AIMS Mathematics

AIMS Mathematics, 10(8): 19693–19711.
DOI: 10.3934/math.2025878
Received: 05 July 2025
Revised: 06 August 2025
Accepted: 13 August 2025
Published: 27 August 2025

https://www.aimspress.com/journal/Math

Research article

Reliability and aging properties of novel nonparametric lifetime distribution classes

Mohamed Kayid* **and Mutairah Alanazi**

Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: drkayid@ksu.edu.sa; Tel: +966558727457; Fax: +96614676274.

Abstract: This paper introduces novel nonparametric classes of lifetime distributions characterized by the shape of the failure rate function. Particular reputable parametric distributions are examined to lie within these novel classes. Preservation property of one of these classes under order statistics is established. An empirical estimator of the failure rate ratio is proposed and in a simulation study, its attributes are explored. Then, a data set of strength of glass fibers is analyzed.

Keywords: failure rate function; empirical estimation; relative aging; order statistics; nonparametric class; aging properties
Mathematics Subject Classification: 62N05, 62G05, 62G30

1. Introduction

Lifetime distributions play a fundamental role to analyze the stochastic behavior of time-to-failure of systems as they provide insights into system reliability and its aging properties under uncertainty. In reliability theory, aging refers to the progressive increase in failure probability as a system ages, contrasting with systems that exhibit constant rates of failure, where aging is absent. To classify lifetime distributions within some nonparametric classes is beneficial as it is much more informative about the time-to-failure. These classes are mainly constructed by means of the underlying characteristics of lifetime distributions such as their survival function (sf) or the failure rate (FR) function. Over time, several non-parametric classes of lifetime distributions have been developed, each characterized by unique properties to model specific aspects of aging.

The sf (also called the reliability function), denoted by $\bar{F}(x)$, represents the probability that a system operates successfully without failure up to time x. From a modern reliability perspective, refined concepts have been introduced to better describe aging behavior, particularly in biological and engineering systems. Several nonparametric classes of lifetime distributions have been developed to capture different aspects of aging, allowing for more flexible and detailed modeling. A fundamental concept in this context is the FR, which quantifies the instantaneous risk of failure at time x, conditional on survival until that point. The rate of aging is commonly described using this function in both reliability theory and survival analysis. For a non-negative random variable (*rv*) X, representing the time to failure, the FR function $h_X(t)$ is defined as:

$$h_X(t) = \frac{f_X(t)}{\bar{F}_X(t)}, t \geq 0: \bar{F}_X(t) > 0,$$

where $f_X(t)$ is the probability density function (pdf) of X. The function $h_X(\cdot)$ provides a direct measure of the system's vulnerability at time x; a higher value of $h_X(\cdot)$ indicates a greater likelihood of failure at that moment. The FR function has been extensively applied in the study of various lifetime distributions (see, for example, Navarro and Sarabia [1]).

An important aspect of aging phenomenon is the speed or the rate of aging, which describes how rapidly the FR changes over time. This concept reflects how quickly the risk of failure increases or decreases, and can be quantified by analyzing the derivative of the FR function. As we shall state, the speed of aging is captured by the sign and magnitude of $t \cdot \frac{d}{dt} ln[h_X(t)]$ whenever $h_X(t)$ is differentiable in t. Note that $h_X(t)$ is itself equal with the derivative of $-Ln(\bar{F}_X(t))$ in t. A preliminary interpretation is as follows:

- $\frac{d}{dt} h_X(t) > 0$ reveals that the FR is increasing, indicating positive aging- that is, the system becomes more failure-prone over time.
- $\frac{d}{dt} h_X(t) < 0$ shows that the FR is decreasing, suggesting negative aging, where the system becomes more reliable with age.
- $\frac{d}{dt} h_X(t) = 0$ means that the system exhibits constant aging speed, and the FR remains unchanged over time.

In reliability engineering, the behavior of the FR function over time offers critical insights into the aging characteristics of a system or component. For instance, service life distributions are commonly classified into three major categories based on the trend of the FR function:

- Increasing Failure Rate (IFR): If $h_X(t)$ is increasing, it indicates that the system becomes more vulnerable to failure as time progresses. This behavior is typical of aging systems in which wear and tear accumulate over time.
- Decreasing Failure Rate (DFR): If $h_X(t)$ is decreasing, it suggests that the system is progressively less likely to fail. This pattern is often observed in systems that undergo a "burn-in" phase, during which early failures are weeded out and reliability improves.
- Constant Failure Rate (CFR): If $h_X(t)$ is constant, failures occur at a uniform rate over time. This is characteristic of memoryless systems, such as those modeled by the exponential distribution, where no aging effects are present.

Probability distributions play a fundamental role in reliability engineering by providing rigorous tools for modeling a wide range of reliability data and failure mechanisms. Among these, distributions with IFR properties are especially important for modeling the lifetimes of products and components in fields such as mechanical engineering, electronics, and materials science, where aging and wear substantially influence system performance. Identifying IFR behavior has significant practical implications. It enables reliability analysts to optimize preventive maintenance schedules, design more effective warranty policies, and implement strategies to minimize operational risk. Systems exhibiting IFR characteristics typically require more frequent inspections and proactive interventions to avoid unexpected breakdowns. Understanding whether a system adheres to the IFR class thus provides critical insights for developing maintenance protocols and ensuring sustained operational reliability. For comprehensive discussions in this area, we refer the reader to Lai and Xie [2], Finkelstein [3], and Navarro [4].

This paper introduces novel classes of lifetime distributions motivated by the relative behavior of the FR function. These new classes move beyond the traditional focus on the monotonicity of the FR function, whether increasing or decreasing, and instead highlight the variation in the intensity of FR ratios over time. The key question is how quickly or slowly an item ages over time, and the new classes categorize lifetime distributions based on the concept of relative aging. The rate of a device's aging may not be constant, and the speed of its aging process can be either accelerated or diminished over time. This broader framework provides a more refined approach to modeling aging processes and capturing subtle reliability characteristics that are not sufficiently addressed by classical aging classifications. The proposed classes are formally analyzed within the framework of distribution theory and reliability modeling. Their structural properties, preservation behavior, and potential applications in lifetime data analysis are also investigated, providing valuable tools for advancing nonparametric reliability theory.

The remainder of this paper is organized as follows. Section 2 presents the definitions and foundational properties of the proposed nonparametric classes, with emphasis on the concept of aging speed and measures of relative aging, which are central to stochastic reliability analysis. Section 3 investigates the preservation properties of these classes under order statistics. In Section 4, an estimator for the FR ratio is proposed, and its performance is evaluated through a simulation study. This section also includes a real-data application involving the strength of glass fibers, illustrating the practical utility of the new lifetime classes and the proposed estimator. Section 5 concludes the paper by summarizing the main contributions and suggesting directions for future research.

2. Some novel nonparametric classes

This section introduces the definitions, motivations, and preliminary properties of novel classes of lifetime distributions that incorporate the concept of aging speed, a fundamental aspect in stochastic reliability analysis. The framework is built upon the idea of relative aging, which provides a basis for establishing ordering relations between lifetime distributions and enables the comparison of aging intensities across different components or systems. By capturing changes in the FR beyond simple monotonicity, these classes allow for a more refined analysis of aging behavior. In particular, they extend the classical classification of life distributions and offer deeper insights into the reliability characteristics of individual units. The following definition serves as the foundation for the construction of these new distributional classes.

Definition 2.1. Let X be a non-negative rv with FR function h_X. Then, it is said that X has:

(i). Increasing (Decreasing) failure rate ratio property [denoted by $X \in IFRR(X \in DFRR)$], whenever $h_X(t)/h_X(\alpha t)$ is increasing (decreasing) in $t \geq 0$, for all $\alpha \in (0,1)$.

(ii). Increasing (Decreasing) failure rate relative to average failure rate property [denoted as $X \in IFR/A$ $(X \in DFR/A)$], whenever $th_X(t)/\int_0^t h_X(x)dx$ is increasing (decreasing) in $t \geq 0$.

The IFR/A and DFR/A properties in Definition 2.1(ii) were introduced and discussed in Righter et al. [5]. However, the $IFRR$ and $DFRR$ properties proposed in Definition 2.1(i) are novel in this paper. The IFR/A and $IFRR$ classes are useful for describing the lifetime distributions of devices where the rate of aging increases over time. Conversely, the DFR/A and $DFRR$ classes can be used to model the lifetime of devices where the rate of aging decreases over time. Table 1 presents several well-known parametric lifetime distributions that exhibit either the $IFRR$ or $DFRR$ properties, thereby illustrating the applicability of the proposed classification to widely used reliability models.

Table 1. Some distributions with their classes of life distributions.

Distribution	sf	FR	Class
Weibull $(\lambda > 0, b > 0)$	$\exp(-(\lambda t)^b), t \geq 0$	$b\lambda^b t^{b-1}$	$DFRR$
Gompertz $(\eta > 0, b > 0)$	$\exp(-\eta(\exp(bt) - 1)), t \geq 0$	$\eta b\exp(bt)$	$IFRR$
Pareto $(\theta > 0, b > 0)$	$\left(\frac{\theta}{t}\right)^b, t \geq \theta$	$\frac{b}{t}$	$DFRR$
Lomax $(a > 0, b > 0)$	$\left(1 + \frac{t}{b}\right)^{-a}, t \geq 0$	$\frac{a}{b+t}$	$DFRR$
Loglogistic $(\beta > 0, \theta > 0)$	$\frac{\left(\frac{t}{\theta}\right)^{-\beta}}{1 + \left(\frac{t}{\theta}\right)^{-\beta}}, t \geq 0$	$\frac{\beta}{\theta} \cdot \frac{\left(\frac{t}{\theta}\right)^{\beta-1}}{1 + \left(\frac{t}{\theta}\right)^{\beta}}$	$DFRR$
Burr$(k > 0, c > 0)$	$\frac{1}{(1+t^c)^k}, t \geq 0,$	$\frac{kct^{c-1}}{1+t^c}$	$DFRR$
Makeham$(\delta > 0, \lambda > 0, k > 1)$	$\exp\left(-\frac{\gamma t\ln(k) + \delta k^t - \delta}{\ln(k)}\right), t \geq 0$	$\gamma + \delta k^t$	$IFRR$

Remark 2.2. It is noteworthy that the classes defined in parts (i) and (ii) of Definition 2.1 are closely related to the concept of the aging intensity (AI) function. Let X be a non-negative *rv* with sf $\bar{F}_X$ and FR function h_X, then $L_X(t) = th_X(t)/-\ln(\bar{F}_X(t))$ is known as the AI function of X. In this case $X \in IFR/A (X \in DFR/A)$ is equivalent to saying that X has an increasing (a decreasing) AI function. For further details, we refer the reader to Kayid and Alshehri [6]. There are two important very well-known classes based on the failure rate (FR) function namely IFR and DFR classes. The classes $IFRR$ and $DFRR$ actually describe the rate of increments (or reductions) in the FR function. For example, when for two lifetime distributions, the amount of the ratio $h_X(t)/h_X(\alpha t)$ does not depend on $t \geq 0$, it means that the rates of increments (or reductions) in the FR functions of those distributions are similar. As such, when two IFR (or DFR) distributions have similar constant rates of increase (or decrease) in their FR functions, their FRs will increase (or decrease) in a similar manner. In this case, none of these distributions has a failure rate that increases faster or slower than the other. For example in Table 1, the Weibull distribution with parameter $\lambda > 0$ and $b < 1$ and the Pareto distribution with parameters $\theta > 0$ and $b > 0$ both are DFR and further they have a constant rate of reduction in their FR functions. In fact, the $IFRR$ and $DFRR$ classes refer to the rate or degree to which the FR function increases or decreases over time.

The relationships among the lifetime distribution classes introduced in Definition 2.1 are established through the following results.

Proposition 2.3. The following implications hold:

$$X \in IFRR(X \in DFRR) \Longrightarrow X \in IFR/A(X \in DFR/A).$$

Proof. Suppose X has AI function L_X. From Remark 2.1, it is sufficient to show that L_X is an increasing [a decreasing] function. We have

$$L_X(t) = h_X(t)/\left(\frac{1}{t}\int_0^t h_X(x)dx\right) = 1/\left(\frac{1}{t}\int_0^t \frac{h_X(x)}{h_X(t)}dx\right) = 1/\left(\int_0^1 \frac{h_X(\alpha t)}{h_X(t)}d\alpha\right),$$

where the last expression is due to the change of variable $\alpha = x/t$. Now, assume that $X \in IFRR$ $[X \in DFRR]$. Then, according to Definition 2.1(i), $h_X(\alpha t)/h_X(t)$ is decreasing [increasing] in $t > 0$, for all $\alpha \in (0,1)$. Thus, $\int_0^1 h_X(\alpha t)/h_X(t)d\alpha$ is also decreasing [increasing] in $t > 0$.

The following proposition provides a necessary and sufficient condition for an *rv* X to belong to the class $IFRR$ (or $DFRR$). This result formally demonstrates that the $IFRR$ and $DFRR$ properties are fully characterized by the shape and behavior of the FR function.

Proposition 2.4. Let h_X be a differentiable function. Then, $X \in IFRR(X \in DFRR)$ if, and only if, $t(h_X'(t)/h_X(t))$ is increasing (decreasing) in $t > 0$.

Proof. We prove the $IFRR$ case; the $DFRR$ case follows analogously. From Definition 2.1.(i), $X \in IFRR$ if, and only if, $h_X(t)/h_X(\alpha t)$ is increasing in $t > 0$ for all $0 < \alpha < 1$. Let $h_X(t)$ be differentiable in $t > 0$. It then follows that $\frac{\partial}{\partial t}\ln\left(\frac{h_X(t)}{h_X(\alpha t)}\right) \geq 0$, for all $t > 0$ and for all $0 < \alpha < 1$, i.e., $\frac{h_X'(t)}{h_X(t)} - \frac{\alpha h_X'(\alpha t)}{h_X(\alpha t)} \geq 0$, for all $t > 0$ and for every $0 < \alpha < 1$. This is equivalent to having $\frac{th_X'(t)}{h_X(t)} \geq \frac{\alpha t h_X'(\alpha t)}{h_X(\alpha t)}$, for all $t > 0$ and for every $0 < \alpha < 1$, which means that $\frac{th_X'(t)}{h_X(t)}$ is increasing in $t > 0$. The proof is finalized.

The result of Proposition 2.4 does not depend on properties of the hazard rate function $h_X(\cdot)$. As such, it is remarkable that if either cumulative distribution function (cdf), sf or even the logarithms of these functions is used in place of $h_X(\cdot)$, then some known classes of lifetime distributions such as increasing generalized failure rate ($IGFR$) or other ones can be reproduced (see e.g., Righter et al. [5] and Oliveira and Torrado [7]). In general, the FR function in Definition 2.1 can be replaced by other aging characteristics of distribution. The following example illustrates a probability distribution with a non-monotonic FR function that nevertheless satisfies the criteria for membership in the $DFRR$ class.

Example 2.5. Let X be a non-negative *rv* with inverse exponential distribution having sf $\bar{F}_X(t) = 1 - \exp\left(-\frac{1}{t}\right), t > 0$. The FR function of X is $h_X(t) = t^{-2}/\left(\exp\left(\frac{1}{t}\right) - 1\right), t > 0$. Figure 1 (left) exhibits the graph of the plot of $h_X(t)$ for $0 < t < 20$. However, X is DFRR as $th_X'(t)/h_X(t) = \left(t^{-1}\exp\left(\frac{1}{t}\right) - 2\exp\left(\frac{1}{t}\right) + 2\right)/\left(\exp\left(\frac{1}{t}\right) - 1\right)$ is decreasing in $t > 0$ (see Figure 1 (right) where the graph of $th_X'(t)/h_X(t)$ is plotted for $t \in (0,50)$).

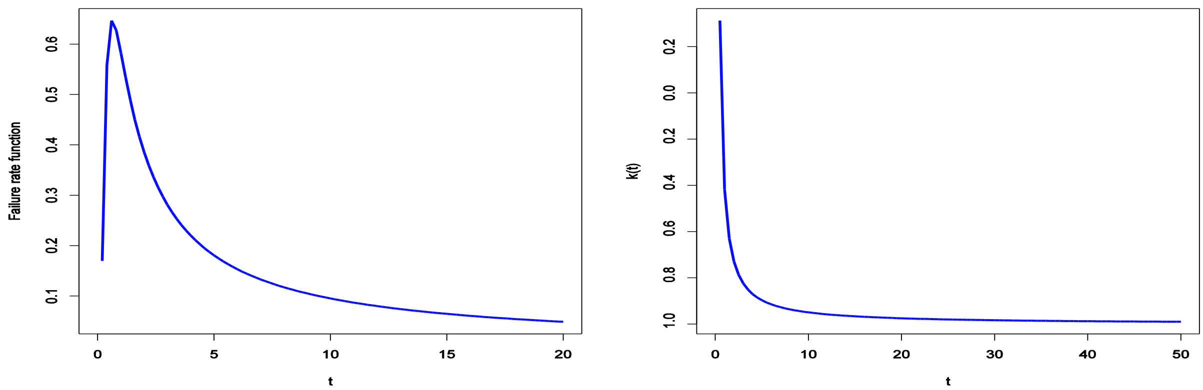

Figure 1. (left) Plot of the function $h_X(t)$ for $0 < t < 20$ in Example 2.5. (right) Plot of the function $k(t) = \frac{t h'_X(t)}{h_X(t)}$ for $0 < t < 50$ in Example 2.5.

Proposition 2.6.

(i). If $h_X(t)$ is increasing (i.e., if X is IFR) and also log-convex in $t > 0$, then $X \in IFRR$.

(ii). If $h_X(t)$ is decreasing (i.e., if X is DFR) and log-concave in $t > 0$, then $X \in DFRR$.

Proof. Assume that $h_X(t)$ is differentiable in $t > 0$ and denote $\gamma(t) := t\frac{h'_X(t)}{h_X(t)}$. To prove (i), note that if $h_X(t)$ is increasing and log-convex, then $\frac{d}{dt}\left(\ln\left(h_X(t)\right)\right)$ is non-negative and increasing in $t \geq 0$. Thus, $\gamma(t) = t \cdot \frac{d}{dt}\left(\ln\left(h_X(t)\right)\right)$ as the multiplication of two non-negative increasing functions, is also increasing in $t \geq 0$. From the non-parenthetical part of Proportion 2.4, it is deduced that $X \in IFRR$.

The log-convexity of the FR function has been adopted in Burkschat [8] where multivariate dependence of spacings of generalized order statistics is studied. Next, we give an example where it is shown that the log-convexity condition in Proposition 2.6(i) as well as the log-concavity condition in Proposition 2.6(ii) are not necessary.

Example 2.7. Let X be a non-negative *r.v.* with sf $\bar{F}_X(x) = \exp\left(-2 - 2\left(\sqrt{x} - 1\right)e^{\sqrt{x}}\right), x \geq 0$. Then, the FR of X is obtained as $h_X(t) = e^{\sqrt{t}}, t \geq 0$. It is seen that $X \in IFR$ and that $\ln\left(h_X(t)\right) = \sqrt{t}$ is not convex in $t \geq 0$. It can, however, be verified that $\gamma(t) = \frac{\sqrt{t}}{2}$ is increasing in $t \geq 0$, i.e., by Proposition 2.4, $X \in IFRR$. We showed that the log-convexity condition in Proposition 2.6(i) can be relaxed in some situations. Now, let X be a non-negative *r.v.* with sf $\bar{F}_X(x) = \frac{1}{x+1}, x \geq 0$. The FR of X is then derived as $h_X(t) = \frac{1}{t+1}, t \geq 0$, and thus it has a DFR. We can see that $\ln\left(h_X(t)\right) = -\ln(t+1)$ which is not concave in $t \geq 0$. However, $\gamma(t) = -\frac{t}{t+1}$ which is decreasing in $t \geq 0$, that is by Proposition 2.4, X has a $DFRR$. Therefore, it was indicated that the log-concavity condition in Proposition 2.6(ii) can be omitted in this case.

The following example illustrates an application of Proposition 2.6(i) to verify the membership of a distribution within a specified reliability class.

Example 2.8. Let X be uniformly distributed on $(0, b)$ where $b > 0$. We have $h_X(t) = \frac{1}{b-t}$ in which $t < b$. Clearly, X is IFR. We can see that $\frac{\partial}{\partial t}\ln(h_X(t)) = \frac{1}{b-t}$ which is increasing in $0 < t < b$. Therefore, $h_X(t)$ is log-convex in $0 < t < b$. By Proposition 2.6(i), we conclude that X is $IFRR$.

3. Preservation of the $DFRR$ class under order statistics

The behavior of the $DFRR$ property under transformations such as order statistics plays a vital role in the assessment of system reliability, particularly in systems composed of multiple components or redundant structures. This section explores whether the $DFRR$ class is preserved when applied to the distribution of order statistics drawn from a random sample. Special emphasis is placed on extreme order statistics, which are commonly encountered in reliability studies, life testing, and quality control applications.

3.1. Extreme order statistics

The behavior of the $DFRR$ property is examined for extreme order statistics, specifically the minimum and maximum order statistics. These cases are particularly important in reliability theory, where system performance is often governed by the first or last component to fail. Let us consider the order statistic $X_{k:n}$, in two special cases $k = n$ and $k = 1$. Since $\bar{F}_{X_{1:n}}(t) = \bar{F}^n_{X_1}(t)$, for all $t \geq 0$, thus $h_{X_{1:n}}(t) = nh_{X_1}(t)$, for all $t \geq 0$. Therefore, $\frac{h_{X_{1:n}}(t)}{h_{X_{1:n}}(\alpha t)} = \frac{h_{X_1}(t)}{h_{X_1}(\alpha t)}$, for all $t \geq 0$ and for all $\alpha \in (0,1)$. From Definition 2.1(i), this means that $X_{1:n}$ is $IFRR\,(DFRR)$ if, and only if, X_1 is $IFRR\,(DFRR)$. However, as we discuss in the sequel, the $DFRR$ class is preserved under the kth order statistic when $1 < k < n$.

3.2. Order statistics

In this subsection, we establish the preservation property of the $DFRR$ class under order statistics in a general scenario. Let X and Y be two non-negative *rv*s with respective sf's $\bar{F}_X$ and $\bar{F}_Y$ and also respective pdfs f_X and f_Y. Then, it is said that X is smaller (bigger) than Y in the usual stochastic order (denoted by $X \leq_{st} Y (X \geq_{st} Y)$) if $\bar{F}_X(t) \leq (\geq) \bar{F}_Y(t)$, for all $t \geq 0$. Equivalently, $X \leq_{st} Y$ if for all increasing functions $\phi, E[\phi(X)] \leq E[\phi(Y)]$ or for all decreasing functions $\psi, E[\psi(X)] \geq E[\psi(Y)]$ where all expectations exist and are finite. Furthermore, it is said that X is smaller (bigger) than Y in the likelihood ratio order (denoted by $X \leq_{lr} Y (X \geq_{lr} Y)$) whenever $f_Y(t)/f_X(t)$ is increasing (decreasing) in $t > 0$. From Shaked and Shanthikumar [9], $X \leq_{lr} Y$ (resp. $X \geq_{lr} Y$) implies that $X \leq_{st} Y$ (resp. $X \geq_{st} Y$). Before stating the main result of this subsection, we state a technical lemma that will be useful.

Lemma 3.1. Let $w_k(p) = \frac{\int_0^p x^{n-k}(1-x)^{k-1}dx}{p^{n-k+1}(1-p)^{k-1}}$ for $p \in (0,1)$, where $k = 1, \dots, n$. Then $\frac{-p\ln(p)w'_k(p)}{w_k(p)}$ is increasing in $p \in (0,1)$.

Proof. We see that

$$\frac{w_k'(p)}{w_k(p)} = \frac{d}{dp}\ln\left(w_k(p)\right)$$
$$= \frac{d}{dp}\ln\left(\int_0^p x^{n-l}(1-x)^{k-1}dx\right) - \frac{d}{dp}\ln\left(p^{n-l+1}(1-p)^{k-1}\right)$$
$$= \frac{p^{n-k}(1-p)^{k-1}}{\int_0^p x^{n-k}(1-x)^{k-1}dx} - \frac{n(1-p)-(k-1)}{p(1-p)}, \text{ for all } p \in (0,1).$$

Therefore,

$$p\frac{w_k'(p)}{w_k(p)} = \frac{p}{1-p}\left(\frac{p^{n-k}(1-p)^k}{\int_0^p x^{n-k}(1-x)^{k-1}dx} - \frac{n(1-p)-(k-1)}{p}\right).$$

For all $p \in (0,1)$ one has

$$p^{n-k+1}(1-p)^k = \int_0^p (n-(n+1)x-(k-1))x^{n-k}(1-x)^{k-1}dx.$$

By substituting $p^{n-k+1}(1-p)^k$ form the above expression, we obtain

$$p\frac{w_k'(p)}{w_k(p)} = \frac{p\,\mho}{1-p}\left(\frac{1}{p\int_0^p x^{n-k}(1-x)^{k-1}dx}\right)$$
$$= \frac{p}{1-p}\frac{\int_0^p \left(n-(n+1)\frac{x}{p}\right)x^{n-k}(1-x)^{k-1}dx}{\int_0^p x^{n-k}(1-x)^{k-1}dx}$$
$$= \frac{p}{1-p}\frac{\int_0^1 (n-(n+1)w)w^{n-k}(1-pw)^{k-1}dw}{\int_0^1 w^{n-k}(1-uw)^{k-1}dw},$$

where

$$\mho = \int_0^p (n-(n+1)x-(k-1))x^{n-k}(1-x)^{k-1}dx$$
$$-(n(1-p)-(k-1))\int_0^p x^{n-k}(1-x)^{k-1}dx.$$

Hence, as a result,

$$\frac{-p\ln(p)w_k'(p)}{w_k(p)} = \frac{-p\ln(p)}{1-p}\cdot\frac{\int_0^1 (n-(n+1)w)w^{n-k}(1-pw)^{k-1}dw}{\int_0^1 w^{n-k}(1-pw)^{k-1}dw}, \tag{1}$$

and it is enough to show that

$$k(p) = \frac{\int_0^1 (n-(n+1)w)w^{n-k}(1-pw)^{k-1}dw}{\int_0^1 w^{n-k}(1-pw)^{k-1}dw}, \tag{2}$$

is a non-negative increasing function because $\frac{-p\ln(p)}{1-p}$ is non-negative and increasing in $p \in (0,1)$ from Example 5 in Kayid and Almohsen [10]. Since $w_k(p)$ is increasing in $p \in (0,1)$ for all $k = 1,\dots,n$, thus in the spirit of (1), $k(p)$ is non-negative for all $p \in (0,1)$. Now, we show that $k(p)$ is increasing in $p \in (0,1)$. Notice that from (2), $k(p) = E\left[n-(n+1)W_p\right]$ where W_p is a non-negative *rv* with pdf

$$f_k(w \mid p) = \frac{w^{n-k}(1-pw)^{k-1}}{\int_0^1 w^{n-k}(1-pw)^{k-1}dw}, 0 < w < 1.$$

It is straightforward to see that $\frac{f_k(w|p_1)}{f_k(w|p_2)}$ is increasing in $w \in (0,1)$, for all $0 < p_1 < p_2 < 1$. That is $W_{p_1} \geq_{lr} W_{p_2}$, for all $0 < p_1 < p_2 < 1$. Since likelihood ratio order implies the usual stochastic order, thus we further conclude that $W_{p_1} \geq_{st} W_{p_2}$, for all $0 < p_1 < p_2 < 1$. Now, since $n-(n+1)w$ is decreasing function in $w \in (0,1)$, for all $n = 1,2,\ldots$, thus, by definition we have $k(p_1) = E[n-(n+1)W_{p_1}] \leq k(p_2) = E[n-(n+1)W_{p_2}]$, for all $0 < p_1 < p_2 < 1$, i.e., $k(p)$ is increasing in $p \in (0,1)$.

The preservation of aging (nonparametric) classes of lifetime distributions under the structure of coherent systems has been extensively studied in the literature (see, for example, Navarro et al. [11], Navarro [12], Lindqvist and Samaniego [13], and Badía et al. [14]). The first result in this section, which can be interpreted as a preservation of the $DFRR$ class under *(n-k+1)*-out-of-n system structures, is stated as follows:

Theorem 3.2. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) *rv*s with common sf $\bar{F}_{X_1}$ and pdf f_{X_1}. Then, $X_1 \in DFRR$ implies $X_{k:n} \in DFRR$ where $k = 1, \ldots, n$.

Proof. The FR function of $X_{k:n}$ is $h_{X_{k:n}}(t) = f_{X_{k:n}}(t)/\bar{F}_{X_{k:n}}(t)$. We need to prove that $\frac{h_{X_{k:n}}(t)}{h_{X_{k:n}}(\alpha t)}$ is decreasing in $t > 0$, for all $\alpha \in (0,1)$. First note that

$$\frac{h_{X_{k:n}}(t)}{h_{X_{k:n}}(\alpha t)} = \frac{h_{X_1}(t)}{h_{X_1}(\alpha t)} \frac{F_{X_1}^{k-1}(t)\bar{F}_{X_1}^{n-k+1}(t) \int_0^{\bar{F}_{X_1}(\alpha t)} u^{n-k}(1-u)^{k-1}du}{F_{X_1}^{k-1}(\alpha t)\bar{F}_{X_1}^{n-k+1}(\alpha t) \int_0^{\bar{F}_{X_1}(t)} u^{n-k}(1-u)^{k-1}du}.$$

As $X_1 \in DFRR$, the ratio $\frac{h_{X_1}(t)}{h_{X_1}(\alpha t)}$ is decreasing in $t \geq 0$, for all $\alpha \in (0,1)$. Therefore, $X_{k:n} \in DFRR$ and the proof is obtained if we show in addition that

$$\frac{w_k(\bar{F}_{X_1}(\alpha t))}{w_k(\bar{F}_{X_1}(t))} = \frac{\bar{F}_{X_1}^{n-k+1}(t)F_{X_1}^{k-1}(t) \int_0^{\bar{F}_{X_1}(\alpha t)} u^{n-k}(1-u)^{k-1}du}{\bar{F}_{X_1}^{n-k+1}(\alpha t)F_{X_1}^{k-1}(\alpha t) \int_0^{\bar{F}_{X_1}(t)} u^{n-k}(1-u)^{k-1}du},$$

is also decreasing in $t \geq 0$, for all $\alpha \in (0,1)$ where $w_k(u) = \frac{\int_0^u x^{n-k}(1-x)^{k-1}dx}{u^{n-k+1}(1-u)^{k-1}}$ for every $u \in (0,1)$. To this end, we show that $\frac{d}{dt}\frac{w_k(\bar{F}_{X_1}(\alpha t))}{w_k(\bar{F}_{X_1}(t))} \leq 0$, for all $t \geq 0$, and for all $\alpha \in (0,1)$. From the parenthetical part of Proposition 2.3(i), since $X_1 \in DFRR$, thus $X_1 \in DFR/A$. As a result,

$$\alpha h_{X_1}(\alpha t) \geq h_{X_1}(t) \frac{\int_0^{\alpha t} h_{X_1}(x)dx}{\int_0^t h_{X_1}(x)dx}, \text{ for all } t \geq 0, \text{ and for all } \alpha \in (0,1). \quad (3)$$

It is not hard to prove that $w_k(u)$ is increasing in $u \in (0,1)$ which further implies that $w_k'(\bar{F}_{X_1}(t)) \geq 0$, for all $t \geq 0$, and further $\bar{F}_{X_1}(t)w_k'(\bar{F}_{X_1}(t))w_k(\bar{F}_{X_1}(\alpha t)) \geq 0$, for all $t \geq 0$. By multiplying both sides of (3) in $\bar{F}_{X_1}(t)w_k'(\bar{F}_{X_1}(t))w_k(\bar{F}_{X_1}(\alpha t))$, for all $t \geq 0$, and for all $\alpha \in (0,1)$ we derive

$$\alpha h_{X_1}(\alpha t)\bar{F}_{X_1}(t)w_k'\left(\bar{F}_{X_1}(t)\right)w_k\left(\bar{F}_{X_1}(\alpha t)\right)$$

$$\geq h_{X_1}(t)\frac{\int_0^{\alpha t} h_{X_1}(x)dx}{\int_0^t h_{X_1}(x)dx}\bar{F}_{X_1}(t)w_k'\left(\bar{F}_{X_1}(t)\right)w_k\left(\bar{F}_{X_1}(\alpha t)\right). \tag{4}$$

For all $t \geq 0$ and for all $\alpha \in (0,1)$, we obtain

$$\frac{d}{dt}\frac{w_k\left(\bar{F}_{X_1}(\alpha t)\right)}{w_k\left(\bar{F}_{X_1}(t)\right)} = \frac{A}{w_k^2\left(\bar{F}_{X_1}(t)\right)} = \frac{B}{w_k^2\left(\bar{F}_{X_1}(t)\right)},$$

where

$$A = f_{X_1}(t)w_k'\left(\bar{F}_{X_1}(t)\right)w_k\left(\bar{F}_{X_1}(\alpha t)\right) - \alpha f_{X_1}(\alpha t)w_k'\left(\bar{F}_{X_1}(\alpha t)\right)w_k\left(\bar{F}_{X_1}(t)\right),$$

$$B = h_{X_1}(t)\bar{F}_{X_1}(t)w_k'\left(\bar{F}_{X_1}(t)\right)w_k\left(\bar{F}_{X_1}(\alpha t)\right) - \alpha h_{X_1}(\alpha t)\bar{F}_{X_1}(\alpha t)w_k'\left(\bar{F}_{X_1}(\alpha t)\right)w_k\left(\bar{F}_{X_1}(t)\right),$$

and the inequality is due to (4). It is notable that $\bar{F}_X(t) = e^{-\int_0^t h_{X_1}(x)dx}$ which yields $\bar{F}_{X_1}(\alpha t) = e^{-\int_0^{\alpha t} h_{X_1}(x)dx}$, for all $t \geq 0$. Therefore, $\frac{d}{dt}\frac{w_k\left(\bar{F}_{X_1}(\alpha t)\right)}{w_k\left(\bar{F}_{X_1}(t)\right)}$ is non-positive if, and only if,

$$\frac{\int_0^t h_{X_1}(x)dx e^{-\int_0^t h_{X_1}(x)dx} w_k'\left(e^{-\int_0^t h_{X_1}(x)dx}\right)}{w_k\left(e^{-\int_0^t h_{X_1}(x)dx}\right)}$$

$$\leq \frac{\int_0^{\alpha t} h_{X_1}(x)dx e^{-\int_0^{\alpha t} h_{X_1}(x)dx} w_k'\left(e^{-\int_0^{\alpha t} h_{X_1}(x)dx}\right)}{w_k\left(e^{-\int_0^{\alpha t} h_{X_1}(x)dx}\right)}, \text{ for all } t \geq 0. \tag{5}$$

Since $\bar{F}_{X_1}(\alpha t) \geq \bar{F}_{X_1}(t)$, for all $t \geq 0$ and for all $\alpha \in (0,1)$, from which one gets $\int_0^{\alpha t} h_{X_1}(x)dx \geq \int_0^t h_{X_1}(x)dx$, for all $t \geq 0$. Let us define $\delta_k(y) = \frac{ye^{-y}w_k'(e^{-y})}{w_k(e^{-y})}$ which by Lemma 3.1 it is a decreasing function in $y > 0$. The inequality given in (5) can be written accordingly and equivalently as $\delta_k\left(\int_0^t h_{X_1}(x)dx\right) \leq \delta_k\left(\int_0^{\alpha t} h_{X_1}(x)dx\right)$, for all $t \geq 0$. The proof of the theorem is now complete.

The following example is provided to verify the validity of the result established in Theorem 3.2 and to illustrate its practical applicability.

Example 3.3. Let $X_i, i = 1,2,3,4$ be non-negative i.i.d *rv*s with common sf $\bar{F}_{X_1}(t) = (t+1)e^{-t}, t \geq 0$. Note that the FR function of X_1 is differentiable and $\frac{th'_{X_1}(t)}{h_{X_1}(t)} = \frac{1}{t+1}$, which is a decreasing function in $t > 0$. From the parenthetical part of Proposition 2.4(i), X_1 is $DFRR$. Consider the second order statistic $X_{2:4}$ and note that $h_{X_{2:4}}(t) = h_{X_1}(t) \cdot \frac{F_{X_1}(t)\bar{F}_{X_1}(t)}{\int_0^{\bar{F}_{X_1}(t)} y^2(1-y)dy}$ which gives

$$\frac{th'_{X_{2:4}}(t)}{h_{X_{2:4}}(t)} = \frac{th'_{X_1}(t)}{h_{X_1}(t)} + t\frac{d}{dt}\ln\left(\frac{F_{X_1}(t)\bar{F}_{X_1}(t)}{\int_0^{\bar{F}_{X_1}(t)} y^2(1-y)dy}\right)$$
$$= \frac{th'_{X_1}(t)}{h_{X_1}(t)} + th_{X_1}(t)\left(1 + \bar{F}^2_{X_1}(t)F^2_{X_1}(t)\right) - th_{X_1}(t)$$
$$= \frac{1}{t+1} + \frac{t^2}{e^t - (t+1)}\left(1 + (t+1)^2 e^{-2t}\left(1 - (t+1)e^{-t}\right)^2\right) - \frac{t^2}{t+1}.$$

Figure 2 displays the graph of $th'_{X_{2:4}}(t)/h_{X_{2:4}}(t)$, which clearly is decreasing for $t \in (0,50)$. This behavior supports the parenthetical claim in Proposition 2.4, confirming that $X_{2:4}$ belongs to the $DFRR$ class.

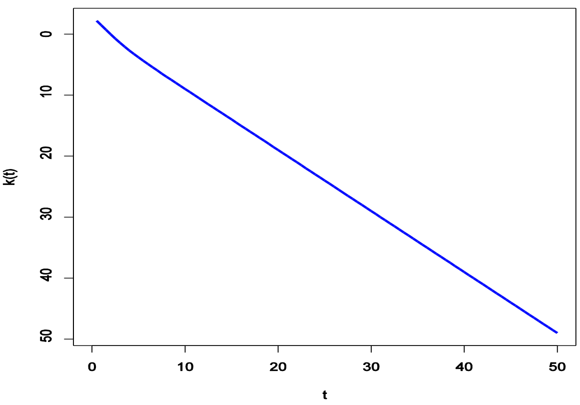

Figure 2. Plot of the function $k(t) = th'_{X_{2:4}}(t)/h_{X_{2:4}}(t)$ for $0 < t < 50$ in Example 3.3.

4. Empirical failure rate ratio

Empirical methods in reliability analysis of lifetime distributions have been widely utilized by both statisticians and practitioners (see, e.g., Kvam et al. [15] and Fernández [16]). Let $t_1, t_2, \ldots, t_n$ represent a sample of observed lifetimes, and let $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ denote their corresponding ordered values arranged in ascending order. In this section, we propose an empirical estimator for the failure rate ratio (FRR) function, defined as:

$$\eta_\alpha(t) = \frac{h(t)}{h(\alpha t)}, \text{ for a fixed } \alpha \in (0,1).$$

In the first step, for each sufficiently large i, such that $t_{(i)} > \frac{t_{(1)}}{\alpha}$, we calculate

$$t_{(\alpha,i)} = \text{Nearest observation to } \alpha t_{(i)},$$

and set $k < i$ such that $t_{(\alpha,i)} = t_{(k)}$. Then, we take a suitable bandwidth

$$\Delta = \max\{(t_{(i)} - t_{(i-1)}), (t_{(i+1)} - t_{(i)}), (t_{(k)} - t_{(k-1)}), (t_{(k+1)} - t_{(k)})\}.$$

Now, the empirical density ratio function which is defined by the following equation,

$$\xi_{\alpha,n}(t_{(i)}) = \frac{\#(t_{(i)},\Delta)}{\#(t_{(k)},\Delta)},$$

where $\#(t_{(i)},\Delta)$ is the number of observations around the $t_{(i)}$ which their distance to $t_{(i)}$ does not exceed Δ. Since Δ is the maximum of the distances in the set, both expressions $\#(t_{(i)},\Delta)$ and $\#(t_{(k)},\Delta)$ will be greater than zero, which makes $\xi_{\alpha,n}(t_{(i)})$ computable. It could be considered as an empirical estimate of the density ratio function

$$\xi_{\alpha}(t_{(i)}) = f(t_{(i)})/f(\alpha t_{(i)}).$$

So, the empirical FRR function could be defined by

$$\eta_{\alpha,n}(t_{(i)}) = \xi_{\alpha,n}(t_{(i)})\frac{R_n(t_{(k)})}{R_n(t_{(i)})},$$

where R_n is a suitable estimator of the reliability function R at the given point. Applying Parzen's consistency theorem (refer to Parzen [17]), it results that $\xi_{\alpha,n}(t_{(i)})$ converges to $\xi_{\alpha}(t_{(i)})$ almost surely. On the other hand, it is a well known result of the Glivenko-Cantelli theorem that R_n converges to R, almost surely. Thus, we can conclude that $\eta_{\alpha,n}(t_{(i)})$ converges to $\eta_{\alpha}(t_{(i)})$ almost surely. To investigate the behavior of the estimator graphically, we consider the Gompertz, Weibull, Lomax and Makeham distributions and samples of size 100 are generated from these models. Figures 3 to 6 show the proposed empirical FRR function for these models.

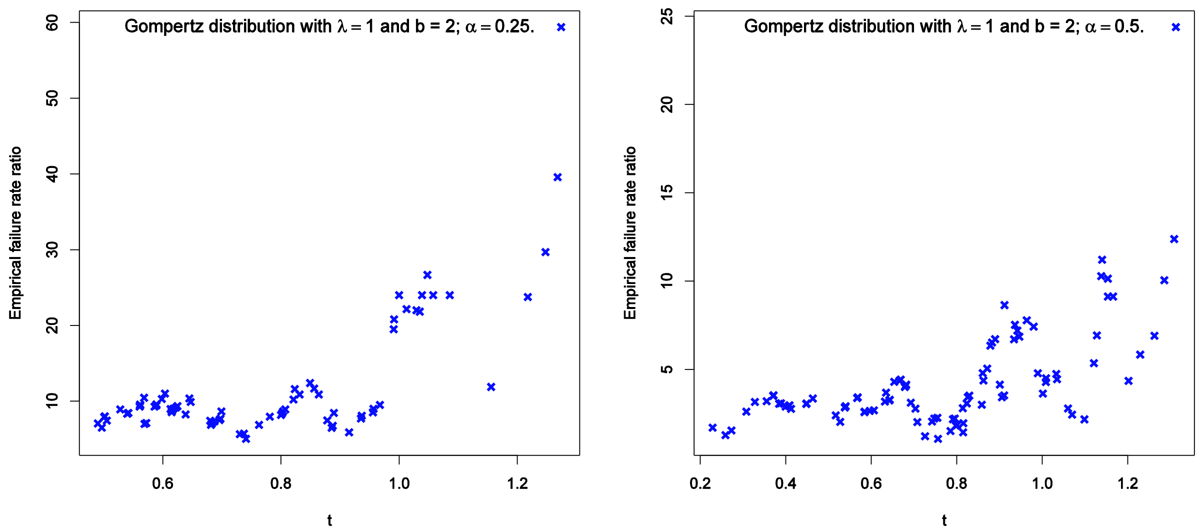

Figure 3. Empirical FRR functions for samples of size 100 of Gompertz distribution which show increasing forms.

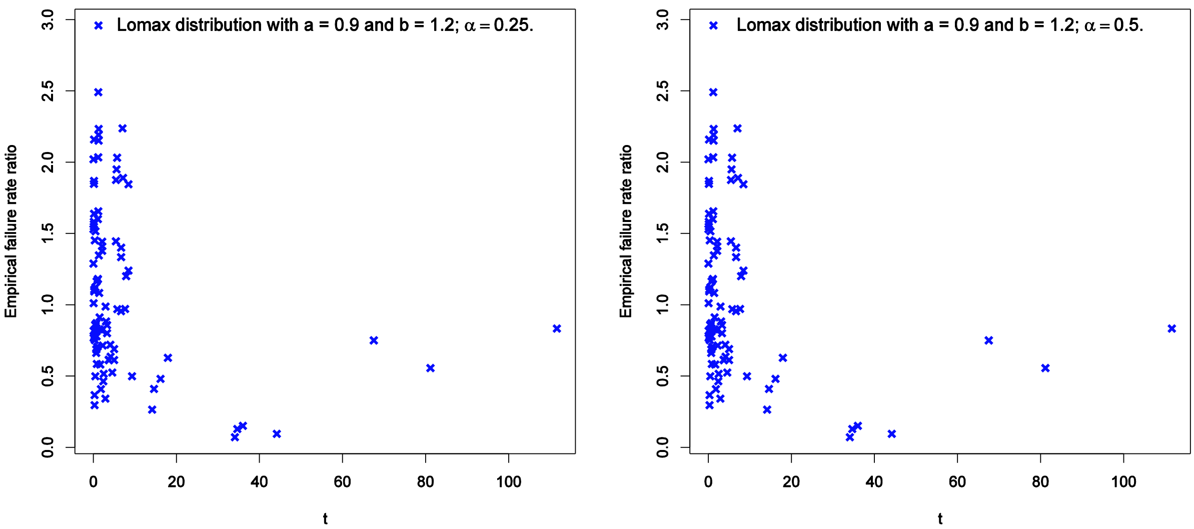

Figure 4. Empirical FRR functions for samples of size 100 of Lomax distribution.

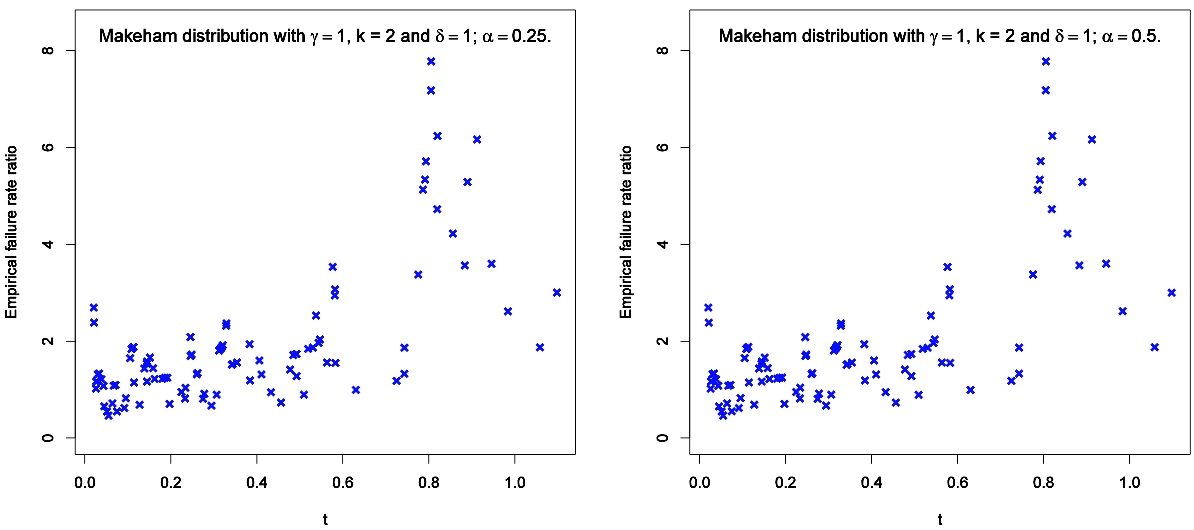

Figure 5. Empirical FRR functions for samples of size 100 of Makeham distribution.

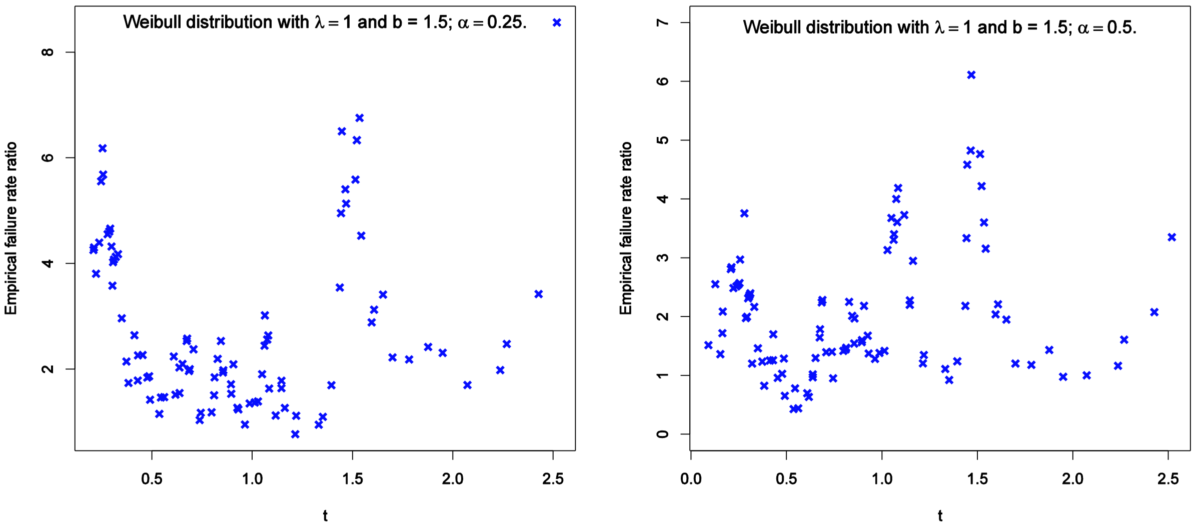

Figure 6. Empirical FRR functions for samples of size 100 from Weibull distribution.

4.1. Simulation *study*

To evaluate the performance of the proposed empirical estimator for the FRR, we conduct a simulation study under various lifetime distribution scenarios. The objective is to assess the estimator's behavior in terms of bias, mean squared error (MSE), and sensitivity to sample size and bandwidth selection. In this simulation, we explore the behavior of the proposed estimator using two commonly studied lifetime models: the Gompertz and Makeham distributions. Simulated samples from these distributions are generated using their respective inverse distribution functions. Two values of the FRR parameter are considered, $\alpha = 0.25$ and $\alpha = 0.50$, along with two sample sizes $n = 25$ and $n = 50$. For each configuration, 1000 independent samples of size *n* are generated. Within each sample, the target function $\eta_{\alpha,n}(t)$ is evaluated at three quantiles of the underlying distribution: $t = q_{0.25}, q_{50}$, and $q_{0.75}$. The bias and mean squared error (MSE) of the estimator are then computed for each run. One may also check the results under big sizes of n (n=150, 300 and 500).

The results, summarized in Tables 2 and 3, show that the MSE decreases as *n* increases, indicating that the estimator is consistent. Additionally, comparisons between the true values and estimated values demonstrate that the biases are negligible across all considered scenarios. These findings confirm that the estimator is capable of accurately capturing the behavior of the FRR function. All simulations and computations were performed using the R statistical programming language.

Table 2. Simulation results for the Gompertz distribution.

			n				
			25		50		
parameters	α	point	B	MSE	B	MSE	real value
$\lambda = 1,\ b = 1.2$	0.25	q_1	-0.3764	3.9276	0.1575	2.5998	1.6193
		q_2	0.2144	4.3924	0.2429	2.6348	2.0219
		q_3	0.4762	5.8227	0.5352	3.3727	2.6704
	0.5	q_1	0.1054	0.9374	0.2196	0.7980	1.3650
		q_2	0.2201	0.9395	0.1853	0.7498	1.5465
		q_3	0.4022	1.1257	0.4458	0.8604	1.8772
$\lambda = 1,\ b = 0.8$	0.25	q_1	0.2876	1.2633	0.1971	0.6128	0.8977
		q_2	0.2797	1.1920	0.2079	0.5743	1.0785
		q_3	0.4096	1.2102	0.3552	0.6429	1.3574
	0.5	q_1	0.2208	0.5265	0.1215	0.3704	0.9694
		q_2	0.1707	0.4879	0.2067	0.3557	1.0893
		q_3	0.2502	0.4826	0.3028	0.3844	1.2605

Table 3. Simulation results for the Makeham distribution.

			n				
			25		50		
parameters	α	point	B	MSE	B	MSE	real value
$\gamma = 1,\ k = 2, \delta = 2$	0.25	q_1	0.0900	1.7860	0.2149	0.8514	1.0335
		q_2	0.2926	1.7796	0.2496	0.8331	1.0815
		q_3	0.4271	1.7797	0.3316	0.8122	1.1655
	0.5	q_1	0.2138	0.6163	0.1623	0.4507	1.0222
		q_2	0.2046	0.6037	0.1728	0.4414	1.0539
		q_3	0.2615	0.5868	0.2326	0.4301	1.1087
$\gamma = 1,\ k = 1.5, \delta = 1$	0.25	q_1	0.0949	1.6261	0.2155	0.8202	1.0221
		q_2	0.3513	1.6210	0.2192	0.8073	1.0544
		q_3	0.4348	1.6171	0.3304	0.7895	1.1122
	0.5	q_1	0.1742	0.5219	0.1770	0.4655	1.0147
		q_2	0.1792	0.5148	0.2064	0.4583	1.0363
		q_3	0.2656	0.5046	0.2664	0.4479	1.0746

4.2. Strength *of glass fibers*

Smith and Naylor [18] studied an experimental data set concerning the strength of glass fibers with a length of 1.5 cm, as reported by the National Physical Laboratory in England. Table 4 presents the data.

Table 4. Strength of $1.5cm$ length glass fibers.

.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68
.73	1.81	2.00	0.74	1.04	1.27	1.39	1.49	1.53	1.59
.61	1.66	1.68	1.76	1.82	2.01	0.77	1.11	1.28	1.42
.50	1.54	1.60	1.62	1.66	1.69	1.76	1.84	2.24	0.81
.13	1.29	1.48	1.50	1.55	1.61	1.62	1.66	1.70	1.77
.84	0.84	1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67
.70	1.78	1.89							

The empirical FRR function was evaluated for three values of the scaling parameter, $\alpha = 0.35$, 0.50 and 0.65, and the corresponding plots are displayed in Figures 7 and 8. All of these plots exhibit an increasing pattern, which graphically suggests that the data conform to a model with an $IFRR$. Furthermore, the Total Time on Test (TTT) plot shown in the right panel of Figure 8 also indicates an increasing FR structure for the given data. The observation that the data follow an $IFRR$ model supports the conclusion that the tested glass fibers exhibit limited flexibility under stress. As a final remark, the empirical FRR methodology appears to be a useful tool for evaluating the strength and reliability characteristics of glass fibers or similar materials.

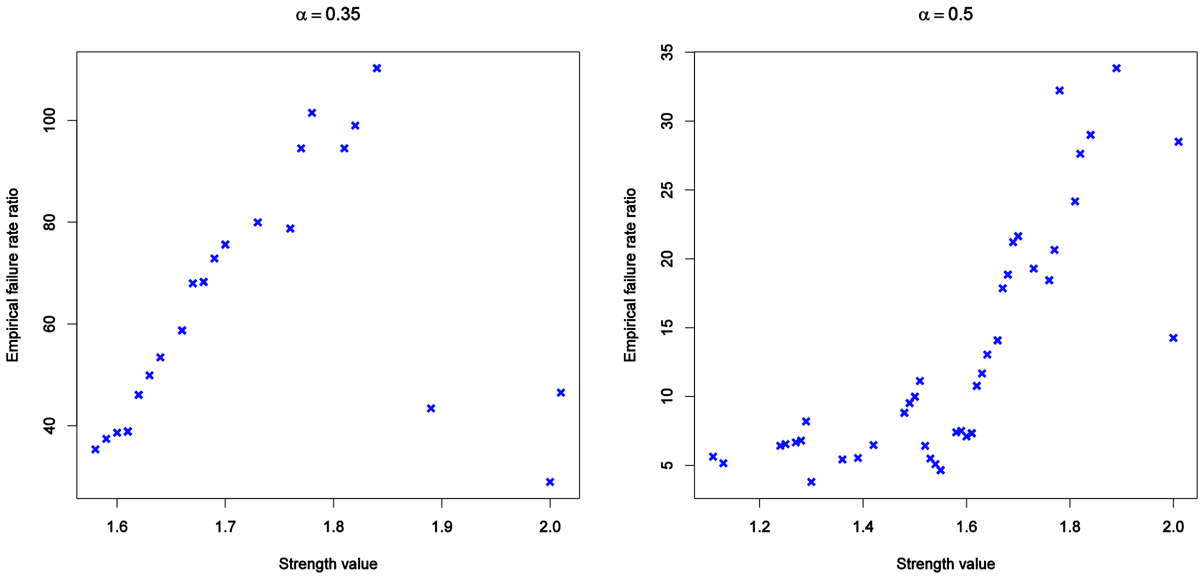

Figure 7. Empirical FRR functions for glass fibers data set that show increasing forms.

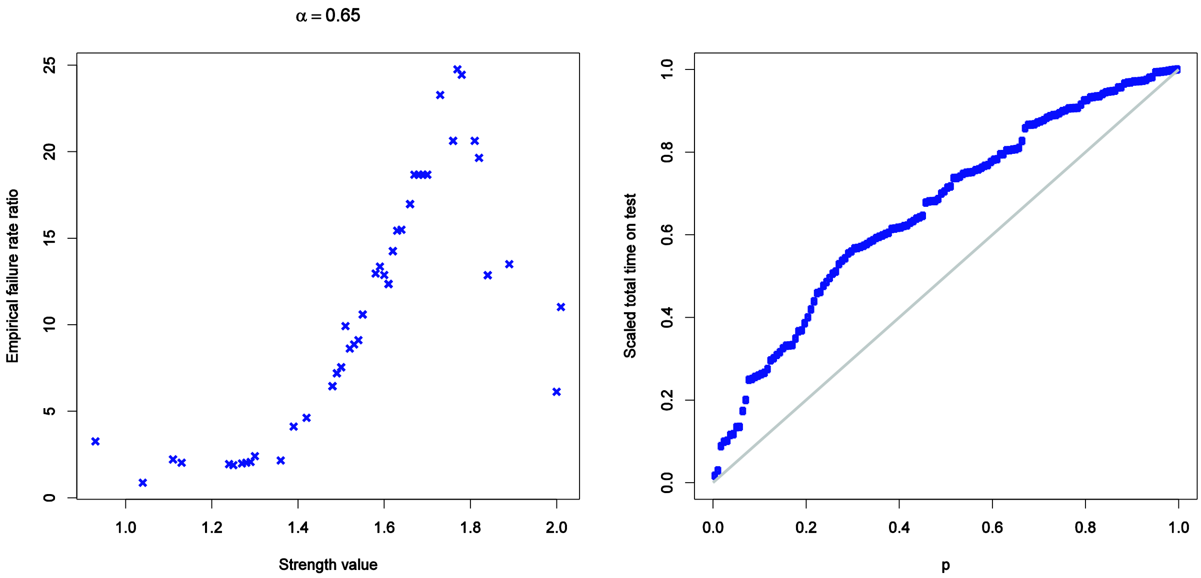

Figure 8. (left) Empirical FRR function of glass fiber data set. (Right) TTT plot indicates increasing FR function.

It is notable that the TTT plot (right panel) is a standard tool for detecting IFR behavior as employed by many authors in literature.

5. Conclusions

This study introduced the $IFRR$ and $DFRR$ classes as two novel nonparametric lifetime distribution classes that capture aging behavior through the relative dynamics of the FR function at different time points. By analyzing the monotonicity of the ratio $h_X(t)/h_X(\alpha t)$, where $\alpha \in (0,1)$, with respect to t, we established a unified framework for characterizing various forms of aging phenomena in reliability systems. We demonstrated that the preservation property of the $DFRR$ class under order statistics supports the practical relevance of these classes in modeling complex system reliability. This result is particularly significant since the lifetime of a coherent system with n components, operating under an $(n-k+1)$-out-of-n structure, corresponds to the k-th order statistic $X_{k:n}$. The behavior of the FR ratio, by comparing its value at a given time to earlier points, was thoroughly explored using empirical data. Our findings clarify that, for lifetime distributions possessing both IFR and $IFRR$ properties, the FR increases more rapidly over time. Conversely, for distributions exhibiting IFR and $DFRR$ properties, the FR continues to increase but at a decelerating pace. Furthermore, we established a strong theoretical link between the $IFRR/DFRR$ classes and the monotone classes defined via the AI function (see Proposition 2.3), confirming that the $IFRR$ and $DFRR$ classes are closely related to the IFR/A and DFR/A classes, respectively.

Table 1 summarized several well-known distributions belonging to the $IFRR$ and $DFRR$ classes. Proposition 2.4 provided necessary and sufficient conditions for a lifetime distribution to exhibit these properties. Additionally, Proposition 2.6 characterized conditions under which the $IFRR/DFRR$ property follows from the more classical IFR/DFR assumptions. The preservation result for the $DFRR$ class under order statistics, rigorously established in Theorem 3.2, further reinforces the robustness of these new aging classes in reliability modeling. By addressing both theoretical aspects and data-driven validation, this study contributes to the advancement of stochastic modeling in reliability engineering, especially in the nuanced analysis of aging and system performance variability.

Future research will focus on extending the applicability of the $IFRR$ and $DFRR$ classes to record values, including upper records, k-records, and lower records. Investigating whether these classes are preserved under such structures will further enhance their utility in modeling complex systems and reliability phenomena. We also investigate similar classes of lifetime distributions by using the mean residual life and the mean inactivity time functions instead of the FR function.

Author contributions

M. Kayid: Conceptualization, methodology, writing-original draft; M. Alanazi: writing-review and editing, software, validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are deeply thankful to four anonymous reviewers for their helpful comments and suggestions, which greatly strengthened the overall manuscript. This work is supported by Ongoing Research Funding program, (ORF-2025-392), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. Navarro, J. M. Sarabia, A note on the limiting behavior of hazard rate functions of generalized mixtures, *J. Comput. Appl. Math.*, **435** (2024), 114653. https://doi.org/10.1016/j.cam.2022.114653
2. C. D. Lai, M. Xie, *Stochastic ageing and dependence for reliability*, New York: Springer, 2006. https://doi.org/10.1007/0-387-34232-X
3. M. Finkelstein, *Failure rate modelling for reliability and risk*, Springer, New York, 2008. https://doi.org/10.1007/978-1-84800-986-8
4. J. Navarro, Aging properties, In: *Introduction to system reliability theory*, Cham: Springer, 2022.
5. R. Righter, M. Shaked, J. G. Shanthikumar, Intrinsic aging and classes of nonparametric distributions, *Probab. Eng. Inf. Sci.*, **23** (2009), 563–582. https://doi.org/10.1017/S0269964809990015
6. M. Kayid, M. A. Alshehri, Stochastic aspects of reversed aging intensity function of random quantiles, *J. Inequal. Appl.*, **2024** (2024), 119. https://doi.org/10.1186/s13660-024-03198-y
7. P. E. Oliveira, N. Torrado, On proportional reversed failure rate class, *Stat. Pap.*, **56** (2015), 999–1013. https://doi.org/10.1007/s00362-014-0620-8

8. M. Burkschat, Multivariate dependence of spacings of generalized order statistics, *J. Multivar. Anal.*, **100** (2009), 1093–1106. https://doi.org/10.1016/j.jmva.2008.10.008
9. M. Shaked, J. G. Shanthikumar, *Stochastic orders*, New York: Springer, 2007. https://doi.org/10.1007/978-0-387-34675-5
10. M. Kayid, R. A. Almohsen, Preservation of relative failure rate and relative reversed failure rate orders by distorted distributions, *Acta Appl. Math.*, **194** (2024), 1–15. https://doi.org/10.1007/s10440-024-00704-8
11. J. Navarro, Y. del Águila, M. A. Sordo, A. Suárez-Llorens, Preservation of reliability classes under the formation of coherent systems, *Appl. Stoch. Models Bus. Ind.*, **30** (2014), 444–454. https://doi.org/10.1002/asmb.1985
12. J. Navarro, Preservation of DMRL and IMRL aging classes under the formation of order statistics and coherent systems, *Stat. Probab. Lett.*, **137** (2018), 264–268. https://doi.org/10.1016/j.spl.2018.02.005
13. B. H. Lindqvist, F. J. Samaniego, Some new results on the preservation of the NBUE and NWUE aging classes under the formation of coherent systems, *Naval Res. Logist.*, **66** (2019), 430–438. https://doi.org/10.1002/nav.21849
14. F. G. Badía, J. H. Cha, H. Lee, C. Sangüesa, Preservation of the log concavity by Bernstein operator with an application to ageing properties of a coherent system, *J. Comput. Appl. Math.*, **443** (2024), 115748. https://doi.org/10.1016/j.cam.2023.115748
15. P. H. Kvam, H. Singh, L. R. Whitaker, Estimating distributions with increasing failure rate in an imperfect repair model, *Lifetime Data Anal.*, **8** (2002), 53–67. https://doi.org/10.1023/A:1013570832286
16. A. J. Fernández, Computing optimal confidence sets for Pareto models under progressive censoring, *J. Comput. Appl. Math.*, **258** (2014), 168–180. https://doi.org/10.1016/j.cam.2013.09.014
17. E. Parzen, On estimation of a probability density function and mode, *Ann. Math. Stat.*, **33** (1962), 1065–1076.
18. R. L. Smith, J. C. Naylor, A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, *J. R. Stat. Soc. Ser. C (Appl. Stat.)*, **36** (1987), 358–369. https://doi.org/10.2307/2347795

AIMS Press