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Research article

Asymptotic formulas for Dirichlet convolutions

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Abstract: In this paper, we use convoluting prime-number-theorem-related functions (Mobius, von Mangoldt, and Liouville) by the Piltz divisor function to research an asymptotic formula for the convolution sum ($x \geq 1$). Our main result is Theorem 1, which separates the error terms for the PNT-related functions and the Piltz divisor function. Under the assumption of the conjectural estimate for the latter, we obtain the expected error term as in PNT.

Keywords: prime-number-theorem; Piltz divisor function; asymptotic formula; Dirichlet convolution; Riemann zeta-function
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1. Introduction

The Riemann zeta-function is defined by

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1}$$

for $\sigma > 1$, where $s = \sigma + it$, $\sigma, t \in \mathbb{R}$ in the first instance. It is continued meromorphically over the whole plane with a simple pole at $s = 1$ with residue 1 by way of the functional equation

$$\zeta(s) = \frac{1}{\pi}(2\pi)^s \sin\frac{\pi}{2}s\Gamma(1-s)\zeta(1-s), \tag{1.2}$$

which is [1, p.13, (2.1.1)] and consideration in the critical strip $0 < \sigma < 1$. For $\zeta(s)$, we refer to [2] and [1].

The Euler function $\varphi(n)$ is generated by

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}, \quad \sigma > 1, \tag{1.3}$$

which implies

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = \sum_{dx=n} \mu(d) x, \tag{1.4}$$

where μ denotes the Möbius function (1.9).

We refer to [3, p.188, Remark] for the following information on correcting the error term. By the ingenuous method of Vinogradov, Walfisz [4], and then Saltykov [5] established the asymptotic formula

$$\Phi(x) = \sum_{n \le x} \varphi(n) = \frac{1}{2\zeta(2)} x^2 + O(\mathcal{E}(x)), \tag{1.5}$$

where

$$\mathcal{E}(x) = x(\log x)^{\frac{2}{3}} (\log \log x)^{1+\varepsilon}. \tag{1.6}$$

1.1. Prime number theorem related functions

The density of primes has been an object of research by many authors since Gauss (cf. e.g., [1–5]). The prime number theorem may be stated in a few ways. Letting, as usual

$$\pi(x) = \#\{y : y \quad prime, y < x\},$$

where $\#(\cdot)$ denotes the cardinality, the *prime counting function*, and the celebrated *prime number theorem* states that

$$\pi(x) \sim \frac{x}{\log x} \tag{1.7}$$

(cf. e.g., [6]). We refer to a more refined form of PNT

$$\pi(x) = \mathrm{li}(x) + O(x\delta(x)), \tag{1.8}$$

which is equivalent to the three asymptotic formulas in Theorem 1. Therefore, $\mathrm{li}(x)$ is expressed as the main term (non-error term) of the asymptotic formulas for these three functions.

Let

$$\mu(n) = \begin{cases} 1, & n = 1 \\ 0, & p^2 | n \\ (-1)^k, & n = p_1 p_2 \cdots p_k, p_i \ distinct \end{cases} \tag{1.9}$$

be the Möbius function, and we consider its summatory functions

$$\mathrm{M}(x) = \sum_{n \le x} \mu(n), \tag{1.10}$$

(1.9) is generated by

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \tag{1.11}$$

We define the von Mangoldt function by

$$\Lambda(n) = \begin{cases} \log p, n = p^k, \quad p \quad is \quad prime, k \geq 1 \\ 0, \qquad\qquad otherwise \end{cases} \tag{1.12}$$

we consider its summatory functions, the Chebyshev function

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \log p. \tag{1.13}$$

(1.12) is generated by

$$-\frac{\zeta'}{\zeta}(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \tag{1.14}$$

We define the Liouville function by

$$\lambda(n) = (-1)^r, \tag{1.15}$$

where if n has r prime factors, a factor of degree k is counted k times.

We consider its summatory functions,

$$\mathrm{L}(x) = \sum_{n \leq x} \lambda(n). \tag{1.16}$$

(1.15) is generated by

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}. \tag{1.17}$$

Proposition 1. (PNT) *We have the following asymptotic formulas:*

$$\mathrm{M}(x) = \sum_{n \leq x} \mu(n) = O(x\delta(x)), \tag{1.18}$$

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \log p = x + O(x\delta(x)), \tag{1.19}$$

$$\mathrm{L}(x) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = x + O(x\delta(x)), \tag{1.20}$$

where

$$\delta(x) = \delta_c(x) = e^{-c(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{3}}}, \tag{1.21}$$

and where $c > 0$ is a constant, not necessarily the same at each occurrence.

Proposition 2. *Let $\mathcal{M}(x)$ denote any one of the summatory functions in Proposition 1. Then*

$$\mathcal{M}(x) = \sum_{n \leq x} \mathfrak{m}(n) = Cx + O(x\delta(x)), \tag{1.22}$$

where C is 1 or 0, as the case may be.

1.2. The Piltz divisor problem

This section assembles necessary data for the determination of coefficients of Theorem 1, especially Proposition 3 and Lemma 2.

Let $d_\varkappa(n)$ denote the number of ways of expressing the integer n as a product of $\varkappa$ factors $d_\varkappa(n) = \sum_{d_1\cdots d_\varkappa=n} 1$, which is called the $\varkappa$-fold divisor function and is generated by $\zeta^\varkappa(s) = \sum_{n=1}^{\infty} \frac{d_\varkappa(n)}{n^s}, \sigma > 1$. The case $\varkappa = 2$, $d_2(n) = d(n)$, is the ordinary divisor function. The Piltz divisor problem, a generalization of the Dirichlet divisor problem for $d(n)$, is to the effect that in the asymptotic formula

$$D_\varkappa(x) = \sum_{n\le x} d_\varkappa(n) = xP_\varkappa(\log x) + \Delta_\varkappa(x), \tag{1.23}$$

where $x \cdot P_\varkappa(\log x) = \operatorname*{Res}_{s=1} \frac{\zeta^\varkappa(s)x^s}{s}$, the remainder $\Delta_\varkappa(x)$ is to be estimated. Here $P_\varkappa(t)$ is a polynomial of degree $\varkappa - 1$.

For the remainder term $\Delta_\varkappa(x)$, we refer to [2, Chapter 13], [1, Chapter 12].

Lavrik [7] was the first to express the coefficients of $P_\varkappa(\log x)$ explicitly in terms of generalized Euler constants γ_k's.

The kth generalized Euler constant γ_k $(k \ge 0)$ is defined by

$$\gamma_k = \frac{(-1)^k}{k!} \lim_{x\to\infty} \left(\sum_{n\le x} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{k+1} \right), \gamma_0 = \gamma. \tag{1.24}$$

Then the Laurent expansion of $\zeta(s)$ at $s = 1$ is given by $\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \gamma_k (s-1)^k$. For numerical values of γ_k's, see [8]. E.g., $-\gamma_1 = -0.07281\cdots$.

Proposition 3. *We have the closed form for the coefficients of the polynomial in* (1.23)*:*

$$P_\varkappa(\log x) = a_{\varkappa-1}^{(\varkappa)} (\log x)^{\varkappa-1} + \cdots + a_1^{(\varkappa)} \log x + a_0^{(\varkappa)}, \tag{1.25}$$

we have

$$a_j^{(\varkappa)} = \frac{1}{j!} \beta_{\varkappa-1-j}^{(\varkappa)}, \ 0 \le j \le \varkappa - 1 \tag{1.26}$$

and

$$\beta_n^{(\varkappa)} = (-1)^n \left(1 + \sum_{m=1}^{n} \frac{(-1)^m}{m!} S_{\varkappa,m} \right) \tag{1.27}$$

and where $S_{\varkappa,m}$ is given by

$$S_{\varkappa,m} = m! \sum_{s=1}^{m} \binom{\varkappa}{s} \sum_{\substack{i_1,\cdots,i_s\ge 0 \\ i_1+\cdots+i_s=m-s}} \gamma_{i_1} \cdots \gamma_{i_s} \tag{1.28}$$

([9]).

1.3. The hyperbola method

In order to improve the error term in the divisor problem, Dirichlet introduced the *hyperbola method*, which is another expression for the *inclusion-exclusion principle* and is sometimes the underlying principle of important identities, including the Eratosthenes sieve. We state it as the well-known identity in set operations

$$A \cup B = A \cup (B \backslash (A \cap B)), \tag{1.29}$$

where $B \backslash C$ is the difference set, which consists of elements of B not belonging to C.

The method turns out to be very useful in deriving asymptotic formulas for the sum of other arithmetic functions, too. [2, Chap. 14], [3, Chap. 14], and [10–12]. We are concerned with the power reduction theorem of Stronina [13] ([3, p.289, Theorem]).

Let μ_m and ν_n be increasing sequences with $\mu_m, \nu_n \geq 1$. Let $a(m), b(n)$ be arithmetical functions whose summatory functions are

$$A(x) = \sum_{m \leq x} a(m), \quad B(x) = \sum_{n \leq x} b(n). \tag{1.30}$$

$$\tilde{A}(x) = \sum_{\mu_m \leq x} a(m), \quad \tilde{B}(x) = \sum_{\nu_n \leq x} b(n). \tag{1.31}$$

Let $x > 0$ be a large variable, and let $\rho = \rho(x)$, $0 < \rho \leq x$, be chosen suitably, and consider the dissection of the sum

$$\begin{aligned} S(x) :&= \sum_{\mu_m \nu_n \leq x} a(m)b(n) \\ &= \sum_{\mu_m \leq \rho} a(m) \sum_{\nu_n \leq \frac{x}{\mu_m}} b(n) + \sum_{\nu_n \leq \rho^{-1}x} b(n) \sum_{\mu_m \leq \frac{x}{\nu_n}} a(m) - \sum_{\mu_m \leq \rho, \nu_n \leq \rho^{-1}x} a(m)b(n) \\ &= \sum_{\mu_m \leq \rho} a(m)\tilde{B}\left(\frac{x}{\mu_m}\right) + \sum_{\nu_n \leq \rho^{-1}x} b(n)\tilde{A}\left(\frac{x}{\nu_n}\right) - \tilde{A}(\rho)\tilde{B}(\rho^{-1}x). \end{aligned} \tag{1.32}$$

This follows from (1.29) with

$$A = \{(\mu_m, \nu_n) | \mu_m \leq \rho, \nu_n \leq \frac{x}{\mu_m}\}, \quad B = \{(\mu_m, \nu_n) | \nu_n \leq \rho^{-1}x, \mu_m \leq \frac{x}{\nu_n}\}. \tag{1.33}$$

2. Main result

In this paper, we research an asymptotic formula for the convolution sum ($x \geq 1$)

$$S(x) = \sum_{m^2 n \leq x} \mathfrak{m}(m) d_\varkappa(n), \tag{2.1}$$

where $\mathfrak{m}(m)$ resp. by proposition 2 in §1.1 , $d_\varkappa(n)$ is the prime-number-theorem (PNT) related function in §1.1, the Piltz divisor function in §1.2, and $S(x) := \sum_{\mu_m \nu_n \leq x} a(m)b(n)$ is a sum in §1.3.

If $a(m) = \mathfrak{m}(m), b(n) = d_\varkappa(n), \mu_m = m^2, \nu_n = n$, where $\mathfrak{m}$ is as in Proposition 2. Then

$$S(x) = \sum_{m \leq \sqrt{\rho}} \mathfrak{m}(m) D_\varkappa\left(\frac{x}{m^2}\right) + \sum_{n \leq \rho^{-1}x} d_\varkappa(n) \mathcal{M}\left(\sqrt{\frac{x}{n}}\right) - \mathcal{M}(\sqrt{\rho}) D_\varkappa(\rho^{-1}x). \tag{2.2}$$

Theorem 1. *From §1.3 and the asymptotic formula for the convolution sum* ($x \geq 1$)*, we have*

$$\begin{aligned} S(x) &= \sum_{m^2 n \leq x} \mathfrak{m}(m) d_\varkappa(n) \\ &= \sum_{m \leq \sqrt{\rho}} \mathfrak{m}(m) D_\varkappa\left(\frac{x}{m^2}\right) + \sum_{n \leq \rho^{-1} x} d_\varkappa(n) \mathcal{M}\left(\sqrt{\frac{x}{n}}\right) - \mathcal{M}(\sqrt{\rho}) D_\varkappa(\rho^{-1} x) \\ &= \sum_{m \leq \sqrt{\rho}} \mathfrak{m}(m) D_\varkappa\left(\frac{x}{m^2}\right) + C\sqrt{x} \sum_{n \leq \rho^{-1} x} \frac{d_\varkappa(n)}{\sqrt{n}} - C\frac{x}{\sqrt{\rho}} P_\varkappa\left(\log\frac{x}{\rho}\right) \\ &+ O\left(\sum_{n \leq \rho^{-1} x} d_\varkappa(n) \sqrt{\frac{x}{n}} \delta\left(\sqrt{\frac{x}{n}}\right)\right) + O\left(\sqrt{\rho}\delta(\sqrt{\rho})\right) D_\varkappa(\rho^{-1} x) \\ &= x \sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)} \sum_{l=0}^{j} \binom{j}{l} 2^l Z^{(l)}(2) (\log x)^{j-l} + O\left(\sqrt{x}\delta(x)^{-1}\right), \end{aligned} \tag{2.3}$$

where

$$Z^{(l)}(2) = \sum_{m=1}^{\infty} \frac{\mathfrak{m}(m)(-\log m)^l}{m^2},$$

$$\mathcal{M}(x) = \sum_{n \leq x} \mathfrak{m}(n) = Cx + O(x\delta(x)),$$

$$D_\varkappa(x) = \sum_{n \leq x} d_\varkappa(n) = x P_\varkappa(\log x) + \Delta_\varkappa(x),$$

and where C is 1 *or* 0 *,* $x \cdot P_\varkappa(\log x) = \operatorname{Res}_{s=1} \frac{\zeta^\varkappa(s) x^s}{s}$*, the remainder $\Delta_\varkappa(x)$ is to be estimated. Here $P_\varkappa(t)$ is a polynomial of degree* $\varkappa - 1$*.*

We state a special case of Theorem 1 as

Corollary 1. *In the above and in some other type of convolutions, the choice of $\sqrt{x}$ may not be optimal. We may consider other choices and may incorporate the results, e.g., for k-type integers [14]. E.g., for the Euler function $\varphi(n)$, we have*

$$\begin{aligned} S^E(x) &:= \sum_{m^2 n \leq x} \varphi(m) d_\varkappa(n) \\ &= x \sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)} \sum_{l=0}^{j} \binom{j}{l} (\log x)^{j-l} T_l(x) - A x P_\varkappa\left(\log\frac{x}{\rho}\right) \\ &+ O\left(x \mathcal{E}^{1-\alpha_\varkappa-\epsilon}\right), \end{aligned}$$

where

$$A = \frac{1}{2\zeta(2)}, \quad \mathcal{E}(x) = (\log x)^{\frac{2}{3}} (\log\log x)^{1+\varepsilon}, \quad T_l(x) = \sum_{m \leq X} \frac{\varphi(m)(-\log m)^l}{m^2}.$$

Example 1. *For the Möbius function, we have the result of Stronina [13], [3, p.289, Theorem], [2]*

$$S_3(x) = \sum_{n\le x} d(n^2) = \sum_{m^2 n\le x} \mu(m) d_3(n) = xQ_2(\log x) + O\left(\sqrt{x}\delta(x)\right), \tag{2.4}$$

where

$$Q_2(x) = \nu_0 x^2 + \nu_1 x + \nu_2, \tag{2.5}$$

and

$$\nu_0 = Z(2)a_2^{(3)}, \quad \nu_1 = Z(2)a_1^{(3)} + 4Z'(2)a_2^{(3)}, \quad \nu_2 = Z(2)a_0^{(3)} + 2Z'(2)a_1^{(3)} + 4Z''(2)a_2^{(3)}. \tag{2.6}$$

Here

$$a_0^{(3)} = 1 - 3\gamma + 3\gamma_1 + 3\gamma^2, \quad a_1^{(3)} = -1 + 3\gamma, \quad a_2^{(3)} = \frac{1}{2}, \tag{2.7}$$

see §1.2 and

$$Z(s) = \frac{1}{\zeta(s)}, \quad Z(2) = \frac{6}{\pi^2}, \quad Z'(2) = -\frac{\zeta'(2)}{(\frac{\pi^2}{6})^2} = -\frac{6}{\pi^2}(\gamma + \log(2\pi) - 12\log A), \tag{2.8}$$
$$Z''(2) = -\frac{\zeta''(2)\zeta^2(2) + 2\zeta(2)\zeta'(2)}{\zeta^4(2)},$$

where A is the Glaisher-Kinkelin constant [15, p.39, (2)].

The values in (2.8) *can be found from [15, p.558, (21)]*

$$\zeta'(2) = \frac{\pi^2}{6}(\gamma + \log(2\pi) - 12\log A) = -0.93754825431\cdots, \tag{2.9}$$
$$\zeta''(2) = -2\int_0^\infty \bar{B}_1(t)t^{-3}\cdot[\log^2(t+1) - \log t]\,\mathrm{d}t.$$

Corollary1 includes ([2, p. 394, (14.25) and (14.27)]). By interpreting $D_1(x) = \sqrt{x} = x + O(1)$*, we can include the case of ([2, p. 394, (14.24) with k = 2]).*

We may generalize the situation with the kth power of m and separate the error terms, to which we return elsewhere.

3. Some lemmas

To prove our Theorem 1, we need some lemmas as follows:

Lemma 1. *We choose* $\rho = x\delta_{c'}(x), 0 < c' << c$*. Therefore, we have*

$$\sqrt{\rho} = \sqrt{x}\delta_{c'}(\sqrt{x}). \tag{3.1}$$

We state the error estimate in §1.2 as the following, then we have

$$S(x) = S_1(x) + O(S_2(x)) + O(S_3(x)), \tag{3.2}$$

where

$$S_1(x) = S_{11}(x) + S_{12}(x) - C\frac{x}{\sqrt{\rho}}P_\varkappa\left(\log\frac{x}{\rho}\right) \tag{3.3}$$

and where

$$S_{11}(x) = \sum_{m\le\sqrt{\rho}} \mathfrak{m}(m)D_\varkappa\left(\frac{x}{m^2}\right) \tag{3.4}$$

and

$$S_{12}(x) = C\sqrt{x}\sum_{n\le\rho^{-1}x}\frac{d_\varkappa(n)}{\sqrt{n}}. \tag{3.5}$$

Further

$$S_2(x) = \sum_{n\le\rho^{-1}x} d_\varkappa(n)\sqrt{\frac{x}{n}}\delta\left(\sqrt{\frac{x}{n}}\right), \tag{3.6}$$

$$S_3(x) = O(\sqrt{\rho}\delta(\sqrt{\rho}))D_\varkappa(\rho^{-1}x).$$

Proof. Lemma 1 follows on substituting (1.22) and (1.23) in (2.2). □

Lemma 2. *For the error term* $\Delta_\varkappa(x)$ *in* (1.23)*, we have*

$$\alpha_\varkappa = \inf\{a_\varkappa | \Delta_\varkappa(x) = O(x^{a_\varkappa+\epsilon}), \forall\epsilon > 0\}, \tag{3.7}$$

so that

$$\Delta_\varkappa(x) = O(x^{\alpha_\varkappa+\epsilon}), \quad \forall\epsilon > 0 \tag{3.8}$$

for which we impose either a conjectural estimate

$$\alpha_\varkappa \le \frac{\varkappa-1}{2\varkappa} < \frac{1}{2}, \tag{3.9}$$

or a known estimate

$$\alpha_\varkappa \le 1 - \frac{1}{\varkappa}. \tag{3.10}$$

Remark 1. The case where (3.9) holds is only for $\varkappa = 3$ due to Atkinson [16]. The best result known is due to Kolesnik, as stated in [2, p.354, (13.14)]. [2, p.355, Theorem 13.2] gives a bound for bigger $\varkappa$, and in $\varkappa = 4$ case, $\alpha_4 = \frac{1}{2}$. It may be expected that this case could be improved slightly by the method of exponent pairs [17].

Lemma 3. *For the sum* $S_{11}(x)$ *in* (4.2) *we have*

$$S_{11}(x) = x\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)} \sum_{l=0}^{j}\left(\binom{j}{l}2^l Z^{(l)}(2)(\log x)^{j-l} + 2Ca(l,X)\right) + O\left(\sqrt{x}\delta(\sqrt{x})^{1-2\alpha_\varkappa}\right) \tag{3.11}$$
$$+ O\left(\sqrt{x}\delta(x)\right),$$

where $Z(s) = Z_\mu(s) = \frac{1}{\zeta(s)}$, $Z(s) = Z_\Lambda(s) = -\frac{\zeta'}{\zeta}(s)$, $Z(s) = Z_\lambda(s) = \frac{\zeta(2s)}{\zeta(s)}$ *and*

$$a(l, X) := (-1)^{l+1} l \int_X^\infty \frac{(\log t)^{l-1}}{t^2} dt = (-1)^{l+1} \sum_{j=0}^{l-1} l(l-1)\cdots(l-j) \frac{(\log X)^{l-1-j}}{X}. \tag{3.12}$$

Here

$$X = \sqrt{\rho} = \sqrt{x}\delta_{c'}(\sqrt{x}). \tag{3.13}$$

Proof. Substituting Lemma 2 and Proposition 3, we obtain

$$S_{11}(x) = x \sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)} \sum_{l=0}^{j} \binom{j}{l} 2^l (\log x)^{j-l} T_l(x) + O(\sqrt{x}\delta(\sqrt{x})^{1-2\alpha_\varkappa}), \tag{3.14}$$

where

$$T_l(x) = \sum_{m \le X} \frac{\mathfrak{m}(m)(-\log m)^l}{m^2} = T_l^{(1)}(x) - T_l^{(2)}(x), \tag{3.15}$$

and where

$$T_l^{(1)}(x) = \sum_{m=1}^{\infty} \frac{\mathfrak{m}(m)(-\log m)^l}{m^2} = Z^{(l)}(2), \tag{3.16}$$

$$T_l^{(2)}(x) = \sum_{m > X} \frac{\mathfrak{m}(m)(-\log m)^l}{m^2}.$$

By partial summation, we have

$$\mathcal{M}_1(x) := \sum_{n > x} \frac{\mathfrak{m}(m)}{m^2} = \frac{2C}{x} + O\left(\frac{\delta(x)}{x}\right). \tag{3.17}$$

Using this, we obtain

$$T_l^{(2)}(x) = 2Ca(l, X). \tag{3.18}$$

Hence

$$T_l(x) = Z^{(l)}(2) + 2Ca(l, X) + O\left(\frac{\delta(X)}{X}\right), \tag{3.19}$$

so that

$$S_{11}(x) = x \sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)} \sum_{l=0}^{j} \left(\binom{j}{l} 2^l Z^{(l)}(2)(\log x)^{j-l} + 2Ca(l, X)\right) + O\left(\frac{x\delta(X)}{X}\right) \tag{3.20}$$
$$+ O(\sqrt{x}\delta(\sqrt{x})^{1-2\alpha_\varkappa}).$$

Incorporating $X = \sqrt{\rho}$, we complete the proof. □

Remark 2. We note the following.

$$\delta(\sqrt{x})^{1-2\alpha_\varkappa} = \delta(x) \tag{3.21}$$

because of (3.9) and

$$Cxa(l, X) = CO\left(\frac{x(\log X)^l}{X}\right) = CO\left(\sqrt{x}\delta(x)^{-1}\right). \tag{3.22}$$

To prove Lemma 5, we use

Lemma 4. *For $b \geq \frac{1}{2}$ and*

$$Y = \frac{x}{\rho} = \delta^{-1} = \delta^{-1}(x), \tag{3.23}$$

we have

$$\sum_{n\leq Y} \frac{d_\varkappa(n)}{n^b} = \sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)} \left(Y^{1-b}(\log Y)^j + b\int_1^Y \frac{(\log t)^j}{t^b}\,\mathrm{d}t \right) + O\left(\delta^{b-\alpha_\varkappa-\epsilon}\right), \tag{3.24}$$

where δ is the reducing factor (1.21). *For $b = \frac{1}{2}$ we have*

$$\begin{aligned}\int_1^Y \frac{(\log t)^j}{\sqrt{t}}\,\mathrm{d}t &= \frac{2(j!)^2}{(2j+1)!}\sum_{r=0}^{j} 4^{j-r}\frac{(2r)!}{(r!)^2}\sqrt{t}(\log t)^r\Big|_1^Y \\ &= \frac{2(j!)^2}{(2j+1)!}\left(\sqrt{Y}\sum_{r=0}^{j} 4^{j-r}\frac{(2r)!}{(r!)^2}(\log Y)^r - 4^j\right)\end{aligned} \tag{3.25}$$

and for $b = 1$ we have

$$\int_1^Y \frac{(\log t)^j}{t}\,\mathrm{d}t = \frac{1}{j+1}(\log t)^{j+1}\Big|_1^Y = \frac{1}{j+1}(\log Y)^{j+1}. \tag{3.26}$$

Proof. By partial summation and Lemma 2 successively, we derive that

$$\sum_{n\leq Y} \frac{d_\varkappa(n)}{n^b} = Y^{1-b}\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)}(\log Y)^j + O(Y^{\alpha_\varkappa+\epsilon-b}) + b\int_1^Y \frac{\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)}(\log t)^j}{t^b}\,\mathrm{d}t + U(Y), \tag{3.27}$$

where

$$U(Y) = O\left(\int_1^Y \frac{\Delta_\varkappa(t)}{t^{b+1}}\,\mathrm{d}t\right) = O\left(Y^{\alpha_\varkappa+\epsilon-b}\right) = O\left(\delta^{b-\alpha_\varkappa-\epsilon}\right). \tag{3.28}$$

Thus, (3.24) follows. □

Lemma 5. *For the sum $S_{12}(x)$ in* (4.3) *we have*

$$\begin{aligned}S_{12}(x) = C\sqrt{x}\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)}\left(\sqrt{Y}(\log Y)^j + \frac{2(j!)^2}{(2j+1)!}\left(\sum_{r=0}^{j} 4^{j-r}\frac{(2r)!}{(r!)^2}\sqrt{Y}(\log Y)^r - 4^j\right)\right) \\ + C\sqrt{x}O\left(\delta(x)^{\frac{1}{2}-\alpha_\varkappa-\epsilon}\right),\end{aligned} \tag{3.29}$$

where Y is defined by (3.23).

Proof. By (3.24) with $b = \frac{1}{2}$, $S_{12}(x)$ reads

$$\begin{aligned}S_{12}(x) = C\sqrt{x}\left(\sqrt{Y}\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)}(\log Y)^j + \int_1^Y \frac{\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)}(\log t)^j}{\sqrt{t}}dt\right) \\ + CO\left(\sqrt{x}\delta^{1-\alpha_\varkappa-\epsilon}\right).\end{aligned} \tag{3.30}$$

Substituting (3.25) completes the proof of (3.29). □

4. Proof of theorem

In this section, we substitute the three lemmas in section 2, and we complete the proof of Theorem 1.
Proof of Theorem 1. With the help of Lemma 1, where

$$S_1(x) = S_{11}(x) + S_{12}(x) - C\frac{x}{\sqrt{\rho}}P_\varkappa\left(\log\frac{x}{\rho}\right) \tag{4.1}$$

and where

$$S_{11}(x) = \sum_{m\le\sqrt{\rho}} \mathfrak{m}(m)D_\varkappa\left(\frac{x}{m^2}\right) \tag{4.2}$$

and

$$S_{12}(x) = C\sqrt{x}\sum_{n\le\rho^{-1}x}\frac{d_\varkappa(n)}{\sqrt{n}}. \tag{4.3}$$

Further

$$S_2(x) = \sum_{n\le\rho^{-1}x} d_\varkappa(n)\sqrt{\frac{x}{n}}\delta\left(\sqrt{\frac{x}{n}}\right), \tag{4.4}$$
$$S_3(x) = O(\sqrt{\rho}\delta(\sqrt{\rho}))D_\varkappa(\rho^{-1}x).$$

We have

$$S(x) = S_1(x) + O(S_2(x)) + O(S_3(x)). \tag{4.5}$$

Then applying Lemma 2, we can obtain

$$S_{11}(x) = x\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)}\sum_{l=0}^{j}\left(\binom{j}{l}2^l Z^{(l)}(2)(\log x)^{j-l} + 2Ca(l,X)\right) + O\left(\sqrt{x}\delta(\sqrt{x})^{1-2\alpha_\varkappa}\right) + O\left(\sqrt{x}\delta(x)\right), \tag{4.6}$$

where $Z(s) = Z_\mu(s) = \frac{1}{\zeta(s)}$, $Z(s) = Z_\Lambda(s) = -\frac{\zeta'}{\zeta}(s)$, $Z(s) = Z_\lambda(s) = \frac{\zeta(2s)}{\zeta(s)}$ and

$$a(l,X) := (-1)^{l+1}l\int_X^\infty \frac{(\log t)^{l-1}}{t^2}dt = (-1)^{l+1}\sum_{j=0}^{l-1} l(l-1)\cdots(l-j)\frac{(\log X)^{l-1-j}}{X}. \tag{4.7}$$

Then applying Lemma 4, we can obtain

$$S_{12}(x) = C\sqrt{x}\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)}\left(\sqrt{Y}(\log Y)^j + \frac{2(j!)^2}{(2j+1)!}\left(\sum_{r=0}^{j} 4^{j-r}\frac{(2r)!}{(r!)^2}\sqrt{Y}(\log Y)^r - 4^j\right)\right) + C\sqrt{x}O\left(\delta(x)^{\frac{1}{2}-\alpha_\varkappa-\epsilon}\right), \tag{4.8}$$

where Y is defined by (3.23).

Then by Lemma 4 and (3.1), we can obtain

$$S_2(x) = \sqrt{x}\delta(\sqrt{\rho}) \sum_{n\le\rho^{-1}x} \frac{d_\varkappa(n)}{\sqrt{n}} = \sqrt{x}\delta(x), \tag{4.9}$$

and

$$S_3(x) = O(\sqrt{\rho}\delta_c(\sqrt{\rho}))\rho^{-1}x = \sqrt{x}\delta_c(\sqrt{\rho})\delta_{c'}(\sqrt{x}) = O(\sqrt{x}\delta(x)). \tag{4.10}$$

Then, substituting (4.6), (4.8), (4.9), and (4.10) and applying the above estimates to (4.1) and (4.5), we have

$$\begin{aligned} S(x) &= \sum_{m\le\sqrt{\rho}} \mathfrak{m}(m) D_\varkappa\left(\frac{x}{m^2}\right) + \sum_{n\le\rho^{-1}x} d_\varkappa(n) \mathcal{M}\left(\sqrt{\frac{x}{n}}\right) - \mathcal{M}(\sqrt{\rho}) D_\varkappa(\rho^{-1}x) \\ &= x\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)} \sum_{l=0}^{j} \binom{j}{l} 2^l Z^{(l)}(2)(\log x)^{j-l} + M(x) + O\left(\sqrt{x}\delta(\sqrt{x})^{1-2\alpha_\varkappa}\right) + O\left(\sqrt{x}\delta(x)\right), \end{aligned} \tag{4.11}$$

where $\alpha_\varkappa < \frac{1}{2}$, $M(x)$ is given by

$$\begin{aligned} M(x) &= Cxa(l, X) + \\ &C\sqrt{x}\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)} \left(\sqrt{Y}(\log Y)^j + \frac{2(j!)^2}{(2j+1)!}\left(\sum_{r=0}^{j} 4^{j-r}\frac{(2r)!}{(r!)^2}\sqrt{Y}(\log Y)^r - 4^j\right)\right) \\ &- C\frac{x}{\sqrt{\rho}} P_\varkappa\left(\log\frac{x}{\rho}\right), \end{aligned} \tag{4.12}$$

where X is defined by (3.13) and Y is defined by (3.23). $M(x)$ can be estimated as

$$M(x) = O\left(\sqrt{x}\delta(x)^{-1}\right). \tag{4.13}$$

The error term $O\left(\sqrt{x}\delta(\sqrt{x})^{1-2\alpha_\varkappa}\right)$ is controllable by the values of $\alpha_\varkappa$, which is the lower bound for the error term for the Piltz divisor problem. See (3.7). Under the condition (3.9), the penultimate error term is absorbed in $O\left(\sqrt{x}\delta(x)\right)$. Under the other condition (3.10), we have the error term $O\left(\sqrt{x}\delta(x)^{-1}\right)$.

Since we have

$$S(x) = x\sum_{j=0}^{\varkappa-1} a_j^{(\varkappa)} \sum_{l=0}^{j} \binom{j}{l} 2^l Z^{(l)}(2)(\log x)^{j-l} + O\left(\sqrt{x}\delta(x)^{-1}\right), \tag{4.14}$$

which completes the proof of Theorem 1. □

5. Conclusions

In this paper, we are using convoluting prime-number-theorem-related functions (Mobius, von Mangoldt, and Liouville) by the Piltz divisor function to research an asymptotic formula for the convolution sum ($x \ge 1$) and managed to extract the error term, which decides the error term to be the best one with the reduction factor or one with a small power of x under a conjectural upper bound for the Piltz divisor problem.

Author contributions

Ruiyang Li: Writing–review and editing, Writing–original draft, Validation, Resources, Methodology, Formal analysis, Conceptualization; Hai Yang: Writing–review and editing, Resources, Methodology, Supervision, Validation, Formal analysis, Funding acquisition.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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