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Research article

Stationary distribution, extinction and probability density function of a stochastic tuberculosis model with Ornstein-Uhlenbeck process

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Abstract: In order to investigate the impact of environmental disturbances on disease transmission, this paper established a stochastic tuberculosis model, where the infection rate satisfied the Ornstein-Uhlenbeck (OU) process. For the corresponding deterministic model, the endemic equilibrium and its stability were investigated. For the stochastic system under OU noise perturbation, we first established the existence and uniqueness of the global positive solution. Subsequently, properly constructed Lyapunov functions were used to deduce sufficient criteria to ensure the existence of stationary distribution and the eradication of the disease. When $R_0^s > 1$, the system has a stationary distribution, which means that the disease will persist. When $R_0^E < 1$, the disease becomes extinct. Furthermore, an analytical expression for the probability density in the vicinity of the endemic equilibrium was obtained by solving the five-dimensional Fokker-Planck equation. Finally, the accuracy of these theoretical conclusions was corroborated through numerical simulations.

Keywords: stochastic tuberculosis model; Ornstein-Uhlenbeck process; stationary distribution; extinction; probability density function
Mathematics Subject Classification: 34D05, 60H10, 92D30

1. Introduction

Mycobacterium tuberculosis causes tuberculosis (TB), a long-standing infectious disease that primarily affects the lungs (known as pulmonary tuberculosis) but may also involve other organs, such as lymph nodes, bones, kidneys, etc. The main mode of TB transmission is airborne spread. When TB patients cough, sneeze, or speak, they release tiny droplet nuclei (1–5 microns in diameter) containing bacteria. These nuclei can remain suspended in the air for extended periods and infect others when inhaled. According to the report of the World Health Organization [1]: In recent years, more than 10 million people have suffered from TB every year, and the number is rising year by year. In 2023, 1.25

million people died from TB—nearly twice the number of deaths caused by AIDS. Despite significant advancements in TB diagnostics and therapeutics driven by improved diagnosis/treatment levels and medical technology progress, the prevalence of drug-resistant TB and complex socio-economic factors continue to pose serious challenges to disease prevention and control [2]. Various mathematical frameworks have been developed in recent decades to present the spread and management of infectious diseases. These frameworks can serve as important tools to study transmission patterns and evaluate the effectiveness of intervention strategies (see [3–6]).

In the process of infectious disease transmission, it is inevitable to be affected by various environmental disturbances [7–9]. For example, the influenza virus survives and spreads more readily in low temperature and low humidity environments, while high temperature and humidity environments are conducive to the reproduction and survival of Vibrio cholerae. High temperature and rainy seasons are conducive to the growth and reproduction of Aedes albopictus, which will cause a high incidence of dengue fever. The spread of pulmonary TB is closely related to living conditions. In poorly ventilated and crowded living conditions, respiratory droplets containing TB bacteria exhaled by pulmonary TB patients stay in the air for a long time, which is more likely to be inhaled by healthy people, making them infected. When building epidemic models, a key challenge is accounting for the impact of unpredictable, randomly driven fluctuations. Through the drift diffusion framework, stochastic differential equations transform environmental disturbances into mathematically treatable noise terms, providing a more realistic dynamic description for the spread of infectious diseases. Introducing environmental disturbances into the deterministic model can be achieved by modifying model parameters. In current studies, there are mainly two ways to modify parameters. One way is to replace the parameter with a Gaussian white noise function [10, 11]. For example, Xin et al. [12] used this method to establish a stochastic TB model. They examined different noise intensity levels, finding that increased white noise promotes disease extinction. Another way is to assume that the parameter satisfies a mean-reverting stochastic process. Environmental factors such as temperature, humidity, and population mobility fluctuate, but these fluctuations follow certain patterns and are not completely random or unbounded, for example, temperature fluctuations caused by seasonal changes; although the daily temperatures vary, they generally fluctuate around the seasonal average. This kind of mean-reverting fluctuation characteristic is suitable to be described by the Ornstein-Uhlenbeck (OU) process [13–16].

Studies indicate that TB incidence demonstrates periodicity linked to seasonal changes and temperature shifts. Increased indoor activities during winter, coupled with cold weather induced immune suppression, contribute to higher TB incidence in colder months. However, as TB incidence is affected by complex interactions involving regional economic levels, healthcare access, population density and other factors, its seasonal patterns are not strictly cyclical. Therefore, we propose a stochastic TB model incorporating the OU process.

The structure of this paper is organized as follows: Section 2 presents the stochastic TB model with the OU process and analyzes the endemic equilibrium and its local asymptotic stability for the corresponding deterministic model. Section 3 establishes the existence and uniqueness of the global solution for the stochastic model. In Section 4, a critical threshold R_0^s for the existence of a stationary distribution is derived by constructing a series of Lyapunov functions. Section 5 provides a sufficient condition for the exponential extinction of the disease. In Section 6, the solution to the five-dimensional Fokker-Planck equation yields an analytical expression for the probability density function of the

system state variables. Finally, Section 7 employs numerical simulations to validate the theoretical results. Lastly, discussion and conclusion are drawn.

2. The model formulation

In 2010, Liu et al. [17] introduced a TB transmission model that incorporates standard incidence rate and considers the cyclical change of infection rate

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \dfrac{\beta(t)S(t)I(t)}{N(t)} - \mu S(t), \\ \dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = (1-q)\dfrac{\beta(t)S(t)I(t)}{N(t)} - (k(t)+\mu)E(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = q\dfrac{\beta(t)S(t)I(t)}{N(t)} + k(t)E(t) - (r+\sigma+\mu)I(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = rI(t) - \mu R(t), \\ N(t) = S(t) + E(t) + I(t) + R(t). \end{cases} \tag{2.1}$$

Table 1 displays the interpretations of all dynamic variables and parameters, where all parameters are positive constants and $q < 1$. Here, $\beta(t)$ and $k(t)$ denote the periodic infection rate and reactivation rate, respectively. Assuming $\beta(t) \equiv \bar{\beta}$ and $k(t) \equiv k$, model (2.1) reduces to the autonomous system

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \dfrac{\bar{\beta}S(t)I(t)}{N(t)} - \mu S(t), \\ \dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = (1-q)\dfrac{\bar{\beta}S(t)I(t)}{N(t)} - (k+\mu)E(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = q\dfrac{\bar{\beta}S(t)I(t)}{N(t)} + kE(t) - (r+\sigma+\mu)I(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = rI(t) - \mu R(t), \\ N(t) = S(t) + E(t) + I(t) + R(t). \end{cases} \tag{2.2}$$

If we replace the parameter $\bar{\beta}$ with a Gaussian white noise function, i.e.,

$$\beta(t) = \bar{\beta} + \theta\frac{\mathrm{d}B(t)}{\mathrm{d}t},$$

where $B(t)$ represents standard Brownian motion defined on a complete probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$, with filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions, and θ^2 represents the intensity of $B(t)$, then

$$\langle\beta(t)\rangle = \frac{1}{t}\int_0^t \beta(s)\mathrm{d}s = \bar{\beta} + \theta\frac{B(t)}{t} \sim N\left(\bar{\beta}, \frac{\theta^2}{t}\right),$$

where $\langle\beta(t)\rangle$ is the time average of $\beta(t)$. Obviously,

$$Var(\langle\beta(t)\rangle) \to \infty \text{ as } t \to 0,$$

which means $\langle\beta(t)\rangle$ changes dramatically as time tends to zero. This is unreasonable in a continuous biological environment.

Table 1. The interpretations of variables and parameters.

Variable or arameter	Interpretation
$S(t)$	The number of susceptible individuals at time t
$E(t)$	The number of latent individuals at time t
$I(t)$	The number of infectious individuals at time t
$R(t)$	The number of recovered individuals at time t
$N(t)$	The total population at time t
Λ	Population input rate
$\bar{\beta}$	The infection rate
μ	The natural death rate
q	The proportion of susceptible individuals direct to the infected
k	The rate of transfer from latent to infectious individuals
r	The rate of transfer from infectious to recovered individuals
σ	The disease-induced mortality rate

If $\bar{\beta}$ satisfies a mean-reverting stochastic process, i.e.,

$$\mathrm{d}(\beta(t)) = \alpha(\bar{\beta} - \beta(t))\mathrm{d}t + \theta \mathrm{d}B(t), \tag{2.3}$$

where α quantifies the reversion speed and θ denotes the fluctuation intensity, both of which are positive constants. $\bar{\beta}$ is the mean reversion level of $\beta(t)$. By solving (2.3), we obtain

$$\beta(t) = \bar{\beta} + (\beta(0) - \bar{\beta})e^{-\alpha t} + \theta \int_0^t e^{-\alpha(t-s)}\mathrm{d}B(s),$$

where $\beta(0)$ serves as the initial condition for $\beta(t)$. Since the stochastic integral term $\theta \int_0^t e^{-\alpha(t-s)}\ \mathrm{d}B(s)$ follows $N(0, \frac{\theta^2}{2\alpha}(1 - e^{-2\alpha t}))$($N(.,.)$ denotes the normal distribution), when $t \to +\infty$, we have

$$E[\beta(t)] = \bar{\beta} + (\beta(0) - \bar{\beta})e^{-\alpha t} \to \bar{\beta},$$

$$Var[\beta(t)] = \frac{\theta^2}{2\alpha}(1 - e^{-2\alpha t}) \to \frac{\theta^2}{2\alpha}.$$

Thus, for any initial $\beta(0)$, $\beta(t)$ asymptotically converges to $N(\bar{\beta}, \frac{\theta^2}{2\alpha})$ when $t \to +\infty$. Consequently, the invariant density of $\beta(t)$ at infinity is given by

$$\pi(x) = \frac{\sqrt{\alpha}}{\sqrt{\pi\theta^2}} e^{-\frac{\alpha(x-\bar{\beta})^2}{\theta^2}}.$$

Supposing $\beta(0) = \bar{\beta}$, we obtain

$$\langle\beta(t)\rangle = \frac{1}{t}\int_0^t \beta(s)\mathrm{d}s = \bar{\beta} + \frac{1}{t}\int_0^t \frac{\theta}{\alpha}(1 - e^{\alpha(s-t)})\mathrm{d}B(s) \sim N(\bar{\beta}, \frac{\theta^2 t}{3} + o(t^2)),$$

where $o(t^2)$ represents high order infinitesimal of t^2. Obviously,

$$Var(\langle\beta(t)\rangle) \to 0 \text{ as } t \to 0,$$

which is more reasonable in a continuous biological environment. Thus, in this article, we introduce OU process in the infection rate $\bar{\beta}$. Based on the biological significance of $\bar{\beta}$, $\bar{\beta}$ should be a positive constant. Given the normal distribution property of $\beta(t)$, it has a high probability of taking positive values. Therefore, we define $\beta^+(t) := max\{\beta(t), 0\}$ to make sure that $\beta^+(t)$ always stays nonnegative. We proposed the following stochastic system:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \dfrac{\beta^+(t)S(t)I(t)}{N(t)} - \mu S(t), \\ \dfrac{\mathrm{d}E(t)}{\mathrm{d}t} = (1-q)\dfrac{\beta^+(t)S(t)I(t)}{N(t)} - (k+\mu)E(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = q\dfrac{\beta^+(t)S(t)I(t)}{N(t)} + kE(t) - (r+\sigma+\mu)I(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = rI(t) - \mu R(t), \\ \mathrm{d}\beta(t) = \alpha(\bar{\beta} - \beta(t))\mathrm{d}t + \theta \mathrm{d}B(t). \end{cases} \tag{2.4}$$

According to literature [17], system (2.2) has a basic reproductive number

$$R_0 = \frac{\bar{\beta}(q\mu + k)}{(\mu + k)(r + \sigma + \mu)},$$

and when $R_0 < 1$, the disease-free equilibrium $P_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$ is globally asymptotically stable. Next, we discuss the endemic equilibrium and its stability of system (2.2).

First, assume

$$(H) : Q_1 > 0, Q_2 > 0, Q_3 > 0, Q_4 > 0, Q_1Q_2 - Q_3 > 0, Q_1Q_2Q_3 - Q_1^2Q_4 - Q_3^2 > 0,$$

where $Q_i (i = 1, 2, 3, 4)$ will be given in the proof of Theorem 1.

Theorem 1. *When $1 < R_0 < \frac{\bar{\beta}}{\sigma}$, the deterministic system* (2.2) *admits a unique endemic equilibrium $P^*(S^*, E^*, I^*, R^*)$. Furthermore, when conditions (H) holds, P^* is locally asymptotically stable.*

Proof. At the endemic equilibrium, we have

$$I^* \neq 0, \frac{\mathrm{d}S}{\mathrm{d}t}\mid_{P^*} = \frac{\mathrm{d}E}{\mathrm{d}t}\mid_{P^*} = \frac{\mathrm{d}I}{\mathrm{d}t}\mid_{P^*} = \frac{\mathrm{d}R}{\mathrm{d}t}\mid_{P^*} = 0,$$

then the total population dynamic $\frac{\mathrm{d}N}{\mathrm{d}t}\mid_{P^*} = \Lambda - \mu N^* - \sigma I^* = 0$, i.e.,

$$N^* = \frac{\Lambda - \sigma I^*}{\mu}. \tag{2.5}$$

By solving $\frac{\mathrm{d}S}{\mathrm{d}t}\mid_{P^*} = 0$, we get

$$\frac{\bar{\beta}S^*I^*}{N^*} = \Lambda - \mu S^*. \tag{2.6}$$

Substituting (2.6) into $\frac{\mathrm{d}E}{\mathrm{d}t}\mid_{P^*} = 0$, we obtain

$$(1-q)(\Lambda - \mu S^*) = (\mu + k)E^* \Rightarrow E^* = \frac{(1-q)(\Lambda - \mu S^*)}{\mu + k}. \tag{2.7}$$

Inserting (2.6) and (2.7) into $\frac{\mathrm{d}I}{\mathrm{d}t}\mid_{P^*}=0$, we derive

$$\begin{aligned} &q(\Lambda-\mu S^*)+k\frac{(1-q)(\Lambda-\mu S^*)}{\mu+k}=(r+\sigma+\mu)I^* \\ \Rightarrow &S^*=\frac{\Lambda}{\mu}-\frac{(\mu+k)(r+\sigma+\mu)I^*}{\mu(q\mu+k)}. \end{aligned} \tag{2.8}$$

From (2.5) and (2.6), we can get

$$\mu(\bar{\beta}-\sigma)S^*I^*+\Lambda\mu S^*+\Lambda\sigma I^*-\Lambda^2=0. \tag{2.9}$$

Substituting (2.8) into (2.9), we obtain

$$\begin{aligned} I^*=&\frac{\bar{\beta}\Lambda(q\mu+k)-\Lambda(\mu+k)(r+\sigma+\mu)}{(\bar{\beta}-\sigma)(\mu+k)(r+\sigma+\mu)} \\ =&\frac{\Lambda[\bar{\beta}(q\mu+k)-(\mu+k)(r+\sigma+\mu)]}{(\bar{\beta}-\sigma)(\mu+k)(r+\sigma+\mu)} \\ =&\frac{\Lambda(R_0-1)}{\bar{\beta}-\sigma}, \end{aligned}$$

then

$$S^*=\frac{\Lambda}{\mu}-\frac{(\mu+k)(r+\sigma+\mu)I^*}{\mu(q\mu+k)}, E^*=\frac{(1-q)(\Lambda-\mu S^*)}{\mu+k}, R^*=\frac{r}{\mu}I^*.$$

From condition $1<R_0<\frac{\bar{\beta}}{\sigma}$, system (2.2) has a unique endemic equilibrium P^*.

Next, we will discuss the stability of P^*. The Jacobian matrix of model (2.2) at point P^* is

$$J\mid_{P^*}=\begin{pmatrix} -a_{11} & a_{12} & -a_{13} & a_{14} \\ a_{21} & -a_{22} & a_{23} & -a_{24} \\ a_{31} & -a_{32} & -a_{33} & -a_{34} \\ 0 & 0 & a_{43} & -a_{44} \end{pmatrix}, \tag{2.10}$$

where

$$\begin{aligned} &a_{11}=\frac{\bar{\beta}I^*}{N^*}(1-\frac{S^*}{N^*})+\mu, a_{12}=\frac{\bar{\beta}S^*I^*}{(N^*)^2}, \\ &a_{13}=\frac{\bar{\beta}S^*}{N^*}(1-\frac{I^*}{N^*}), a_{14}=\frac{\bar{\beta}S^*I^*}{(N^*)^2}=a_{12}, \\ &a_{21}=(1-q)\frac{\bar{\beta}I^*}{N^*}(1-\frac{S^*}{N^*}), a_{22}=(1-q)\frac{\bar{\beta}S^*I^*}{(N^*)^2}+(k+\mu), \\ &a_{23}=(1-q)\frac{\bar{\beta}S^*}{N^*}(1-\frac{I^*}{N^*}), a_{24}=(1-q)\frac{\bar{\beta}S^*I^*}{(N^*)^2}, \\ &a_{31}=q\frac{\bar{\beta}I^*}{N^*}(1-\frac{S^*}{N^*}), a_{32}=q\frac{\bar{\beta}S^*I^*}{(N^*)^2}-k, a_{33}=(r+\sigma+\mu)-q\frac{\bar{\beta}S^*}{N^*}(1-\frac{I^*}{N^*}), \\ &a_{34}=q\frac{\bar{\beta}S^*I^*}{(N^*)^2}, a_{43}=r, a_{44}=\mu. \end{aligned}$$

The corresponding characteristic equation is

$$\begin{vmatrix} \lambda + a_{11} & -a_{12} & a_{13} & -a_{14} \\ -a_{21} & \lambda + a_{22} & -a_{23} & a_{24} \\ -a_{31} & a_{32} & \lambda + a_{33} & a_{34} \\ 0 & 0 & -a_{43} & \lambda + a_{44} \end{vmatrix} = \lambda^4 + Q_1\lambda^3 + Q_2\lambda^2 + Q_3\lambda + Q_4 = 0, \tag{2.11}$$

where

$$\begin{aligned} Q_1 =& a_{11} + a_{22} + a_{33} + a_{44}, \\ Q_2 =& (a_{11} + a_{22})(a_{33} + a_{44}) + a_{11}a_{22} + a_{33}a_{44} + a_{13}a_{31} + a_{23}a_{32} + a_{34}a_{43} - a_{12}a_{21}, \\ Q_3 =& a_{11}a_{22}a_{33} + a_{44}(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}) - a_{32}(a_{21}a_{13} + a_{24}a_{43}) \\ & - a_{31}(a_{12}a_{23} + a_{14}a_{43}) + a_{13}a_{31}(a_{22} + a_{44}) + a_{23}a_{32}(a_{11} + a_{44}) \\ & - a_{12}a_{21}(a_{33} + a_{44}) + a_{34}a_{43}(a_{11} + a_{22}), \\ Q_4 =& a_{44}[a_{13}(a_{22}a_{31} - a_{21}a_{32}) + a_{23}(a_{11}a_{32} - a_{12}a_{31}) + a_{33}(a_{11}a_{22} - a_{12}a_{21})] \\ & + a_{43}[a_{31}(a_{12}a_{24} - a_{14}a_{22}) + a_{32}(a_{14}a_{21} - a_{11}a_{24}) + a_{34}(a_{11}a_{22} - a_{12}a_{21})]. \end{aligned}$$

Based on Routh-Hurwitz criteria,when (H) holds, $P^*(S^*, E^*, I^*, R^*)$ is locally asymptotically stable. □

3. Existence and uniqueness of global positive solution for model (2.4)

From system (2.4), it is clearly seen that

$$\begin{aligned} \mathrm{d}N = \mathrm{d}(S + E + I + R) &= (\Lambda - \mu N - \sigma I)\mathrm{d}t \\ &\leq (\Lambda - \mu N)\mathrm{d}t. \end{aligned}$$

Thus, there is a positively invariant set of model (2.4).

$$\Omega^* = \left\{(S, E, I, R, \beta) \in R_+^4 \times R : S + E + I + R < \frac{\Lambda}{\mu}\right\}. \tag{3.1}$$

In addition, it is necessary to give Itŏ's formula for the following contents.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Define $\mathbb{R}_+^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, 1 \leq i \leq n\}$. Let $\mathbb{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$ denote the family of all real-valued functions $V(x, t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ such that they are continuously twice differentiable in x and once in t. Let $L^n(\mathbb{R}_+; \mathbb{R}^{n\times m})$ represent the family of all $n \times m$-matrix-valued measurable processes $f(t)$ such that

$$\int_0^T |f(t)|^n \, \mathrm{d}t \leq \infty \text{ a.s. for every } T \geq 0.$$

Lemma 1. *(Itŏ's formula) [18]. Let $x(t)$ be an n-dimensional Itŏ's process on $t \geq 0$ with the stochastic differential*

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}B(t),$$

where $B(t) = (B_1(t), B_2(t), \cdots, B_m(t))$, $f \in L^1(\mathbb{R}_+; \mathbb{R}^n)$, and $g \in L^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$. If $V \in \mathbb{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, then $V(x(t), t)$ is a real-valued Itŏ's process and its stochastic differential is given by

$$\begin{aligned} \mathrm{d}V(x(t),t) =&[V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}trace(g^T(t)V_{xx}(x(t),t)g(t))]\mathrm{d}t \\ &+ V_x(x(t),t)g(t)\mathrm{d}B(t) \\ =&\mathcal{L}V(x(t),t)\mathrm{d}t + V_x(x(t),t)g(t)\mathrm{d}B(t), a.s., \end{aligned}$$

in which $\mathcal{L}V(x(t),t) = V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}trace(g^T(t)V_{xx}(x(t),t)g(t))$ denotes the differential operator, and

$$V_t = \frac{\partial V}{\partial t}, V_x = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}), V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j}).$$

Theorem 2. *For any given initial value* $(S(0), E(0), I(0), R(0), \beta(0)) \in \Omega^*$*, system* (2.4) *admits a unique positive solution that is global and lies almost surely within* Ω^*.

Proof. The coefficients of system (2.4) satisfy the local Lipschitz condition. Consequently, for arbitrary initial data in Ω^*, there exists a unique global positive solution $(S(t), E(t), I(t), R(t), \beta(t))$ defined over $t \in [0, \tau_e)$ that remains almost surely in Ω^*, where τ_e refers to the explosion time. So if we want to prove Theorem 2, we only need to prove that almost everywhere $\tau_e = \infty$.

Assuming $n_0 \geq 1$ and sufficiently large to include initial values $(S(0), E(0), I(0), R(0), \beta(0))$ in $\left[\frac{1}{n_0}, n_0\right]$, for any integer $n \geq n_0$, the stopping time is defined as:

$$\tau_n = \inf\{t \in [0, \tau_e) : \min\{S(t), E(t), I(t), R(t), \beta(t)\} \leq \frac{1}{n} \text{ or } \max\{S(t), E(t), I(t), R(t), \beta(t)\} \geq n\}.$$

Obviously, as $n \to \infty$, τ_n increases. If $\tau_\infty = \lim_{n\to\infty} \tau_n$, then $\tau_\infty \leq \tau_e$ is almost everywhere. So if we want to prove that almost everywhere $\tau_e = \infty$, we only need to prove that $\tau_\infty = \infty$ is almost everywhere.

Next, we use the method of contradiction to prove that $\tau_\infty = \infty$. Assuming $\tau_\infty \neq \infty$, there exist constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon,$$

therefore, there exists a positive integer $n_1 \geq n_0$ satisfying

$$P\{\tau_n \leq T\} \geq \varepsilon \text{ for all } n \geq n_1.$$

Construct

$$\begin{aligned} W =&S - 1 - \ln S + E - 1 - \ln E + I - 1 - \ln I + R - 1 - \ln R \\ &+ (\frac{\Lambda}{\mu} - S - E - I - R) - 1 - \ln(\frac{\Lambda}{\mu} - S - E - I - R) + \frac{\beta^2(t)}{2}. \end{aligned}$$

Applying Itŏ's formula,we have

$$
\begin{aligned}
\mathcal{L}W =& -\frac{1}{S}(\Lambda - \frac{\beta^+(t)SI}{N} - \mu S) - \frac{1}{E}\left[(1-q)\frac{\beta^+(t)SI}{N} - (k+\mu)E\right] \\
& - \frac{1}{I}\left[q\frac{\beta^+(t)SI}{N} + kE - (r+\sigma+\mu)I\right] - \frac{1}{R}(rI - \mu R) \\
& + \frac{\Lambda - \mu N - \sigma I}{\frac{\Lambda}{\mu} - N} + \alpha\beta(t)(\bar{\beta} - \beta(t)) + \frac{1}{2}\theta^2 \\
=& -\frac{\Lambda}{S} - (1-q)\frac{\beta^+(t)SI}{EN} - \frac{q\beta^+(t)S}{N} - \frac{kE}{I} - \frac{rI}{R} + \frac{\beta^+(t)I}{N} + 4\mu \\
& + k + \sigma + r + \frac{\Lambda - \mu N - \sigma I}{\frac{\Lambda}{\mu} - N} + \alpha\beta(t)(\bar{\beta} - \beta(t)) + \frac{1}{2}\theta^2 \\
\leq& 5\mu + k + \sigma + r + \frac{1}{2}\theta^2 + \beta^+(t) + \alpha\beta(t)(\bar{\beta} - \beta(t)) \\
\leq& 5\mu + k + \sigma + r + \frac{1}{2}\theta^2 + (\alpha\bar{\beta} + 1)\,|\beta(t)| - \alpha\,|\beta(t)|^2 \\
\leq& 5\mu + k + \sigma + r + \frac{1}{2}\theta^2 + \sup_{\beta(t)\in R}\left\{(\alpha\bar{\beta} + 1)\,|\beta(t)| - \alpha\,|\beta(t)|^2\right\} \\
:=& \tilde{C},
\end{aligned}
\tag{3.2}
$$

where $\tilde{C}$ is a positive constant independent of $S(t), E(t), I(t), R(t)$, and $\beta(t)$. Integrating and taking expectation on both sides of (3.2), we have

$$
\begin{aligned}
0 \leq & E[W(S(\tau_n \wedge T), E(\tau_n \wedge T), I(\tau_n \wedge T), R(\tau_n \wedge T), \beta(\tau_n \wedge T))] \\
= & E[W(S(0), E(0), I(0), R(0), \beta(0))] + E[\int_0^{\tau_n \wedge T} \mathcal{L}W(S(\tau), E(\tau), I(\tau), R(\tau), \beta(\tau))d\tau] \\
\leq & E[W(S(0), E(0), I(0), R(0), \beta(0))] + \tilde{C}T.
\end{aligned}
$$

Let $F_n = \{\tau_n \leq T\}$, where $n \geq n_1$, so $P(F_n) \geq \varepsilon$. For any $\xi \in F_n$, we can know

$$
W(S(\tau_n, \xi), E(\tau_n, \xi), I(\tau_n, \xi), R(\tau_n, \xi), \beta(\tau_n, \xi)) \geq (n - 1 - \ln n) \wedge (\frac{1}{n} - 1 - \ln\frac{1}{n}) \wedge \frac{n^2}{2}.
$$

So,

$$
\begin{aligned}
& E[W(S(0), E(0), I(0), R(0), \beta(0))] + \tilde{C}T \\
\geq & E[W(S(\tau_n \wedge T), E(\tau_n \wedge T), I(\tau_n \wedge T), R(\tau_n \wedge T), \beta(\tau_n \wedge T))] \\
\geq & E[\mathbb{I}_{F_n(\xi)} W(S(\tau_n \wedge T), E(\tau_n \wedge T), I(\tau_n \wedge T), R(\tau_n \wedge T), \beta(\tau_n \wedge T))] \\
\geq & P(F_n(\xi))W(S(\tau_n, \xi), E(\tau_n, \xi), I(\tau_n, \xi), R(\tau_n, \xi), \beta(\tau_n, \xi)) \\
\geq & \varepsilon[(n - 1 - \ln n) \wedge (\frac{1}{n} - 1 - \ln\frac{1}{n}) \wedge \frac{n^2}{2}].
\end{aligned}
$$

Letting $n \to +\infty$, there is

$$
+\infty \leq E[W(S(0), E(0), I(0), R(0), \beta(0))] + \tilde{C}T < +\infty.
$$

This is a contradiction, so $\tau_\infty = \infty$ holds. □

4. Existence of stationary distribution of system (2.4)

This section aims to construct the sufficient condition guaranteeing the existence of a stationary distribution. The stationary distribution, representing stochastic weak stability of the system, implies that the disease will persist long-term with persistent fluctuations under environmental noise.

Define

$$R_0^s = \frac{\beta_1 (1-q) k + (\mu + k)\beta_2 q}{(\mu + k)\left(\mu + r + \sigma + (c_1 + d_1)\frac{\theta}{\sqrt{\pi\alpha}}\right)},$$

where

$$\beta_1 = \left(\int_0^{+\infty} x^{\frac{1}{4}}\pi(x)\,\mathrm{d}x\right)^4, \beta_2 = \left(\int_0^{+\infty} x^{\frac{1}{3}}\pi(x)\,\mathrm{d}x\right)^3, c_1 = \frac{\beta_1(1-q)k}{\mu(\mu+k)}, d_1 = \frac{\beta_2 q}{\mu}.$$

In $\mathbb{R}^d$, a homogeneous Markov process is described by the stochastic differential equation

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}B(t), \tag{4.1}$$

with the initial value $x(0) \in \mathbb{R}^d$. Furthermore, $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable. The following lemma is related to the existence of stationary distribution.

Lemma 2. *[13] Assume that there is a bounded closed region $\mathbb{D} \subset \mathbb{R}^d$ with a regular boundary Γ, for initial value $x(0) \in \mathbb{R}^d$, if*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}(s, x(s), \mathbb{D})\mathrm{d}s > 0, a.s.,$$

where $\mathbb{P}(s, x(s), .)$ denotes the transfer probability of $x(s)$. Then, there exists a solution of system (4.1) which has the Feller property, and system (4.1) admits at least one stationary distribution on $\mathbb{R}^d$.

Theorem 3. *When $R_0^s > 1$, system (2.4) has a stationary distribution on Ω^*.*

Proof. The proof process is divided into three steps.

Step 1: Construct a nonnegative function.

Let

$$V_1 = -\ln I - (c_1 + d_1)\ln S + (c_2 + d_2)(S + E + I + R) - c_3 \ln E,$$

where c_1,c_2,c_3,d_1, and d_2 will be rigorously determined in the following analytical procedures. Through application of the Itŏ formula, it follows that

$$
\begin{aligned}
\mathcal{L}V_1 =& -q\frac{\beta^+(t)S}{N} - \frac{kE}{I} + r + \sigma + \mu - (c_1+d_1)\frac{\Lambda}{S} + (c_1+d_1)\frac{\beta^+(t)I}{N} + (c_1+d_1)\mu \\
&+ (c_2+d_2)\Lambda - (c_2+d_2)\mu N - (c_2+d_2)\sigma I - c_3(1-q)\frac{\beta^+(t)SI}{EN} + c_3(k+\mu) \\
\leq& -(\frac{kE}{I} + c_1\frac{\Lambda}{S} + c_2\mu N + c_3(1-q)\frac{\beta^+(t)SI}{EN}) - (q\frac{\beta^+(t)S}{N} + d_1\frac{\Lambda}{S} + d_2\mu N) \\
&+ c_1\mu + c_2\Lambda + c_3(k+\mu) + d_1\mu + d_2\Lambda + r + \sigma + \mu + (c_1+d_1)\frac{\beta^+(t)I}{N} \\
\leq& -4\sqrt[4]{c_1c_2c_3\beta^+(t)(1-q)\mu k\Lambda} - 3\sqrt[3]{d_1d_2\beta^+(t)q\mu\Lambda} \\
&+ c_1\mu + c_2\Lambda + c_3(k+\mu) + d_1\mu + d_2\Lambda + r + \sigma + \mu + (c_1+d_1)\frac{\beta^+(t)I}{N} \\
=& -4\sqrt[4]{c_1c_2c_3\beta_1(1-q)\mu k\Lambda} - 3\sqrt[3]{d_1d_2\beta_2q\mu\Lambda} \\
&+ c_1\mu + c_2\Lambda + c_3(k+\mu) + d_1\mu + d_2\Lambda + r + \sigma + \mu + (c_1+d_1)\frac{\beta^+(t)I}{N} \\
&+ 4(\sqrt[4]{c_1c_2c_3\beta_1(1-q)\mu k\Lambda} - \sqrt[4]{c_1c_2c_3\beta^+(t)(1-q)\mu k\Lambda}) \\
&+ 3(\sqrt[3]{d_1d_2\beta_2q\mu\Lambda} - \sqrt[3]{d_1d_2\beta^+(t)q\mu\Lambda}),
\end{aligned}
$$

where $\beta_1 = \left(\int_0^{+\infty} x^{\frac{1}{4}}\pi(x)\,\mathrm{d}x\right)^4, \beta_2 = \left(\int_0^{+\infty} x^{\frac{1}{3}}\pi(x)\,\mathrm{d}x\right)^3$. Let c_1, c_2, and c_3 satisfy the following equalities:

$$c_1\mu = c_2\Lambda = c_3(k+\mu) = \frac{\beta_1(1-q)k}{k+\mu},$$

and d_1,d_2 satisfy

$$d_1\mu = d_2\Lambda = \beta_2 q,$$

then

$$
\begin{aligned}
\mathcal{L}V_1 \leq& -\frac{\beta_1(1-q)k}{\mu+k} - \beta_2 q + r + \sigma + \mu + (c_1+d_1)\frac{\beta^+(t)I}{N} + h_1(\beta) + h_2(\beta) \\
=& -\frac{\beta_1(1-q)k}{\mu+k} - \beta_2 q + r + \sigma + \mu + (c_1+d_1)\frac{\bar{\beta}I}{N} + h_1(\beta) + h_2(\beta) + (c_1+d_1)\frac{I}{N}(\beta^+(t) - \bar{\beta}) \\
\leq& -\frac{\beta_1(1-q)k}{\mu+k} - \beta_2 q + r + \sigma + \mu + (c_1+d_1)\frac{\theta}{\sqrt{\pi\alpha}} + (c_1+d_1)\frac{\bar{\beta}I}{N} \\
&+ h_1(\beta) + h_2(\beta) + (c_1+d_1)\left|\beta(t) - \bar{\beta}\right| - (c_1+d_1)\frac{\theta}{\sqrt{\pi\alpha}} \\
=& -(\mu + r + \sigma + (c_1+d_1)\frac{\theta}{\sqrt{\pi\alpha}})(R_0^s - 1) + (c_1+d_1)\frac{\bar{\beta}I}{N} \\
&+ h_1(\beta) + h_2(\beta) + (c_1+d_1)\left|\beta(t) - \bar{\beta}\right| - (c_1+d_1)\frac{\theta}{\sqrt{\pi\alpha}} \\
=& -(\mu + r + \sigma + (c_1+d_1)\frac{\theta}{\sqrt{\pi\alpha}})(R_0^s - 1) + (c_1+d_1)\frac{\bar{\beta}I}{N} + F(\beta(t)),
\end{aligned}
$$

where

$$h_1(\beta) = 4(\sqrt[4]{c_1c_2c_3\beta_1(1-q)\mu k\Lambda} - \sqrt[4]{c_1c_2c_3\beta^+(t)(1-q)\mu k\Lambda}),$$
$$h_2(\beta) = 3(\sqrt[3]{d_1d_2\beta_2 q\mu\Lambda} - \sqrt[3]{d_1d_2\beta^+(t)q\mu\Lambda}),$$
$$F(\beta(t)) = h_1(\beta) + h_2(\beta) + (c_1+d_1)\left|\beta(t)-\bar{\beta}\right| - (c_1+d_1)\frac{\theta}{\sqrt{\pi\alpha}}.$$

Next, we define

$$V_2 = -\ln S - \ln I - \ln R + I,$$
$$V_3 = -\ln(\frac{\Lambda}{\mu} - S - E - I - R),$$
$$V_4 = \frac{\beta^2(t)}{2}.$$

Applying Itŏ's formula yields

$$\begin{aligned}
\mathcal{L}V_2 =& -\frac{1}{S}(\Lambda - \frac{\beta^+(t)SI}{N} - \mu S) - \frac{1}{I}[q\frac{\beta^+(t)SI}{N} + kE - (r+\sigma+\mu)I] - \frac{1}{R}(rI - \mu R)\\
&+ q\frac{\beta^+(t)SI}{N} + kE - (r+\sigma+\mu)I\\
=& -\frac{\Lambda}{S} + \frac{\beta^+(t)I}{N} + \mu - q\frac{\beta^+(t)S}{N} - \frac{kE}{I} + r + \sigma + \mu - \frac{rI}{R} + \mu\\
&+ q\frac{\beta^+(t)SI}{N} + kE - (r+\sigma+\mu)I\\
\leq& -\frac{\Lambda}{S} + \beta^+(t) + \mu - q\frac{\bar{\beta}S}{N} + q|\bar{\beta}-\beta(t)| - \frac{kE}{I} + r + \sigma + \mu - \frac{rI}{R} + \mu + \frac{q\Lambda}{\mu}\beta^+(t) + kE\\
\leq& -\frac{\Lambda}{S} + (1+\frac{q\Lambda}{\mu})|\beta(t)| + \mu + q|\bar{\beta}-\beta(t)| - \frac{kE}{I} + r + \sigma + \mu - \frac{rI}{R} + \mu + kE,
\end{aligned}$$

$$\mathcal{L}V_3 = \frac{\Lambda - \mu N - \sigma I}{\frac{\Lambda}{\mu} - N} = \mu - \frac{\sigma I}{\frac{\Lambda}{\mu} - N},$$
$$\mathcal{L}V_4 = \alpha\beta(t)(\bar{\beta}-\beta(t)) + \frac{1}{2}\theta^2.$$

Denote

$$\overline{V} = MV_1 + V_2 + V_3 + V_4,$$

and

$$B = \sup_{\beta\in R}\left\{(\alpha\bar{\beta} + 1 + \frac{q\Lambda}{\mu} + q)\,|\beta(t)| - \frac{1}{2}\alpha\,|\beta(t)|^2 + q\bar{\beta} + 4\mu + r + \sigma + \frac{1}{2}\theta^2\right\},$$

where M is sufficiently large to satisfy

$$-M(\mu + r + \sigma + (c_1+d_1)\frac{\theta}{\sqrt{\pi\alpha}})(R_0^s - 1) + B \leq -2.$$

Since $\overline{V}(S,E,I,R,\beta(t)) \to +\infty$ as $(S,E,I,R,\beta(t))$ approaches the boundary of Ω^*, we deduce that $\overline{V}(S,E,I,R,\beta(t))$ has a minimal value $\overline{V}_{min}$ in Ω^*. Therefore, we introduce the nonnegative function

$$V(S,E,I,R,\beta(t)) = \overline{V}(S,E,I,R,\beta(t)) - \overline{V}_{min}.$$

Reapplying Itŏ's formula gives

$$\begin{aligned}
\mathcal{L}V \leq & -M(\mu+r+\sigma+(c_1+d_1)\frac{\theta}{\sqrt{\pi\alpha}})(R_0^s-1)+M(c_1+d_1)\frac{\bar{\beta}I}{N}+MF(\beta(t)) \\
& -\frac{\Lambda}{S}(1+\frac{q\Lambda}{\mu})|\beta(t)|+\mu+q|\bar{\beta}-\beta(t)|-\frac{kE}{I}+r+\sigma \\
& +\mu-\frac{rI}{R}+\mu+kE+\mu-\frac{\sigma I}{\frac{\Lambda}{\mu}-N}+\alpha\beta(t)(\bar{\beta}-\beta(t))+\frac{1}{2}\theta^2 \\
\leq & -M(\mu+r+\sigma+(c_1+d_1)\frac{\theta}{\sqrt{\pi\alpha}})(R_0^s-1)+(\alpha\bar{\beta}+1+\frac{q\Lambda}{\mu}+q)\,|\beta(t)| \\
& -\frac{1}{2}\alpha\,|\beta(t)|^2+q\bar{\beta}+4\mu+r+\sigma+\frac{1}{2}\theta^2+M(c_1+d_1)\frac{\bar{\beta}I}{N}+kE \\
& -\frac{\Lambda}{S}-\frac{kE}{I}-\frac{rI}{R}-\frac{\sigma I}{\frac{\Lambda}{\mu}-N}-\frac{1}{2}\alpha\,|\beta(t)|^2+MF(\beta(t)) \\
:= & G(S,E,I,R,\beta(t))+MF(\beta(t)).
\end{aligned} \tag{4.2}$$

Step 2: Construct a compact set.
A compact set D_ε is defined as:

$$D_\varepsilon=\left\{(S,E,I,R,\beta(t))\in\Omega^*\mid S\geq\varepsilon,E\geq\varepsilon,I\geq\varepsilon^2,R\geq\varepsilon^3,0<N\leq\frac{\Lambda}{\mu}-\varepsilon^3,|\beta(t)|\leq\frac{1}{\varepsilon}\right\},$$

where ε is a sufficiently small positive constant satisfying the following conditions:

$$\begin{aligned}
& -2+M(c_1+d_1)\bar{\beta}\varepsilon+k\varepsilon\leq-1, \\
& -2+M(c_1+d_1)\bar{\beta}+\frac{k\Lambda}{\mu}-\frac{\Lambda}{\varepsilon}\leq-1, \\
& -2+M(c_1+d_1)\bar{\beta}+\frac{k\Lambda}{\mu}-\frac{k}{\varepsilon}\leq-1, \\
& -2+M(c_1+d_1)\bar{\beta}+\frac{k\Lambda}{\mu}-\frac{r}{\varepsilon}\leq-1, \\
& -2+M(c_1+d_1)\bar{\beta}+\frac{k\Lambda}{\mu}-\frac{\sigma}{\varepsilon}\leq-1, \\
& -2+M(c_1+d_1)\bar{\beta}+\frac{k\Lambda}{\mu}-\frac{\alpha}{2\varepsilon^2}\leq-1.
\end{aligned} \tag{4.3}$$

The complement D_ε^c partitions into six subsets

$$
\begin{aligned}
D_{1,\varepsilon}^{c} &= \left\{(S,E,I,R,\beta(t)) \in \Omega^{*} | I < \varepsilon^{2}, S \geq \varepsilon, E < \varepsilon\right\},\\
D_{2,\varepsilon}^{c} &= \left\{(S,E,I,R,\beta(t)) \in \Omega^{*} | S < \varepsilon\right\},\\
D_{3,\varepsilon}^{c} &= \left\{(S,E,I,R,\beta(t)) \in \Omega^{*} | E \geq \varepsilon, I < \varepsilon^{2}\right\},\\
D_{4,\varepsilon}^{c} &= \left\{(S,E,I,R,\beta(t)) \in \Omega^{*} | I \geq \varepsilon^{2}, R < \varepsilon^{3}\right\},\\
D_{5,\varepsilon}^{c} &= \left\{(S,E,I,R,\beta(t)) \in \Omega^{*} | I \geq \varepsilon^{2}, N > \frac{\Lambda}{\mu} - \varepsilon^{3}\right\},\\
D_{6,\varepsilon}^{c} &= \left\{(S,E,I,R,\beta(t)) \in \Omega^{*} | |\beta(t)| > \frac{1}{\varepsilon}\right\}.
\end{aligned}
$$

From inequalities (4.3), we have the following result.
Case 1: In subset $D_{1,\varepsilon}^{c}$,

$$G(S,E,I,R,\beta(t)) \leq -2 + M(c_1 + d_1)\frac{\bar{\beta} I}{S} + kE \leq -2 + M(c_1 + d_1)\bar{\beta}\varepsilon + k\varepsilon \leq -1.$$

Case 2: In subset $D_{2,\varepsilon}^{c}$,

$$G(S,E,I,R,\beta(t)) \leq -2 + M(c_1 + d_1)\bar{\beta} + \frac{k\Lambda}{\mu} - \frac{\Lambda}{\varepsilon} \leq -1.$$

Case 3: In subset $D_{3,\varepsilon}^{c}$,

$$G(S,E,I,R,\beta(t)) \leq -2 + M(c_1 + d_1)\bar{\beta} + \frac{k\Lambda}{\mu} - \frac{k}{\varepsilon} \leq -1.$$

Case 4: In subset $D_{4,\varepsilon}^{c}$,

$$G(S,E,I,R,\beta(t)) \leq -2 + M(c_1 + d_1)\bar{\beta} + \frac{k\Lambda}{\mu} - \frac{r}{\varepsilon} \leq -1.$$

Case 5: In subset $D_{5,\varepsilon}^{c}$,

$$G(S,E,I,R,\beta(t)) \leq -2 + M(c_1 + d_1)\bar{\beta} + \frac{k\Lambda}{\mu} - \frac{\sigma}{\varepsilon} \leq -1.$$

Case 6: In subset $D_{6,\varepsilon}^{c}$,

$$G(S,E,I,R,\beta(t)) \leq -2 + M(c_1 + d_1)\bar{\beta} + \frac{k\Lambda}{\mu} - \frac{\alpha}{2\varepsilon^{2}} \leq -1.$$

Analysis of these six cases implies that for sufficiently small ε, the following holds:

$$G(S,E,I,R,\beta(t)) \leq -1, \forall (S,E,I,R,\beta(t)) \in \Omega^{*} \setminus D_{\varepsilon}.$$

Moreover, there exists a positive constant H such that

$$G(S,E,I,R,\beta(t)) \leq H < +\infty, \forall (S,E,I,R,\beta(t)) \in \Omega^{*}. \tag{4.4}$$

Step 3: Existence of stationary distribution.
For any initial value $(S(0), E(0), I(0), R(0), \beta(0)) \in \Omega^*$, integrating (4.2) from 0 to t, dividing by t, and taking the expectation yields

$$\begin{aligned}
0 \leq & \frac{E\left[V\left(S(t), E(t), I(t), R(t), \beta(t)\right)\right]}{t} \\
= & \frac{E\left[V\left(S(0), E(0), I(0), R(0), \beta(0)\right)\right]}{t} + \frac{1}{t}\int_0^t E\left[\mathcal{L}V\left(S(\tau), E(\tau), I(\tau), R(\tau), \beta(\tau)\right)\right]\mathrm{d}\tau \\
\leq & \frac{E\left[V\left(S(0), E(0), I(0), R(0), \beta(0)\right)\right]}{t} + \frac{1}{t}\int_0^t E\left[G\left(S(\tau), E(\tau), I(\tau), R(\tau), \beta(\tau)\right)\right]\mathrm{d}\tau \\
& + 4M\sqrt[4]{c_1c_2c_3(1-q)\mu k\Lambda}\frac{1}{t}\int_0^t E[\sqrt[4]{\beta_1} - \sqrt[4]{\beta^+(\tau)}]\mathrm{d}\tau \\
& + 3M\sqrt[3]{d_1d_2q\mu\Lambda}\frac{1}{t}\int_0^t E[\sqrt[3]{\beta_2} - \sqrt[3]{\beta^+(\tau)}]\mathrm{d}\tau + M(c_1+d_1)\frac{1}{t}\int_0^t E[|\beta(\tau) - \bar{\beta}| - \frac{\theta}{\sqrt{\pi\alpha}}]\mathrm{d}\tau.
\end{aligned} \tag{4.5}$$

According to [19–21], it is clear to know that $\beta(t)$ has ergodic property, so one has

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \left|\beta(s) - \bar{\beta}\right|\mathrm{d}s = \int_{-\infty}^{+\infty}\left|x - \bar{\beta}\right|\pi(x)\mathrm{d}x = \frac{\theta}{\sqrt{\pi\alpha}}, \tag{4.6}$$

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \sqrt[4]{\beta^+(s)}\mathrm{d}s = \int_{-\infty}^{+\infty}\sqrt[4]{\max\{0,x\}}\pi(x)\mathrm{d}x = \int_0^{+\infty}\sqrt[4]{x}\pi(x)\mathrm{d}x = \sqrt[4]{\beta_1}, \tag{4.7}$$

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \sqrt[3]{\beta^+(s)}\mathrm{d}s = \int_{-\infty}^{+\infty}\sqrt[3]{\max\{0,x\}}\pi(x)\mathrm{d}x = \int_0^{+\infty}\sqrt[3]{x}\pi(x)\mathrm{d}x = \sqrt[3]{\beta_2}. \tag{4.8}$$

Taking the limit inferior of (4.5) and applying (4.6), (4.7), and (4.8) yields

$$\begin{aligned}
0 \leq & \liminf_{t\to+\infty}\frac{1}{t}\int_0^t E[G(S(\tau), E(\tau), I(\tau), R(\tau), \beta(\tau))]\mathrm{d}\tau \\
= & \liminf_{t\to+\infty}\frac{1}{t}\int_0^t E[G(S(\tau), E(\tau), I(\tau), R(\tau), \beta(\tau))\mathbb{I}_{\{(S(\tau),E(\tau),I(\tau),R(\tau),\beta(\tau))\in D_\varepsilon\}}]\mathrm{d}\tau \\
& + \liminf_{t\to+\infty}\frac{1}{t}\int_0^t E[G(S(\tau), E(\tau), I(\tau), R(\tau), \beta(\tau))\mathbb{I}_{\{(S(\tau),E(\tau),I(\tau),R(\tau),\beta(\tau))\in\Omega^*\setminus D_\varepsilon\}}]\mathrm{d}\tau \\
\leq & H\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{I}_{\{(S(\tau),E(\tau),I(\tau),R(\tau),\beta(\tau))\in D_\varepsilon\}}\mathrm{d}\tau \\
& - \liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{I}_{\{(S(\tau),E(\tau),I(\tau),R(\tau),\beta(\tau))\in\Omega^*\setminus D_\varepsilon\}}\mathrm{d}\tau \\
= & -1 + (H+1)\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{I}_{\{(S(\tau),E(\tau),I(\tau),R(\tau),\beta(\tau))\in D_\varepsilon\}}\mathrm{d}\tau,
\end{aligned}$$

where $\mathbb{I}$ denotes the indicator function. Consequently,

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{I}_{\{(S(\tau),E(\tau),I(\tau),R(\tau),\beta(\tau))\in D_\varepsilon\}}\mathrm{d}\tau \geq \frac{1}{H+1} > 0 \ a.s..$$

Let $P(t, S(t), E(t), I(t), R(t), \beta(t), D_\varepsilon)$ represent the transition probability of $(S(t), E(t), I(t), R(t), \beta(t))$. Through rigorous analysis, we establish

$$\liminf_{t\to+\infty} \frac{1}{t} \int_0^t P(\tau, S(\tau), E(\tau), I(\tau), R(\tau), \beta(\tau), D_\varepsilon) \mathrm{d}\tau \geq \frac{1}{H+1} > 0.$$

By Lemma 2, system (2.4) has a stationary distribution on Ω^*, which concludes the proof. □

5. Extinction of system (2.4)

Define

$$R_0^E = R_0 + \frac{(q\mu + k)\theta}{(\mu + k)(\mu + \sigma + r)\sqrt{\pi\alpha}}.$$

Theorem 4. *Let $(S(t), E(t), I(t), R(t), \beta(t))$ denote the solution to system (2.4) with arbitrary initial conditions. If $R_0^E < 1$, then*

$$\limsup_{t\to+\infty} \frac{\ln(kE + (\mu + k)I)}{t} \leq (r + \sigma + \mu)(R_0^E - 1) < 0, a.s.,$$

implying that the disease becomes extinct with probability one.

Proof. Define

$$Q(E, I) = kE + (\mu + k)I.$$

From the above formulation, we derive

$$k\frac{E}{Q} + (\mu + k)\frac{I}{Q} = 1,$$

implying $\frac{I}{Q} < \frac{1}{\mu+k}$. Consequently,

$$\begin{aligned}
\mathcal{L}(\ln Q) =& \frac{1}{Q}[k(1-q)\frac{\beta^+(t)SI}{N} - k(\mu + k)E + (\mu + k)q\frac{\beta^+(t)SI}{N} \\
&+ (\mu + k)kE - (\mu + k)(r + \sigma + \mu)I] \\
=& \frac{1}{Q}[k\frac{\beta^+(t)SI}{N} + \mu q\frac{\beta^+(t)SI}{N} - (\mu + k)(r + \sigma + \mu)I] \\
\leq& \frac{I}{Q}[(k + \mu q)\beta^+(t) - (\mu + k)(r + \sigma + \mu)] \\
=& \frac{I}{Q}[(k + \mu q)\bar{\beta} - (\mu + k)(r + \sigma + \mu)] + \frac{I}{Q}(k + \mu q)(\beta^+(t) - \bar{\beta}) \\
\leq& \frac{1}{\mu + k}[(k + \mu q)\bar{\beta} - (\mu + k)(r + \sigma + \mu)] + \frac{k + \mu q}{\mu + k}\left|\beta(t) - \bar{\beta}\right| \\
=& (r + \sigma + \mu)(R_0 - 1) + \frac{k + \mu q}{\mu + k}\left|\beta(t) - \bar{\beta}\right|.
\end{aligned} \tag{5.1}$$

Integrating Eq (5.1) yields

$$\frac{\ln Q(t) - \ln Q(0)}{t} \leq (r + \sigma + \mu)(R_0 - 1) + \frac{k + \mu q}{\mu + k}\frac{1}{t}\int_0^t \left|\beta(s) - \bar{\beta}\right| \mathrm{d}s. \tag{5.2}$$

Combining with (4.6), we obtain

$$\begin{aligned}\limsup_{t\to+\infty}\frac{\ln Q(t)}{t} \leq & (r+\sigma+\mu)(R_0-1)+\frac{k+\mu q}{\mu+k}\frac{\theta}{\sqrt{\pi\alpha}}\\ = & (r+\sigma+\mu)(R_0^E-1).\end{aligned}$$

We can obtain the conclusion that $\limsup_{t\to+\infty}\frac{\ln(kE+(\mu+k)I)}{t} \leq (r+\sigma+\mu)(R_0^E-1) < 0, a.s.$ when $R_0^E < 1$. Thus,

$$\lim_{t\to+\infty} E(t) = \lim_{t\to+\infty} I(t) = 0, a.s..$$

System (2.4) shows that if $\lim_{t\to+\infty} I(t) = 0$ almost surely, then $\lim_{t\to+\infty} R(t) = 0$ almost surely. Therefore, the disease dies out exponentially with probability one. □

6. Probability density of stationary distribution

Theorem 3 shows that if $R_0^s > 1$, then the system (2.4) has a stationary distribution. How to get the probability density function of the stationary distribution is the problem to be solved in this section.

From systems (2.2) and (2.4), it is not difficult to find that the corresponding deterministic system of system (2.4) has a unique positive equilibrium $(S^*, E^*, I^*, R^*, \bar{\beta})$, and the Jacobian matrix at this point is

$$A = \begin{pmatrix} -a_{11} & a_{12} & -a_{13} & a_{14} & -a_{15} \\ a_{21} & -a_{22} & a_{23} & -a_{24} & a_{25} \\ a_{31} & -a_{32} & -a_{33} & -a_{34} & a_{35} \\ 0 & 0 & a_{43} & -a_{44} & 0 \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix},$$

where $a_{ij}(i = 1, 2, 3, 4; j = 1, 2, 3, 4)$ are identical to those in matrix J of Eq (2.10) and $a_{15} = \frac{S^*I^*}{N^*}, a_{25} = (1-q)\frac{S^*I^*}{N^*}, a_{35} = q\frac{S^*I^*}{N^*}, a_{55} = \alpha$.

Lemma 3. *If condition (H) holds, then the Jacobian matrix A is a Hurwitz matrix.*

Proof. The characteristic polynomial of matrix A is $\varphi_A(\lambda)$, i.e.,

$$\begin{aligned}\varphi_A(\lambda) &= \begin{vmatrix} \lambda+a_{11} & -a_{12} & a_{13} & -a_{14} & a_{15} \\ -a_{21} & \lambda+a_{22} & -a_{23} & a_{24} & -a_{25} \\ -a_{31} & a_{32} & \lambda+a_{33} & a_{34} & -a_{35} \\ 0 & 0 & -a_{43} & \lambda+a_{44} & 0 \\ 0 & 0 & 0 & 0 & \lambda+a_{55} \end{vmatrix} \\ &= (\lambda+a_{55})\begin{vmatrix} \lambda+a_{11} & -a_{12} & a_{13} & -a_{14} \\ -a_{21} & \lambda+a_{22} & -a_{23} & a_{24} \\ -a_{31} & a_{32} & \lambda+a_{33} & a_{34} \\ 0 & 0 & -a_{43} & \lambda+a_{44} \end{vmatrix}.\end{aligned}$$

From Eq (2.11), we know

$$\begin{vmatrix} \lambda + a_{11} & -a_{12} & a_{13} & -a_{14} \\ -a_{21} & \lambda + a_{22} & -a_{23} & a_{24} \\ -a_{31} & a_{32} & \lambda + a_{33} & a_{34} \\ 0 & 0 & -a_{43} & \lambda + a_{44} \end{vmatrix} = \lambda^4 + Q_1\lambda^3 + Q_2\lambda^2 + Q_3\lambda + Q_4.$$

So,

$$\begin{aligned} \varphi_A(\lambda) &= (\lambda + a_{55})(\lambda^4 + Q_1\lambda^3 + Q_2\lambda^2 + Q_3\lambda + Q_4) \\ &= \lambda^5 + p_1\lambda^4 + p_2\lambda^3 + p_3\lambda^2 + p_4\lambda + p_5, \end{aligned}$$

where $p_1 = Q_1 + a_{55}$, $p_2 = Q_2 + Q_1a_{55}$, $p_3 = Q_3 + Q_2a_{55}$, $p_4 = Q_4 + Q_3a_{55}$, $p_5 = Q_4a_{55}$. The eigenvalues of A can be obtained by $\lambda_1 = -a_{55}$ and $\lambda^4 + Q_1\lambda^3 + Q_2\lambda^2 + Q_3\lambda + Q_4 = 0$, so, when condition (H) holds, all roots of the characteristic equation have negative real parts. A is a Hurwitz matrix. □

By defining $Z(t) = (Z_1(t), Z_2(t), Z_3(t), Z_4(t), Z_5(t))^T = (S(t)-S^*, E(t)-E^*, I(t)-I^*, R(t)-R^*, \beta(t)-\bar{\beta})^T$, the linearization of system (2.4) can be expressed as

$$\begin{cases} \mathrm{d}Z_1 = [-a_{11}Z_1 + a_{12}Z_2 - a_{13}Z_3 + a_{14}Z_4 - a_{15}Z_5]\mathrm{d}t, \\ \mathrm{d}Z_2 = [a_{21}Z_1 - a_{22}Z_2 + a_{23}Z_3 - a_{24}Z_4 + a_{25}Z_5]\mathrm{d}t, \\ \mathrm{d}Z_3 = [a_{31}Z_1 - a_{32}Z_2 - a_{33}Z_3 - a_{34}Z_4 + a_{35}Z_5]\mathrm{d}t, \\ \mathrm{d}Z_4 = [a_{43}Z_3 - a_{44}Z_4]\mathrm{d}t, \\ \mathrm{d}Z_5 = -a_{55}Z_5\mathrm{d}t + \theta\mathrm{d}B(t), \end{cases} \tag{6.1}$$

i.e.,

$$\begin{pmatrix} \mathrm{d}Z_1 \\ \mathrm{d}Z_2 \\ \mathrm{d}Z_3 \\ \mathrm{d}Z_4 \\ \mathrm{d}Z_5 \end{pmatrix} = \begin{pmatrix} -a_{11} & a_{12} & -a_{13} & a_{14} & -a_{15} \\ a_{21} & -a_{22} & a_{23} & -a_{24} & a_{25} \\ a_{31} & -a_{32} & -a_{33} & -a_{34} & a_{35} \\ 0 & 0 & a_{43} & -a_{44} & 0 \\ 0 & 0 & 0 & 0 & -a_{55} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathrm{d}B(t) \end{pmatrix}.$$

Let

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta \end{pmatrix}, \mathbf{B}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ B(t). \end{pmatrix}.$$

Then, Eq (6.1) can be written as:

$$\mathrm{d}Z(t) = AZ(t)\mathrm{d}t + G\mathrm{d}\mathbf{B}(t). \tag{6.2}$$

According to [22], system (6.2) has a unique probability density function $\Phi(z_1, z_2, z_3, z_4, z_5)$ determined by the following Fokker-Planck equation near $(0, 0, 0, 0, 0)$

$$\frac{\partial\Phi(z(t), t)}{\partial t} + \frac{\partial}{\partial z(t)}[Az(t)\Phi(z(t), t)] - \frac{\theta^2}{2}\frac{\partial^2\Phi(z(t), t)}{\partial z_5^2} = 0.$$

According to [13], when the diffusion matrix G is a constant matrix, $\Phi(z_1, z_2, z_3, z_4, z_5)$ can be represented as a Gaussian distribution

$$\Phi(z_1, z_2, z_3, z_4, z_5) = ce^{-\frac{1}{2}(z_1,z_2,z_3,z_4,z_5)P(z_1,z_2,z_3,z_4,z_5)^T},$$

where c is a constant that satisfies $\int_{R^5} \Phi(z_1, z_2, z_3, z_4, z_5)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4\mathrm{d}z_5 = 1$, and P is a symmetric matrix that satisfies $PG^2P + A^TP + PA = 0$. If P is positive definite and $P^{-1} = \Sigma$, we can obtain

$$G^2 + A\Sigma + \Sigma A^T = 0. \tag{6.3}$$

In order to solve the above matrix equation, we first give the following lemma [23].

Lemma 4. *Consider the algebraic equation*

$$G_0^2 + D\Sigma_2 + \Sigma_2 D^T = 0,$$

where $G_0 = diag(1, 0, 0, 0, 0)$, Σ_2 is a real symmetric matrix, and

$$D = \begin{pmatrix} -p_1 & -p_2 & -p_3 & -p_4 & -p_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

If the following conditions hold

$$\begin{aligned} &\Delta_1 = p_1 > 0, \Delta_2 = p_1p_2 - p_3 > 0, \\ &\Delta_3 = p_1(p_2p_3 - p_1p_4) - (p_3^2 - p_1p_5) > 0, \\ &\Delta_4 = (p_1p_2 - p_3)(p_3p_4 - p_2p_5) - (p_1p_4 - p_5)^2 > 0, \\ &\Delta_5 = p_5\Delta_4 > 0, p_3 > 0, p_4 > 0, \end{aligned}$$

then Σ_2 is a positive definite matrix given by

$$\Sigma_2 = \begin{pmatrix} \theta_{11} & 0 & -\theta_{22} & 0 & \theta_{33} \\ 0 & \theta_{22} & 0 & -\theta_{33} & 0 \\ -\theta_{22} & 0 & \theta_{33} & 0 & -\theta_{44} \\ 0 & -\theta_{33} & 0 & \theta_{44} & 0 \\ \theta_{33} & 0 & -\theta_{44} & 0 & \theta_{55} \end{pmatrix}, \tag{6.4}$$

where

$$\begin{aligned} &\theta_{11} = \frac{p_2(p_3p_4 - p_2p_5) - p_4(p_1p_4 - p_5)}{2\Delta_4}, \theta_{22} = \frac{p_3p_4 - p_2p_5}{2\Delta_4}, \\ &\theta_{33} = \frac{p_1p_4 - p_5}{2\Delta_4}, \theta_{44} = \frac{\Delta_2}{2\Delta_4}, \theta_{55} = \frac{\Delta_3}{2\Delta_5}. \end{aligned}$$

Next, our work is to get the probability density function $\Phi(z_1, z_2, z_3, z_4, z_5)$ by solving Eq (6.3), which is extremely difficult to solve directly. We use the similarity transformation as the core processing technology, and change the original equation into a mathematically equivalent but easy to solve form.

Theorem 5. *When $R_0^s > 1$ and condition (H) hold, the solution to system (6.2) follows a normal probability density distribution $\Phi(z_1, z_2, z_3, z_4, z_5)$ near $(0, 0, 0, 0, 0)$. This distribution is explicitly defined as:*

$$\Phi(z_1, z_2, z_3, z_4, z_5) = (2\pi)^{-\frac{5}{2}} |\Sigma|^{-\frac{1}{2}} \exp[-\frac{1}{2}(z_1, z_2, z_3, z_4, z_5)\Sigma^{-1}(z_1, z_2, z_3, z_4, z_5)^T],$$

where Σ is a positive definite matrix and the explicit expression of Σ is given in the subsequent proof.

Proof. First, we define the following matrix to perform similarity transformation on A:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1-q & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & q & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, J_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{a_{42}^{'}}{b_{32}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$J_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{qa_{43}}{b_{32}} & 0 & 1 \end{pmatrix}, J_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{a_{43}(a_{42}^{'} + qb_{33} + qa_{44})}{b_{43}b_{32}} & 1 \end{pmatrix},$$

where

$$\begin{aligned} a_{42}^{'} =& a_{31} + (1-q)a_{32} - qa_{11} - q(1-q)a_{12} + qa_{13} + qa_{33}, \\ b_{22} =& -a_{11} - (1-q)a_{12} + qa_{13}, b_{32} = (1-q)k, \\ b_{23} =& a_{12} - \frac{a_{13}a_{42}^{'}}{b_{32}} - \frac{qa_{14}a_{43}}{b_{32}}, b_{33} = -(k+\mu), b_{43} = \frac{k-r-\sigma)a_{42}^{'}}{b_{32}} + k, \\ b_{24} =& \frac{a_{14}a_{43}(a_{42}^{'} + qb_{33} + qa_{44})}{b_{43}b_{32}} - a_{13}, b_{44} = -(r+\sigma+\mu), \\ b_{54} =& a_{43} - \frac{a_{43}(a_{42}^{'} + qb_{33} + qa_{44})(b_{44} + a_{44})}{b_{43}b_{32}}. \end{aligned}$$

Let $J = J_6J_5J_4J_3J_2J_1$, then

$$JAJ^{-1} = \begin{pmatrix} -a_{55} & 0 & 0 & 0 & 0 \\ -a_{15} & b_{22} & b_{23} & b_{24} & a_{14} \\ 0 & b_{32} & b_{33} & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} & 0 \\ 0 & 0 & 0 & b_{54} & -a_{44} \end{pmatrix} \triangleq B.$$

Thus, Eq (6.3) can be written as:

$$JG^2J^T + (JAJ^{-1})(J\Sigma J^T) + (J\Sigma J^T)(JAJ^{-1})^T = 0. \tag{6.5}$$

From the form of J_6, J_5, J_4, J_3, J_2, and J_1, we can get that $JG^2J^T = diag(\theta^2, 0, 0, 0, 0) := G_1^2$. Let $J\Sigma J^T = \Sigma_1$, therefore, Eq (6.5) can be expressed as:

$$G_1^2 + B\Sigma_1 + \Sigma_1 B^T = 0.$$

Following the method in [24], the standardized transformation matrix M is

$$M = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 & m_5 \\ 0 & b_{54}b_{43}b_{32} & b_{54}b_{43}(b_{33}+b_{44}-a_{44}) & b_{54}(b_{44}^2 - a_{44}b_{44} + a_{44}^2) & -a_{44}^3 \\ 0 & 0 & b_{54}b_{43} & b_{54}(b_{44}-a_{44}) & a_{44}^2 \\ 0 & 0 & 0 & b_{54} & -a_{44} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} m_1 =& -a_{15}b_{54}b_{43}b_{32}, m_2 = b_{54}b_{43}b_{32}b_{22} + b_{54}b_{43}b_{33}b_{32} + b_{54}b_{44}b_{43}b_{32} - b_{54}a_{44}b_{43}b_{32}, \\ m_3 =& b_{54}b_{43}b_{32}b_{23} + b_{54}b_{43}b_{33}^2 + b_{54}b_{44}b_{43}b_{33} - b_{54}a_{44}b_{43}b_{33} + b_{54}b_{44}^2b_{43} - b_{54}a_{44}b_{44}b_{43} + a_{44}^2b_{54}b_{43}, \\ m_4 =& b_{54}b_{44}^3 - b_{54}a_{44}b_{44}^2 + a_{44}^2b_{54}b_{44} - a_{44}^3b_{54}, m_5 = a_{44}^4. \end{aligned}$$

Then, we obtain

$$MBM^{-1} = \begin{pmatrix} -p_1 & -p_2 & -p_3 & -p_4 & -p_5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \triangleq D,$$

where

$$p_1 = Q_1 + a_{55}, p_2 = Q_2 + Q_1a_{55}, p_3 = Q_3 + Q_2a_{55}, p_4 = Q_4 + Q_3a_{55}, p_5 = Q_4a_{55}.$$

Since matrix A and matrix MDM^{-1} are similar, according to the Routh-Hurwitz criterion, we know that

$$\begin{aligned} &\Delta_1 = p_1 > 0, \Delta_2 = p_1p_2 - p_3 > 0, \Delta_3 = p_1(p_2p_3 - p_1p_4) - (p_3^2 - p_1p_5) > 0, \\ &\Delta_4 = (p_1p_2 - p_3)(p_3p_4 - p_2p_5) - (p_1p_4 - p_5)^2 > 0, \Delta_5 = p_5\Delta_4 > 0, p_3 > 0, p_4 > 0. \end{aligned}$$

Furthermore, $MG_1^2M^T = diag(\theta^2p_1^2, 0, 0, 0, 0)$. Let

$$\rho = \theta p_1, \Sigma_2 = \rho^{-2}M\Sigma_1M^T \text{ and } G_0 = diag(1, 0, 0, 0, 0),$$

then

$$MG_1^2M^T + (MBM^{-1})(M\Sigma_1M^T) + (M\Sigma_1M^T)(MBM^{-1})^T = 0$$

can be expressed by

$$G_0^2 + D\Sigma_2 + \Sigma_2D^T = 0.$$

By Lemma 4, the form of Σ_2 can be given as

$$\Sigma_2 = \begin{pmatrix} \theta_{11} & 0 & -\theta_{22} & 0 & \theta_{33} \\ 0 & \theta_{22} & 0 & -\theta_{33} & 0 \\ -\theta_{22} & 0 & \theta_{33} & 0 & -\theta_{44} \\ 0 & -\theta_{33} & 0 & \theta_{44} & 0 \\ \theta_{33} & 0 & -\theta_{44} & 0 & \theta_{55} \end{pmatrix},$$

where $\theta_{11} = \frac{p_2(p_3p_4-p_2p_5)-p_4(p_1p_4-p_5)}{2\Delta_4}, \theta_{22} = \frac{p_3p_4-p_2p_5}{2\Delta_4}, \theta_{33} = \frac{p_1p_4-p_5}{2\Delta_4}, \theta_{44} = \frac{\Delta_2}{2\Delta_4}, \theta_{55} = \frac{\Delta_3}{2\Delta_5}$.

From

$$\Sigma_1 = \rho^2 M^{-1}\Sigma_2(M^{-1})^T \text{ and } \Sigma_1 = J\Sigma J^T,$$

we derive the explicit expression for Σ, a positive defined matrix

$$\Sigma = \rho^2(MJ)^{-1}\Sigma_2[(MJ)^{-1}]^T.$$

Thus, the probability density function near $(0, 0, 0, 0, 0)$ is

$$\Phi(z_1, z_2, z_3, z_4, z_5) = (2\pi)^{-\frac{5}{2}} |\Sigma|^{-\frac{1}{2}} \exp[-\frac{1}{2}(z_1, z_2, z_3, z_4, z_5)\Sigma^{-1}(z_1, z_2, z_3, z_4, z_5)^T].$$

□

From Theorem 5, it is not difficult to get the following.

Corollary 1. *When $R_0^s > 1$ and condition (H) hold, the solution to system (2.4) follows a normal probability density distribution $\Phi(S, E, I, R, \beta(t))$ near $(S^*, E^*, I^*, R^*, \bar{\beta})$. This distribution is explicitly defined as:*

$$\begin{aligned} \Phi(S, E, I, R, \beta(t)) =& (2\pi)^{-\frac{5}{2}} |\Sigma|^{-\frac{1}{2}} \exp \\ & [-\frac{1}{2}(S - S^*, E - E^*, I - I^*, R - R^*, \beta(t) - \bar{\beta})\Sigma^{-1} \\ & (S - S^*, E - E^*, I - I^*, R - R^*, \beta(t) - \bar{\beta})^T]. \end{aligned}$$

7. Numerical simulations

To validate the theoretical findings, we conduct numerical simulations using the Milstein's higher order method described in reference [25]. Let $x(t) = \beta(t) - \bar{\beta}$ and discretize system (2.4) as

$$\begin{cases} S^{j+1} = S^j + [\Lambda - \dfrac{x^j S^j I^j}{S^j + E^j + I^j + R^j} - \dfrac{\bar{\beta} S^j I^j}{S^j + E^j + I^j + R^j} - \mu S^j]\triangle t, \\ E^{j+1} = E^j + [(1-q)\dfrac{x^j S^j I^j}{S^j + E^j + I^j + R^j} + (1-q)\dfrac{\bar{\beta} S^j I^j}{S^j + E^j + I^j + R^j} - (k+\mu)E^j]\triangle t, \\ I^{j+1} = I^j + [q\dfrac{x^j S^j I^j}{S^j + E^j + I^j + R^j} + q\dfrac{\bar{\beta} S^j I^j}{S^j + E^j + I^j + R^j} + kE^j - (r+\sigma+\mu)I^j]\triangle t, \\ R^{j+1} = R^j + [rI^j - \mu R^j]\triangle t, \\ x^{j+1} = x^j - \alpha x^j \triangle t + \theta\sqrt{\triangle t}\xi_j + \dfrac{\theta^2}{2}(\xi_j^2 - 1)\triangle t. \end{cases} \tag{7.1}$$

where $\triangle t > 0$ denotes the time interval, $\xi_j (j = 1, 2, \ldots, n)$ are random variables that follow the normal distribution, and $(S^j, E^j, I^j, R^j, x^j)^T$ is the jth iteration value of the differential system (7.1).

The initial conditions are set to $(S(0), E(0), I(0), R(0), \beta(0)) = (0.3, 0.25, 0.2, 0.1, 0.6)$, and other parameters are selected as follows:

$$\Lambda = 0.09, \mu = 0.1, q = 0.2, \sigma = 0.001, r = 0.1, k = 0.1.$$

Example 7.1. Select $\bar{\beta} = 0.2, \alpha = 2, \theta = 0.2$. By calculation we can get

$$R_0 = 0.5970 < 1 \text{ and } R_0^E = 0.8352 < 1.$$

At this time, the condition of Theorem 4 holds, so the disease will be extinct. The simulation results are shown in Figure 1. By maintaining $\bar{\beta} = 0.2, \alpha = 2$, but varying the fluctuation intensity θ for two values of θ (i.e., $\theta = 0.1, \theta = 0.01$). Figures 2 and 3 show the dynamics of $S(t), E(t), I(t), R(t)$. We can observe that as θ approaches 0, the extinction behaviors of the deterministic model and the stochastic model are almost identical. This is consistent with our theoretical derivation, that is, $R_0^E \to R_0$ as $\theta \to 0$.

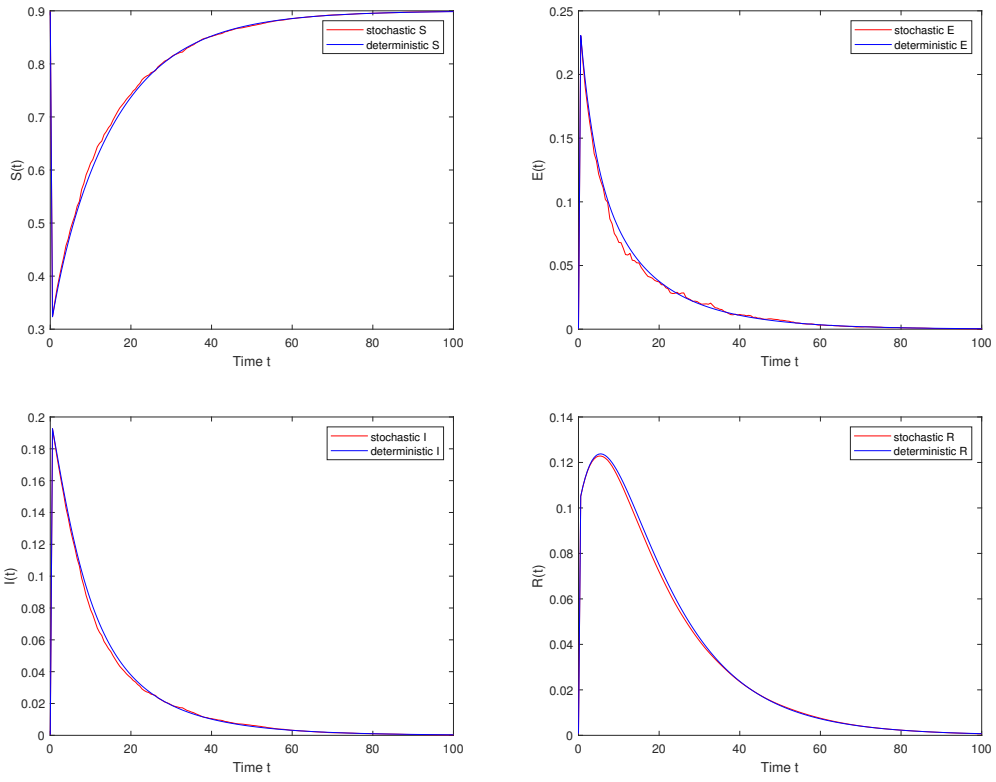

Figure 1. The solution trajectories of stochastic and deterministic systems under the parameter configurations $\bar{\beta} = 0.2, \alpha = 2$, and $\theta = 0.2$.

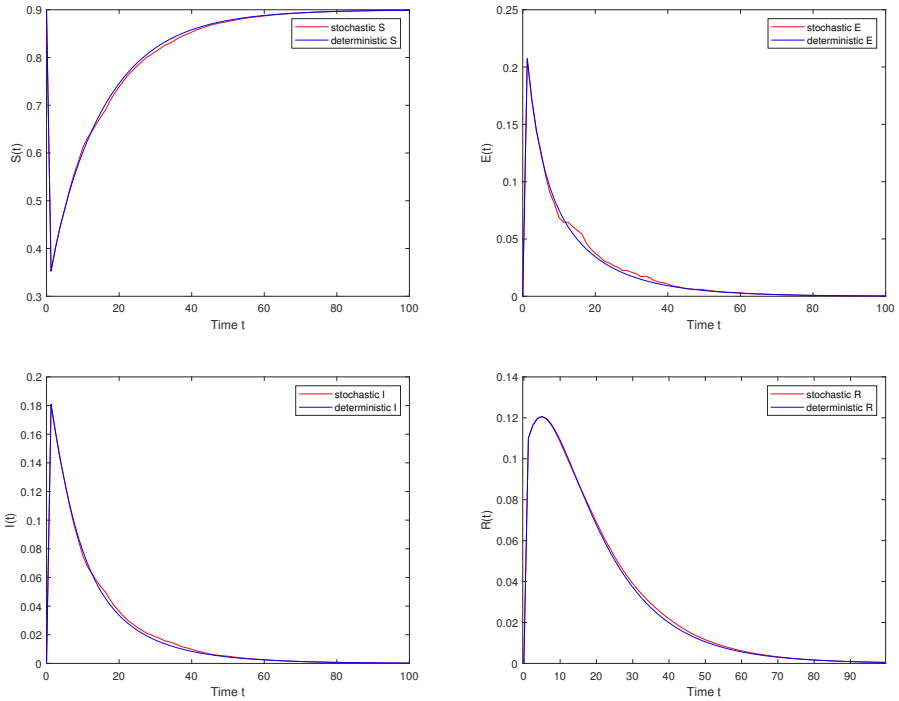

Figure 2. The solution trajectories of stochastic and deterministic systems under the parameter configurations $\bar{\beta} = 0.2, \alpha = 2$, and $\theta = 0.1$.

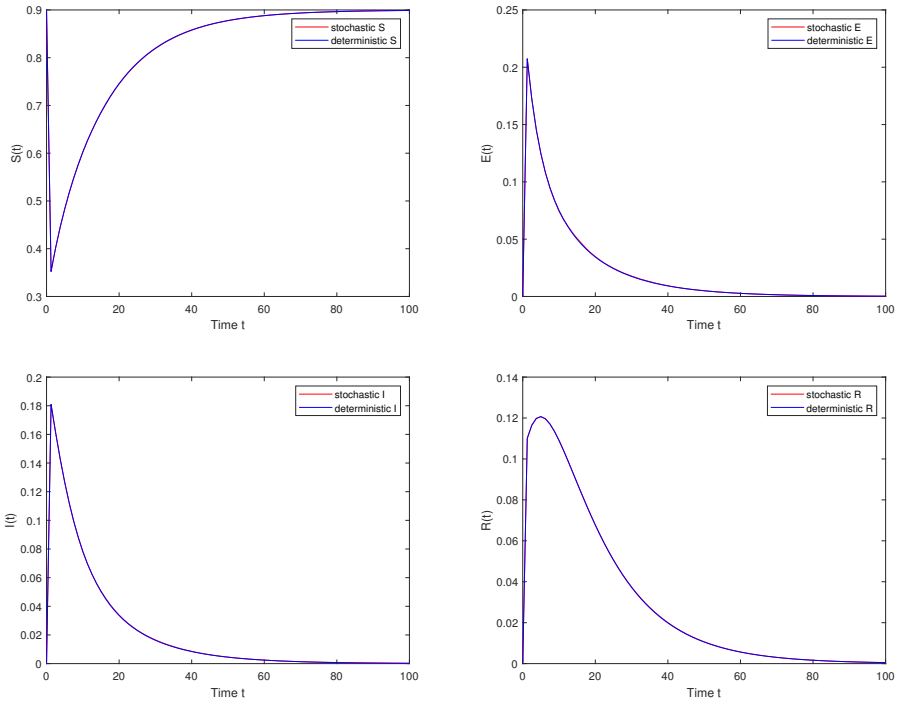

Figure 3. The solution trajectories of stochastic and deterministic systems under the parameter configurations $\bar{\beta} = 0.2, \alpha = 2$, and $\theta = 0.01$.

Example 7.2. Choose $\bar{\beta} = 0.6$, $\alpha = 2$, and $\theta = 0.1$. By calculation, it can be concluded that

$$\beta_1 = (0.8795)^4, \beta_2 = (0.8428)^3,$$

$$R_0 = 1.7910 > 1, R_0^s = 1.0421 > 1,$$

$$(S^*, E^*, I^*, R^*, \beta^*) = (0.5018, 0.1593, 0.1189, 0.1189, 0.6),$$

$$Q_1 = 0.6134 > 0, Q_2 = 0.1166 > 0, Q_3 = 0.0097 > 0, Q_4 = 3.1858 \times 10^{-4} > 0,$$

$$Q_1Q_2 - Q_3 = 0.0618 > 0, Q_1Q_2Q_3 - Q_1^2Q_4 - Q_3^2 = 4.7981 \times 10^{-4} > 0.$$

The numerical simulation results are shown in Figure 4. At this point, both the conditions of Theorems 1 and 3 hold, system (2.2) has a unique endemic equilibrium and it is locally asymptotically stable, and system (2.4) has a stationary distribution. According to Theorem 5, the state variables of system (2.4) have a normal probability density function. In Matlab, by calling the 'Lyap' function, we can obtain

$$\Sigma = \begin{pmatrix} 7.1219 \times 10^{-5} & -3.5201 \times 10^{-5} & -2.2490 \times 10^{-5} & -1.3394 \times 10^{-5} & -7.8874 \times 10^{-5} \\ -3.5201 \times 10^{-5} & 1.9451 \times 10^{-5} & 1.0917 \times 10^{-5} & 4.7849 \times 10^{-6} & 6.0231 \times 10^{-5} \\ -2.2490 \times 10^{-5} & 1.0917 \times 10^{-5} & 7.2474 \times 10^{-6} & 4.2829 \times 10^{-6} & 1.7787 \times 10^{-5} \\ -1.3394 \times 10^{-5} & 4.7849 \times 10^{-6} & 4.2829 \times 10^{-6} & 4.2829 \times 10^{-6} & 8.4702 \times 10^{-7} \\ -7.8874 \times 10^{-5} & 6.0231 \times 10^{-5} & 1.7787 \times 10^{-5} & 8.4702 \times 10^{-7} & 2.5000 \times 10^{-3} \end{pmatrix}.$$

Then,

$$S \sim N(0.5018, 7.1219 \times 10^{-5}), E \sim N(0.1593, 1.9451 \times 10^{-5}),$$
$$I \sim N(0.1189, 7.2474 \times 10^{-6}), R \sim N(0.1189, 4.2829 \times 10^{-6}).$$

So, we get four marginal density functions

$$\Phi_S = 47.2729e^{-7.0206\times10^3(S-0.5018)^2}, \Phi_E = 90.4564e^{-2.5706\times10^4(E-0.1593)^2},$$
$$\Phi_I = 148.1900e^{-6.8990\times10^4(I-0.1189)^2}, \Phi_R = 192.7707e^{-1.1674\times10^5(R-0.1189)^2}.$$

Example 7.3. The dynamic impact of fluctuation intensity and mean-reversion rate on system (2.4).
Case 1. Keep $\bar{\beta} = 0.6, \alpha = 2$ constant, but let the severity of the fluctuations vary. Examine the next two instances of θ:

$$\theta = 0.08, \theta = 0.06.$$

Calculations indicate that in the two cases, the corresponding R_0^s are 1.1395 and 1.2526, respectively. As demonstrated by Figures 5 and 6, the system (2.4) permits a stationary distribution in any situation, in accordance with Theorem 3. The amplitude of the random solution close to the positive equilibrium point decreases with decreasing fluctuation intensity, as can be shown by comparing the left column of Figures 4–6. The marginal density function curve and the distribution range of data points are narrowed with decreasing fluctuation intensity, as indicated by the right columns of Figures 4–6.
Case 2. Maintain $\bar{\beta} = 0.6, \theta = 0.1$, but change the reversion speed. Consider the following three cases of α.

$$\alpha = 2, \alpha = 4, \alpha = 6.$$

The changing trends of $S(t), E(t), I(t)$, and $R(t)$ are shown in Figure 7. The solution of the random system obviously approaches the solution of (2.2) as α rises.

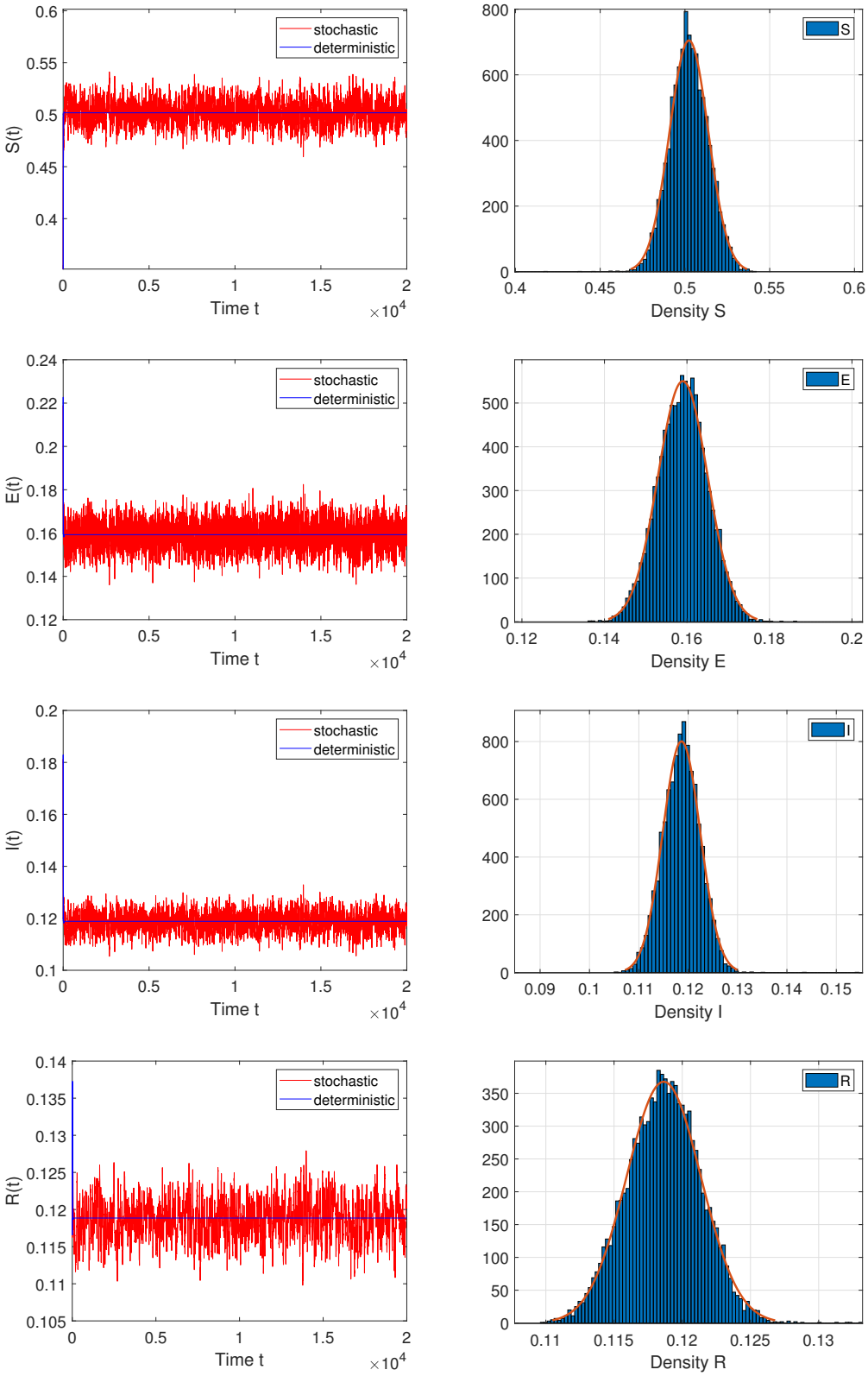

Figure 4. The solution trajectories of stochastic and deterministic systems under the parameter configurations $\bar{\beta} = 0.6, \alpha = 2$, and $\theta = 0.1$.

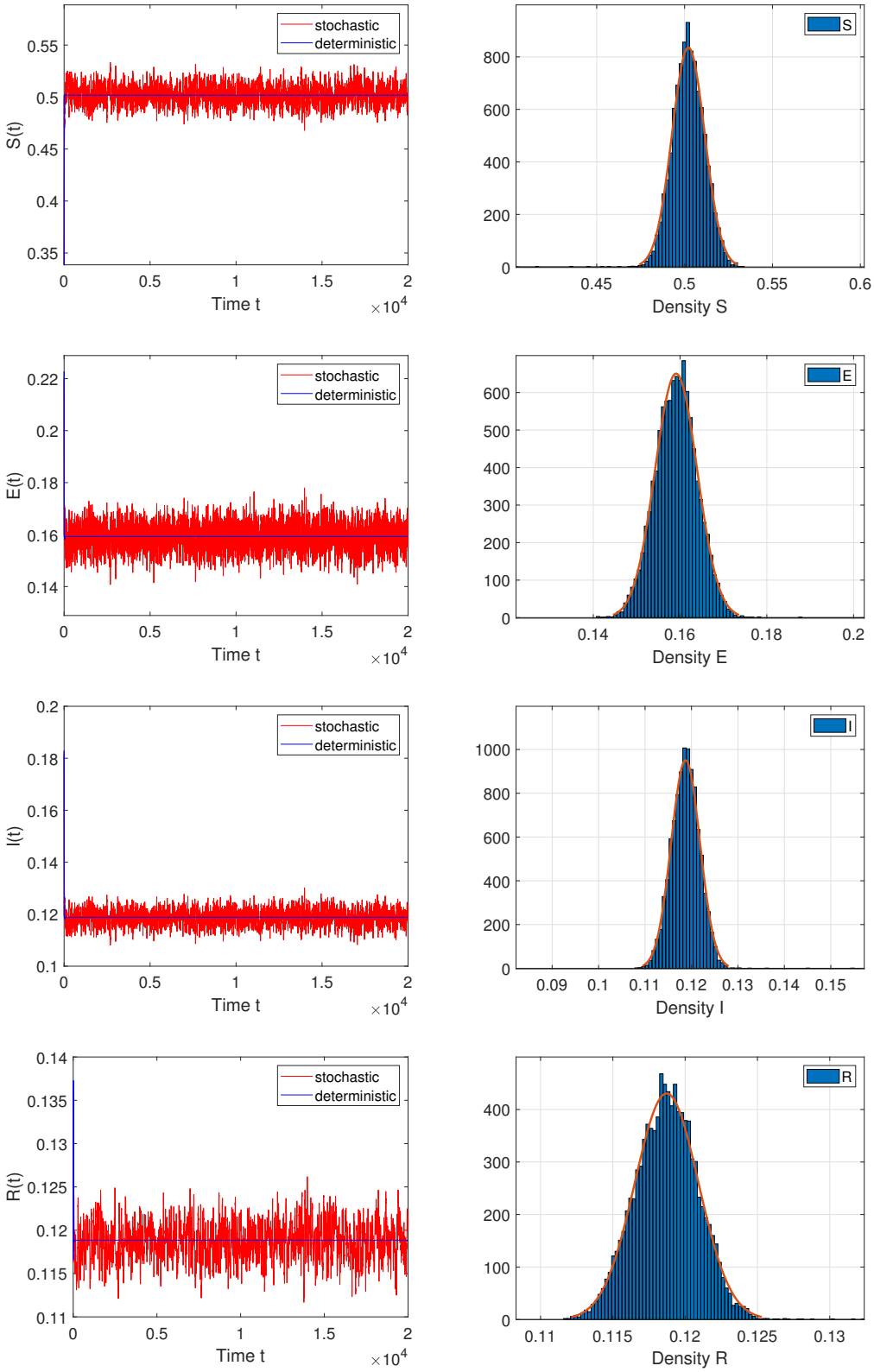

Figure 5. The solution trajectories of stochastic and deterministic systems under the parameter configurations $\bar{\beta} = 0.6, \alpha = 2$, and $\theta = 0.08$.

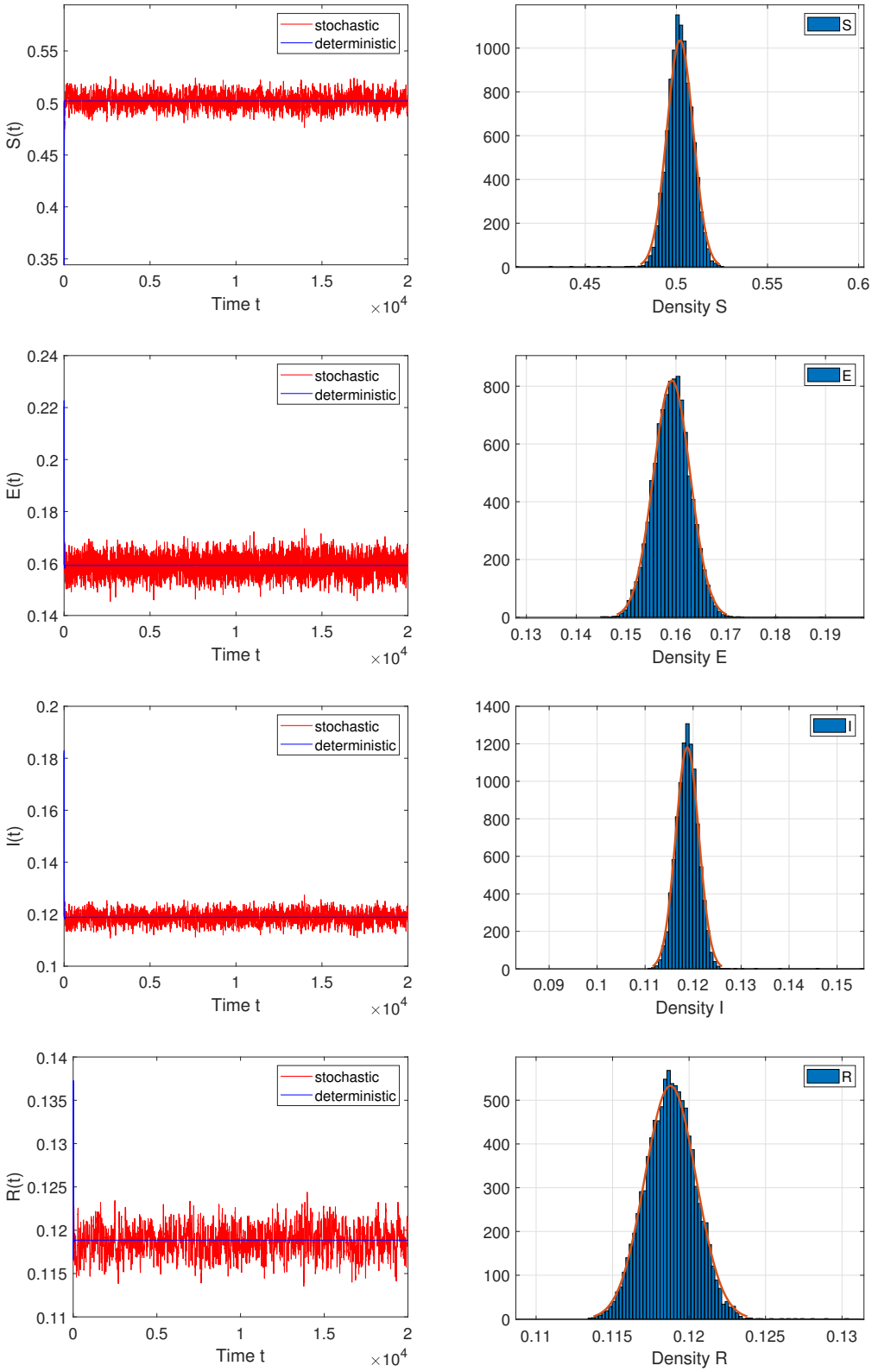

Figure 6. The solution trajectories of stochastic and deterministic systems under the parameter configurations $\bar{\beta} = 0.6, \alpha = 2$, and $\theta = 0.06$.

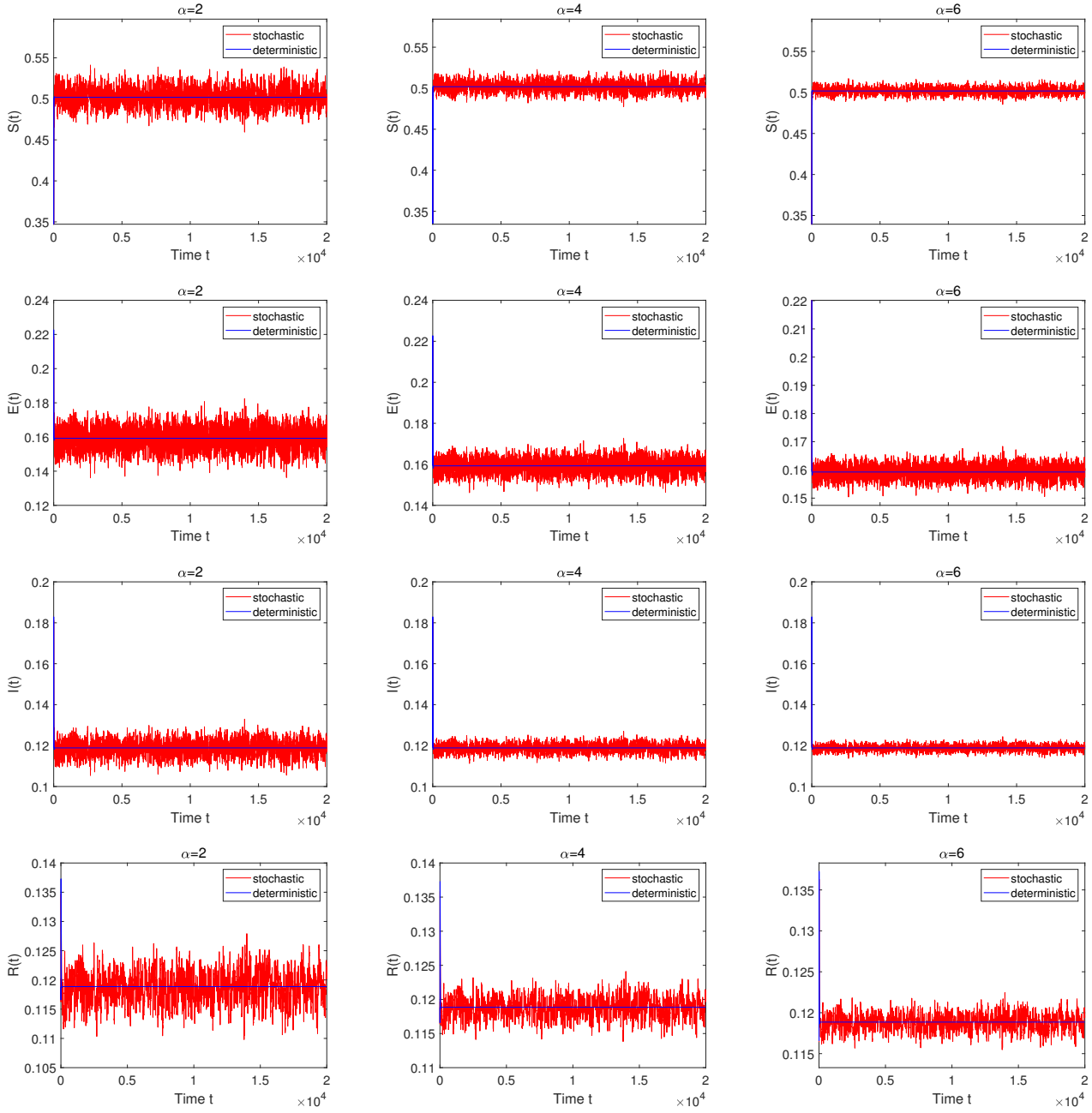

Figure 7. The solution trajectories of stochastic and deterministic systems with $\bar{\beta} = 0.6, \theta = 0.1$, and different α.

8. Discussion and conclusions

Infectious disease transmission is inevitably affected by environmental disturbances. Deterministic epidemic models, however, ignore the ubiquity of stochastic factors and provide only approximate representations of real phenomena, limiting their accuracy in predicting system dynamics. To address this, noise can be introduced into deterministic models. The methods of introducing noise in deterministic models include additive noise, multiplicative noise, parameter randomization, Lévy

jump processes, etc. The appropriate noise model should be selected based on the characteristics of the disturbance. If the influence of environmental disturbances (such as short-term temperature and humidity fluctuations, etc.) on the spread of diseases does not depend on the population size, then additive noise is suitable for modeling [26], for example,

$$\mathrm{d}S(t) = (\Lambda - \frac{\bar{\beta}S(t)I(t)}{N(t)} - \mu S(t))\mathrm{d}t + \sigma \mathrm{d}B(t),$$

where σ quantifies the white noise intensity. If the influence of environmental disturbances (such as population aggregation and mobility, etc.) on the spread of diseases increases with the increase in the number of susceptible individuals, then multiplicative noise is suitable for modeling [27, 28], for example,

$$\mathrm{d}S(t) = (\Lambda - \frac{\bar{\beta}S(t)I(t)}{N(t)} - \mu S(t))\mathrm{d}t + \sigma S(t)\mathrm{d}B(t).$$

Due to the fact that the incidence rate of TB exhibits a mean reversion characteristic under environmental disturbances, in this paper, we study the stochastic TB model with the OU process. First, we analyze the deterministic system (2.2), confirming the existence, uniqueness, and stability of its endemic equilibrium. For the stochastic model (2.4), we prove the existence and uniqueness of a global positive solution. To characterize disease persistence and extinction, we leverage the ergodicity of the OU process and construct Lyapunov functions, deriving threshold conditions. Using matrix similarity transformations, the Routh-Hurwitz criterion, and the Fokker-Planck equation, we obtain the exact probability density function of the stationary distribution. Lastly, the theoretical results were verified through numerical simulations. The main results of this article are as follows:

(1) Theorems 3 and 4 give conditions for persistence and extinction of disease in the stochastic system (2.4).
(a) If $R_0^s > 1$, then system (2.4) has a stationary distribution. In other words, the disease will persist over a long period of time and fluctuate continuously under the influence of environmental noise.
(b) If $R_0^E < 1$, then the disease will go extinct. Furthermore, we find $R_0^E \to R_0$ as $\theta \to 0$. This implies consistency in extinction conditions between stochastic and deterministic systems. This was also confirmed in the numerical simulation part (in Section 7 Example 7.1).

(2) By using matrix similarity transformation, Routh-Hurwitz criterion, and the Fokker-Planck equation, we obtain the exact expression of the density function for the stationary distribution. This result is given in Theorem 5.

(3) We also analyze the impact of the fluctuation intensity and the reversion speed on the dynamics of system (2.4). Example 7.3 in Section 7 indicates that the stochastic system has similar properties to the corresponding deterministic system when the fluctuation intensity is small or the speed of reversion is large.

Author contributions

Huimei Liu: writing-original draft preparation, writing-review and editing, visualization; Wencai Zhao: writing-review and editing, visualization, supervision, project administration, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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