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Research article

The averaging principle for stochastic differential equations with Lévy noise involving conformable fractional derivative

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Abstract: In this paper, the averaging principle for conformable fractional stochastic differential equations with Lévy noise is investigated. Initially, the averaging principle for classical Itô-type conformable fractional stochastic differential equations is presented. Subsequently, the averaging principle is extended to the case involving Lévy noise. Different from the approach of integration by parts or decomposing integral interval to deal with the estimation of integral involving singular kernel, this study introduces a novel method to assess the error between the averaged stochastic equation and the original stochastic differential equations, thereby effectively addressing the challenge posed by singular kernels. Finally, a simulation example is provided to validate the theoretical analysis.

Keywords: averaging principle; conformable fractional derivative; stochastic differential equations; Lévy noise
Mathematics Subject Classification: 26A33, 34A37

1. Introduction

In engineering modeling, fractional derivatives are widely regarded as a powerful mathematical tool for analyzing genetic characteristics [1, 2]. Starting from the establishment of fractional derivatives, many researchers have paid attention to and conducted studies on fractional calculus [3]. The fractional differential equations have developed to this point, and rich achievements have been made in their qualitative theory as well as the qualitative theory of related common coupled equations [4, 5].

In recent years, stochastic fractional modeling has gained increasing importance across various fields of science and industry, owing to its ability to simultaneously account for the memory effects inherent in fractional derivatives and the influence of stochastic disturbances on dynamic systems. The controllability and optimal control [6, 7], existence [8], stability [9, 10], and averaging principle [11, 12] of solutions for Caputo fractional stochastic differential equations have been extensively studied and have attracted considerable attention in the mathematical community. However, the Caputo fractional derivative is a non-local operator and lacks certain mathematical properties associated with local derivatives. For instance, when the order of the fractional derivative is less than zero, it does not guarantee a decreasing behavior of the original function. Therefore, it is of significant importance to develop a new class of fractional stochastic differential equations that exhibit more favorable mathematical properties.

The conformable fractional derivatives, which extend the classical limit-based definitions of derivatives for functions, were first introduced by Khalil et al. [13]. Subsequently, several fundamental properties were established, including Leibniz's law, the chain rule, exponential functions, Gronwall's inequality, integration by parts, and Taylor power series within the framework of fractional calculus [14, 15]. Compared to the Caputo fractional derivative, the conformable fractional derivative possesses simpler and more favorable mathematical properties, thereby facilitating analytical and computational operations [16, 17] (the comparison of conformable fractional derivatives and Caputo fractional derivative see Table 1). Ma et al. [18] observed that simulations based on conformable fractional derivatives are simpler and more appropriate than those using Caputo fractional derivatives in the context of grey system models. The existence and stability of solutions to conformable fractional stochastic differential equations have been investigated in the literature [19–21]. Luo et al. [22, 23] studied the conformable backward stochastic differential equations. Moreover, the regional controllability and observability for conformable systems have also been analyzed [24]. These findings indicate that the qualitative analysis theory for conformable fractional stochastic differential equations has reached a mature stage.

Table 1. Comparison of conformable fractional derivatives and Caputo fractional derivative.

References	**Fractional derivatives**	**Fractional integral**	**Advantage**	**Properties**	**Parameters adjustable?**
conformable fractional derivatives [13–17]	$\mathfrak{D}_0^\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(t+t^{1-\alpha}\varepsilon)-f(t)}{\varepsilon}, \ t > 0, \quad 0 < \alpha \le 1$	$I_0^\alpha f(t) = \int_0^t f(s) d\frac{s^\alpha}{\alpha} = \int_0^t f(s)s^{\alpha-1}ds, \quad t > 0, \quad 0 < \alpha \le 1$	Derivative simpler, more favorable mathematical properties, facilitating analytical	Local derivative, similar to integer derivative	**Yes**
Caputo fractional derivative [1–3]	${}^C D_{0^+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds, \ t \in [0, \infty)$, where $\Gamma(\alpha) := \int_0^\infty s^{\alpha-1}e^{-s}ds$ is Gamma function	$\mathbb{I}_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \ t \in [0, \infty)$	Analyze processes that involve genetic and memory properties	Nonlocal derivatives, unlike integer derivatives, involve the gamma function	**Yes**

Stochastic differential equations serve as essential mathematical instruments for analyzing random phenomena and have garnered significant attention from numerous researchers in the field of mathematics [25]. Stochastic averaging principle for Itô stochastic differential equations was proposed by Khasminskii in [26], where proved that a solution of the average equation can converge to the solution of the complex system under the suitable conditions. However, the averaging principle for conformable fractional stochastic differential equations has not yet been established in the literature. In general, much of the noise encountered in practical systems is non-Gaussian in nature, characterized by more spikes and abrupt fluctuations. In recent years, the qualitative theory of stochastic differential equations driven by non-Gaussian noise has attracted considerable research interest. For instance, properties such as stability [27] and the averaging principle [28–30] have received increasing attention. Shen et al. extended the averaging principle for fractional stochastic differential equations with Lévy noise to the cases of delay systems [31] and hybrid systems [32]. Furthermore, Xu et al. [33] established an averaging principle for fractional stochastic dynamical systems driven by non-Gaussian Lévy noise under a class of weakened Lipschitz conditions. In their approach, the classical method of integration by parts was employed to handle fractional integrals; however, this technique leads to the appearance of singular integrals, which are difficult to evaluate analytically. To address this issue, Guo et al. [34] proposed a novel set of averaging conditions formulated in terms of fractional integral expressions, thereby effectively mitigating the singularity problem.

In this paper, we investigate the averaging principle for conformable fractional stochastic differential equations. Unlike previous approaches based on integration by parts [35] or decomposition of the integral interval [36], we impose growth conditions on the nonlinear stochastic term and employ Hölder's inequality and Chebyshev's inequality to estimate integrals involving singular kernels. We establish the validity of the averaging principle for conformable fractional stochastic differential equations with Lévy noise in the mean-square sense.

The paper is organized as follows. In Section 2, some preliminary definitions and lemmas are introduced. The averaging principle for conformable fractional stochastic differential equations with both white noise and Lévy noise is presented in Section 3. In Section 4, a numerical example is provided to illustrate the theoretical analysis. Section 5 provides a brief concluding summary.

2. Preliminaries

Throughout the paper, let $\mathcal{X}_t := \mathbb{L}^2(\Omega, \mathcal{F}_t, P),\ t \in [0, +\infty)$ be the space of all $\mathcal{F}_t$-measurable process. Moreover, a process x is called $\mathcal{F}$-adapted if $x(t) \in \mathcal{X}_t, t \in [0, +\infty)$. $a \vee b$ represents $\max\{a, b\}$, $E(x)$ denotes the mathematical expectation of x. The space $L_n^2[0, T]$ denotes the family of Borel measurable functions $f : [0, T] \to \mathbb{R}^n$ such that $\int_0^T |h(t)|dt < \infty$, where $|\cdot|$ is the norm of $\mathbb{R}^n$.

Definition 2.1. *(see [13, Definition 2.1]) The conformable fractional derivative with low index* 0 *of a function* $f : [0, \infty) \to \mathbb{R}$ *is defined as*

$$\mathfrak{D}_0^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + t^{1-\alpha}\varepsilon) - f(t)}{\varepsilon}, \quad t > 0, \quad 0 < \alpha \le 1,$$

where $\mathfrak{D}_0^\alpha f(0) = \lim_{t \to 0^+} \mathfrak{D}_0^\alpha f(t)$, $f(\cdot)$ *is differentiable and* $\lim_{t \to 0^+} \mathfrak{D}_0^\alpha f(t)$ *exist.*

Definition 2.2. *(see [14, Notation, p.58]) The conformable fractional integral with low index* α *of a function* $f : [0, \infty) \to \mathbb{R}$ *is defined as*

$$I_0^\alpha f(t) = \int_0^t f(s) d\frac{s^\alpha}{\alpha} = \int_0^t f(s) s^{\alpha-1} ds, \quad t > 0, \quad 0 < \alpha \le 1.$$

Next, we will introduce some important lemmas.

Lemma 2.3. *(see [37, Theorem 1] Gronwall-Bellman inequality) Let $X(\cdot)$ be real continue function on $[t_0, t_1]$, $f(\cdot) \geq 0$ is the integrable function over interval $[t_0, t_1]$, σ is a positive real number. If*

$$X(t) \leq \sigma + \int_{t_0}^{t} f(s)X(s)ds, \quad t \in [t_0, t_1],$$

then

$$X(t) \leq \sigma \exp\left(\int_{t_0}^{t} f(s)ds\right), \quad t \in [t_0, t_1].$$

3. Main results

3.1. Stochastic averaging principle

In this section, we consider the following conformable fractional stochastic differential equations

$$\mathfrak{D}_0^{\alpha} X(t) = f(t, X(t)) + g(t, X(t))\frac{dB(t)}{dt}, \quad \alpha \in (\frac{1}{2}, 1], t \geq 0, \tag{3.1}$$

with the initial value $X(0) = X_0$, where $\mathfrak{D}_0^{\alpha}$ is the conformable fractional derivative, $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous functions. $B(\cdot)$ is a m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $E(|X_0|^2) < \infty$.

We introduce the following assumption.

[H1] For any $(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, there exists two positive constants L_1, L_2 such that

$$|f(t, x) - f(t, y)| \quad \vee \quad |g(t, x) - g(t, y)| \leq L_1|x - y|,$$
$$|f(t, x)| \quad \vee \quad |g(t, x)| \leq L_2(1 + |x|).$$

Lemma 3.1. *(see [19, Theorem 4.3]) If [H1] holds, then Eq* (3.1) *has a unique solution $X(\cdot) \in \mathbb{L}_n^2[0, T]$ provided that $\alpha \in (\frac{1}{2}, 1]$ give by*

$$X(t) = X_0 + \int_0^t f(s, X(s))s^{\alpha-1}ds + \int_0^t g(s, X(s))s^{\alpha-1}dB(s), \tag{3.2}$$

and $\int_0^T |x(t)|^2 dt < \infty$.

Definition 3.2. *If $X(\cdot)$ is an $\mathcal{F}$-adapted process and satisfies the integral equation* (3.2) *almost surely for all $t \geq 0$, then $X(\cdot)$ is called a mild solution to Eq* (3.1)*.*

Now, we investigate the averaging principle of conformable fractional stochastic differential equations. Consider the standard form of Eq (3.1)

$$\mathfrak{D}_0^{\alpha} X(t) = \epsilon f(t, X(t)) + \sqrt{\epsilon} g(t, X(t))\frac{dB(t)}{dt}, \quad \alpha \in (\frac{1}{2}, 1], t \geq 0.$$

where, the initial value X_0 and functions f, g are same as (3.1), ϵ is a positive small parameter. Combining the existence and uniqueness results (Lemma 3.1), we can get its solution

$$X_\epsilon(t) = X_0 + \epsilon \int_0^t f(s, X_\epsilon(s))s^{\alpha-1}ds + \sqrt{\epsilon} \int_0^t g(s, X_\epsilon(s))s^{\alpha-1}dB(s).$$

Next, taking the average of the functions $f(\cdot), g(\cdot)$ with respect to t, and named them by $\bar{f}(\cdot), \bar{g}(\cdot)$ as follows:

$$\begin{aligned} \bar{f}(X(t)) &= \lim_{T_1 \to \infty} \frac{1}{T_1} \int_0^{T_1} f(t, X(t))dt, \\ \bar{g}(X(t)) &= \lim_{T_1 \to \infty} \frac{1}{T_1} \int_0^{T_1} g(t, X(t))dt, \end{aligned} \tag{3.3}$$

with frozen slow components $X(\cdot)$. Therefore, we can get the following conformable fractional time-averaged equation:

$$\mathfrak{D}_0^{\alpha} Y(t) = \epsilon \bar{f}(Y(t)) + \sqrt{\epsilon}\bar{g}(Y(t))\frac{dB(t)}{dt}, \tag{3.4}$$

with initial value $Y(0) = X(0)$. Here, $Y(\cdot)$ is a stochastic process and the solution of the time-averaged equation.

Remark 3.3. *(see [39, p.193]) If all two limits in* (3.3) *exist, the functions functions $\bar{f}(\cdot), \bar{g}(\cdot)$ also satisfy the linear growth condition and Lipschitz condition as functions functions $f(\cdot), g(\cdot)$ with the same constants L_1, L_2.*

[H2] For any $t \in [0, T]$, $X \in \mathbb{R}^n$, there exists two positive bounded functions $\delta_1(\cdot), \delta_2(\cdot)$ such that

$$\begin{aligned} \frac{1}{t} \int_0^t |f(s, X) - \bar{f}(X)|^2 ds &\leq \delta_1(t)(1 + |X|^2), \\ \frac{1}{t} \int_0^t |g(s, X) - \bar{g}(X)|^4 ds &\leq \delta_2(t)(1 + |X|^4), \end{aligned}$$

where $\lim_{t\to\infty} \delta_1(t) = 0, \lim_{t\to\infty} \delta_2(t) = 0$.

Remark 3.4. *It should be emphasized that we restrict $\frac{1}{t}\int_0^t |g(s, X) - \bar{g}(X)|^4 ds \leq \delta_2(t)(1 + |X|^4)$ instead of $\frac{1}{t}\int_0^t |g(s, X) - \bar{g}(X)|^2 ds \leq \delta_2(t)(1 + |X|^2)$ to solve singular integrals when applying the integration by parts formula.*

Let $Y_\epsilon(\cdot)$ be the solution of the time-averaged equation (3.4). Hence, we have

$$Y_\epsilon(t) = X_0 + \epsilon \int_0^t \bar{f}(Y_\epsilon(s))s^{\alpha-1}ds + \sqrt{\epsilon}\int_0^t \bar{g}(Y_\epsilon(s))s^{\alpha-1}dB(s).$$

$E(X)$ represents the expectation of random variable X. Next, we show that $Y_\epsilon(t) \to X_\epsilon(t), \epsilon \to 0^+, t \in [0, T]$ in the sense of mean square.

Theorem 3.5. *Assume that conditions [H1],[H2] are satisfied. For a given arbitrarily real number $\varepsilon > 0$, there exist $L > 0$, $\epsilon > 0$ such that for any $\alpha \in (\frac{3}{4}, 1]$,*

$$E\Big(\sup_{t\in[0,L\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)|^2\Big) \leq \varepsilon.$$

hold for any $\alpha \in (\frac{3}{4}, 1]$.

Proof. For any $t \in [0, h] \subset [0, T]$,

$$X_\epsilon(t) - Y_\epsilon(t) = \epsilon \int_0^t [f(s, X_\epsilon(s)) - \bar{f}(Y_\epsilon(s))] s^{\alpha-1} ds + \sqrt{\epsilon} \int_0^t [g(s, X_\epsilon(s)) - \bar{g}(Y_\epsilon(s))] s^{\alpha-1} dB(s).$$

Note that $|a + b|^2 \leq 2|a|^2 + 2|b|^2$, we have

$$\begin{aligned} E\Big(\sup_{t\in[0,h]} |X_\epsilon(t) - Y_\epsilon(t)|^2\Big) &\leq 2\epsilon^2 E\Big(\sup_{t\in[0,h]} \Big|\int_0^t [f(s, X_\epsilon(s)) - \bar{f}(Y_\epsilon(s))] s^{\alpha-1} ds\Big|^2\Big) \\ &\quad +2\epsilon E\Big(\sup_{t\in[0,h]} \Big|\int_0^t [g(s, X_\epsilon(s)) - \bar{g}(Y_\epsilon(s))] s^{\alpha-1} dB(s)\Big|^2\Big) \\ &=: I_1 + I_2. \end{aligned} \tag{3.5}$$

Similarly, we can get

$$\begin{aligned} I_1 &\leq 4\epsilon^2\Big(E \sup_{t\in[0,h]} \Big|\int_0^t \Big[f(s, X_\epsilon(s)) - f(s, Y_\epsilon(s))\Big] s^{\alpha-1} ds\Big|^2\Big) \\ &\quad +4\epsilon^2 E\Big(\sup_{t\in[0,h]} \Big|\int_0^t \Big[f(s, Y_\epsilon(s)) - \bar{f}(Y_\epsilon(s))\Big] s^{\alpha-1} ds\Big|^2\Big) \\ &=: I_{11} + I_{12}. \end{aligned}$$

Applying the Cauchy-Schwarz inequality and conditions [H1], we have

$$\begin{aligned} I_{11} &\leq 4\epsilon^2 h E\Big(\sup_{t\in[0,h]} \int_0^t \Big|\Big[f(s, X_\epsilon(s)) - f(s, Y_\epsilon(s))\Big] s^{\alpha-1}\Big|^2 ds\Big) \\ &\leq 4h\epsilon^2 L_1^2 \int_0^h E\Big(\sup_{s\in[0,h]} \Big|X_\epsilon(s) - Y_\epsilon(s)\Big|^2\Big) s^{2\alpha-2} ds. \end{aligned} \tag{3.6}$$

Next, by [H2] and applying Cauchy-Schwarz inequality again, we have

$$\begin{aligned} I_{12} &\leq 4\epsilon^2\Big(\sup_{t\in[0,h]} \int_0^t s^{2\alpha-2} ds\Big) E\Big(\sup_{t\in[0,h]} \int_0^h \Big|f(s, Y_\epsilon(s)) - \bar{f}(Y_\epsilon(s))\Big|^2 ds\Big) \\ &\leq 4\epsilon^2 \frac{h^{2\alpha-1}}{2\alpha - 1} E\Big(\sup_{t\in[0,h]} \delta_1(t)(1 + |Y_\epsilon(s)|^2)\Big) \\ &\leq \frac{4\epsilon^2 h^{2\alpha}}{2\alpha - 1} \sup_{t\in[0,h]} \delta_1(t)\Big(1 + E \sup_{t\in[0,h]} |Y_\epsilon(t)|^2\Big). \end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} I_2 &\leq 4\epsilon E\Big(\sup_{t\in[0,h]} \Big|\int_0^t [g(s, X_\epsilon(s)) - g(s, Y_\epsilon(s))] s^{\alpha-1} dB(s)\Big|^2\Big) \\ &\quad +4\epsilon E\Big(\sup_{t\in[0,h]} \Big|\int_0^t [g(s, Y_\epsilon(s)) - \bar{g}(Y_\epsilon(s))] s^{\alpha-1} dB(s)\Big|^2\Big) \\ &\leq 4\epsilon E\Big(\sup_{t\in[0,h]} \int_0^t \Big|[g(s, X_\epsilon(s)) - g(s, Y_\epsilon(s))] s^{\alpha-1}\Big|^2 ds\Big) \end{aligned}$$

$$
\begin{aligned}
&+4\epsilon E\Big(\sup_{t\in[0,h]}\int_0^t\Big|[g(s,Y_\epsilon(s))-\bar{g}(Y_\epsilon(s))]s^{\alpha-1}\Big|^2ds\\
=:&\ \ I_{21}+I_{22}.
\end{aligned}
$$

Using [H1], we have

$$
I_{21} \ \leq\ 4\epsilon L_1^2\int_0^h E(\sup_{s\in[0,h]}\Big|X_\epsilon(s)-Y_\epsilon(s)\Big|^2)s^{2\alpha-2}ds. \tag{3.8}
$$

Now, using the Hölder inequality, one has

$$
\begin{aligned}
I_{22} &\leq 4\epsilon E\Big(\sup_{t\in[0,h]}\int_0^t\Big|[g(s,Y_\epsilon(s))-\bar{g}(Y_\epsilon(s))]s^{\alpha-1}\Big|^2ds\Big)\\
&\leq 4\epsilon E\Big(\sup_{t\in[0,h]}\Big[\Big(\int_0^t s^{4\alpha-4}ds\Big)^{\frac{1}{2}}\Big(\int_0^t\Big|g(s,Y_\epsilon(s))-\bar{g}(Y_\epsilon(s))\Big|^4ds\Big)^{\frac{1}{2}}\Big]\Big)\\
&\leq \frac{4\epsilon\sqrt{4\alpha-3}h^{2\alpha-\frac{3}{2}}}{4\alpha-3}E\Big(\sup_{t\in[0,h]}\int_0^t|g(s,Y_\epsilon(s))-\bar{g}(Y_\epsilon(s))|^4ds\Big)^{\frac{1}{2}}\\
&\leq \frac{4\epsilon\sqrt{4\alpha-3}h^{2\alpha-1}}{4\alpha-3}\sup_{t\in[0,h]}\sqrt{\delta_2(t)}E\sup_{t\in[0,h]}\Big(1+|Y_\epsilon(t)|^4\Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.9}
$$

Thus, we have $I_{12}+I_{22}\leq K(\epsilon)$, where

$$
\begin{aligned}
K(\epsilon) \ =\ & \frac{4\epsilon^2T^{2\alpha}}{2\alpha-1}\sup_{t\in[0,T]}\delta_1(t)\Big(1+E\sup_{t\in[0,T]}|Y_\epsilon(t)|^2\Big)\\
&+\frac{4\epsilon\sqrt{4\alpha-3}T^{2\alpha-1}}{4\alpha-3}\sup_{t\in[0,T]}\sqrt{\delta_2(t)}E\sup_{t\in[0,T]}\Big(1+|Y_\epsilon(t)|^4\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Now, substitute (3.6)–(3.9) into (3.5), we have

$$
E\Big(\sup_{t\in[0,h]}|X_\epsilon(t)-Y_\epsilon(t)|^2\Big)\leq K(\epsilon)+4\epsilon L_1^2(1+h\epsilon)\int_0^h E(\sup_{s\in[0,h]}\Big|X_\epsilon(s)-Y_\epsilon(s)\Big|^2)s^{2\alpha-2}ds.
$$

Applying Gronwall inequality, we have

$$
E\Big(\sup_{t\in[0,h]}|X_\epsilon(t)-Y_\epsilon(t)|^2\Big)\leq K(\epsilon)\cdot\exp\big(4L_1^2(\epsilon+h\epsilon^2)\frac{h^{2\alpha-1}}{2\alpha-1}\big).
$$

This implies that we can select a number $L>0,\alpha\in(\frac{3}{4},1]$, such that for every $t\in[0,L\epsilon^{-\alpha}]\subseteq[0,T]$ having

$$
E\Big(\sup_{t\in[0,L\epsilon^{-\alpha}]}|X_\epsilon(t)-Y_\epsilon(t)|^2\Big)\leq K(\epsilon)\cdot\xi(\epsilon),\quad \epsilon>0.
$$

where

$$
\xi(\epsilon)=\exp\Big(\frac{4L_1^2L^{2\alpha-1}}{2\alpha-1}(\epsilon^{1+\alpha-2\alpha^2}+L\epsilon^{2-2\alpha^2})\Big),\quad \epsilon>0.
$$

Obviously, $K(\epsilon) \cdot \xi(\epsilon), \epsilon > 0$ are increasing function with respect to the variable ϵ. Noting that $\inf_{\epsilon>0} K(\epsilon) \cdot \xi(\epsilon) = 0$. Hence, for any given number $\varepsilon > 0$, we can always find a number $\epsilon > 0$, such that $K(\epsilon) \cdot \xi(\epsilon) = \varepsilon$. This implies that

$$E\Big(\sup_{t\in[0,L\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)|^2\Big) \le \varepsilon.$$

The proof is completed. □

3.2. Stochastic averaging principle with Lévy noise

In this section, we consider the averaging principle for the following standard conformable fractional stochastic differential equations with Lévy noise:

$$\mathfrak{D}_0^\alpha X(t) = \epsilon f(t, X(t)) + \sqrt{\epsilon} g(t, X(t))\frac{dL(t)}{dt},\ X(0) = X_0,\ \alpha \in (\frac{3}{4}, 1],\ t \ge 0, \tag{3.10}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous function, $L(\cdot)$ is a m-dimensional Lévy motion, and X_0 is an $\mathcal{F}_0$-measurable $\mathbb{R}^n$-value random variable, satisfying $E(|X_0|^p) < +\infty, p \ge 2$.

Firstly, using Lévy-Itô decomposition, we can rewrite the standard conformable fractional stochastic differential equations (3.10) and give it with a more general representation

$$\begin{aligned}\mathfrak{D}_0^\alpha X(t) &= \epsilon b(t, X(t-)) + \sqrt{\epsilon}\sigma(t, X(t-))\frac{dB(t)}{dt} + \frac{\sqrt{\epsilon}}{dt}\int_{|x|<c} H(t, X(t-), x)\tilde{N}(dt, dx) \\ &\quad + \frac{\sqrt{\epsilon}}{dt}\int_{|x|\ge c} G(t, X(t-), x)N(dt, dx),\end{aligned} \tag{3.11}$$

where $X(t-)$ is the left limit $\lim_{s<t,s\to t} X(s)$, $b : \mathbb{R}^+\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^+\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$, $H : \mathbb{R}^+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable, constant $c \in (0, \infty]$ is the maximum allowable jump size, $N(t, dx) : \mathbb{R}^+ \times \{\mathbb{R}^m \setminus \mathbf{0}\}$ controlling small jumps, $\tilde{N}(dt, dx) = N(t, dx) - t\nu(dx)$ controlling large jumps, both are compensated Poisson random measure and ν is the jump measure.

To simplify the mathematical model, here we only consider the Lévy motion without large jumps, then (3.11) can be rewritten as

$$\mathfrak{D}_0^\alpha X(t) = \epsilon b(t, X(t-)) + \sqrt{\epsilon}\sigma(t, X(t-))\frac{dB(t)}{dt} + \frac{\sqrt{\epsilon}}{dt}\int_{|x|<c} H(t, X(t-), x)\tilde{N}(dt, dx). \tag{3.12}$$

Let $|\cdot|$ be $\mathbb{R}^n$-norm, $\|\cdot\|$ be matrix norm, and $a(\cdot, x_1, x_2) = \sigma(\cdot, x_1)\sigma(\cdot, x_2)^T$ is a $n \times n$ matrix. We introduce two assumptions on functions $b(\cdot), \sigma(\cdot)$ and $H(\cdot)$.

$[A3]$: For all $x_1, x_2 \in \mathbb{R}^n$, $t \in [0, T]$ and constant $C_1 > 0$. Assume

$$\begin{aligned}&|b(t, x_1) - b(t, x_2)|^2 \vee \|a(t, x_1, x_1) - 2a(t, x_1, x_2) + a(t, x_2, x_2)\| \vee \\ &\int_{|x|<c} |H(t, x_1, x) - H(t, x_2, x)|^2\nu(dx) \le C_1|x_1 - x_2|^2.\end{aligned}$$

$[A4]$: For all $x_1 \in \mathbb{R}^n$, $t \in [0, T]$ and constant $C_2 > 0$. Assume

$$|b(t, x_1)|^2 \vee \|a(t, x_1, x_1)\| \vee \int_{|x|<c} |H(t, x_1, x)|^2\nu(dx) \le C_2(1 + |x_1|^2).$$

For the conformable fractional stochastic differential equation (3.12) with Lévy noise, according to Definition 2.2, we can easily conclude as follows.

Lemma 3.6. *If $\alpha \in (\frac{1}{2}, 1]$ and Assumptions $[H1]$ and $[H2]$ hold, then conformable fractional stochastic differential equation* (3.12) *has a unique, adapted and càdlàg (right continuous with left limits) mild solution*

$$\begin{aligned} X(t) := X_\epsilon(t) &= X_0 + \epsilon \int_0^t b(s, X_\epsilon(s-))s^{\alpha-1}ds + \sqrt{\epsilon} \int_0^t \sigma(s, X_\epsilon(s-))s^{\alpha-1}dB(s) \\ &\quad + \sqrt{\epsilon} \int_0^t s^{\alpha-1} \int_{|x|<c} H(s, X(s-), x)\tilde{N}(ds, dx), \end{aligned}$$

and $E(\int_0^T |X(t)|^2 dt) < +\infty$ with $T < \infty$.

Proof. It is worth noting that conformable fractional integration is a special form of Caputo fractional integration. Hence, replace $\frac{1}{\Gamma(\alpha)} \int_0^t \sigma(s, X_\epsilon(s-))(t-s)^{\alpha-1}dB(s)$ with $\int_0^t \sigma(s, X_\epsilon(s-))s^{\alpha-1}dB(s)$, $\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_{|x|<c} H(s, X(s-), x)\tilde{N}(ds, dx)$ with $\int_0^t s^{\alpha-1} \int_{|x|<c} H(s, X(s-), x)\tilde{N}(ds, dx)$. It should be noted that both functions $(t-s)^{\alpha-1}$, and $s^{\alpha-1}$ are continuous, and their properties remain invariant throughout the integration process. Consequently, repeating the proof of [38, Theorem 3.1], the conclusion can be established. In addition, $E(\int_0^T |X(t)|^2 dt) < +\infty$ with $T < \infty$. The conclusion has been rigorously established. □

Before presenting the main results, we first establish a lemma concerning the estimate on p-th moments on the solution, which serves as a foundation for the subsequent analysis.

Lemma 3.7. *(Estimate on p-th moments) Let $p \geq 2$, $\alpha \in (\frac{p-1}{p}, 1]$. If the Assumptions $[A3]$ and $[A4]$ hold, then the solution of FSDEs* (3.10) *satisfies*

$$E\Big(\sup_{t\in[0,T]} |X(t)|^p\Big) \leq \Big(8^{p-1}E\Big|1 + X_0\Big|^p\Big)\exp\Big(\frac{\Psi T^{p\alpha-p+1}}{p\alpha - p + 1}\Big) + 2^{p-1},$$

where $\Psi = \frac{4^{p-1}C_2^{\frac{p}{2}}\epsilon^{\frac{p}{2}}}{\Gamma(\alpha)^p}(2T^{\frac{p-2}{2}} + \epsilon^{\frac{p}{2}}T^{p-1})$.

Proof. According to Lemma 3.6 and inequality $(a + b + c + d)^p \leq 4^{p-1}(a^p + b^p + c^p + d^p)$, we have

$$\begin{aligned} |1 + X(t)|^p &= 4^{p-1}\Big|1 + X_0\Big|^p + 4^{p-1}\Big(\frac{\epsilon}{\Gamma(\alpha)} \int_0^t b(s, X_\epsilon(s-))s^{\alpha-1}ds\Big)^p \\ &\quad +4^{p-1}\Big(\frac{\sqrt{\epsilon}}{\Gamma(\alpha)} \int_0^t \sigma(s, X_\epsilon(s-))s^{\alpha-1}dB(s)\Big)^p \\ &\quad +4^{p-1}\Big(\frac{\sqrt{\epsilon}}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} \int_{|x|<c} H(s, X(s-), x)\tilde{N}(ds, dx)\Big)^p \end{aligned}$$

hold for all $p \geq 2,\ t \in [0, T]$. Now, applying the Höder's inequality and $[A4]$, we have

$$\begin{aligned} E\Big(\int_0^t b(s, X_\epsilon(s-))s^{\alpha-1}ds\Big)^p &\leq T^{p-1}E\Big(\int_0^t b^p(s, X_\epsilon(s-))s^{p\alpha-p}ds\Big) \\ &\leq C_2^{\frac{p}{2}}T^{p-1}E\Big(\int_0^t (1 + |X_\epsilon(s-)|)^p s^{p\alpha-p}ds\Big). \end{aligned}$$

Next, using Itô's isometry and [$A4$], we have

$$E\Big(\int_0^t \sigma(s,X_\epsilon(s-))s^{\alpha-1}dB(s)\Big)^p \leq E\Big(\int_0^t \sigma^2(s,X_\epsilon(s-))s^{2\alpha-2}ds\Big)^{\frac{p}{2}}$$
$$\leq E\Big(C_2\int_0^t (1+|X_\epsilon(s-)|^2)s^{2\alpha-2}ds\Big)^{\frac{p}{2}}.$$

From Höder's inequality, one can get

$$\begin{aligned} E\Big(C_2\int_0^t (1+|X_\epsilon(s-)|^2)s^{2\alpha-2}ds\Big)^{\frac{p}{2}} &\leq C_2^{\frac{p}{2}}E\Big(\int_0^t (1+|X_\epsilon(s-)|)^p s^{p\alpha-p}ds\Big)\Big(\int_0^t 1^{\frac{2}{p-2}}ds\Big)^{\frac{p-2}{2}} \\ &= C_2^{\frac{p}{2}}T^{\frac{p-2}{2}}E\Big(\int_0^t (1+|X_\epsilon(s-)|)^p s^{p\alpha-p}ds\Big). \end{aligned} \tag{3.13}$$

Next, applying the Burkholedr's inequality and [$A4$], we have

$$\begin{aligned} & E\Big(\int_0^t s^{\alpha-1}\int_{|x|<c} H(s,X(s-),x)\tilde{N}(ds,dx)\Big)^p \\ \leq\ & E\Big(\int_0^t s^{2\alpha-2}\int_{|x|<c}|H(t,x_1,x)|^2 v dx ds\Big)^{\frac{p}{2}} \\ \leq\ & E\Big(C_2\int_0^t (1+|X_\epsilon(s-)|^2)s^{2\alpha-2}ds\Big)^{\frac{p}{2}}. \end{aligned}$$

Repeating the proof of (3.13), we have

$$\begin{aligned} & E\Big(\int_0^t s^{\alpha-1}\int_{|x|<c} H(s,X(s-),x)\tilde{N}(ds,dx)\Big)^p \\ \leq\ & C_2^{\frac{p}{2}}T^{\frac{p-2}{2}}E\Big(\int_0^t (1+|X(s-)|)^p s^{p\alpha-p}ds\Big). \end{aligned}$$

Thus, we have

$$\begin{aligned} E\Big(\sup_{t\in[0,T]}|1+X(t)|^p\Big) &\leq 4^{p-1}E\Big|1+X_0\Big|^p + \Psi E\Big(\sup_{t\in[0,T]}\int_0^t (1+|X_\epsilon(s-)|)^p s^{p\alpha-p}ds\Big) \\ &\leq 4^{p-1}E\Big|1+X_0\Big|^p + \Psi E\Big(\int_0^t \sup_{\tau\in[0,s]}(1+|X_\epsilon(\tau)|)^p \tau^{p\alpha-p}d\tau\Big), \end{aligned}$$

where $\Psi = \frac{4^{p-1}C_2^{\frac{p}{2}}\epsilon^{\frac{p}{2}}}{\Gamma(\alpha)^p}(2T^{\frac{p-2}{2}} + \epsilon^{\frac{p}{2}}T^{p-1})$ is a positive constant. Now, using Gronwall-Bellman inequality (Lemma 2.3), we obtain

$$E\Big(\sup_{t\in[0,T]}|1+X(t)|^p\Big) \leq \Big(4^{p-1}E\Big|1+X_0\Big|^p\Big)\exp\Big(\frac{\Psi T^{p\alpha-p+1}}{p\alpha-p+1}\Big).$$

Note that $E(|X_0|^p) < +\infty$, we can get $E(\big|1+X_0\big|^p) \leq 2^{p-1} + 2^{p-1}E(|X_0|^p) < +\infty$. Therefore, we can easily deduce that

$$E\Big(\sup_{t\in[0,T]}|X(t)|^p\Big) \leq \Big(8^{p-1}E\Big|1+X_0\Big|^p\Big)\exp\Big(\frac{\Psi T^{p\alpha-p+1}}{p\alpha-p+1}\Big) + 2^{p-1} < +\infty.$$

The proof is completed. □

Now, we will establish the averaging principle for conformable fractional stochastic differential equations (3.12) in the sense of *pth* moment. Firstly, taking the average of the functions $b(\cdot), \sigma(\cdot)$, and $H(\cdot)$ with respect to t, named them by $\bar{b}(\cdot), \bar{\sigma}(\cdot)$, and $\bar{H}(\cdot)$ as follows:

$$\begin{aligned} \bar{b}(X(t)) &= \lim_{T_1\to\infty} \frac{1}{T_1}\int_0^{T_1} b(t, X(t))dt, \\ \bar{\sigma}(X(t)) &= \lim_{T_1\to\infty} \frac{1}{T_1}\int_0^{T_1} \sigma(t, X(t))dt, \\ \bar{H}(X(t), x) &= \lim_{T_1\to\infty} \frac{1}{T_1}\int_0^{T_1} H(t, X(t), x)dt, \end{aligned} \tag{3.14}$$

with frozen slow components $X(\cdot)$. Hence, we can get the following time-averaged equation

$$\mathfrak{D}_0^\alpha Y(t) = \epsilon\bar{b}(Y(t-)) + \sqrt{\epsilon}\bar{\sigma}(Y(t-))\frac{dB(t)}{dt} + \frac{\sqrt{\epsilon}}{dt}\int_{|x|<c} \bar{H}(Y(t-), x)\tilde{N}(dt, dx), \tag{3.15}$$

with initial value $Y(0) = X(0)$, where $\bar{b} : \mathbb{R}^n \to \mathbb{R}^n$, $\bar{\sigma} : \mathbb{R}^n \to \mathbb{R}^n$, $\bar{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions. Here, $Y(\cdot)$ is a stochastic process.

Remark 3.8. *(see [39, p.193]) If the limits in* (3.14) *exist, the functions functions $\bar{b}(\cdot), \bar{\sigma}(\cdot)$ and $\bar{H}(\cdot)$ also satisfy the linear growth condition and Lipschitz condition as functions functions $b(\cdot), \sigma(\cdot)$ and $H(\cdot)$ with the same constant C_1. In addition, the limit* $\lim_{T_1\to\infty} \frac{1}{T_1}\int_0^{T_1} b(\xi_t, X(t))dt = \bar{b}(X(t))$ *exists, for example, if ξ_t is periodic or is a sum of periodic functions.*

Suppose the functions $\bar{b}(\cdot), \bar{\sigma}(\cdot)$, and $\bar{H}(\cdot)$ satisfy the following conditions:

$[A5]$: For any $t \geq 0$, $x \in \mathbb{R}^n$, there exist three positive bounded functions $K_1(\cdot), K_2(\cdot), K_3(\cdot)$ such that the functions $\bar{b}(\cdot)$, $\bar{\sigma}(\cdot)$, and $\bar{H}(\cdot)$ satisfy

$$\begin{aligned} \frac{1}{t}\int_0^t |b(s, X) - \bar{b}(X)|^2 ds &\leq K_1(t)(1 + |X(t)|^2), \\ \frac{1}{t}\int_0^t |\sigma(s, X) - \bar{\sigma}(X)|^4 ds &\leq K_2(t)(1 + |X(t)|^4), \\ \frac{1}{t}\int_0^t \left|\int_{|x|<c} |H(s, Y(s-), x) - \bar{H}(Y(s-), x)|^2 v(dx)\right|^2 ds &\leq K_3(t)(1 + |X(t)|^4), \end{aligned}$$

where $\lim_{t\to\infty} K_i(t) = 0, i = 1, 2, 3$.

Let Y_ϵ be the unique solution of (3.15), $\kappa = \max\{\frac{p-1}{p}, \frac{3}{4}\}$, $p \geq 2$. Now we are ready to present the main result.

Theorem 3.9. *If conditions $[A3] - [A5]$ are satisfied, then there exists $L_1 > 0, \epsilon_1 \in [0, \epsilon^*]$ and $\beta \in (0, 1)$ such that for all $p \geq 2, \epsilon \in (0, \epsilon_1]$, $\alpha \in (\kappa, 1]$, $\kappa = \max\{\frac{p-1}{p}, \frac{3}{4}\}$*

$$\lim_{\epsilon\to 0} E\Big(\sup_{t\in[0, L_1\epsilon^{-\beta}]} |X_\epsilon(t) - Y_\epsilon(t)|^p\Big) = 0.$$

Proof. For any $t \in [0, u]$, we have

$$X_\epsilon(t) - Y_\epsilon(t) = \epsilon\int_0^t \Big[b(s, X_\epsilon(s-)) - \bar{b}(Y_\epsilon(s-))\Big]s^{\alpha-1}ds$$

$$+\sqrt{\epsilon}\int_0^t\Big[\sigma(s,X_\epsilon(s-))-\bar{\sigma}(Y_\epsilon(s-))\Big]s^{\alpha-1}dB(s)$$
$$+\sqrt{\epsilon}\int_0^t s^{\alpha-1}\int_{|x|<c}\Big[H(s,X_\epsilon(s-),x)-\bar{H}(Y_\epsilon(s-),x)\Big]\tilde{N}(ds,dx).$$

Note that $E\big(\sup_{t\in[0,T]}|X(t)|^p\big)<+\infty$ and $E\big(\int_0^T|X(s)|^p ds\big)<+\infty$. Let $u<T$ be a variable. Thus, for any $p\geq 2$, using the inequality $|a+b+c|^p\leq 3^{p-1}(|a|^p+|b|^p+|c|^p)$ and computing the expectation, we have

$$\begin{aligned}
& E\Big(\sup_{t\in[0,u]}|X_\epsilon(t)-Y_\epsilon(t)|^p\Big)\\
\leq\ & 3^{p-1}\epsilon^p E\Big(\sup_{t\in[0,u]}\Big|\int_0^t[b(s,X_\epsilon(s-))-\bar{b}(Y_\epsilon(s-))]s^{\alpha-1}ds\Big|^p\Big)\\
& +3^{p-1}\epsilon^{\frac{p}{2}}E\Big(\sup_{t\in[0,u]}\Big|\int_0^t[\sigma(s,X_\epsilon(s-))-\bar{\sigma}(Y_\epsilon(s-))]s^{\alpha-1}dB(s)\Big|^p\Big)\\
& +3^{p-1}\epsilon^{\frac{p}{2}}E\Big(\sup_{t\in[0,u]}\int_0^t s^{\alpha-1}\int_{|x|<c}\Big[H(s,X_\epsilon(s-),x)-\bar{H}(Y_\epsilon(s-),x)\Big]\tilde{N}(ds,dx)\Big)^p\\
=:\ & I_1+I_2+I_3.
\end{aligned}\tag{3.16}$$

Note that $|a+b|^p\leq 2^{p-1}|a|^p+|b|^p$, we can get

$$\begin{aligned}
I_1\ \leq\ & 6^{p-1}\epsilon^p E\Big(\sup_{t\in[0,u]}\Big|\int_0^t[b(s,X_\epsilon(s-))-b(s,Y_\epsilon(s-))]s^{\alpha-1}ds\Big|^p\Big)\\
& +6^{p-1}\epsilon^p E\Big(\sup_{t\in[0,u]}\Big|\int_0^t\Big[b(s,Y_\epsilon(s-))-\bar{b}(Y_\epsilon(s-))\Big]s^{\alpha-1}ds\Big|^p\Big)\\
=:\ & I_{11}+I_{12}.
\end{aligned}$$

Applying the Höder's inequality and [$A3$], we have

$$\begin{aligned}
I_{11}\ \leq\ & 6^{p-1}\epsilon^p u^{p-1}E\Big(\sup_{t\in[0,u]}\int_0^t\Big|\Big[b(s,X_\epsilon(s-))-b(s,Y_\epsilon(s-))\Big]s^{\alpha-1}\Big|^p ds\Big)\\
\leq\ & 6^{p-1}\epsilon^p u^{p-1}C_1^p\sup_{t\in[0,u]}\int_0^u E\big(\big|X_\epsilon(s-)-Y_\epsilon(s-)\big|^p\big)s^{p\alpha-p}ds\\
\leq\ & 6^{p-1}\epsilon^p u^{p-1}C_1^p\int_0^u E\Big(\sup_{\tau\in[0,s]}\big|X_\epsilon(\tau)-Y_\epsilon(\tau)\big|^p\Big)s^{p\alpha-p}ds.
\end{aligned}\tag{3.17}$$

Applying the Cauchy-Schwarz inequality and [$A5$], we have

$$\begin{aligned}
I_{12}\ \leq\ & 6^{p-1}\epsilon^p\Big[\sup_{t\in[0,u]}\int_0^t s^{2\alpha-2}ds E\Big(\sup_{t\in[0,u]}\int_0^t\Big|b(s,Y_\epsilon(s-))-\bar{b}(Y_\epsilon(s-))\Big|^2 ds\Big)\Big]^{\frac{p}{2}}\\
\leq\ & 6^{p-1}\epsilon^p\Big(\frac{u^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{p}{2}}E\Big(\sup_{t\in[0,u]}[t\cdot K_1(t)(1+|Y_\epsilon(t)|^2)]\Big)^{\frac{p}{2}}\\
\leq\ & \frac{6^{p-1}\epsilon^p u^{2p\alpha}}{(2\alpha-1)^{\frac{p}{2}}}\Big[\sup_{t\in[0,u]}K_1(t)\Big(1+E\big(\sup_{t\in[0,u]}|Y_\epsilon(t)|^2\big)\Big)\Big]^{\frac{p}{2}}
\end{aligned}$$

$$= \frac{6^{p-1}\epsilon^p u^{2p\alpha}}{(2\alpha-1)^{\frac{p}{2}}}\Upsilon_1, \tag{3.18}$$

where $\Upsilon_1 = \Big[\sup_{t\in[0,u]} K_1(t)(1+E(\sup_{t\in[0,u]}|Y_\epsilon(t)|^2))\Big]^{\frac{p}{2}}$ is a constant. Note that $|a+b|^p \le 2^{p-1}|a|^p + |b|^p$, applying Itô's isometry, we have

$$\begin{aligned}
I_2 &\le 6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\Big|\int_0^t \Big[\sigma(s,X_\epsilon(s-)) - \sigma(s,Y_\epsilon(s-))\Big]s^{\alpha-1}dB(s)\Big|^p\Big)\\
&\quad +6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\Big|\int_0^t \Big[\sigma(s,Y_\epsilon(s-)) - \bar{\sigma}(Y_\epsilon(s-))\Big]s^{\alpha-1}dB(s)\Big|^p\Big)\\
&= 6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\int_0^t \Big|\Big[\sigma(s,X_\epsilon(s-)) - \sigma(s,Y_\epsilon(s-))\Big]s^{\alpha-1}\Big|^2 ds\Big)^{\frac{p}{2}}\\
&\quad +6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\int_0^t \Big|\Big[\sigma(s,Y_\epsilon(s-)) - \bar{\sigma}(Y_\epsilon(s-))\Big]s^{\alpha-1}\Big|^2 ds\Big)^{\frac{p}{2}}\\
&=: I_{21} + I_{22}.
\end{aligned}$$

Applying [$A3$], we have

$$\begin{aligned}
I_{21} &= 6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\int_0^t \Big|[\sigma(s,X_\epsilon(s-)) - \sigma(s,Y_\epsilon(s-))]s^{\alpha-1}\Big|^2 ds\Big)^{\frac{p}{2}}\\
&\le 6^{p-1}\epsilon^{\frac{p}{2}} C_1^p E\Big(\sup_{t\in[0,u]}\int_0^t \Big|[X_\epsilon(s-) - Y_\epsilon(s-)]s^{\alpha-1}\Big|^2 ds\Big)^{\frac{p}{2}}.
\end{aligned} \tag{3.19}$$

Now, using the Höder's inequality, we get

$$\begin{aligned}
I_{21} &\le 6^{p-1}\epsilon^{\frac{p}{2}} C_1^p E\Big(\sup_{t\in[0,u]}\int_0^t \Big|[X_\epsilon(s-) - Y_\epsilon(s-)]s^{\alpha-1}\Big|^2 ds\Big)^{\frac{p}{2}}\\
&\le 6^{p-1}\epsilon^{\frac{p}{2}} C_1^p E\Big(\sup_{t\in[0,u]}\int_0^t \Big|[X_\epsilon(s-) - Y_\epsilon(s-)]s^{\alpha-1}\Big|^p ds\Big)\sup_{t\in[0,u]}\Big(\int_0^t 1ds\Big)^{\frac{p-2}{2}}\\
&\le 6^{p-1}\epsilon^{\frac{p}{2}} u^{\frac{p-2}{2}} C_1^p \int_0^u E(\sup_{\tau\in[0,s]}\Big|X_\epsilon(\tau) - Y_\epsilon(\tau)\Big|^p)s^{p\alpha-p}ds.
\end{aligned} \tag{3.20}$$

Next, using Höder's inequality and [$A5$], one can show that

$$\begin{aligned}
I_{22} &\le 6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\int_0^t \Big|\sigma(s,Y_\epsilon(s-)) - \bar{\sigma}(Y_\epsilon(s-))\Big|^2 s^{2\alpha-2}ds\Big)^{\frac{p}{2}}\\
&\le 6^{p-1}\epsilon^{\frac{p}{2}} E\sup_{t\in[0,u]}\Big[\Big(\int_0^t s^{4\alpha-4}ds\Big)^{\frac{p}{4}}\Big(\int_0^t \Big|\sigma(s,Y_\epsilon(s-)) - \bar{\sigma}(Y_\epsilon(s-))\Big|^4 ds\Big)^{\frac{p}{4}}\Big]\\
&\le 6^{p-1}\epsilon^{\frac{p}{2}}\Big(\frac{u^{4\alpha-3}}{4\alpha-3}\Big)^{\frac{p}{4}}\Big[\sup_{t\in[0,u]} t\cdot K_2(t)\Big(1+E(\sup_{t\in[0,u]}|Y_\epsilon(t)|^4)\Big)\Big]^{\frac{p}{4}}\\
&= \frac{6^{p-1}\epsilon^{\frac{p}{2}} u^{\frac{p(2\alpha-1)}{2}}}{(4\alpha-3)^{\frac{p}{4}}}\Upsilon_2,
\end{aligned} \tag{3.21}$$

where $\Upsilon_2 = \Big[\sup_{t\in[0,u]} K_2(t)\Big(1 + E(\sup_{t\in[0,u]} |Y_\epsilon(t)|^4)\Big)\Big]^{\frac{p}{4}}$ is a constant. In the sequel, for I_3, using Burkholder's inequality, we have

$$\begin{aligned}
I_3 &\le 6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\int_0^t s^{\alpha-1}\int_{|x|<c}\Big|H(s, X_\epsilon(s-), x) - H(s, Y_\epsilon(s-), x)\Big|\tilde{N}(ds, dx)\Big)^p \\
&\quad +6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\int_0^t s^{\alpha-1}\int_{|x|<c}\Big|H(s, Y_\epsilon(s-), x) - \bar{H}(Y_\epsilon(s-), x)\Big|\tilde{N}(ds, dx)\Big)^p \\
&\le 6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\int_0^t s^{2\alpha-2}\int_{|x|<c}\Big|H(s, X_\epsilon(s-), x) - H(s, Y_\epsilon(s-), x)\Big|^2 v(dx)ds\Big)^{\frac{p}{2}} \\
&\quad +6^{p-1}\epsilon^{\frac{p}{2}} E\Big(\sup_{t\in[0,u]}\int_0^t s^{2\alpha-2}\int_{|x|<c}\Big|H(s, Y_\epsilon(s-), x) - \bar{H}(Y_\epsilon(s-), x)\Big|^2 v(dx)ds\Big)^{\frac{p}{2}} \\
&\le I_{31} + I_{32}.
\end{aligned}$$

Obviously, from $[A3]$, we can get

$$I_{31} \le 6^{p-1}\epsilon^{\frac{p}{2}} C_1^p E\Big(\sup_{t\in[0,u]}\int_0^t \Big|[X_\epsilon(s-) - Y_\epsilon(s-)]s^{\alpha-1}\Big|^2 ds\Big)^{\frac{p}{2}},$$

which is the same as (3.19). Therefore, according to (3.20), we have

$$I_{31} \le 6^{p-1}\epsilon^{\frac{p}{2}} u^{\frac{p-2}{2}} C_1^p \int_0^u E(\sup_{\tau\in[0,s]}\Big|X_\epsilon(\tau) - Y_\epsilon(\tau)\Big|^p)s^{p\alpha-p}ds. \tag{3.22}$$

From condition $[A3]$ and Höder's inequality, employing the same method as I_{22}, we have

$$\begin{aligned}
I_{32} &\le 6^{p-1}\epsilon^{\frac{p}{2}} E\sup_{t\in[0,u]}\Big[(\int_0^t s^{4\alpha-4}ds)^{\frac{p}{4}} \\
&\quad \times\Big(\int_0^t\Big|\int_{|x|<c}|H(s, Y_\epsilon(s-), x) - \bar{H}(Y_\epsilon(s-), x)|^2 v(dx)\Big|^2 ds\Big)^{\frac{p}{4}}\Big] \\
&\le \frac{6^{p-1}\epsilon^{\frac{p}{2}} u^{\frac{p(2\alpha-1)}{2}}}{(4\alpha-3)^{\frac{p}{4}}}\Upsilon_3,
\end{aligned} \tag{3.23}$$

where $\Upsilon_3 = [\sup_{t\in[0,u]} K_3(t)(1 + E(\sup_{t\in[0,u]} |Y_\epsilon(t)|^4))]^{\frac{p}{4}}$ is a constant. Now, substitute (3.17), (3.18) and (3.20)–(3.23) into (3.16), we have

$$\begin{aligned}
&E\Big(\sup_{t\in[0,u]}|X_\epsilon(t) - Y_\epsilon(t)|^p\Big) \\
&\le \Big(\frac{6^{p-1}\epsilon^p u^{2p\alpha}}{(2\alpha-1)^{\frac{p}{2}}}\Upsilon_1 + \frac{6^{p-1}\epsilon^{\frac{p}{2}} u^{\frac{p(2\alpha-1)}{2}}}{(4\alpha-3)^{\frac{p}{4}}}(\Upsilon_2 + \Upsilon_3)\Big) \\
&\quad +6^{p-1}C_1^p\epsilon^{\frac{p}{2}}\Big(\epsilon^{\frac{p}{2}}u^{p-1} + 2u^{\frac{p-2}{2}}\Big)\int_0^u E(\sup_{\tau\in[0,s]}\Big|X_\epsilon(\tau) - Y_\epsilon(\tau)\Big|^p)s^{p\alpha-p}ds.
\end{aligned}$$

Here, in order to make the integrals $\int_0^t s^{4\alpha-4}ds$ and $\int_0^t s^{p\alpha-p}ds$ solvable, we have to restrict $\alpha \in (\kappa, 1]$, $\kappa = \max\{\frac{p-1}{p}, \frac{3}{4}\}$. Thus, using Gronwall-Bellman inequality (Lemma 2.3), we can get

$$E\Big(\sup_{t\in[0,u]}|X_\epsilon(t) - Y_\epsilon(t)|^p\Big) \le \Phi(u)\epsilon, \tag{3.24}$$

where

$$\begin{aligned}\Phi(u) &= \Big(\frac{6^{p-1}\epsilon^{p-1}u^{2p\alpha}}{(2\alpha-1)^{\frac{p}{2}}}\Upsilon_1 + \frac{6^{p-1}\epsilon^{\frac{p}{2}-1}u^{\frac{p(2\alpha-1)}{2}}}{(4\alpha-3)^{\frac{p}{4}}}(\Upsilon_2+\Upsilon_3)\Big)\\ &\quad \times \exp\Big[6^{p-1}C_1^p\epsilon^{\frac{p}{2}}\Big(\epsilon^{\frac{p}{2}}u^{p-1} + 2u^{\frac{p-2}{2}}\Big)\frac{u^{p\alpha-p+1}}{p\alpha-p+1}\Big].\end{aligned}$$

This implies that we can select $\beta \in (0,1)$ and $L_1 > 0$, such that for all $t \in [0, L_1\epsilon^{-\beta}] \subset [0,T]$ having

$$E\Big(\sup_{t\in[0,L_1\epsilon^{-\beta}]} |X_\epsilon(t) - Y_\epsilon(t)|^p\Big) \le \Phi_1(\epsilon)\epsilon^{1-\beta},$$

where

$$\begin{aligned}\Phi_1(\epsilon) &= \Big(\frac{6^{p-1}L_1^{2p\alpha}\epsilon^{p+\beta-1-2p\alpha\beta}}{(2\alpha-1)^{\frac{p}{2}}}\Upsilon_1 + \frac{6^{p-1}L_1^{\frac{p(2\alpha-1)}{2}}\epsilon^{\frac{p+2\beta-2-p\beta(2\alpha-1)}{2}}}{(4\alpha-3)^{\frac{p}{4}}}(\Upsilon_2+\Upsilon_3)\Big)\\ &\quad \times \exp\Big[6^{p-1}C_1^p\epsilon^{\frac{p}{2}}\frac{L_1^{p\alpha-p+1}\epsilon^{\beta(p-p\alpha-1)}}{p\alpha-p+1}\Big(L_1^{p-1}\epsilon^{\frac{p+2\beta-2p\beta}{2}} + 2L_1^{\frac{p-2}{2}}\epsilon^{\frac{\beta(2-p)}{2}}\Big)\Big],\end{aligned}$$

is a positive constant.

Note that $\Phi_1(\epsilon)$ is an increasing continuous function with respect to variable ϵ, and $\inf_{\epsilon\ge 0}\Phi_1(\epsilon)\epsilon^{1-\beta} = 0$. Thus, we have, $\lim_{\epsilon\to 0}\Phi_1(\epsilon)\epsilon^{1-\beta} = 0$ and $\lim_{\epsilon\to 0} L_1\epsilon^{-\beta} \to +\infty$.

The proof is completed. □

Remark 3.10. *Theorem 3.9 can be regarded as a generalization of a special Caputo derivative [28, Theorem 1] . The calculation is relatively simple and there is no Gamma function* $\Gamma(\cdot)$ *involved, but it still needs to handle the singular integral term* $\int_0^t b(s, Y_\epsilon(s-))s^{\alpha-1}ds$.

4. An example

In this section, we present an example to demonstrate the procedure of the averaging principle.

Example 4.1. *Consider the following conformable fractional stochastic differential equations with Lévy noise.*

$$\begin{cases}\mathfrak{D}_0^\alpha X_\epsilon(t) = 2\epsilon(\frac{\cos(t+X_\epsilon(t))}{e^t}+1) + 2\sqrt{\epsilon}\frac{dB(t)}{dt} + \frac{\sqrt{\epsilon}}{dt}\int_{|x|<c} x^4\sin(X_\epsilon(t))\frac{e^t+1}{e^t}\nu_\lambda(dx),\ t\ge 0,\\ X_\epsilon(0) = 1.\end{cases} \tag{4.1}$$

Choose $\alpha \in (0.75, 1], \lambda \in (0,2), \xi > 0$, p=2, λ-stable Lévy jump measure $\nu_\lambda(dx) = \frac{\xi}{x^{1+\lambda}}dx$. Set $b(t, X(t)) = 2(\frac{\cos(t+X_\epsilon(t))}{e^t}+1)$, $\sigma(t, X(t)) = 2$, $H(t, X(t), x) = x^4\sin(X_\epsilon(t))\frac{e^t+1}{e^t}$, $t \ge 0$. Thus, frozen slow component X_ϵ, one can compute

$$\begin{aligned}\bar{b}(X(t)) &= \lim_{t\to\infty}\frac{1}{t}\int_0^t 2\Big(\frac{\cos(t+X_\epsilon(t))}{e^t}+1\Big)ds = 2,\\ \bar{\sigma}(X(t)) &= \lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma(t, X(t))ds = 2,\end{aligned}$$

and

$$
\begin{aligned}
& \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_{|x|<c}\frac{e^s+1}{e^s}\frac{\xi}{x^{1+\lambda}}x^4\sin(X_\epsilon(t))dxds \\
= & \lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{e^s+1}{e^s}\sin(X_\epsilon(t))\int_{|x|<c}\frac{\xi x^4}{x^{1+\lambda}}dxds \\
= & \frac{\xi c^{4-\lambda}}{4-\lambda}\sin(X_\epsilon(t))\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{e^s+1}{e^s}ds \\
= & \frac{\xi c^{4-\lambda}}{4-\lambda}\sin(X_\epsilon(t)).
\end{aligned}
$$

Therefore, we obtain the following averaged equation

$$
\begin{cases}
\mathfrak{D}_0^\alpha Y_\epsilon(t) = 2\epsilon + \frac{\sqrt{\epsilon}\xi c^{4-\lambda}}{4-\lambda}\sin(Y_\epsilon(t)) + \sqrt{\epsilon}\frac{dB(t)}{dt},\ t\geq 0, \\
Y_\epsilon(0) = 1.
\end{cases}
\tag{4.2}
$$

Now, we check that condition $[A3]$ is satisfied. For function $b(\cdot)$, we have

$$
\begin{aligned}
\frac{1}{t}\int_0^t|b(t,X_\epsilon)-\bar{b}(X_\epsilon)|^2dt &= \frac{2}{t}\int_0^t\left|\frac{\cos^2(t+X_\epsilon)}{e^{2t}}\right|dt \\
&\leq \frac{2}{t}\int_0^t\frac{1}{e^{2t}}dt \\
&= \frac{2}{t}\cdot\frac{1}{2}(1-e^{-2t}) \\
&\leq \frac{1-e^{-2t}}{t}(1+|X_\epsilon|^2).
\end{aligned}
$$

For function $H(\cdot)$, we can calculate

$$
\begin{aligned}
& \frac{1}{t}\int_0^t\left|\int_{|x|<c}|H(t,X,x)-\tilde{H}(X,x)|^2\nu(dx)\right|^2ds \\
= & \frac{1}{t}\int_0^t\left|\int_{|x|<c}\xi e^{-2s}|\sin(X_\epsilon(t))|^2x^{7-\lambda}dx\right|^2ds \\
\leq & \frac{1-e^{-4t}}{4t}\frac{\xi^2c^{15-2\lambda}}{15-2\lambda}(1+|X_\epsilon(t)|^2).
\end{aligned}
$$

Therefore, $[A3]$ is satisfied with $K_1(t)=\frac{1-e^{-2t}}{t}$, $K_3(t)=\frac{1-e^{-4t}}{4t}\frac{\xi^2c^{15-2\lambda}}{15-2\lambda}$, $t\geq 0$. Next, let $E_r=|X_\epsilon-Y_\epsilon|^2$ be the approximation error. With the help of MATLAB software, we show the averaged solution of (4.2) can converge to the original solution of (4.1) on the interval $[0,50]$, see Figure 1. This means that $E_r=|X_\epsilon-Y_\epsilon|^2\to 0, \epsilon\to 0$ is also verified by simulation.

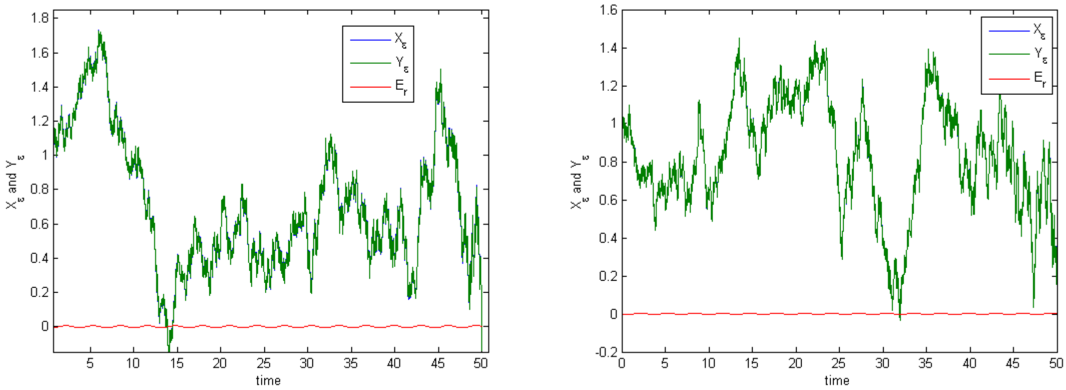

Figure 1. Comparison of X_ϵ and Y_ϵ with $\alpha = 0.8, \epsilon = 0.01$ and $\alpha = 0.8, \epsilon = 0.001$ on the time interval $[0, 50]$.

5. Conclusions

We employ the method proposed in [40] to establish sufficient conditions that ensure the validity of the averaging principle for conformable fractional stochastic differential equations with Lévy noise. Our theoretical analysis demonstrates that conformable fractional calculus is significantly simpler in formulation and application compared to Caputo fractional calculus. Based on this finding, the averaging principle for stochastic differential equations incorporating delay effects can also be explored.

Author contributions

All authors contributed to the study conception and design. Yuan Yuan and Guanli Xiao wrote the main manuscript text and prepared figures. Lulu Ren reviewed the manuscript. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest in this paper.

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