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Research article

A diagonal transformation-based algorithm for computing the minimum eigenvalue of irreducible M-matrices

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Abstract: This study presents a novel algorithm for computing the minimum eigenvalue of irreducible M-matrices. Firstly, we introduce a new diagonal transformation technique and establish sharper two-sided bounds for the spectral radius of irreducible nonnegative matrices with positive diagonal entries. Building upon these theoretical foundations, we develop an efficient iterative algorithm to compute the minimum eigenvalue of irreducible M-matrices. The convergence of the proposed algorithm is rigorously proved, ensuring both computational stability and asymptotic accuracy. Numerical experiments demonstrate the effectiveness of the method, showing that it achieves high precision with relatively low computational cost compared to existing approaches.

Keywords: M-matrix; minimum eigenvalue; diagonal transformation; irreducible
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1. Introduction

Throughout this paper, $\mathbb{R}^{n\times n}$ represents the set of real matrices of order n. A real matrix $A = (a_{ij})$ is said to be nonnegative, denoted by $A \geq 0$, if every entry satisfies $a_{ij} \geq 0$. The spectral radius $\rho(A)$ of a matrix A is defined as the largest absolute value among its eigenvalues. Mathematically, if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A, then

$$\rho(A) = \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}.$$

According to nonnegative matrix theory [1], the spectral radius $\rho(A)$, also known as the Perron root, is a nonnegative eigenvalue corresponding to a nonnegative eigenvector.

The class of M-matrices, introduced next, is closely related to nonnegative matrices. Recall that a matrix $K \in \mathbb{R}^{n\times n}$ is defined to be a nonsingular M-matrix if K can be written as

$$K = sI - P, s > \rho(P),$$

where $P \geq 0$ and I represents the $n \times n$ identity matrix. According to the theory of M-matrices, if K is an M-matrix, then K has a positive characteristic root with a nonnegative eigenvector. This positive eigenvalue, which is denoted by $q(K)$, is defined to be the minimum eigenvalue of K. M-matrices have many applications in physics, biology, economics, and engineering. The minimum eigenvalue of M-matrices, for example, can be used to describe atomic structures [2].

In addition, many engineering research problems can be attributed to the calculation of characteristic roots of special matrices. The computation of the characteristic roots of a matrix and its implementation on a computer have received much attention, and many algorithms have been developed [3–5]. In practical problems, it is sometimes necessary to compute only the minimum or maximum eigenvalues of a special matrix or to estimate lower and upper bounds on the minimum and maximum eigenvalues with sufficiently narrow bounds, i.e., to the level of accuracy required in practice. Extensive research has been conducted [6–8].

In this paper, we develop a new approach for computing the minimum eigenvalue of irreducible M-matrices by exploiting their relationship with nonnegative matrices, and we provide theoretical convergence guarantees for the algorithm. The new algorithm, which provides the required accuracy, is computationally efficient and convenient to implement in a computer program. Finally, we conduct numerical experiments based on this algorithm, and the results show that the algorithm is feasible and effective.

2. Preliminaries

We begin this section with some standard notations and definitions.

Definition 1. *Let $A \in \mathbb{R}^{n\times n}$. Then A is defined as an irreducible matrix if either*
(a) $n = 1$ and $A \neq 0$; or
(b) $n \geq 2$, there does not exist a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix},$$

where $A_{11} \in \mathbb{R}^{k\times k}, A_{12} \in \mathbb{R}^{k\times(n-k)}, A_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$, and O is an $(n-k) \times k$ zero matrix, $1 \leq k \leq n-1$. Otherwise, A is reducible.

Definition 2. *Let $A = \left(a_{ij}\right), B = \left(b_{ij}\right) \in \mathbb{R}^{n\times n}$. The matrix B is said to have the same zero pattern as A if for all $1 \leq i, j \leq n$,*

$$b_{ij} = 0 \ if \ and \ only \ if \ a_{ij} = 0.$$

Let $K = \left(k_{ij}\right) \in \mathbb{R}^{n\times n}$ be an irreducible M-matrix and set

$$R > \max_i k_{ii} \ for \ i = 1, 2, \cdots, n.$$

Then $A = RI - K$ is a nonnegative irreducible matrix with positive main diagonal entries. Furthermore, we have

$$q(K) = R - \rho(A).$$

Therefore, the algorithm for computing the minimum eigenvalue of irreducible M-matrices depends on the calculation of the spectral radius of the nonnegative irreducible matrices with positive diagonal elements.

There are many ways to calculate the spectral radius of nonnegative matrices, among which diagonal similar transformations are summarized by Bunse [9]. The convergence of fundamental diagonal transformations is demonstrated, and 13 different types of diagonal transformation equations are provided. Duan and Zhang [10] present an enhanced diagonal similar transformation approach for nonnegative irreducible matrices with positive main diagonal entries. The algorithm of [10] was modified by Wen [11], and the calculation efficiency was significantly increased.

3. A new method for computing the minimum eigenvalue

In this section, our goal is to find sharper two-sided bounds for the spectral radius of nonnegative irreducible matrices with positive main diagonal entries using a new diagonal similarity transformation while reducing computational effort. Based on this, we further derive an algorithm to compute the minimum eigenvalue of irreducible M-matrices.

Let $A = A_0 = \left(a_{ij}^{(0)}\right) \in \mathbb{R}^{n\times n}$ be an irreducible nonnegative matrix with $a_{ii}^{(0)} > 0$. We compute the column sum

$$c_i^{(0)} = \sum_{t=1}^{n} a_{ti}^{(0)} \ \text{ for } \ i = 1, 2, \cdots, n.$$

We can find the maximum column sum $\max\limits_i c_i^{(0)}$ and the minimum column sum $\min\limits_i c_i^{(0)}$. According to the Perron-Frobenius theorem [1], we obtain

$$\min_i c_i^{(0)} \leq \rho(A) \leq \max_i c_i^{(0)}.$$

We now define

$$Q_0 = \operatorname{diag}\Big(\frac{1}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(0)} a_{t1}^{(0)}}}, \frac{1}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(0)} a_{t2}^{(0)}}}, \cdots, \frac{1}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(0)} a_{tn}^{(0)}}}\Big).$$

For $k = 0, 1, 2, \cdots$, we define

$$A_{k+1} = \left(a_{ij}^{(k+1)}\right) = Q_k{}^{-1} A_k Q_k = \begin{pmatrix} a_{11}^{(k)} & \frac{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{t1}^{(k)}}}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{t2}^{(k)}}} a_{12}^{(k)} & \cdots & \frac{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{t1}^{(k)}}}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{tn}^{(k)}}} a_{1n}^{(k)} \\ \frac{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{t2}^{(k)}}}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{t1}^{(k)}}} a_{21}^{(k)} & a_{22}^{(k)} & \cdots & \frac{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{t2}^{(k)}}}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{tn}^{(k)}}} a_{2n}^{(k)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{tn}^{(k)}}}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{t1}^{(k)}}} a_{n1}^{(k)} & \frac{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{tn}^{(k)}}}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)} a_{t2}^{(k)}}} a_{n2}^{(k)} & \cdots & a_{nn}^{(k)} \end{pmatrix}, \tag{1}$$

where $c_t^{(k)} (t = 1, 2, \cdots, n)$ is the sum of the t-th column of the matrix $A_k = \left(a_{ij}^{(k)}\right)$. Furthermore, this similar diagonalization operation has the following important properties:

(a) $a_{ii}^{(k)} = a_{ii},\ k = 0, 1, 2, \cdots$;
(b) If $a_{ij} > 0$, then $a_{ij}^{(k)} > 0$; if $a_{ij} = 0$, then $a_{ij}^{(k)} = 0$. Therefore, A_k is irreducible and has the same zero pattern as the matrix A.

In addition, we obtain the sequence of similar matrices $\{A_k\}$, the sequence of the maximum column sum $\left\{\max_i c_i^{(k)}\right\}$, and the sequence of the minimum column sum $\left\{\min_i c_i^{(k)}\right\}$. What follows are the important properties of the sequences of column sums mentioned above.

Theorem 1. *Let $A = A_0 = \left(a_{ij}^{(0)}\right) \in \mathbb{R}^{n\times n}$ be a nonnegative irreducible matrix such that $a_{ii}^{(0)} > 0$ and*

$$Q_k = diag\left(\frac{1}{\sum_{t=1}^{n} \sqrt{c_t^{(k)} a_{t1}^{(k)}}}, \frac{1}{\sum_{t=1}^{n} \sqrt{c_t^{(k)} a_{t2}^{(k)}}}, \cdots, \frac{1}{\sum_{t=1}^{n} \sqrt{c_t^{(k)} a_{tn}^{(k)}}}\right), k = 0, 1, 2, \cdots, \tag{2}$$

where $c_t^{(k)}(t = 1, 2, \cdots, n)$ is the sum of the t-th column of A_k. If $A_{k+1} = Q_k{}^{-1} A_k Q_k$, then

$$\max_i c_i^{(k+1)} \le \max_i c_i^{(k)}, \tag{3}$$

$$\min_i c_i^{(k)} \le \min_i c_i^{(k+1)}. \tag{4}$$

Equality holds if and only if $\min_i c_i^{(k)} = \max_i c_i^{(k)}$.

Proof. Note that since A_0 is a nonnegative irreducible matrix with positive main diagonal entries, we have

$$c_t^{(0)} > 0 \text{ for } t = 1, 2, \cdots, n,$$

and

$$\sum_{t=1}^{n} \sqrt{c_t^{(0)} a_{ti}^{(0)}} > 0 \text{ for } i = 1, 2, \cdots, n.$$

It follows that the matrix Q_0 is nonsingular. Furthermore, the diagonal similar transformation

$$A_{k+1} = Q_k{}^{-1} A_k Q_k \, (k = 0, 1, 2, \cdots)$$

does not change the position of the zero elements. We have

$$c_t^{(k)} > 0 \text{ for } t = 1, 2, \cdots, n, k = 0, 1, 2, \cdots,$$

and

$$\sum_{t=1}^{n} \sqrt{c_t^{(k)} a_{ti}^{(k)}} > 0 \text{ for } i = 1, 2, \cdots, n, k = 0, 1, 2, \cdots.$$

Therefore, the matrix Q_k is nonsingular. It follows from Eq (1) that

$$\max_i c_i^{(k+1)} = \max_i \sum_{p=1}^{n} \left(\frac{\sum_{t=1}^{n} \sqrt{c_t^{(k)} a_{tp}^{(k)}}}{\sum_{t=1}^{n} \sqrt{c_t^{(k)} a_{ti}^{(k)}}} a_{pi}^{(k)} \right)$$

$$
\begin{aligned}
&= \max_i \frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)}} a_{ti}^{(k)}} \sum_{p=1}^{n} \left(\sum_{t=1}^{n} \sqrt{c_t^{(k)}} a_{tp}^{(k)} \cdot \ a_{pi}^{(k)} \right) \\
&\leq \max_t \sqrt{c_t^{(k)}} \cdot \max_i \frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)}} a_{ti}^{(k)}} \sum_{p=1}^{n} \left(\sum_{t=1}^{n} a_{tp}^{(k)} \cdot \ a_{pi}^{(k)} \right) \\
&= \max_t \sqrt{c_t^{(k)}} \cdot \max_i \frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)}} a_{ti}^{(k)}} \sum_{p=1}^{n} \left(c_p^{(k)} \cdot a_{pi}^{(k)} \right) \\
&= \max_t \sqrt{c_t^{(k)}} \cdot \max_i \frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)}} a_{ti}^{(k)}} \sum_{p=1}^{n} \left(\sqrt{c_p^{(k)}} \cdot \sqrt{c_p^{(k)}} \cdot a_{pi}^{(k)} \right) \\
&\leq \max_t \sqrt{c_t^{(k)}} \cdot \max_p \sqrt{c_p^{(k)}} \cdot \max_i \frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)}} a_{ti}^{(k)}} \sum_{p=1}^{n} \left(\sqrt{c_p^{(k)}} \cdot a_{pi}^{(k)} \right) \\
&= \max_i c_i^{(k)}.
\end{aligned}
$$

This is the desired result (3). Equality holds if and only if for all $i = 1, 2, \cdots, n$, $c_i^{(k)} = \max\limits_i c_i^{(k)}$, i.e., $\min\limits_i c_i^{(k)} = \max\limits_i c_i^{(k)}$ for $i = 1, 2, \cdots, n$.

In the same spirit, we can demonstrate that $\min\limits_i c_i^{(k)} \leq \min\limits_i c_i^{(k+1)}$. Equality holds if and only if for all $i = 1, 2, \cdots, n$, $\min\limits_i c_i^{(k)} = \max\limits_i c_i^{(k)}$. This completes the proof. □

Theorem 2. *Let $A = A_0 = \left(a_{ij}^{(0)}\right) \in \mathbb{R}^{n\times n}$ be a nonnegative irreducible matrix such that $a_{ii}^{(0)} > 0$ and*

$$
Q_k = diag\left(\frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)}} a_{t1}^{(k)}}, \frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)}} a_{t2}^{(k)}}, \cdots, \frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)}} a_{tn}^{(k)}}\right), k = 0, 1, 2, \cdots,
$$

where $c_t^{(k)} (t = 1, 2, \cdots, n)$ is the sum of the t-th column of A_k. If $A_{k+1} = {Q_k}^{-1} A_k Q_k$, then

$$
\min_i c_i^{(k+1)} \leq \rho(A) \leq \max_i c_i^{(k+1)}, k = 0, 1, 2, \cdots. \tag{5}
$$

Proof. It follows from $A_{k+1} = {Q_k}^{-1} A_k Q_k$ that A and A_{k+1} have the same characteristic roots. Therefore, we have

$$
\rho(A) = \rho(A_{k+1}), k = 0, 1, 2, \cdots.
$$

According to the Perron-Frobenius theorem, we obtain

$$
\min_i c_i^{(k+1)} \leq \rho(A_{k+1}) \leq \max_i c_i^{(k+1)}, k = 0, 1, 2, \cdots,
$$

which implies the desired result. □

Based on the results of Theorems 1 and 2, we conclude

$$\min_i c_i^{(k)} \le \min_i c_i^{(k+1)} \le \rho(A) \le \max_i c_i^{(k+1)} \le \max_i c_i^{(k)}.$$

The above inequality shows that the sequence $\left\{\min_i c_i^{(k)}\right\}_{k=0}^{\infty}$ is monotonically increasing with an upper bound $\rho(A)$, and the sequence $\left\{\max_i c_i^{(k)}\right\}_{k=0}^{\infty}$ is monotonically decreasing with a lower bound $\rho(A)$. This implies that these sequences converge. Next, we will demonstrate that both sequences have the same limits, i.e.,

$$\lim_{k\to\infty} \min_i c_i^{(k)} = \lim_{k\to\infty} \max_i c_i^{(k)}.$$

Theorem 3. *Let* $A = A_0 = \left(a_{ij}^{(0)}\right) \in \mathbb{R}^{n\times n}$ *be a nonnegative irreducible matrix such that* $a_{ii}^{(0)} > 0$ *and*

$$Q_k = diag\left(\frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)} a_{t1}^{(k)}}}, \frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)} a_{t2}^{(k)}}}, \cdots, \frac{1}{\sum\limits_{t=1}^{n} \sqrt{c_t^{(k)} a_{tn}^{(k)}}}\right), k = 0, 1, 2, \cdots,$$

where $c_t^{(k)} (t = 1, 2, \cdots, n)$ *is the sum of the t-th column of* A_k*. If* $A_{k+1} = {Q_k}^{-1} A_k Q_k$*, then*

$$\lim_{k\to\infty} \left(\max_i c_i^{(k)} - \min_i c_i^{(k)}\right) = 0.$$

Proof. We now divide the problem into two cases.
Case 1. $\max_i c_i^{(k)} = \min_i c_i^{(k)}$. It is quite evident that the conclusion holds.
Case 2. $\max_i c_i^{(k)} \ne \min_i c_i^{(k)}$. According to Theorem 1, we have

$$0 < \max_i c_i^{(k+1)} < \max_i c_i^{(k)}, 0 < \min_i c_i^{(k)} < \min_i c_i^{(k+1)}.$$

It holds that

$$0 < \max_i c_i^{(k+1)} - \min_i c_i^{(k+1)} < \max_i c_i^{(k)} - \min_i c_i^{(k)}.$$

Therefore, we obtain

$$0 < \frac{\max_i c_i^{(k+1)} - \min_i c_i^{(k+1)}}{\max_i c_i^{(k)} - \min_i c_i^{(k)}} < 1. \tag{6}$$

Now, we define

$$\frac{\max_i c_i^{(k+1)} - \min_i c_i^{(k+1)}}{\max_i c_i^{(k)} - \min_i c_i^{(k)}} = L_k, k = 0, 1, 2, \cdots. \tag{7}$$

Here L_k is a constant independent of k, whose value is completely determined by

$$\max_i c_i^{(k)}, \min_i c_i^{(k)}, \max_i c_i^{(k+1)}, \text{ and } \min_i c_i^{(k+1)}.$$

Thus, we obtain the iterative formula

$$\max_i c_i^{(k+1)} - \min_i c_i^{(k+1)} = L_k \left(\max_i c_i^{(k)} - \min_i c_i^{(k)}\right), k = 0, 1, 2, \cdots.$$

Therefore,

$$\max_i c_i^{(k)} - \min_i c_i^{(k)} = L_{k-1}\left(\max_i c_i^{(k-1)} - \min_i c_i^{(k-1)}\right) = L_{k-1}L_{k-2}L_{k-3}\cdots L_0\left(\max_i c_i^{(0)} - \min_i c_i^{(0)}\right).$$

Let $L = \max_{0\le t\le k-1} L_t$, we have

$$0 < \max_i c_i^{(k)} - \min_i c_i^{(k)} \le L^k\left(\max_i c_i^{(0)} - \min_i c_i^{(0)}\right). \tag{8}$$

From (6), it follows that $0 < L < 1$, and consequently $\lim_{k\to\infty} L^k = 0$. By inequality (8), we conclude

$$\lim_{k\to\infty}\left(\max_i c_i^{(k)} - \min_i c_i^{(k)}\right) = 0.$$

The proof is completed. □

Thus, it is straightforward to deduce the following result by combining Theorems 2 and 3.

Theorem 4. *Let $A = A_0 = \left(a_{ij}^{(0)}\right) \in \mathbb{R}^{n\times n}$ be a nonnegative irreducible matrix such that $a_{ii}^{(0)} > 0$ and*

$$Q_k = diag\Big(\frac{1}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)}}a_{t1}^{(k)}}, \frac{1}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)}}a_{t2}^{(k)}}, \cdots, \frac{1}{\sum\limits_{t=1}^{n}\sqrt{c_t^{(k)}}a_{tn}^{(k)}}\Big), k = 0, 1, 2, \cdots,$$

where $c_t^{(k)}(t = 1, 2, \cdots, n)$ is the sum of the t-th column of A_k. If $A_{k+1} = Q_k{}^{-1}A_kQ_k$, then

$$\lim_{k\to\infty}\min_i c_i^{(k)} = \rho(A) = \lim_{k\to\infty}\max_i c_i^{(k)}.$$

According to Theorem 4 and by combining the relationship between M-matrices and nonnegative matrices, we immediately obtain the following theorem for computing the minimum eigenvalue of an M-matrix.

Theorem 5. *Let $K = (k_{ij}) \in \mathbb{R}^{n\times n}$ be an irreducible M-matrix with the minimum eigenvalue $q(K)$. If $A = RI - K = A_0 = \left(a_{ij}^{(0)}\right)$ and $A_{k+1} = Q_k{}^{-1}A_kQ_k$, $k = 0, 1, 2, \cdots$, then*

$$R - \lim_{k\to\infty}\min_i c_i^{(k)} = q(K) = R - \lim_{k\to\infty}\max_i c_i^{(k)},$$

where $R > \max_i k_{ii}$, and $c_i^{(k)}(i = 1, 2, \cdots, n)$ is the sum of the i-th column of A_k, with Q_k defined as in Eq (2).

4. Algorithm and numerical examples

In practical applications, for the convenience of calculation, we usually set

$$R = 1 + \max_i k_{ii}$$

in Theorem 5. According to Theorems 4 and 5, only a finite number of iterations are needed to make $\left(\max_i c_i^{(k)} - \min_i c_i^{(k)}\right)$ small enough to obtain an approximation of the minimum eigenvalue:

$$q(K) \approx R - \frac{\max_i c_i^{(k)} + \min_i c_i^{(k)}}{2},$$

and the approximation satisfies the accuracy requirements.

Based on the previous analysis, we design the algorithm as follows:
Step 1. Given an irreducible M-matrix $K = \left(k_{ij}\right) \in \mathbb{R}^{n\times n}$ and a sufficiently small positive number ε;
Step 2. Let $R = 1 + \max_i k_{ii}$ and $A = RI - K = A_k = \left(a_{ij}^{(k)}\right), k = 0$;
Step 3. Compute

$$c_i^{(k)} = \sum_{t=1}^{n} a_{ti}^{(k)}\,(i = 1, 2, \cdots, n)\,, c_{\max}^{(k)} = \max_i c_i^{(k)}, c_{\min}^{(k)} = \min_i c_i^{(k)};$$

Step 4. If $c_{\max}^{(k)} - c_{\min}^{(k)} < \varepsilon$, go to Step 6; Otherwise, compute

$$Q_k = \operatorname{diag}\Big(\frac{1}{\sum_{t=1}^{n}\sqrt{c_t^{(k)}}a_{t1}^{(k)}}, \frac{1}{\sum_{t=1}^{n}\sqrt{c_t^{(k)}}a_{t2}^{(k)}}, \cdots, \frac{1}{\sum_{t=1}^{n}\sqrt{c_t^{(k)}}a_{tn}^{(k)}}\Big);$$

Step 5. Update. Let $A_{k+1} = {Q_k}^{-1}A_kQ_k$. Set $k = k + 1$ and go back to Step 3;

Step 6. Output $q(K) = R - \dfrac{c_{\max}^{(k)} + c_{\min}^{(k)}}{2}$. Stop.

Following this, we report some numerical experiments to test our algorithm and compare the results with previous research.

Example 1. [6, 7] Consider an irreducible M-matrix:

$$K = \begin{pmatrix} 8 & 0 & -2 & -1 & 0 & -1 & 0 & 0 \\ -2 & 7 & -1 & 0 & 0 & -2 & -1 & -1 \\ -2 & 0 & 8 & 0 & -3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 5 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & 7 & -2 & 0 & -1 \\ -2 & -2 & -1 & 0 & -1 & 9 & -2 & -1 \\ 0 & -4 & 0 & -1 & 0 & -2 & 6 & 0 \\ -1 & 0 & -2 & 0 & 0 & -1 & 0 & 5 \end{pmatrix}.$$

The minimum eigenvalue and the comparison of the number of iterations for Example 1 are shown in Tables 1 and 2, which display the computation time of each algorithm.

Table 1. Comparison of number of iterations.

ε	Number of iterations			$q(K)$
	This paper	[6]	[7]	
10^{-4}	11	15	17	0.9444
10^{-8}	22	30	33	0.94440470
10^{-12}	32	46	48	0.944404695029

Table 2. Comparison of computation time.

ε	Computation time(s)			$q(K)$
	This paper	[6]	[7]	
10^{-4}	0.583573	0.0167943	0.0108782	0.9444
10^{-8}	0.0073441	0.0132855	0.004649	0.94440470
10^{-12}	0.0054896	0.0046742	0.0040324	0.944404695029

Example 2. [7] Calculate the minimum eigenvalue of the M-matrix:

$$K = \begin{pmatrix} 1+\frac{2}{n} & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2+\frac{4}{n} & -2 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3+\frac{6}{n} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-2+\frac{2\,(n-2)}{n} & -(n-2) & 0 \\ 0 & 0 & 0 & \cdots & -1 & n-1+\frac{2\,(n-1)}{n} & -(n-1) \\ 0 & 0 & 0 & \cdots & 0 & -1 & n+2 \end{pmatrix}.$$

The approximate value of $q(K)$, computation time, and the number of iterations are presented in Table 3.

Table 3. Estimation of the minimum eigenvalue.

n	ε	Number of iterations		Computation time(s)		$q(K)$
		This paper	[7]	This paper	[7]	
10	10^{-5}	83	153	0.6061397	0.6028088	0.36667
	10^{-10}	161	268	0.03208	0.0266929	0.3666667064
20	10^{-5}	166	335	0.0305944	0.015159	0.19091
	10^{-10}	322	568	0.0432937	0.0154594	0.1909090910
50	10^{-5}	412	923	0.2566481	0.0473879	0.07847
	10^{-10}	799	1503	0.5088301	0.0454216	0.0784615385
100	10^{-5}	819	1977	2.6597557	0.1064548	0.03961
	10^{-10}	1591	3133	4.9898833	0.1667462	0.0396078432

Remark 1. *The algorithm presented in this paper uses a new diagonal transformation for iteration. From the computed data above, it can be observed that our algorithm reduces the number of iterations and allows calculation of the minimum eigenvalue with arbitrary precision. In addition, the nonnegative matrix constructed from the M-matrix retains the main diagonal elements unchanged, while the positions of the zero elements remain unchanged during the iteration process, and only the non-zero off-diagonal elements are modified. Therefore, the algorithm in this paper has the advantages of low memory usage, low computational cost, and high operability.*

Remark 2. *If K is a reducible M-matrix, then there exists a permutation matrix P of order n such that PKP^T has an upper block triangular form:*

$$PKP^T = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1m} \\ O & K_{22} & \cdots & K_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & K_{mm} \end{pmatrix}, m \geq 2,$$

where K_{ii} $(1 \leq i \leq m)$ is irreducible and square. The algorithm presented in this paper can be used to compute the minimum eigenvalue $q(K_{ii})$ for each $i = 1, 2, \cdots, m$, and it holds that

$$q(K) = q(PKP^T) = \min_i q(K_{ii}).$$

Thus, the proposed method can compute the minimum eigenvalue of any reducible M-matrix K after applying a suitable permutation similar transformation.

Remark 3. *Given that the minimum column sum sequence $\left\{\min_i c_i^{(k)}\right\}_{k=0}^{\infty}$ is monotonically increasing while the maximum column sum sequence $\left\{\max_i c_i^{(k)}\right\}_{k=0}^{\infty}$ is monotonically decreasing in the iterative process, and the iterative process exhibits a smooth tendency, the algorithm is called a smoothing algorithm.*

Remark 4. *For computational convenience, we set* $R = 1+\max_i k_{ii}$ *in the algorithm. Note that* $1+\max_i k_{ii}$ *is not always the optimal value. If a different positive number* R *satisfying* $R > \max_i k_{ii}$ *is chosen, the number of iterations may be reduced in some cases. The following example illustrates this situation. Consider the M-matrix below:*

$$K = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ -3 & 0 & 2 \end{pmatrix}.$$

We set $R = 1 + \max_i k_{ii} = 3$ *and* $3.5 = R > \max_i k_{ii} = 2$*, respectively. The comparative results are presented in Table 4.*

Table 4. Comparison of number of iterations for different R.

ε	Number of iterations		$q(K)$
	$R = 3$	$R = 3.5$	
10^{-8}	18	17	0.18287941
10^{-10}	22	21	0.1828794072
10^{-12}	26	25	0.182879407168
10^{-14}	30	29	0.18287940716786

We can observe from this example that when $R = 3.5$, the number of iterations of the algorithm decreases by a small amount. This demonstrates that the number of iterations depends on the value of R. However, how to choose the most appropriate value of R? It is directly related to the actual matrix. Finding a function that can determine the optimal value of R for all M-matrices is challenging. For researchers interested in this topic, this issue warrants further investigation.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no competing interests.

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