\|^2 \, \nu(d\mathfrak{X}) \Big) d\mathfrak{G} &\leq \mathbb{P}_3(\tilde{\rho})(1 + \|\upsilon\|^2 + \|\hbar\|^2), \\
\frac{1}{\tilde{\rho}} \left\| \sum_{0<\mathfrak{Z}<\tilde{\rho}} \mathbb{C}_k(\upsilon) - \tilde{\rho}\mathfrak{I}(\upsilon) \right\|^2 &\leq \mathbb{P}_4(\tilde{\rho})(1 + \|\upsilon\|^2).
\end{aligned}$$

Define the operator Υ as follows:

$$\begin{aligned}
(\Upsilon\varpi)(t) &= \frac{(\psi(t) - \psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)} \Big(\beta - \zeta(\iota, \chi(\iota)) + \sum_{\iota<t_k<t} \mathbb{C}_k(\varpi_{t_k^-}) \Big) + \zeta(t, \varpi_t) \\
&\quad + \frac{1}{\delta\Gamma_\delta(\mathfrak{A})} \int_\iota^t \psi'(\mathfrak{G})(\psi(t) - \psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1} \tilde{\xi}(\mathfrak{G}, \varpi(\mathfrak{G}), \varpi_{\mathfrak{G}}) d\mathfrak{G} \\
&\quad + \frac{1}{\delta\Gamma_\delta(\mathfrak{A})} \int_\iota^t \psi'(\mathfrak{G})(\psi(t) - \psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1} \tilde{\varrho}(\mathfrak{G}, \varpi(\mathfrak{G}), \varpi_{\mathfrak{G}}) d\mathfrak{W}(\mathfrak{G}) \\
&\quad + \frac{1}{\delta\Gamma_\delta(\mathfrak{A})} \int_\iota^t \psi'(\mathfrak{G})(\psi(t) - \psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1} \int_{\|\mathfrak{X}\|<s} \mathfrak{R}(\mathfrak{G}, \varpi(\mathfrak{G}), \varpi_{\mathfrak{G}}, \mathfrak{X}) \tilde{\aleph}(d\mathfrak{G}, d\mathfrak{X}), \quad t \in \mathfrak{J}.
\end{aligned}$$

Remark 3.2. *Consider the bounded, closed, and convex set $\mathbb{Q}_r = \Big\{ \varpi \in \mathbb{S} : \|\varpi\|_{\mathbb{S}}^2 \leq r, \quad r > 0, \quad r \geq \frac{\varsigma_1}{1-\varsigma_2} \Big\}$.*

Theorem 3.1. *Assuming that $\mathfrak{K} < 1$ and that the conditions ($P1$) − ($P4$) are met, on $\mathbb{Q}_r$, there exists a unique mild solution to system (2.4), where*

$$\begin{aligned}
\mathfrak{K} &= 5\gamma_\zeta(\psi(\mu) - \psi(\iota))^{2(1-\frac{A_\delta}{\delta})} + \frac{5}{\Gamma_\delta(A_\delta)} \sum_{\iota<t_k<t} \check{\mathfrak{P}} + 10 \frac{\gamma_2(\psi(\mu) - \psi(\iota))^{1-2\mathfrak{B}+2\frac{\mathfrak{A}\mathfrak{B}}{\delta}} \mathbb{U}}{(2\frac{\mathfrak{A}}{\delta} - 1)\delta^2\Gamma_\delta^2(\mathfrak{A})} \\
&\quad + \frac{80\mathbb{U}^{2\mathfrak{A}_1}(\psi(\mu) - \psi(\iota))^{2(1+\frac{\mathfrak{A}}{\delta}-\mathfrak{B}-\mathfrak{A}_1)}}{\delta^2\Gamma_\delta^2(\mathfrak{A})} \left(\frac{1 - \mathfrak{A}_g}{\frac{\mathfrak{A}}{\delta} - \mathfrak{A}_1} \right)^{2-2\mathfrak{A}_1} \|G_2\|_{\mathcal{L}^{\frac{1}{2\mathfrak{A}_1-\delta}}}.
\end{aligned}$$

Proof. We provide proof for this in the next three phases.
Step 1. Throughout $\mathbb{Q}_r$, Υ is continuous. This step's validity may be proven employing simple arguments from $(P1)$–$(P3)$.
Step 2. The mapping of $\mathbb{Q}_r$ into itself is accomplished via the operator Υ.

$\forall\ \varpi \in \mathbb{Q}_r$, $t \in \mathfrak{J}$, we are able to receive

$$\begin{aligned}\|\Upsilon\varpi\|_{\mathbb{S}}^2 &\leq 5\sup_{t\in\mathfrak{J}} \frac{1}{\Gamma_\delta^2(A_\delta)}\mathbb{E}\left\|\beta - \zeta(\iota,\chi(\iota)) + \sum_{\iota<t_k<t}\mathbb{C}_k(\varpi_{t_k^-})\right\|^2 + 5\sup_{t\in\mathfrak{J}}(\psi(t)-\psi(\iota))^{2(1-\frac{A_\delta}{\delta})}\mathbb{E}\left\|\zeta(t,\varpi_t)\right\|^2 \\ &+ 5\sup_{t\in\mathfrak{J}}\mathbb{E}\left\|\frac{(\psi(t)-\psi(\iota))^{1-\frac{A_\delta}{\delta}}}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t \psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\tilde{\xi}(\mathfrak{G},\varpi(\mathfrak{G}),\varpi_{\mathfrak{G}})d\mathfrak{G}\right\|^2 \\ &+ 5\sup_{t\in\mathfrak{J}}\mathbb{E}\left\|\frac{(\psi(t)-\psi(\iota))^{1-\frac{A_\delta}{\delta}}}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t \psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\tilde{\varrho}(\mathfrak{G},\varpi(\mathfrak{G}),\varpi_{\mathfrak{G}})d\mathfrak{W}(\mathfrak{G})\right\|^2 \\ &+ 5\sup_{t\in\mathfrak{J}}\mathbb{E}\left\|\frac{(\psi(t)-\psi(\iota))^{1-\frac{A_\delta}{\delta}}}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t \psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\int_{\|\mathfrak{X}\|<s}\mathfrak{R}(\mathfrak{G},\varpi(\mathfrak{G}),\varpi_{\mathfrak{G}},\mathfrak{X})\tilde{\mathfrak{N}}(d\mathfrak{G},d\mathfrak{X})\right\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\|\Upsilon\varpi\|_{\mathbb{S}}^2 &\leq \frac{15}{\Gamma_\delta^2(A_\delta)}\Big(\|\beta\|^2 + \|\zeta(\iota,\chi(\iota))\|^2 + \sum_{\iota<t_k<t} d_k\Big) + 5(\psi(\mu)-\psi(\iota))^{2(1-\frac{A_\delta}{\delta})}\gamma_2(1+\|\varpi\|_{\mathbb{S}}^2) \\ &+ \frac{5\gamma_1(\psi(\mu)-\psi(\iota))^{1-2\mathfrak{B}+2\frac{\mathfrak{A}\mathfrak{B}}{\delta}}\mathbb{U}}{(2\frac{\mathfrak{A}}{\delta}-1)\delta^2\Gamma_\delta^2(\mathfrak{A})}\Big(1+2\|\varpi\|_{\mathbb{S}}^2\Big) \\ &+ \frac{40\mathbb{U}^{2\mathfrak{A}_1}(\psi(\mu)-\psi(\iota))^{2(1+\frac{\mathfrak{A}}{\delta}-\mathfrak{B}-\mathfrak{A}_1)}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\left(\frac{1-\mathfrak{A}_g}{\frac{\mathfrak{A}}{\delta}-\mathfrak{A}_1}\right)^{2-2\mathfrak{A}_1}\|G_1\|_{\mathcal{L}^{\frac{1}{2\mathfrak{A}_1-\delta}}}\Big\{1+2\|\varpi\|_{\mathbb{S}}^2\Big\},\end{aligned}$$

where $\mathbb{U} = \sup_{t\in[-\theta,\mu]}\psi'(t)$. Hence, we get

$$\begin{aligned}\|\Upsilon\varpi\|_{\mathbb{S}}^2 &\leq \frac{15}{\Gamma_\delta^2(A_\delta)}\Big(\|\beta\|^2 + \|\zeta(\iota,\chi(\iota))\|^2 + \sum_{\iota<t_k<t} d_k\Big) + 5\gamma_2(\psi(\mu)-\psi(\iota))^{2(1-\frac{A_\delta}{\delta})}(1+r) \\ &+ \frac{5\gamma_1(\psi(\mu)-\psi(\iota))^{1-2\mathfrak{B}+2\frac{\mathfrak{A}\mathfrak{B}}{\delta}}\mathbb{U}}{(2\frac{\mathfrak{A}}{\delta}-1)\delta^2\Gamma_\delta^2(\mathfrak{A})}\Big(1+2r\Big) \\ &+ \frac{40\mathbb{U}^{2\mathfrak{A}_1}(\psi(\mu)-\psi(\iota))^{2(1+\frac{\mathfrak{A}}{\delta}-\mathfrak{B}-\mathfrak{A}_1)}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\left(\frac{1-\mathfrak{A}_g}{\frac{\mathfrak{A}}{\delta}-\mathfrak{A}_1}\right)^{2-2\mathfrak{A}_1}\|G_1\|_{\mathcal{L}^{\frac{1}{2\mathfrak{A}_1-\delta}}}\Big\{1+2r\Big\} \\ &= \mathfrak{Q}_1 + \mathfrak{Q}_2 r \leq r,\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{Q}_1 &= \frac{15}{\Gamma_\delta^2(A_\delta)}\Big(\|\beta\|^2 + \|\zeta(\iota,\chi(\iota))\|^2 + \sum_{\iota<t_k<t} d_k\Big) + 5\gamma_2(\psi(\mu)-\psi(\iota))^{2(1-\frac{A_\delta}{\delta})} \\
&\quad + \frac{5\gamma_1(\psi(\mu)-\psi(\iota))^{1-2\mathfrak{B}+2\frac{\mathfrak{A}\mathfrak{B}}{\delta}}\mathbb{U}}{(2\frac{\mathfrak{A}}{\delta}-1)\delta^2\Gamma_\delta^2(\mathfrak{A})} \\
&\quad + \frac{40\mathbb{U}^{2\mathfrak{A}_1}(\psi(\mu)-\psi(\iota))^{2(1+\frac{\mathfrak{A}}{\delta}-\mathfrak{B}-\mathfrak{A}_1)}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\left(\frac{1-\mathfrak{A}_g}{\frac{\mathfrak{A}}{\delta}-\mathfrak{A}_1}\right)^{2-2\mathfrak{A}_1}\|G_1\|_{\mathcal{L}^{\frac{1}{2\mathfrak{A}_1-\delta}}}, \\
\mathfrak{Q}_2 &= 5\gamma_2(\psi(\mu)-\psi(\iota))^{2(1-\frac{A_\delta}{\delta})} + \frac{10\gamma_1(\psi(\mu)-\psi(\iota))^{1-2\mathfrak{B}+2\frac{\mathfrak{A}\mathfrak{B}}{\delta}}\mathbb{U}}{(2\frac{\mathfrak{A}}{\delta}-1)\delta^2\Gamma_\delta^2(\mathfrak{A})} \\
&\quad + \frac{80\mathbb{U}^{2\mathfrak{A}_1}(\psi(\mu)-\psi(\iota))^{2(1+\frac{\mathfrak{A}}{\delta}-\mathfrak{B}-\mathfrak{A}_1)}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\left(\frac{1-\mathfrak{A}_g}{\frac{\mathfrak{A}}{\delta}-\mathfrak{A}_1}\right)^{2-2\mathfrak{A}_1}\|G_1\|_{\mathcal{L}^{\frac{1}{2\mathfrak{A}_1-\delta}}}.
\end{aligned}$$

Step 3. Υ is a contraction on $\mathbb{Q}_r$.
$\forall\ \varpi_1, \varpi_2 \in \mathbb{Q}_r$ and $\mathfrak{Z} \in \mathfrak{J}$, we get

$$\begin{aligned}
\|(\Upsilon\varpi_1) - (\Upsilon\varpi_2)\|_{\mathbb{S}}^2 &\le 5\gamma_\zeta(\psi(\mu)-\psi(\iota))^{2(1-\frac{A_\delta}{\delta})}\|\varpi_1-\varpi_2\|_{\mathbb{S}}^2 + \frac{5}{\Gamma_\delta(A_\delta)}\sum_{\iota<t_k<t}\check{\mathfrak{P}}\,\|\varpi_1-\varpi_2\|_{\mathbb{S}}^2 \\
&+ \frac{80\mathbb{U}^{2\mathfrak{A}_1}(\psi(\mu)-\psi(\iota))^{2(1+\frac{\mathfrak{A}}{\delta}-\mathfrak{B}-\mathfrak{A}_1)}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\left(\frac{1-\mathfrak{A}_g}{\frac{\mathfrak{A}}{\delta}-\mathfrak{A}_1}\right)^{2-2\mathfrak{A}_1}\|G_2\|_{\mathcal{L}^{\frac{1}{2\mathfrak{A}_1-\delta}}}\|\varpi_1-\varpi_2\|_{\mathbb{S}}^2 \\
&+ 10\frac{\gamma_2(\psi(\mu)-\psi(\iota))^{1-2\mathfrak{B}+2\frac{\mathfrak{A}\mathfrak{B}}{\delta}}\mathbb{U}}{(2\frac{\mathfrak{A}}{\delta}-1)\delta^2\Gamma_\delta^2(\mathfrak{A})}\|\varpi_1-\varpi_2\|_{\mathbb{S}}^2 \le \mathfrak{K}\,\|\varpi_1-\varpi_2\|_{\mathbb{S}}^2,
\end{aligned}$$

which signifies that Υ is a contraction mapping. □

The definition of the system's standard form (2.4) is

$$\begin{aligned}
{}^{\delta,\mathcal{H}}\mathbb{D}_{\iota+}^{\mathfrak{A},\mathfrak{B};\psi}\Big(\varpi_\varsigma(t)-\zeta(t,\varpi_{\varsigma,t})\Big) &= \varsigma\tilde{\xi}(t,\varpi_\varsigma(t),\varpi_{\varsigma,t}) + \sqrt{\varsigma}\tilde{\varrho}(t,\varpi_\varsigma(t),\varpi_{\varsigma,t})\frac{d\mathfrak{W}(t)}{dt} \\
&\quad + \frac{\sqrt{\varsigma}}{dt}\int_{\|\mathfrak{X}\|<s}\mathfrak{R}(t,\varpi_\varsigma(t),\varpi_{\varsigma,t},\mathfrak{X})\tilde{\aleph}(dt,d\mathfrak{X}), \quad t\in\mathfrak{J}:=(\iota,\mu], \quad t\neq t_k, \\
\Delta{}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_\delta;\psi}\varpi\,|_{t=t_k} &= \mathbb{C}_k(\varpi(t_k^-)), \quad t=t_k, \quad k=1,2,\cdots,\mathcal{T}, \\
\varpi(t) &= \chi(t), \quad -\theta\le t\le\iota, \\
{}^{\delta}\mathbb{I}_{\iota+}^{\delta-A_\delta;\psi}\varpi(\iota) &= \beta, \qquad (3.1)
\end{aligned}$$

where $\tilde{\varrho}$, $\tilde{\vartheta}$, and $\mathfrak{R}$ possess identical requirements to those of system (2.4). $0 < \varsigma \in (0,\varsigma_1]$ with $0 < \varsigma_1 \ll 1$. The mild solution $\varpi_\varsigma(t)$ of system (3.1) is capable of being provided by

$$\begin{aligned}
\varpi_\varsigma(t) &= \frac{(\psi(t)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)}\Big(\beta-\zeta(\iota,\chi(\iota))\Big) \\
&\quad + \varsigma\frac{(\psi(t)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)}\Big(\sum_{\iota<t_k<t}\mathbb{C}_k(\varpi_{\varsigma,t_k^-})\Big) + \zeta(t,\varpi_{\varsigma,t})
\end{aligned}$$

$$+\frac{\varsigma}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t \psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\tilde{\xi}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}})d\mathfrak{G}$$
$$+\frac{\sqrt{\varsigma}}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t \psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\tilde{\varrho}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}})d\mathfrak{W}(\mathfrak{G})$$
$$+\frac{\sqrt{\varsigma}}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t \psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\int_{\|\mathfrak{X}\|<s}\mathfrak{R}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}},\mathfrak{X})\tilde{\aleph}(d\mathfrak{G},d\mathfrak{X}). \quad (3.2)$$

Consequently, we have to deduce that when $\varsigma \to 0$, the initial system solution $\varpi_\varsigma(t)$ approaches the process $\Phi_\varsigma(t)$ which is the average system's solution:

$$\Phi_\varsigma(t) := \frac{(\psi(t)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)}\Big(\beta-\zeta(\iota,\chi(\iota))\Big)+\varsigma\frac{(\psi(t)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)}\Big(\int_\iota^t \eth(\Phi_{\varsigma,\mathfrak{G}})d\mathfrak{G}\Big)+\zeta(t,\Phi_{\varsigma,t})$$
$$+\frac{\varsigma}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t \psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\mathfrak{f}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}})d\mathfrak{G}$$
$$+\frac{\sqrt{\varsigma}}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t \psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\mathfrak{h}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}})d\mathfrak{W}(\mathfrak{G})$$
$$+\frac{\sqrt{\varsigma}}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t \psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\int_{\|\mathfrak{X}\|<s}\mathfrak{Y}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}},\mathfrak{X})\tilde{\aleph}(d\mathfrak{G},d\mathfrak{X}). \quad (3.3)$$

Theorem 3.2. *Presume that* $(P1)$–$(P6)$ *are verified. When an indeterminate tiny number* $\tau > 0$ *is supplied,* $\exists\, \rho_0 > 0$, $\varsigma_2 \in (0,\varsigma_1]$ *and* $0<\varepsilon<1$ *s.t.* $\forall\, \varsigma \in (0,\varsigma_1]$,

$$\mathbb{E}\Big\{\sup_{t\in[-\theta,\rho_0\varsigma^{-\varepsilon}]}\big\|\varpi_\varsigma(t)-\Phi_\varsigma(t)\big\|^2\Big\}\le\tau.$$

Remark 3.3. *The parameter* ε *in Theorem 3.2 represents a scaling exponent that controls the time horizon* $\rho_0\varsigma^{-\varepsilon}$ *over which the averaged approximation remains valid. Its precise value is not fixed a priori, but can be chosen arbitrarily small within* $(0,1)$ *depending on the desired approximation accuracy* τ *and the regularity conditions imposed in assumptions* $(P1)$–$(P6)$*. The flexibility in* ε *allows the framework to adapt to various magnitudes of stochasticity and memory effects, particularly when the fractional operator or Lévy noise induces long-range dependencies. In practical applications, a smaller* ε *enhances the approximation window but may require stronger conditions on the initial data or noise intensity.*

Proof. Given $(P1)$, it can be inferred from (3.2) and (3.3) that

$$\mathbb{E}\sup_{\iota\le t\le u}\big\|\varpi_\varsigma(t)-\Phi_\varsigma(t)\big\|^2\le\frac{\mathbb{E}\sup_{\iota\le t\le u}\big\|\varpi_\varsigma(t)-\Phi_\varsigma(t)-(\zeta(t,\varpi_{\varsigma,t})-\zeta(t,\Phi_{\varsigma,t}))\big\|^2}{(1-\gamma_\zeta)^2},\ t\in\mathfrak{J}, \quad (3.4)$$

$$\varpi_\varsigma(t)-\Phi_\varsigma(t)=\varsigma\frac{(\psi(t)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)}\bigg(\sum_{\iota<t_k<t}\mathbb{C}_k(\varpi_{\varsigma,t_k^-})-\int_\iota^t \eth(\Phi_{\varsigma,\mathfrak{G}})d\mathfrak{G}\bigg)+\zeta(t,\varpi_{\varsigma,t})-\zeta(t,\Phi_{\varsigma,t})$$
$$+\frac{\varsigma}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big(\tilde{\xi}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}})-\mathfrak{f}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}})\Big)d\mathfrak{G}$$
$$+\frac{\sqrt{\varsigma}}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big(\tilde{\varrho}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}})-\mathfrak{h}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}})\Big)d\mathfrak{W}(\mathfrak{G})$$
$$+\frac{\sqrt{\varsigma}}{\delta\Gamma_\delta(\mathfrak{A})}\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\int_{\|\mathfrak{X}\|<s}\Big(\mathfrak{R}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}},\mathfrak{X})-\mathfrak{Y}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}},\mathfrak{X})\Big)\tilde{\aleph}(d\mathfrak{G},d\mathfrak{X}).$$

$\forall\ t\in(\iota,u]\subset\mathfrak{J}$, we receive

$$\mathbb{E}\sup_{\iota\le t\le u}\left\|\varpi_\varsigma(t)-\Phi_\varsigma(t)-(\zeta(t,\varpi_{\varsigma,t})-\zeta(t,\Phi_{\varsigma,t}))\right\|^2$$
$$\le 4\frac{\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\mathbb{E}\sup_{\iota\le t\le u}\left\|\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big(\tilde{\xi}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}})-\mathfrak{f}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}})\Big)d\mathfrak{G}\right\|^2$$
$$+4\frac{\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\mathbb{E}\sup_{\iota\le t\le u}\left\|\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big(\tilde{\varrho}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}})-\mathfrak{h}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}})\Big)d\mathfrak{W}(\mathfrak{G})\right\|^2$$
$$+4\frac{\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\mathbb{E}\sup_{\iota\le\mathfrak{Z}\le u}$$
$$\times\left\|\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\int_{\|\mathfrak{X}\|<s}\Big(\mathfrak{R}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}},\mathfrak{X})-\mathfrak{Y}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}},\mathfrak{X})\Big)\tilde{\aleph}(d\mathfrak{G},d\mathfrak{X})\right\|^2$$
$$+4\varsigma^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{\Gamma_\delta^2(A_\delta)}\mathbb{E}\sup_{\iota\le t\le u}\left\|\sum_{\iota<t_k<t}\mathbb{C}_k(\varpi_{\varsigma,t_k^-})-\int_\iota^t\eth(\Phi_{\varsigma,\mathfrak{G}})d\mathfrak{G}\right\|^2:=\mathbb{T}_1+\mathbb{T}_2+\mathbb{T}_3+\mathbb{T}_4.$$

For the term $\mathbb{T}_1$, we obtain

$$\begin{aligned}\mathbb{T}_1\ \le\ & 8\frac{\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\mathbb{E}\sup_{\iota\le t\le u}\left\|\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big(\tilde{\xi}(\mathfrak{G},\varpi_\varsigma(\mathfrak{G}),\varpi_{\varsigma,\mathfrak{G}})-\tilde{\xi}(\mathfrak{G},\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}})\Big)d\mathfrak{G}\right\|^2\\ &+8\frac{\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\mathbb{E}\sup_{\iota\le t\le u}\left\|\int_\iota^t\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big(\tilde{\xi}(\mathfrak{G},\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}})-\mathfrak{f}(\Phi_\varsigma(\mathfrak{G}),\Phi_{\varsigma,\mathfrak{G}})\Big)d\mathfrak{G}\right\|^2\\ &\le\mathbb{T}_{11}+\mathbb{T}_{12},\end{aligned}$$

where

$$\begin{aligned}\mathbb{T}_{11}\ =\ & 8\frac{\varsigma^2(u-\iota)}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\int_\iota^u\Big(\psi'(\mathfrak{G})(\psi(u)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big)^2\\ &\times\mathbb{G}(\mathfrak{G},\mathbb{E}(\sup_{\iota<\mathfrak{G}_1\le\mathfrak{G}}\left\|\varpi_\varsigma(\mathfrak{G}_1)-\Phi_\varsigma(\mathfrak{G}_1)\right\|^2,\mathbb{E}(\sup_{\iota<\mathfrak{G}_1\le\mathfrak{G}}\left\|\varpi_{\varsigma,\mathfrak{G}_1}-\Phi_{\varsigma,\mathfrak{G}_1}\right\|^2)d\mathfrak{G},\\ \mathbb{T}_{12}\ =\ & 8\frac{\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{2\frac{\mathfrak{A}}{\delta}-1}\sup_{\iota<t\le u}\mathbb{P}_1(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big).\end{aligned}$$

For the term $\mathbb{T}_2$, we can get

$$\begin{aligned}\mathbb{T}_2 &\leq 32\frac{\varsigma(u-\iota)}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\int_\iota^u \left(\psi'(\mathfrak{G})(\psi(u)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\right)^2 \\ &\quad \times \mathbb{G}(\mathfrak{G}, \mathbb{E}(\sup_{\iota<\mathfrak{G}_1\leq\mathfrak{G}} \left\|\varpi_\varsigma(\mathfrak{G}_1)-\Phi_\varsigma(\mathfrak{G}_1)\right\|^2, \mathbb{E}(\sup_{\iota<\mathfrak{G}_1\leq\mathfrak{G}} \left\|\varpi_{\varsigma,\mathfrak{G}_1}-\Phi_{\varsigma,\mathfrak{G}_1}\right\|^2)d\mathfrak{G} \\ &\quad + 32\frac{\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{2\frac{\mathfrak{A}}{\delta}-1}\sup_{\iota<t\leq u}\mathbb{P}_2(t)\Big(1+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_{\varsigma,t}\right\|^2\Big).\end{aligned}$$

For the term $\mathbb{T}_3$, we have

$$\begin{aligned}\mathbb{T}_3 &\leq 32\frac{\varsigma(u-\iota)}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\int_\iota^u \left(\psi'(\mathfrak{G})(\psi(u)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\right)^2 \\ &\quad \times \mathbb{G}(\mathfrak{G}, \mathbb{E}(\sup_{\iota<\mathfrak{G}_1\leq\mathfrak{G}} \left\|\varpi_\varsigma(\mathfrak{G}_1)-\Phi_\varsigma(\mathfrak{G}_1)\right\|^2, \mathbb{E}(\sup_{\iota<\mathfrak{G}_1\leq\mathfrak{G}} \left\|\varpi_{\varsigma,\mathfrak{G}_1}-\Phi_{\varsigma,\mathfrak{G}_1}\right\|^2)d\mathfrak{G} \\ &\quad + 32\frac{\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{2\frac{\mathfrak{A}}{\delta}-1}\sup_{\iota<t\leq u}\mathbb{P}_3(t)\Big(1+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_{\varsigma,t}\right\|^2\Big).\end{aligned}$$

Remark 3.4. *(1)*

$$\sum_{\iota<t_k<t}\mathbb{C}_k(\varpi_{\varsigma,t_k^-}) = \sum_{i=1}^{\beta}\Big(\sum_{\iota<t_k<t_{i+1}}\mathbb{C}_k(\varpi_{\varsigma,t_k^-}) - \sum_{\iota<t_k<t_i}\mathbb{C}_k(\varpi_{\varsigma,t_k^-})\Big).$$

(2)

$$\int_\iota^t \eth(\varpi_{\varsigma,t_k^-})d\mathfrak{G} = \sum_{i=1}^{\beta}(t_{i+1}-t_i)\eth(\varpi_{\varsigma,t_k^-}),$$

where the number of impulses on $[\iota,\mu]$ *is symbolized by* β*.*

(3) $\exists\ M>0$ *involving* $\mathfrak{P}^*$ *s.t.,* $\eth$ *fulfills*

$$\|\eth(\eta_1)-\eth(\eta_2)\|^2 \leq M\|\eta_1-\eta_2\|^2, \quad \forall \eta_1,\eta_2\in\mathbb{S}.$$

For the last term, we get

$$\begin{aligned}T_4 &\leq 16\varsigma^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{\Gamma_\delta^2(A_\delta)}\mathbb{E}\sup_{\iota<t\leq u}\left\|\sum_{\iota<t_k<t}\mathbb{C}_k(\varpi_{\varsigma,t_k^-})-\int_\iota^t\eth(\varpi_{\varsigma,t_k^-})d\mathfrak{G}\right\|^2 \\ &\quad +16\varsigma^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{\Gamma_\delta^2(A_\delta)}\mathbb{E}\sup_{\iota<t\leq u}\left\|\int_\iota^t\eth(\varpi_{\varsigma,t_k^-})-\int_\iota^t\eth(\Phi_{\varsigma,t_k^-})d\mathfrak{G}\right\|^2 \\ &\quad +16\varsigma^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{\Gamma_\delta^2(A_\delta)}\mathbb{E}\sup_{\iota<t\leq u}\left\|\int_\iota^t\eth(\Phi_{\varsigma,t_k^-})-\int_\iota^t\eth(\Phi_{\varsigma,\mathfrak{G}})d\mathfrak{G}\right\|^2.\end{aligned}$$

Consequently, we obtain

$$
\begin{aligned}
T_4 \quad \leq \quad & 16(2\varsigma(u-\iota))^2 \frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{\Gamma_\delta^2(A_\delta)}(\beta+1)\sup_{\iota<t\leq u}\mathbb{P}_4\Big(1+\mathbb{E}\left\|\varpi_{\varsigma,t_k^-}\right\|^2\Big) \\
& + 16\varsigma^2 \frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{\Gamma_\delta^2(A_\delta)}(u-\iota)M\int_\iota^u \mathbb{E}\left\|\varpi_{\varsigma,\mathfrak{G}}-\Phi_{\varsigma,\mathfrak{G}}\right\|^2 d\mathfrak{G} \\
& + 16\Big(\varsigma(u-\iota)^{\frac{\mathfrak{A}}{\delta}+1}\frac{(\psi(u)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)}\Big)^2 M\mathbb{Y}_1 \\
& + 32\Big(\varsigma\frac{(\psi(u)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{\Gamma_\delta(A_\delta)}\Big)^2(u-\iota)^{2\frac{\mathfrak{A}}{\delta}+1}M\mathbb{Y}_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{Y}_1 &= \Big\{\frac{\varsigma^2\mathbb{U}(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\Big\}\mathbb{F}_1, \\
\mathbb{Y}_2 &= \Big\{\frac{\varsigma\mathbb{U}(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\Big\}\mathbb{F}_2,
\end{aligned}
$$

$\mathbb{F}_1 > 0$, and $\mathbb{F}_2 > 0$. Utilizing $(P5)$, $\exists\, \mathfrak{I}(\mathfrak{Z}) > 0$ and $\mathfrak{E}(\mathfrak{Z}) > 0$ s.t.

$$
\mathbb{G}(t,\varpi) \leq \mathfrak{I}(t) + \mathfrak{E}(t)\varpi, \quad \int_\iota^u \mathfrak{I}(t)dt < \infty, \quad \int_\iota^u \mathfrak{E}(t)dt < \infty.
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\sup_{\iota<t\leq u}\left\|\varpi_\varsigma(t)-\Phi_\varsigma(t)\right\|^2 \leq & \Bigg(\frac{\frac{8\mathbb{U}\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}+\frac{64\varsigma\mathbb{U}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(2\frac{\mathfrak{A}}{\delta}-1)(1-\gamma_\zeta)^2}\Bigg)\check{\mathfrak{I}} \\
& + \Bigg(\frac{\frac{8\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)+\frac{64\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)}{(1-\gamma_\zeta)^2}\Bigg)\check{\mathfrak{E}}\int_\iota^u\Big[\psi^{'}(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big]^2 \\
& \times\Big(\mathbb{E}\sup_{\iota<\mathfrak{G}_1\leq\mathfrak{G}}\left\|\varpi_\varsigma(\mathfrak{G}_1)-\Phi_\varsigma(\mathfrak{G}_1)\right\|^2+\mathbb{E}(\sup_{\iota<\mathfrak{G}_1\leq\mathfrak{G}}\left\|\varpi_{\varsigma,\mathfrak{G}_1}-\Phi_{\varsigma,\mathfrak{G}_1}\right\|^2)\Big)d\mathfrak{G} \\
& + \frac{8\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(2\frac{\mathfrak{A}}{\delta}-1)(1-\gamma_\zeta)^2}\sup_{\iota<t\leq u}\mathbb{P}_1(\mathfrak{Z})\Big(1+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_{\varsigma,t}\right\|^2\Big) \\
& + \frac{32\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(1-\gamma_\zeta)^2(2\frac{\mathfrak{A}}{\delta}-1)}\sup_{\iota<t\leq u}\mathbb{P}_2(t)\Big(1+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_{\varsigma,t}\right\|^2\Big) \\
& + \frac{32\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(1-\gamma_\zeta)^2(2\frac{\mathfrak{A}}{\delta}-1)}\sup_{\iota<t\leq u}\mathbb{P}_3(t)\Big(1+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\leq u}\left\|\Phi_{\varsigma,t}\right\|^2\Big) \\
& + 16(2\varsigma(u-\iota))^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{(1-\gamma_\zeta)^2\Gamma_\delta^2(A_\delta)}(\beta+1)\Xi \\
& + 16\varsigma^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{\Gamma_\delta^2(A_\delta)}(u-\iota)M\int_\iota^u\mathbb{E}\left\|\varpi_{\varsigma,\mathfrak{G}}-\Phi_{\varsigma,\mathfrak{G}}\right\|^2 d\mathfrak{G}
\end{aligned}
$$

$$+\,16\Big(\varsigma(u-\iota)^{\frac{\mathfrak{A}}{\delta}+1}\frac{(\psi(u)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{(1-\gamma_\zeta)\Gamma_\delta(A_\delta)}\Big)^2 M\mathbb{Y}_1+32\Big(\varsigma\frac{(\psi(u)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{(1-\gamma_\zeta)\Gamma_\delta(A_\delta)}\Big)^2(u-\iota)^{2\frac{\mathfrak{A}}{\delta}+1}M\mathbb{Y}_2,$$

where $\check{\mathfrak{I}}=\sup_{\iota<t\le u}\mathfrak{I}(t)$, $\check{\mathfrak{E}}=\sup_{\iota<t\le u}\mathfrak{E}(t)$, and $\Xi=\sup_{\iota<t\le u}\mathbb{P}_4(t)\Big(1+\mathbb{E}\left\|\varpi_{\varsigma,t_k^-}\right\|^2\Big)$. Using the fact that $\mathbb{E}\Big(\sup_{-\theta\le t<\iota}\left\|\varpi_\varsigma(t)-\Phi_\varsigma(t)\right\|^2\Big)=0$, and let $\Lambda(u)=\mathbb{E}\Big(\sup_{\iota<t\le u}\left\|\varpi_\varsigma(t)-\Phi_\varsigma(t)\right\|^2\Big)$, then we get

$$\mathbb{E}\Big(\sup_{\iota<\mathfrak{G}_1\le\mathfrak{G}}\left\|\varpi_\varsigma(\mathfrak{G}_1)-\Phi_\varsigma(\mathfrak{G}_1)\right\|^2\Big)=\Lambda(\mathfrak{G}-\kappa),\quad 0<\kappa\le\theta.$$

Thus,

$$\begin{aligned}
\Lambda(u)\le&\Bigg(\frac{\frac{8\mathbb{U}\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}+\frac{64\varsigma\mathbb{U}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(2\frac{\mathfrak{A}}{\delta}-1)(1-\gamma_\zeta)^2}\Bigg)\check{\mathfrak{I}}\\
&+\Bigg(\frac{\frac{8\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)+\frac{64\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)}{(1-\gamma_\zeta)^2}\Bigg)\check{\mathfrak{E}}\int_\iota^u\Big[\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big]^2\Big(\Lambda(\mathfrak{G})-\Lambda(\mathfrak{G}-\kappa)\Big)d\mathfrak{G}\\
&+\frac{8\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(2\frac{\mathfrak{A}}{\delta}-1)(1-\gamma_\zeta)^2}\sup_{\iota<t\le u}\mathbb{P}_1(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big)\\
&+\frac{32\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(1-\gamma_\zeta)^2(2\frac{\mathfrak{A}}{\delta}-1)}\sup_{\iota<t\le u}\mathbb{P}_2(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big)\\
&+\frac{32\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(1-\gamma_\zeta)^2(2\frac{\mathfrak{A}}{\delta}-1)}\sup_{\iota<t\le u}\mathbb{P}_3(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big)\\
&+16(2\varsigma(u-\iota))^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{(1-\gamma_\zeta)^2\Gamma_\delta^2(A_\delta)}(\beta+1)\Xi+16\Big(\varsigma(u-\iota)^{\frac{\mathfrak{A}}{\delta}+1}\frac{(\psi(u)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{(1-\gamma_\zeta)\Gamma_\delta(A_\delta)}\Big)^2M\mathbb{Y}_1+\\
&+16\varsigma^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{(1-\gamma_\zeta)^2\Gamma_\delta^2(A_\delta)}(u-\iota)M\int_\iota^u\mathbb{E}\left\|\varpi_{\varsigma,\mathfrak{G}}-\Phi_{\varsigma,\mathfrak{G}}\right\|^2d\mathfrak{G}\\
&+32\Big(\varsigma\frac{(\psi(u)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{(1-\gamma_\zeta)\Gamma_\delta(A_\delta)}\Big)^2(u-\iota)^{2\frac{\mathfrak{A}}{\delta}+1}M\mathbb{Y}_2.
\end{aligned}$$

By setting $\Delta_u=\sup_{-\theta\le t\le u}\Lambda(t)$, $\forall\ t\in\mathfrak{I}$, then $\Lambda(\mathfrak{G})\le\Delta_\mathfrak{G}$, and $\Lambda(\mathfrak{G}-\kappa)\le\Delta_\mathfrak{G}$, $0<\kappa\le\theta$. Hence,

$$\begin{aligned}
\Lambda(u)\le&\Bigg(\frac{\frac{8\mathbb{U}\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}+\frac{64\varsigma\mathbb{U}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(2\frac{\mathfrak{A}}{\delta}-1)(1-\gamma_\zeta)^2}\Bigg)\check{\mathfrak{I}}\\
&+\Bigg(\frac{\frac{8\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)+\frac{64\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)}{(1-\gamma_\zeta)^2}\Bigg)\check{\mathfrak{E}}\int_\iota^u\Big[\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big]^2 2\Delta_\mathfrak{G}d\mathfrak{G}\\
&+\frac{8\varsigma^2}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(2\frac{\mathfrak{A}}{\delta}-1)(1-\gamma_\zeta)^2}\sup_{\iota<t\le u}\mathbb{P}_1(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big)\\
&+\frac{32\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(1-\gamma_\zeta)^2(2\frac{\mathfrak{A}}{\delta}-1)}\sup_{\iota<t\le u}\mathbb{P}_2(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{32\varsigma}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{(1-\gamma_\zeta)^2(2\frac{\mathfrak{A}}{\delta}-1)}\sup_{\iota<t\le u}\mathbb{P}_3(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big)\\
&+16(2\varsigma(u-\iota))^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{(1-\gamma_\zeta)^2\Gamma_\delta^2(A_\delta)}(\beta+1)\Xi+16\Big(\varsigma(u-\iota)^{\frac{\mathfrak{A}}{\delta}+1}\frac{(\psi(u)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{(1-\gamma_\zeta)\Gamma_\delta(A_\delta)}\Big)^2M\mathbb{Y}_1\\
&+16\varsigma^2\frac{(\psi(u)-\psi(\iota))^{2(\frac{A_\delta}{\delta}-1)}}{(1-\gamma_\zeta)^2\Gamma_\delta^2(A_\delta)}(u-\iota)M\int_\iota^u\mathbb{E}\left\|\varpi_{\varsigma,\mathfrak{G}}-\Phi_{\varsigma,\mathfrak{G}}\right\|^2d\mathfrak{G}\\
&+32\Big(\varsigma\frac{(\psi(u)-\psi(\iota))^{\frac{A_\delta}{\delta}-1}}{(1-\gamma_\zeta)\Gamma_\delta(A_\delta)}\Big)^2(u-\iota)^{2\frac{\mathfrak{A}}{\delta}+1}M\mathbb{Y}_2\\
\le{}&\Big(\wp_1+\wp_2\Big)\varsigma^2(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}+(8\wp_1+\wp_3+\wp_4)\varsigma(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}+\wp_5\varsigma^2(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-2}\\
&+2\wp_6\varsigma^2(\psi(u)-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-2}\int_\iota^u\mathbb{E}\left\|\varpi_{\varsigma,\mathfrak{G}}-\Phi_{\varsigma,\mathfrak{G}}\right\|^2d\mathfrak{G}\\
&+2\Big(\wp_7\varsigma^2+8\wp_7\varsigma\Big)\int_\iota^u\Big[\psi'(\mathfrak{G})(\psi(t)-\psi(\mathfrak{G}))^{\frac{\mathfrak{A}}{\delta}-1}\Big]^2\Delta_{\mathfrak{G}}d\mathfrak{G},
\end{aligned}
$$

where

$$
\begin{aligned}
\wp_1 &= \frac{\frac{8\mathbb{U}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)}{(2\frac{\mathfrak{A}}{\delta}-1)(1-\gamma_\zeta)^2}\check{\mathfrak{J}},\\
\wp_2 &= \frac{8}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)}{(2\frac{\mathfrak{A}}{\delta}-1)(1-\gamma_\zeta)^2}\sup_{\iota<t\le u}\mathbb{P}_1(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big),\\
\wp_3 &= \frac{32}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)}{(1-\gamma_\zeta)^2(2\frac{\mathfrak{A}}{\delta}-1)}\sup_{\iota<t\le u}\mathbb{P}_2(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big),\\
\wp_4 &= \frac{32}{\delta^2\Gamma_\delta^2(\mathfrak{A})}\frac{\mathbb{U}(u-\iota)}{(1-\gamma_\zeta)^2(2\frac{\mathfrak{A}}{\delta}-1)}\sup_{\iota<t\le u}\mathbb{P}_3(t)\Big(1+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_\varsigma(t)\right\|^2+\mathbb{E}\sup_{\iota<t\le u}\left\|\Phi_{\varsigma,t}\right\|^2\Big),\\
\wp_5 &= \frac{64(u-\iota)^2}{(1-\gamma_\zeta)^2\Gamma_\delta^2(A_\delta)}(\beta+1)\Xi+16\frac{\Big((u-\iota)^{\frac{\mathfrak{A}}{\delta}+1}}{(1-\gamma_\zeta)\Gamma_\delta(A_\delta)}\Big)^2M\mathbb{Y}_1+\frac{32}{(1-\gamma_\zeta)^2\Gamma_\delta^2(A_\delta)}(u-\iota)^{2\frac{\mathfrak{A}}{\delta}+1}M\mathbb{Y}_2,\\
\wp_6 &= \frac{8(u-\iota)M}{(1-\gamma_\zeta)^2\Gamma_\delta^2(A_\delta)},\ \wp_7=\frac{\frac{8\breve{\mathfrak{E}}}{\delta^2\Gamma_\delta^2(\mathfrak{A})}(u-\iota)}{(1-\gamma_\zeta)^2}.
\end{aligned}
$$

Then, $\exists\,\rho_0>0$ and $\varepsilon\in(0,1)$ such that

$$
\mathbb{E}\sup_{-\theta\le t<\rho_0\varsigma^{-\varepsilon}}\left\|\varpi_\varsigma(t)-\Phi_\varsigma(t)\right\|^2\le\mathfrak{R}\varsigma^{1-\varepsilon}
$$

holds for all $t\in(\iota,\rho_0\varsigma^{-\varepsilon}]\subset\mathfrak{J}$, where

$$
\begin{aligned}
\mathfrak{R}={}&\Big\{\Big(\wp_1+\wp_2\Big)\varsigma^2(\psi(\rho_0\varsigma^{-\varepsilon})-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}+(8\wp_1+\wp_3+\wp_4)\varsigma(\psi(\rho_0\varsigma^{-\varepsilon})-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}\\
&+\wp_5\varsigma^2(\psi(\rho_0\varsigma^{-\varepsilon})-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-2}\Big\}\times\exp\Big(2\wp_6\varsigma^2(\psi(\rho_0\varsigma^{-\varepsilon})-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}\\
&+\frac{2\Big(\wp_7\varsigma^2+8\wp_7\varsigma\Big)(\psi(\rho_0\varsigma^{-\varepsilon})-\psi(\iota))^{2\frac{\mathfrak{A}}{\delta}-1}}{2\frac{\mathfrak{A}}{\delta}-1}\Big)
\end{aligned}
$$

is a constant. In light of the previously indicated analysis, for $\tau > 0$, $\exists\ \varsigma_2 \in (0, \varsigma_1]$ s.t. for every $\varsigma \in (0, \varsigma_2]$, and $\forall\ t \in [-\theta, \rho_0 \varsigma^{-\varepsilon}]$,

$$\mathbb{E}\Big\{ \sup_{t\in[-\theta,\rho_0\varsigma^{-\varepsilon}]} \big\| \varpi_\varsigma(t) - \Phi_\varsigma(t) \big\|^2 \Big\} \leq \tau.$$

□

4. Applications

Take into consideration the impulsive fractional stochastic delay evolution equations involving the (δ, ψ)-Hilfer fractional derivative as follows:

$$\begin{aligned}
&{}^{1,\mathcal{H}}\mathbb{D}_{0+}^{1/4,3/4;t}\Big(\varpi_\varsigma(t,\iota) - \iota^{1/8} - 1/2\cos(\varpi_\varsigma(t,\iota))\Big) = \varsigma\hat{\lambda}_1\varpi_\varsigma(t,\iota)\sin^2(t) + \sqrt{\varsigma}\hat{\lambda}_2\sin^2(t)\varpi_\varsigma(t,\iota)\frac{d\mathfrak{W}(t)}{dt} \\
&\quad + \sqrt{\varsigma}\frac{1}{dt}\int_{\|\mathfrak{X}\|<s} 2\mathfrak{X}^2\cos^2(t)\varpi_\varsigma(t,\iota)\tilde{\aleph}(dt, d\mathfrak{X}), \quad t \in (0,\pi], \quad \iota \in [0,\pi], \quad t \neq t_k, \\
&\varpi_\varsigma(t,0) = \varpi_\varsigma(t,\pi) = 0, \\
&\Delta\, {}^{1}\mathbb{I}_{0+}^{3/16;t}\varpi_\varsigma\,|_{t=t_k} = \mathbb{C}_k(\varpi_\varsigma(t_k^-)) = |\cos(t)| + \frac{\big|\varpi_\varsigma(t_k^-)\big|}{1+\varpi_\varsigma(t_k^-)}, \quad t = t_k, \quad k = 1,2, \\
&\varpi_\varsigma(x,\iota) = \gamma(x,\iota), \quad -\theta \leq x \leq 0, \\
&{}^{1}\mathbb{I}_{0+}^{3/16;t}\varpi_\varsigma(0) = \gamma_0,
\end{aligned} \tag{4.1}$$

where ${}^{1,\mathcal{H}}\mathbb{D}_{0+}^{1/4,3/4;t}$ is a (δ, ψ)-Hilfer fractional derivative with $\mathfrak{A} = 1/4, \mathfrak{B} = 3/4, \delta = 1, \iota = 0$, and $\psi(t) = t$. In the above, $\mu > 0$, $\hat{\lambda}_1, \hat{\lambda}_2$ are constants, $\mathcal{A} = \mathcal{L}^2([0,\pi])$, and $\mathfrak{W}$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{D}, \mathcal{F})$. Consider

$$\begin{aligned}
\zeta &= \iota^{1/8} + 1/2\cos(\varpi_\varsigma(t,\iota)),\ \tilde{\xi} = \hat{\lambda}_1\varpi_\varsigma(t,\iota)\sin^2(t), \\
\tilde{\varrho} &= \hat{\lambda}_2\sin^2(t)\varpi_\varsigma(t,\iota),\ \ \mathfrak{R} = 2\mathfrak{X}^2\cos^2(t)\varpi_\varsigma(t,\iota).
\end{aligned}$$

Take $\tilde{\rho} = \pi$ and assume

$$\begin{aligned}
\mathfrak{f}(\varpi_\varsigma(t), \varpi_{\varsigma,t}) = \frac{1}{\pi}\int_0^\pi \tilde{\xi}(\mathfrak{G}, \varpi_\varsigma, \varpi_{\varsigma,\mathfrak{G}})d\mathfrak{G} &= \frac{\hat{\lambda}_1}{2}\varpi_\varsigma, \\
\mathfrak{h}(\varpi_\varsigma(t), \varpi_{\varsigma,t}) = \frac{1}{\pi}\int_0^\pi \tilde{\varrho}(\mathfrak{G}, \varpi_\varsigma(\mathfrak{G}), \varpi_{\varsigma,\mathfrak{G}})d\mathfrak{G} &= \frac{\hat{\lambda}_2}{2}\varpi_\varsigma, \\
\mathfrak{Y}(\varpi_\varsigma(t), \varpi_{\varsigma,t}, \mathfrak{X}) = \frac{1}{\pi}\int_0^\pi \mathfrak{R}(\mathfrak{G}, \varpi_\varsigma(\mathfrak{G}), \varpi_{\varsigma,\mathfrak{G}}, \mathfrak{X})d\mathfrak{G} &= \mathfrak{X}^2\varpi_\varsigma.
\end{aligned}$$

Consequently, it is easy to confirm that all of the conditions listed in Theorem 3.2 have been met given the choices provided above. Thus, the averaged equation for (4.1) can be expressed as

$$
\begin{aligned}
&{}^{1,\mathcal{H}}\mathbb{D}_{0+}^{1/4,3/4;t}\Big(\Phi_{\varsigma}(t)(t,\iota)-\iota^{1/8}-1/2\cos(\Phi_{\varsigma}(t,\iota))\Big)=\frac{\varsigma}{2}\hat{\lambda}_1\Phi_{\varsigma}(t,\iota)+\frac{\sqrt{\varsigma}}{2}\hat{\lambda}_2\Phi_{\varsigma}(t,\iota)\frac{d\mathfrak{W}(t)}{dt}\\
&\quad+\sqrt{\varsigma}\frac{1}{dt}\int_{\|\mathfrak{X}\|<s}\mathfrak{X}^2\Phi_{\varsigma}(t,\iota)\tilde{\aleph}(dt,d\mathfrak{X}),\quad t\in(0,\pi],\quad \iota\in[0,\pi],\quad t\neq t_k,\\
&\Phi_{\varsigma}(t,0)=\Phi_{\varsigma}(t,\pi)=0,\\
&\Delta^{1}\mathbb{I}_{0+}^{3/16;t}\Phi_{\varsigma}\mid_{t=t_k}=\mathbb{C}_k(\Phi_{\varsigma}(t_k^-))=|\cos(t)|+\frac{\left|\Phi_{\varsigma}(t_k^-)\right|}{1+\Phi_{\varsigma}(t_k^-)},\quad t=t_k,\quad k=1,2,\\
&\Phi_{\varsigma}(x,\iota)=\gamma(x,\iota),\quad -\theta\le x\le 0,\\
&{}^{1}\mathbb{I}_{0+}^{3/16;t}\Phi_{\varsigma}(0)=\gamma_0.
\end{aligned}
\tag{4.2}
$$

Clearly, compared with the original system (4.1), the time-averaged system (4.2) is a much simpler system. Additionally, Theorem 3.2 guarantees the existence of a very small error in their responses.

To further corroborate the analytical findings of Theorem 3.2, we now simulate the original (non-averaged) impulsive (δ,ψ)-Hilfer FSDE (4.1) and compare it against the averaged model (4.2). The parameter choices are preserved as:

$$\varsigma=1,\quad \hat{\lambda}_1=0.5,\quad \hat{\lambda}_2=0.25,\quad \theta=0.5,\quad \gamma(x,\iota)=\sin(x\iota),\quad \gamma_0=0.$$

Both systems are numerically integrated using a uniform time step $\Delta t=0.01$. The stochastic integrals are approximated via a truncated Euler–Maruyama method, while the Lévy noise is modeled through a compound Poisson process with intensity $\nu=5$ and jump amplitudes sampled from a truncated normal distribution. For the Brownian motion $\mathfrak{W}(t)$, 500 sample paths are generated via Monte Carlo sampling. The impulsive effects at $t_1=\pi/3$ and $t_2=2\pi/3$ are incorporated according to the prescribed jump condition. To quantify the precision of the averaging approximation, we compute the absolute error

$$E(t,\iota):=\left|\varpi_{\varsigma}(t,\iota)-\Phi_{\varsigma}(t,\iota)\right|$$

across representative trajectories. The resulting plots in Figures 1 and 2 confirm that the maximum deviation remains confined within the theoretically predicted bounds, thus validating the robustness of the averaging method applied to impulsive stochastic Hilfer-type systems driven by Lévy noise.

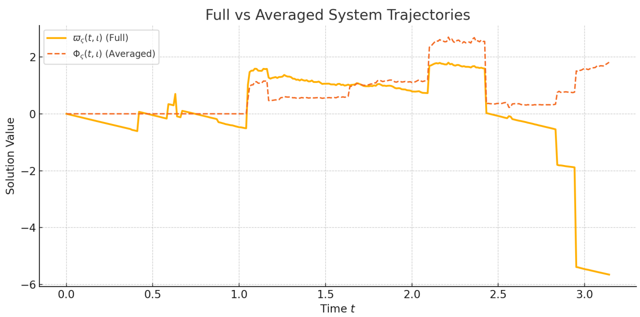

Figure 1. Comparison between the trajectories of the full system $\varpi_{\varsigma}(t,\iota)$ and the averaged system $\Phi_{\varsigma}(t,\iota)$ for a fixed ι. Both exhibit qualitatively similar stochastic dynamics, supporting the averaging theory.

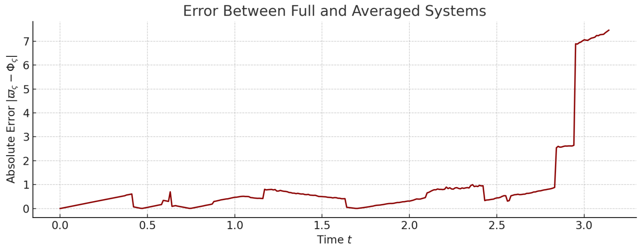

Figure 2. Evolution of the absolute error between $\varpi_\varsigma(t,\iota)$ and $\Phi_\varsigma(t,\iota)$ over time. The maximum deviation stays within the theoretical bounds guaranteed by Theorem 3.2.

These simulations demonstrate the following.

(1) The qualitative agreement between the stochastic behavior of the original and averaged systems.
(2) The empirical verification of the asymptotic closeness between $\varpi_\varsigma(t,\iota)$ and $\Phi_\varsigma(t,\iota)$.
(3) The practical efficacy of averaging in simplifying the analysis of complex impulsive FSDEEs with Lévy noise.

Thus, this simulation serves as concrete numerical evidence of the theoretical results established in Theorem 3.2.

5. Conclusions

This work has presented a novel stochastic averaging theorem for a parameterized family of impulsive fractional stochastic delay differential equations governed by the (δ,ψ)-Hilfer fractional derivative and driven by a Lévy process. Distinct from extant investigations in the field, the present study orchestrates a concurrent analytical treatment of both discontinuous stochastic perturbations induced by Lévy noise and the nonlocal, memory-dependent dynamics encapsulated by the (δ,ψ)-Hilfer-type fractional differential operator. The theoretical framework developed herein furnishes a new mathematical apparatus for examining the asymptotic dynamics of generalized fractional-order systems under impulsive and stochastic influences, thus extending the applicability of the stochastic averaging method to a broader class of fractional stochastic functional systems. Furthermore, considering that fractional Brownian motion and classical Brownian motion frequently arise in modeling temporal phenomena exhibiting self-similarity and long-range dependence, future research avenues should explore the extension of the proposed averaging paradigm to impulsive (δ,ψ)-Hilfer-type fractional stochastic delay differential equations perturbed by such Gaussian processes. Such an extension would provide deeper insight into the interplay between hereditary dynamics, abrupt discontinuities, and long-memory stochastic fluctuations in complex dynamical systems.

Author contributions

A.M. Sayed Ahmed: Formal analysis, Software; Hamdy M. Ahmed: Validation, Methodology; Taher A. Nofal: Resources, Writing–review & editing; Soliman Alkhatib: Software, Writing–review & editing; Hisham H. Hussein: Software, Writing–original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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