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Research article

Kelvin transform on the Heisenberg group revisited and applications to the best constant of Hardy-Sobolev type inequality

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Abstract: In this paper we study the inversion map and the Kelvin transform on the Heisenberg group $\mathbb{H}^n$. We first analyze the invariance of the Kelvin transform and provide an algebraic proof to the formula involving the sub-Laplacian. Furthermore, we apply the formula to seek the cylindrically symmetric solution to a sub-elliptic equation on $\mathbb{H}^n$ and determine the best constant of the Hardy-Sobolev type inequality.

Keywords: Heisenberg group; Kelvin transform; sub-Laplacian; Hardy-Sobolev type inequality; Best constant
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1. Introduction

The Kelvin transform, introduced by Lord Kelvin in the context of electrostatics, is a powerful tool in Euclidean space $\mathbb{R}^n$ for analyzing harmonic functions and geometric properties via sphere inversion. Its applications span multiple areas of mathematics and physics, including potential theory and partial differential equations. In contrast, the Heisenberg group $\mathbb{H}^n$ plays a fundamental role in sub-Riemannian geometry, harmonic analysis, and quantum mechanics. Unlike Euclidean space, $\mathbb{H}^n$ exhibits a non-trivial group structure and a stratified Lie algebra, features that complicate the analysis of differential operators.

The method of moving planes, originally developed by Alexandrov [1] in his study of surfaces with constant mean curvature, has since been widely employed to analyze symmetry properties of solutions to partial differential equations. When combined with the Kelvin transform, this method becomes a powerful tool for proving symmetry and radiality of solutions to elliptic PDEs in Euclidean spaces; see also [2]. Building upon this approach, an asymptotic symmetry method [3] was developed to investigate the asymptotic behavior and local properties of solutions to semilinear elliptic equations through transformation and inversion analysis. In the Heisenberg group setting, [4] extended the method to this framework, while [5, 6] made the adaptation of the Kelvin transform to this non-commutative geometric context.

The Hardy inequality, first proposed by Hardy [7], stands as a fundamental tool in mathematical analysis. It has inspired numerous important variants, including Hardy-Sobolev type inequalities, whose theory in Euclidean spaces is relatively well-established [8,9]. This study focuses on the Hardy-Sobolev type inequality and its constant on the Heisenberg group, extending existing results [10–13] for this setting.

In the Euclidean space, the inversion map $\sigma : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ is defined on by

$$\sigma(x) := -\frac{x}{|x|^2}.$$

This map is used to define the Kelvin transform u^* of a function u, namely,

$$u^*(x) := |x|^{-n+2}\, u\left(\sigma(x)\right).$$

We should mention that the Kelvin transform satisfies that

$$\Delta u^*(x) = |x|^{-n-2}\, \Delta u\left(\sigma(x)\right),$$

which is used to seek the cylindrically symmetric solution to the following elliptic equation

$$-\Delta u = (n-s)(n-2)\frac{u^{2^*(s)-1}}{|x|^s} \tag{1.1}$$

in $\mathbb{R}^n$, where $s \in [0, 2)$ and $2^*(s) = \dfrac{2(n-s)}{n-2}$. Such equation has connection to the Hardy-Sobolev type inequality

$$\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*(s)}}{|x|^s}\, \mathrm{d}x \le C(n, s)\left(\int_{\mathbb{R}^n} |\nabla u|^2\, \mathrm{d}x\right)^{\frac{n-s}{n-2}}, \tag{1.2}$$

see [14]. The cylindrically symmetric solution of (1.1) is of the form

$$u(x) = \frac{1}{(1+|x|^{2-s})^{\frac{n-2}{2-s}}},$$

which makes the inequality (1.2) achieves the sharp constant

$$\left[\frac{2\pi^{\frac{n}{2}}}{(2-s)\Gamma(\frac{n}{2})\Gamma\left(\frac{2(n-s)}{2-s}\right)}\right]^{\frac{s-2}{n-2}} \cdot \frac{\Gamma\left(\frac{n-s}{2-s}\right)^2}{\left[(n-2)^2\, \Gamma\left(\frac{n-2s+2}{2-s}\right)\Gamma\left(\frac{n-2}{2-s}\right)\right]^{\frac{n-s}{n-2}}}.$$

In this paper, we study the Kelvin transform on $\mathbb{H}^n$ and its interplay with the sub-Laplacian, providing a conformally invariant framework. Let $\sigma : \mathbb{H}^n \backslash \{0\} \to \mathbb{H}^n \backslash \{0\}$ be the inversion map defined by

$$\sigma(z, t) = (\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n, \tau), \tag{1.3}$$

where $(z, t) = (x_1 + \mathrm{i}\, y_1, x_2 + \mathrm{i}\, y_2, \dots, x_n + \mathrm{i}\, y_n, t) \in \mathbb{H}^n$ with the homogeneous norm $\rho = \left(|z|^4 + 16t^2\right)^{\frac{1}{4}}$, and

$$\varphi_i = \frac{4ty_i - x_i|z|^2}{|z|^4 + 16t^2}, \quad \psi_i = \frac{-4tx_i - y_i|z|^2}{|z|^4 + 16t^2}, \quad \tau = \frac{-t}{|z|^4 + 16t^2} \tag{1.4}$$

for $i = 1, 2, \ldots, n$. Let $u : \mathbb{H}^n \to \mathbb{R}$ be a function; then the Kelvin transform of u is defined as

$$u^*(z,t) = \rho^{-Q+2}\, u\left(\sigma(z,t)\right), \tag{1.5}$$

where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$.

Let $\Delta_{\mathbb{H}^n}$ be the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$. The following theorem presents the relation between u and u^*, which generalizes the classical Euclidean Kelvin transform to the Heisenberg group and reveals conformal invariance in sub-Riemannian settings.

Theorem 1.1 (The formula revisited)**.** *For any function $u \in C^2(\mathbb{H}^n \setminus \{0\})$ the Kelvin transform u^* satisfies*

$$\Delta_{\mathbb{H}^n}\left(u^*(z,t)\right) \;=\; \rho^{-(Q+2)}\left(\Delta_{\mathbb{H}^n} u\right)\left(\sigma(z,t)\right). \tag{1.6}$$

In [15], Li and Monticelli prove this formula on the Heisenberg group by calculating the trace of the Hessian matrix and using properties of the Jacobian determinant. Their work deeply relies on the CR structure and nonlinear equation characteristics. Relevant research is elaborated in [16, 17]. In this paper, we shall give an algebraic and direct proof of Theorem 1.1. We shall use the chain rule of the sub-Laplacian and the inner product of the horizontal gradient, which reduces the problem to matrix multiplications.

As in the Euclidean setting, we consider the application of Theorem 1.1 and establish relevant conclusions in the Heisenberg group. One has the following Hardy-Sobolev type inequality:

$$\int_{\mathbb{H}^n} \frac{|z|^2}{\rho^2} \frac{|u|^{2^*(s)}}{\rho^s}\, \mathrm{d}z\, \mathrm{d}t \;\le\; C\left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2\, \mathrm{d}z\, \mathrm{d}t\right)^{\frac{Q-s}{Q-2}}, \tag{1.7}$$

with $s \in [0, 2)$ and $2^*(s) = \frac{2(Q-s)}{Q-2}$, which is related to the following generalized sub-elliptic equation

$$-\Delta_{\mathbb{H}^n} u = (Q-s)(Q-2)\frac{|z|^2}{\rho^2}\frac{u^{\frac{Q-2s+2}{Q-2}}}{\rho^s} \tag{1.8}$$

on $\mathbb{H}^n \setminus \{0\}$. We state the following theorem regarding the connection between (1.7) and (1.8).

Theorem 1.2. *Assume that $u \in C^2(\mathbb{H}^n \setminus \{0\})$ is a cylindrically symmetric solution to* (1.8)*, then u is of the form*

$$u(z,t) = \frac{1}{\left[1 + \left(|z|^4 + 16t^2\right)^{\frac{2-s}{4}}\right]^{\frac{Q-2}{2-s}}} = \frac{1}{\left(1 + \rho^{2-s}\right)^{\frac{Q-2}{2-s}}} \tag{1.9}$$

up to group translation and dilation. Then the best constant to the Hardy-Sobolev type inequality (1.7) *is*

$$\left[\frac{\pi^{\frac{Q-1}{2}}\Gamma\left(\frac{Q}{4}\right)}{2(2-s)\Gamma\left(\frac{Q-2}{2}\right)\Gamma\left(\frac{Q+2}{4}\right)\Gamma\left(\frac{2(Q-s)}{2-s}\right)}\right]^{\frac{s-2}{Q-2}} \cdot \frac{\Gamma\left(\frac{Q-s}{2-s}\right)^2}{\left[(Q-2)^2\,\Gamma\left(\frac{Q-2s+2}{2-s}\right)\Gamma\left(\frac{Q-2}{2-s}\right)\right]^{\frac{Q-s}{Q-2}}}. \tag{1.10}$$

2. Preliminaries

In this section, we first present basic definitions and tools in the Heisenberg group and then calculate first-order and second-order derivatives of (1.4).

2.1. Heisenberg group

The Heisenberg group is a connected and simply connected Lie group. Let us first review the following basic definitions and present useful properties; see [18].

Definition 2.1 (Heisenberg group)**.** *We identify the Heisenberg group $\mathbb{H}^n$ with $\mathbb{R}^{2n+1}$, endowed with the group multiplication*

$$(x_1, \cdots, x_n, y_1, \cdots, y_n, t) \circ (x'_1, \cdots, x'_n, y'_1, \cdots, y'_n, t') \tag{2.1}$$
$$= \left(x_1 + x'_1, \cdots, x_n + x'_n, y_1 + y'_1, \cdots, y_n + y'_n, t + t' + \frac{1}{2} \sum_{i=1}^{n} (x_i y'_i - x'_i y_i) \right).$$

This multiplication reflects the non-commutative structure of $\mathbb{H}^n$. The canonical left invariant vector fields (horizontal vector fields) are defined by

$$X_i = \frac{\partial}{\partial x_i} + \frac{y_i}{2} \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - \frac{x_i}{2} \frac{\partial}{\partial t} \tag{2.2}$$

for $1 \le i \le n$. The only non-trivial commutator (vertical vector field) is given as

$$T = \frac{\partial}{\partial t}. \tag{2.3}$$

We should mention that (2.2) and (2.3) satisfy the following commutator relations

$$\left[X_i \, , \, Y_j \right] = -\delta_{ij} T \, , \tag{2.4}$$

and

$$\left[X_i \, , \, X_j \right] = \left[Y_i \, , \, Y_j \right] = \left[T \, , \, X_j \right] = \left[T \, , \, Y_j \right] = 0 \, .$$

Unlike some literature, the presence of the coefficient $1/2$ in (2.1) and (2.2) does not alter the group structure or the left invariance of the vector fields.

We define the horizontal gradient of a function $u : \mathbb{H}^n \to \mathbb{R}$ by

$$\nabla_{\mathbb{H}^n} u = (X_1 u, \ldots, X_n u, Y_1 u, \ldots, Y_n u) \, , \tag{2.5}$$

where its length is given via

$$|\nabla_{\mathbb{H}^n} u|^2 = \sum_{i=1}^{n} \left[(X_i u)^2 + (Y_i u)^2 \right]. \tag{2.6}$$

For a vector-valued function $F = (f_1, \ldots, f_n, g_1, \ldots, g_n) : \mathbb{H}^n \to \mathbb{R}^{2n}$, the horizontal divergence is defined as

$$\mathrm{div}_{\mathbb{H}^n}(F) = \sum_{i=1}^{n} (X_i f_i + Y_i \, g_i). \tag{2.7}$$

We denote by $\Delta_{\mathbb{H}^n}$ the sub-Laplacian operator on $\mathbb{H}^n$ of function $u : \mathbb{H}^n \to \mathbb{R}$ by

$$\Delta_{\mathbb{H}^n} u = \mathrm{div}_{\mathbb{H}^n} \left(\nabla_{\mathbb{H}^n} u \right) = \sum_{i=1}^{n} \left(X_i^2 u + Y_i^2 u \right). \tag{2.8}$$

It is easy to check that

$$\Delta_{\mathbb{H}^n} u = \sum_{i=1}^{n} \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} + y_i \frac{\partial^2 u}{\partial t \partial x_i} - x_i \frac{\partial^2 u}{\partial t \partial y_i} + \frac{x_i^2 + y_i^2}{4} \frac{\partial^2 u}{\partial t^2} \right). \tag{2.9}$$

By (2.9), one obtains directly that

$$(\Delta_{\mathbb{H}^n} u)\,(\sigma(z,t)) = \sum_{i=1}^{n} \left(\frac{\partial^2 u}{\partial \varphi_i^2} + \frac{\partial^2 u}{\partial \psi_i^2} + \psi_i \frac{\partial^2 u}{\partial \tau \partial \varphi_i} - \varphi_i \frac{\partial^2 u}{\partial \tau \partial \psi_i} + \frac{\varphi_i^2 + \psi_i^2}{4} \frac{\partial^2 u}{\partial \tau^2} \right), \tag{2.10}$$

which plays a fundamental role in the proof of the Theorem 1.1. In this work, we consider the Sobolev-type space

$$D^{1,q}(\mathbb{H}^n) = \left\{ u \in L^q(\mathbb{H}^n) \,\middle|\, X_i u, Y_i u \in L^q(\mathbb{H}^n), i = 1, \ldots, n \right\}.$$

2.2. *First order information*

To establish Theorem (1.1), we first derive formulas of first-order differential operators acting on the functions φ_i, ψ_i, and τ, which reflect the non-commutative structure of the Heisenberg group.

Lemma 2.2. *Let the functions φ_i, ψ_i, and τ be defined in* (1.4). *Then the gradients are of the form*

$$\nabla_{\mathbb{H}^n} \varphi_i = \frac{1}{2} \mathbf{M}_i \, \mathbf{z} + \mathbf{U}_i, \tag{2.11}$$

$$\nabla_{\mathbb{H}^n} \psi_i = \frac{1}{2} \mathbf{N}_i \, \mathbf{z} + \mathbf{V}_i, \tag{2.12}$$

$$\nabla_{\mathbb{H}^n} \tau = \frac{1}{2 \left(|z|^4 + 16t^2 \right)^2} \mathbf{P} \, \mathbf{z}, \tag{2.13}$$

where $\mathbf{z} = (x_1, \cdots x_n, y_1, \cdots y_n)^{\mathrm{T}}$, $\mathbf{M}_i$, $\mathbf{N}_i$, *and* $\mathbf{P}$ *are* $2n \times 2n$ *matrices given by*

$$\mathbf{M}_i = \begin{pmatrix} -T\psi_i \mathbf{I}_n & T\varphi_i \mathbf{I}_n \\ -T\varphi_i \mathbf{I}_n & -T\psi_i \mathbf{I}_n \end{pmatrix}, \quad \mathbf{N}_i = \begin{pmatrix} T\varphi_i \mathbf{I}_n & T\psi_i \mathbf{I}_n \\ -T\psi_i \mathbf{I}_n & T\varphi_i \mathbf{I}_n \end{pmatrix}, \tag{2.14}$$

$$\mathbf{P} = \begin{pmatrix} 8t|z|^2 \mathbf{I}_n & \left(-|z|^4 + 16t^2 \right) \mathbf{I}_n \\ \left(|z|^4 - 16t^2 \right) \mathbf{I}_n & 8t|z|^2 \mathbf{I}_n \end{pmatrix}, \tag{2.15}$$

and $\mathbf{U}_i$, $\mathbf{V}_i$ *are* $2n$*-dimensional column vectors expressed by*

$$\mathbf{U}_i = \left(\frac{-|z|^2}{|z|^4 + 16t^2} \mathbf{e_i}, \ \frac{4t}{|z|^4 + 16t^2} \mathbf{e_i} \right)^{\mathrm{T}}, \tag{2.16}$$

$$\mathbf{V}_i = \left(\frac{-4t}{|z|^4 + 16t^2} \mathbf{e_i}, \ \frac{-|z|^2}{|z|^4 + 16t^2} \mathbf{e_i} \right)^{\mathrm{T}} \tag{2.17}$$

for $i = 1, \ldots, n$. *Here* $\mathbf{I}_n$ *is the* $n \times n$ *identity matrix, and the set of* n*-dimensional row vectors* $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ *denotes the standard basis of* $\mathbb{R}^n$.

Proof. By (2.3), it is clear that

$$T\varphi_i = \frac{\partial \varphi_i}{\partial t} = \frac{32tx_i|z|^2 + 4y_i\left(|z|^4 - 16t^2\right)}{(|z|^4 + 16t^2)^2}, \quad T\psi_i = \frac{\partial \psi_i}{\partial t} = \frac{32ty_i|z|^2 - 4x_i\left(|z|^4 - 16t^2\right)}{(|z|^4 + 16t^2)^2}.$$

By calculating

$$\frac{\partial \varphi_i}{\partial x_j} = -\frac{x_j}{2}T\psi_i + \frac{-|z|^2}{|z|^4 + 16t^2}\delta_{ij}, \qquad \frac{\partial \varphi_i}{\partial y_j} = -\frac{y_j}{2}T\psi_i + \frac{4t}{|z|^4 + 16t^2}\delta_{ij},$$

$$\frac{\partial \psi_i}{\partial x_j} = \frac{x_j}{2}T\varphi_i + \frac{-4t}{|z|^4 + 16t^2}\delta_{ij}, \qquad \frac{\partial \psi_i}{\partial y_j} = \frac{y_j}{2}T\varphi_i + \frac{-|z|^2}{|z|^4 + 16t^2}\delta_{ij},$$

and using (2.2), one obtains the formulas involving X_j, Y_j by

$$X_j\varphi_i = -\frac{x_j}{2}T\psi_i + \frac{y_j}{2}T\varphi_i + \frac{-|z|^2}{|z|^4 + 16t^2}\delta_{ij}, \;\; Y_j\varphi_i = -\frac{x_j}{2}T\varphi_i - \frac{y_j}{2}T\psi_i + \frac{4t}{|z|^4 + 16t^2}\delta_{ij}, \tag{2.18}$$

and

$$X_j\psi_i = \frac{x_j}{2}T\varphi_i + \frac{y_j}{2}T\psi_i + \frac{-4t}{|z|^4 + 16t^2}\delta_{ij}, \;\; Y_j\psi_i = -\frac{x_j}{2}T\psi_i + \frac{y_j}{2}T\varphi_i + \frac{-|z|^2}{|z|^4 + 16t^2}\delta_{ij}. \tag{2.19}$$

By the definition of horizontal gradient (2.5), these yield the (2.11) and (2.12), respectively.

In order to get the expression of the horizontal gradient of τ, one computes that

$$\frac{\partial \tau}{\partial x_j} = \frac{4tx_j|z|^2}{(|z|^4 + 16t^2)^2}, \quad \frac{\partial \tau}{\partial y_j} = \frac{4ty_j|z|^2}{(|z|^4 + 16t^2)^2}, \quad \text{and} \quad \frac{\partial \tau}{\partial t} = \frac{-\left(|z|^4 - 16t^2\right)}{(|z|^4 + 16t^2)^2}.$$

It follows that

$$\begin{aligned} X_j\tau &= \frac{\partial \tau}{\partial x_j} + \frac{y_j}{2}\frac{\partial \tau}{\partial t} = \frac{8tx_j|z|^2 - y_j\left(|z|^4 - 16t^2\right)}{2\,(|z|^4 + 16t^2)^2}, \\ Y_j\tau &= \frac{\partial \tau}{\partial y_j} - \frac{x_j}{2}\frac{\partial \tau}{\partial t} = \frac{8ty_j|z|^2 + x_j\left(|z|^4 - 16t^2\right)}{2\,(|z|^4 + 16t^2)^2}, \end{aligned}$$

which deduce the representation (2.13) of the horizontal gradient $\nabla_{\mathbb{H}^n}\tau$. □

Corollary 2.3. *For* $j, k = 1, 2, \ldots, n$, *we have the following multiplication of* $2n \times 2n$ *matrices by*

$$\mathbf{M}_j^{\mathrm{T}}\mathbf{M}_k = \begin{pmatrix} \left(T\varphi_j \cdot T\varphi_k + T\psi_j \cdot T\psi_k\right)\mathbf{I}_n & \left(T\varphi_j \cdot T\psi_k - T\psi_j \cdot T\varphi_k\right)\mathbf{I}_n \\ \left(-T\varphi_j \cdot T\psi_k + T\psi_j \cdot T\varphi_k\right)\mathbf{I}_n & \left(T\varphi_j \cdot T\varphi_k + T\psi_j \cdot T\psi_k\right)\mathbf{I}_n \end{pmatrix} = \mathbf{N}_j^{\mathrm{T}}\mathbf{N}_k, \tag{2.20}$$

$$\mathbf{M}_j^{\mathrm{T}}\mathbf{N}_k = \begin{pmatrix} \left(T\varphi_j \cdot T\psi_k - T\varphi_k \cdot T\psi_j\right)\mathbf{I}_n & \left(-T\varphi_j \cdot T\varphi_k - T\psi_j \cdot T\psi_k\right)\mathbf{I}_n \\ \left(T\varphi_j \cdot T\varphi_k + T\psi_j \cdot T\psi_k\right)\mathbf{I}_n & \left(T\varphi_j \cdot T\psi_k - T\varphi_k \cdot T\psi_j\right)\mathbf{I}_n \end{pmatrix}, \tag{2.21}$$

$$\mathbf{P}^{\mathrm{T}}\mathbf{P} = \left(|z|^4 + 16t^2\right)^2\mathbf{I}_{2n}, \tag{2.22}$$

$$\mathbf{M}_j^{\mathrm{T}}\,\mathbf{P} = \begin{pmatrix} -\left[8t|z|^2T\psi_j+\left(|z|^4-16t^2\right)T\varphi_j\right]\mathbf{I}_n & -\left[8t|z|^2T\varphi_j+\left(|z|^4-16t^2\right)T\psi_j\right]\mathbf{I}_n \\ \left[8t|z|^2T\varphi_j+\left(|z|^4-16t^2\right)T\psi_j\right]\mathbf{I}_n & -\left[8t|z|^2T\psi_j+\left(|z|^4-16t^2\right)T\varphi_j\right]\mathbf{I}_n \end{pmatrix}$$

$$= 8t|z|^2\begin{pmatrix} -T\psi_j\mathbf{I}_n & -T\varphi_j\mathbf{I}_n \\ T\varphi_j\mathbf{I}_n & -T\psi_j\mathbf{I}_n \end{pmatrix} + \left(|z|^4-16t^2\right)\begin{pmatrix} -T\varphi_j\mathbf{I}_n & -T\psi_j\mathbf{I}_n \\ T\psi_j\mathbf{I}_n & -T\varphi_j\mathbf{I}_n \end{pmatrix}, \tag{2.23}$$

$$\mathbf{N}_j^{\mathrm{T}}\,\mathbf{P} = 8t|z|^2\begin{pmatrix} T\varphi_j\mathbf{I}_n & -T\psi_j\mathbf{I}_n \\ T\psi_j\mathbf{I}_n & T\varphi_j\mathbf{I}_n \end{pmatrix} + \left(|z|^4-16t^2\right)\begin{pmatrix} -T\psi_j\mathbf{I}_n & -T\varphi_j\mathbf{I}_n \\ T\varphi_j\mathbf{I}_n & -T\psi_j\mathbf{I}_n \end{pmatrix}, \tag{2.24}$$

where

$$T\varphi_j\cdot T\varphi_k+T\psi_j\cdot T\psi_k=\frac{16\left(x_jx_k+y_jy_k\right)}{\left(|z|^4+16t^2\right)^2},\quad T\varphi_j\cdot T\psi_k-T\varphi_k\cdot T\psi_j=\frac{16\left(x_jx_k-y_jy_k\right)}{\left(|z|^4+16t^2\right)^2}, \tag{2.25}$$

and

$$8t|z|^2T\psi_j+\left(|z|^4-16t^2\right)T\varphi_j=4y_j,\quad 8t|z|^2T\varphi_j-\left(|z|^4-16t^2\right)T\psi_j=4x_j. \tag{2.26}$$

Corollary 2.4. *One has the multiplication of matrices and vectors by*

$$\mathbf{M}_j^{\mathrm{T}}\,\mathbf{U}_k = \mathbf{N}_j^{\mathrm{T}}\,\mathbf{V}_k = \left(\frac{|z|^2T\psi_j-4tT\varphi_j}{|z|^4+16t^2}\,\mathbf{e}_k,\frac{-|z|^2T\varphi_j-4tT\psi_j}{|z|^4+16t^2}\,\mathbf{e}_k\right)^{\mathrm{T}}, \tag{2.27}$$

$$\mathbf{M}_j^{\mathrm{T}}\,\mathbf{V}_k = -\,\mathbf{N}_j^{\mathrm{T}}\,\mathbf{U}_k = \left(\frac{|z|^2T\varphi_j+4tT\psi_j}{|z|^4+16t^2}\,\mathbf{e}_k,\frac{|z|^2T\psi_j-4tT\varphi_j}{|z|^4+16t^2}\,\mathbf{e}_k\right)^{\mathrm{T}}, \tag{2.28}$$

$$\mathbf{P}^{\mathrm{T}}\,\mathbf{U}_k = \left(-4t\,\mathbf{e}_k,\,|z|^2\,\mathbf{e}_k\right)^{\mathrm{T}},\quad \mathbf{P}^{\mathrm{T}}\,\mathbf{V}_k = \left(-|z|^2\,\mathbf{e}_k,\,-4t\,\mathbf{e}_k\right)^{\mathrm{T}}, \tag{2.29}$$

where

$$|z|^2T\psi_j-4tT\varphi_j=4\varphi_j\quad\text{and}\quad |z|^2T\varphi_j+4tT\psi_j=-4\psi_j. \tag{2.30}$$

Corollary 2.5. *We have the inner product of* $2n$*-dimensional column vectors by*

$$\mathbf{U}_j\cdot\mathbf{U}_k = \mathbf{V}_j\cdot\mathbf{V}_k=\frac{1}{|z|^4+16t^2}\,\delta_{jk}, \tag{2.31}$$

$$\mathbf{U}_j\cdot\mathbf{V}_k = 0. \tag{2.32}$$

2.3. Second-order information

We are in a position to derive formulas of sub-Laplacians of the components of (1.3).

Lemma 2.6. *The sub-Laplacians of* φ_i,ψ_i *and* τ *are presented by*

$$\Delta_{\mathbb{H}^n}\varphi_i=-nT\psi_i,\quad \Delta_{\mathbb{H}^n}\psi_i=nT\varphi_i,\quad \Delta_{\mathbb{H}^n}\tau=\frac{8nt|z|^2}{\left(|z|^4+16t^2\right)^2}. \tag{2.33}$$

Proof. By (2.18) and (2.19) in the proof of previous lemma, one has

$$\begin{cases} X_j\varphi_i-Y_j\psi_i=0,\\ Y_j\varphi_i+X_j\psi_i=0, \end{cases} \tag{2.34}$$

which derives the identities involving the second-order derivatives

$$\begin{cases} X_j^2\varphi_i - X_jY_j\psi_i = 0, \\ Y_j^2\varphi_i + Y_jX_j\psi_i = 0. \end{cases} \tag{2.35}$$

Hence the relation (2.4), i.e., $[X_j, Y_j] = -T$, simplifies the expression of the sub-Laplacian (2.8) by

$$\Delta_{\mathbb{H}^n}\varphi_i = \sum_{j=1}^{n}\left(X_j^2\varphi_i + Y_j^2\varphi_i\right) = \sum_{j=1}^{n}\left(X_jY_j\psi_i - Y_jX_j\psi_i\right) = -nT\psi_i.$$

Similarly, there holds

$$\Delta_{\mathbb{H}^n}\psi_i = nT\varphi_i.$$

Furthermore, by finding the second-order derivatives of τ with respect to x_j, y_j, and t, we have

$$\frac{\partial^2\tau}{\partial x_j^2} = \frac{4t\left(2x_j^2 + |z|^2\right)}{(|z|^4 + 16t^2)^2} - \frac{32tx_j^2|z|^4}{(|z|^4 + 16t^2)^3}, \quad \frac{\partial^2\tau}{\partial y_j^2} = \frac{4t\left(2y_j^2 + |z|^2\right)}{(|z|^4 + 16t^2)^2} - \frac{32ty_j^2|z|^4}{(|z|^4 + 16t^2)^3},$$

$$\frac{\partial^2\tau}{\partial t\partial x_j} = \frac{4x_j|z|^2}{(|z|^4 + 16t^2)^2} - \frac{16^2t^2x_j|z|^2}{(|z|^4 + 16t^2)^3}, \quad \frac{\partial^2\tau}{\partial t\partial y_j} = \frac{4y_j|z|^2}{(|z|^4 + 16t^2)^2} - \frac{16^2t^2y_j|z|^2}{(|z|^4 + 16t^2)^3},$$

and

$$\frac{\partial^2\tau}{\partial t^2} = \frac{96t}{(|z|^4 + 16t^2)^2} - \frac{8\times 16^2t^3}{(|z|^4 + 16t^2)^3}.$$

Consequently, one apply (2.9) to get

$$\begin{aligned} \Delta_{\mathbb{H}^n}\tau &= \sum_{j=1}^{n}\left(\frac{\partial^2\tau}{\partial x_j^2} + \frac{\partial^2\tau}{\partial y_j^2} + y_j\frac{\partial^2\tau}{\partial t\partial x_j} - x_j\frac{\partial^2\tau}{\partial t\partial y_j} + \frac{x_j^2 + y_j^2}{4}\frac{\partial^2\tau}{\partial t^2}\right) \\ &= \sum_{j=1}^{n}\left[\frac{40t|z|^2}{(|z|^4 + 16t^2)^2} - \frac{32t|z|^2\left(|z|^4 + 16t^2\right)}{(|z|^4 + 16t^2)^3}\right] = \frac{8nt|z|^2}{(|z|^4 + 16t^2)^2}, \end{aligned}$$

which completes the proof. □

3. Proof of Theorem 1.1

In this section, we first deduce the following proposition concerning the inner products of the horizontal gradients of components of σ. This result reveals the partial orthogonality of gradients, exploits relations of commutators to simplify higher-order derivatives, and plays a crucial role in the proof of the main theorem.

Proposition 3.1. *The inner products of the horizontal gradients of φ_i, ψ_i and τ are expressed by*

$$\nabla_{\mathbb{H}^n}\varphi_j \cdot \nabla_{\mathbb{H}^n}\varphi_k = \nabla_{\mathbb{H}^n}\psi_j \cdot \nabla_{\mathbb{H}^n}\psi_k = \frac{\delta_{jk}}{|z|^4 + 16t^2}, \tag{3.1}$$

$$\nabla_{\mathbb{H}^n}\varphi_j \cdot \nabla_{\mathbb{H}^n}\psi_k = 0, \tag{3.2}$$

$$\nabla_{\mathbb{H}^n}\tau \cdot \nabla_{\mathbb{H}^n}\varphi_j \;=\; \frac{\psi_j}{2\left(|z|^4+16t^2\right)}, \qquad \nabla_{\mathbb{H}^n}\tau \cdot \nabla_{\mathbb{H}^n}\psi_j \;=\; -\frac{\varphi_j}{2\left(|z|^4+16t^2\right)}, \tag{3.3}$$

$$\nabla_{\mathbb{H}^n}\tau \cdot \nabla_{\mathbb{H}^n}\tau \;=\; \frac{|z|^2}{4\left(|z|^4+16t^2\right)^2}. \tag{3.4}$$

Proof. We shall prove each identity using the matrix representations (2.11)–(2.13) and the tools in Subsection (2.2). Below we begin calculating each sub-item to facilitate the subsequent derivation.
(1) We expand $\nabla_{\mathbb{H}^n}\varphi_j \cdot \nabla_{\mathbb{H}^n}\varphi_k$ via

$$\begin{aligned}\nabla_{\mathbb{H}^n}\varphi_j \cdot \nabla_{\mathbb{H}^n}\varphi_k &= \left(\frac{1}{2}\,\mathbf{M}_j\,\mathbf{z}+\mathbf{U}_j\right)\cdot\left(\frac{1}{2}\,\mathbf{M}_k\,\mathbf{z}+\mathbf{U}_k\right)\\ &= \frac{1}{4}\,\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{M}_k\,\mathbf{z}+\frac{1}{2}\,\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{U}_k+\frac{1}{2}\,\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_k^{\mathrm{T}}\,\mathbf{U}_j+\mathbf{U}_j\cdot\mathbf{U}_k.\end{aligned}$$

By the multiplication of block matrices (2.20) and (2.25), one has

$$\frac{1}{4}\,\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{M}_k\,\mathbf{z} = \frac{4|z|^2\left(x_jx_k+y_jy_k\right)}{\left(|z|^4+16t^2\right)^2}.$$

Adopting (2.27), we calculate of the product of a matrix and a vector $\mathbf{M}_j^{\mathrm{T}}\,\mathbf{U}_k$ by

$$\mathbf{M}_j^{\mathrm{T}}\,\mathbf{U}_k = \left(\frac{4\varphi_j}{|z|^4+16t^2}\,\mathbf{e_k},\quad \frac{4\psi_j}{|z|^4+16t^2}\,\mathbf{e_k}\right)^{\mathrm{T}},$$

which leads to

$$\frac{1}{2}\,\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{U}_k = \frac{1}{\left(|z|^4+16t^2\right)^2}\left[8t\left(x_ky_j-x_jy_k\right)-2|z|^2\left(x_jx_k+y_jy_k\right)\right].$$

Likewise, one obtains that

$$\frac{1}{2}\,\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_k^{\mathrm{T}}\,\mathbf{U}_j = \frac{1}{\left(|z|^4+16t^2\right)^2}\left[8t\left(x_jy_k-x_ky_j\right)-2|z|^2\left(x_jx_k+y_jy_k\right)\right].$$

Via (2.31), one gets that

$$\mathbf{U}_j\cdot\mathbf{U}_k = \frac{1}{|z|^4+16t^2}\,\delta_{jk}.$$

Therefore, one deduces that

$$\nabla_{\mathbb{H}^n}\varphi_j \cdot \nabla_{\mathbb{H}^n}\varphi_k = \frac{1}{|z|^4+16t^2}\,\delta_{jk}.$$

Adopting a similar argument, there holds

$$\nabla_{\mathbb{H}^n}\psi_j \cdot \nabla_{\mathbb{H}^n}\psi_k = \frac{1}{|z|^4+16t^2}\,\delta_{jk},$$

which establishes (3.1).

(2) For the expression of the mixed terms, we combine (2.11) and (2.12) to see that

$$\begin{aligned}\nabla_{\mathbb{H}^n}\varphi_j \cdot \nabla_{\mathbb{H}^n}\psi_k &= \left(\frac{1}{2}\mathbf{M}_j\,\mathbf{z} + \mathbf{U}_j\right)\cdot\left(\frac{1}{2}\mathbf{N}_k\,\mathbf{z} + \mathbf{V}_k\right)\\ &= \frac{1}{4}\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{N}_k\,\mathbf{z} + \frac{1}{2}\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{V}_k + \frac{1}{2}\mathbf{z}^{\mathrm{T}}\,\mathbf{N}_k^{\mathrm{T}}\,\mathbf{U}_j + \mathbf{U}_j\cdot\mathbf{V}_k.\end{aligned}$$

From the multiplication of block matrices (2.21) and (2.25), we have

$$\frac{1}{4}\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{N}_k\,\mathbf{z} = \frac{4|z|^2\left(x_jy_k - y_jx_k\right)}{(|z|^4+16t^2)^2}.$$

By (2.28) and (2.30), one has

$$\frac{1}{2}\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{V}_k = \frac{1}{(|z|^4+16t^2)^2}\left[8t\left(x_jx_k + y_jy_k\right) + 2|z|^2\left(x_ky_j - y_kx_j\right)\right],$$

and

$$\frac{1}{2}\mathbf{z}^{\mathrm{T}}\,\mathbf{N}_k^{\mathrm{T}}\,\mathbf{U}_j = -\frac{1}{(|z|^4+16t^2)^2}\left[8t\left(x_jx_k + y_jy_k\right) + 2|z|^2\left(x_jy_k - y_jx_k\right)\right].$$

It follows from (2.32) that $\mathbf{U}_j\cdot\mathbf{V}_k = 0$. There holds that

$$\nabla_{\mathbb{H}^n}\varphi_j\cdot\nabla_{\mathbb{H}^n}\psi_k = 0,$$

which establishes (3.2).

(3) We pay attention to the mixture term of τ, and continue in this way to obtain

$$\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\varphi_j = \frac{1}{2\,(|z|^4+16t^2)^2}\left(\frac{1}{2}\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{P}\,\mathbf{z} + \mathbf{z}^{\mathrm{T}}\,\mathbf{P}^{\mathrm{T}}\,\mathbf{U}_j\right).$$

By (2.23) and (2.26), one obtains that

$$\frac{1}{2}\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_j^{\mathrm{T}}\,\mathbf{P}\,\mathbf{z} = -2y_j|z|^2.$$

Via (2.29), we deduce that

$$\mathbf{z}^{\mathrm{T}}\,\mathbf{P}^{\mathrm{T}}\,\mathbf{U}_j = -4tx_j + y_j|z|^2,$$

which gives us

$$\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{M}_j^{\mathrm{T}}\mathbf{P}\mathbf{z} + \mathbf{z}^{\mathrm{T}}\,\mathbf{P}^{\mathrm{T}}\,\mathbf{U}_j = -4tx_j - y_j|z|^2 = \left(|z|^4+16t^2\right)\psi_j.$$

Thus, there holds

$$\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\varphi_j = \frac{1}{2\,(|z|^4+16t^2)}\psi_j.$$

An argument similar to the one used in $\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\psi_j$ shows that

$$\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\psi_j = -\frac{1}{2\,(|z|^4+16t^2)}\varphi_j.$$

Hence, (3.3) is obtained.
(4) We investigate the inner product

$$\nabla_{\mathbb{H}^n}\tau \cdot \nabla_{\mathbb{H}^n}\tau = \frac{1}{4\left(|z|^4 + 16t^2\right)^4}\, \mathbf{z}^{\mathrm{T}}\, \mathbf{P}^{\mathrm{T}}\, \mathbf{P}\, \mathbf{z}.$$

By applying (2.22), one has

$$\mathbf{z}^{\mathrm{T}}\, \mathbf{P}^{\mathrm{T}}\, \mathbf{P}\, \mathbf{z} = \left(|z|^4 + 16t^2\right)^2 |z|^2,$$

that is,

$$\nabla_{\mathbb{H}^n}\tau \cdot \nabla_{\mathbb{H}^n}\tau = \frac{|z|^2}{4\left(|z|^4 + 16t^2\right)^2}.$$

Combining (1)–(4), we complete the proof. □

Remark 3.2. Combining (1.4) and (3.4), we obtain that

$$\nabla_{\mathbb{H}^n}\tau \cdot \nabla_{\mathbb{H}^n}\tau = \frac{1}{4\left(|z|^4 + 16t^2\right)} \sum_{i=1}^{n} \left(\varphi_i^2 + \psi_i^2\right). \tag{3.5}$$

Based on the preceding tools and conclusions, we show the identity involving the Kelvin transform and the sub-Laplacian and present the algebraic proof; see [19] for other methods.

Proof of Theorem 1.1. Let $\varpi = \rho^{-(Q-2)}$ for notational simplicity, where $\rho = \left(|z|^4 + 16t^2\right)^{\frac{1}{4}}$ is the homogeneous norm on $\mathbb{H}^n$. Let σ and u^* be defined in (1.3) and (1.5), respectively. By the chain rule for the sub-Laplacian $\Delta_{\mathbb{H}^n}$, we decompose the expression as follows:

$$\Delta_{\mathbb{H}^n}\left(u^*(z,t)\right) = \varpi\, \Delta_{\mathbb{H}^n}\left(u \circ \sigma\right) + 2\, \nabla_{\mathbb{H}^n}\varpi \cdot \nabla_{\mathbb{H}^n}\left(u \circ \sigma\right) + \left(u \circ \sigma\right) \cdot \Delta_{\mathbb{H}^n}\varpi.$$

Since ϖ is harmonic on $\mathbb{H}^n\backslash\{0\}$, i.e., $\Delta_{\mathbb{H}^n}\varpi = 0$, then it remains to prove

$$\varpi\, \Delta_{\mathbb{H}^n}\left(u \circ \sigma\right) + 2\, \nabla_{\mathbb{H}^n}\varpi \cdot \nabla_{\mathbb{H}^n}\left(u \circ \sigma\right) = \rho^{-Q-2}\left(\Delta_{\mathbb{H}^n} u\right)\left(\sigma(z,t)\right). \tag{3.6}$$

(1) We calculate the $\varpi\, \Delta_{\mathbb{H}^n}\left(u \circ \sigma\right)$. By recalling the horizontal gradient and divergence operator on $\mathbb{H}^n\backslash\{0\}$, there holds

$$\Delta_{\mathbb{H}^n}\left(u \circ \sigma\right) = \mathrm{div}_{\mathbb{H}^n}\left[\nabla_{\mathbb{H}^n}\left(u \circ \sigma\right)\right] = \left(X_1, \ldots, X_n, Y_1, \ldots, Y_n\right) \begin{pmatrix} X_1\left(u \circ \sigma\right) \\ \vdots \\ X_n\left(u \circ \sigma\right) \\ Y_1\left(u \circ \sigma\right) \\ \vdots \\ Y_n\left(u \circ \sigma\right) \end{pmatrix}, \tag{3.7}$$

and

$$\begin{pmatrix} X_1(u\circ\sigma) \\ \vdots \\ X_n(u\circ\sigma) \\ Y_1(u\circ\sigma) \\ \vdots \\ Y_n(u\circ\sigma) \end{pmatrix} = \begin{pmatrix} X_1\varphi_1 & \cdots & X_1\varphi_n & X_1\psi_1 & \cdots & X_1\psi_n & X_1\tau \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ X_n\varphi_1 & \cdots & X_n\varphi_n & X_n\psi_1 & \cdots & X_n\psi_n & X_n\tau \\ Y_1\varphi_1 & \cdots & Y_1\varphi_n & Y_1\psi_1 & \cdots & Y_1\psi_n & Y_1\tau \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_n\varphi_1 & \cdots & Y_n\varphi_n & Y_n\psi_1 & \cdots & Y_n\psi_n & Y_n\tau \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial\varphi_1} \\ \vdots \\ \frac{\partial u}{\partial\varphi_n} \\ \frac{\partial u}{\partial\psi_1} \\ \vdots \\ \frac{\partial u}{\partial\psi_n} \\ \frac{\partial u}{\partial\tau} \end{pmatrix}. \tag{3.8}$$

Using (3.7) and (3.8), we get that

$$\begin{aligned} \Delta_{\mathbb{H}^n}(u\circ\sigma) &= \begin{pmatrix} \Delta_{\mathbb{H}^n}\varphi_1 & \cdots & \Delta_{\mathbb{H}^n}\varphi_n & \Delta_{\mathbb{H}^n}\psi_1 & \cdots & \Delta_{\mathbb{H}^n}\psi_1 & \Delta_{\mathbb{H}^n}\tau \end{pmatrix} \\ &\quad \times \begin{pmatrix} \frac{\partial u}{\partial\varphi_1} & \cdots & \frac{\partial u}{\partial\varphi_n} & \frac{\partial u}{\partial\psi_1} & \cdots & \frac{\partial u}{\partial\psi_n} & \frac{\partial u}{\partial\tau} \end{pmatrix}^{\mathrm{T}} \\ &\quad + \begin{pmatrix} X_1\varphi_1 & \cdots & X_1\varphi_n & X_1\psi_1 & \cdots & X_1\psi_n & X_1\tau \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ X_n\varphi_1 & \cdots & X_n\varphi_n & X_n\psi_1 & \cdots & X_n\psi_n & X_n\tau \\ Y_1\varphi_1 & \cdots & Y_1\varphi_n & Y_1\psi_1 & \cdots & Y_1\psi_n & Y_1\tau \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_n\varphi_1 & \cdots & Y_n\varphi_n & Y_n\psi_1 & \cdots & Y_n\psi_n & Y_n\tau \end{pmatrix} \\ &\quad \times \begin{pmatrix} X_1\left(\frac{\partial u}{\partial\varphi_1}\right) \cdots X_1\left(\frac{\partial u}{\partial\varphi_n}\right) X_1\left(\frac{\partial u}{\partial\psi_1}\right) \cdots X_1\left(\frac{\partial u}{\partial\psi_n}\right) X_1\left(\frac{\partial u}{\partial\tau}\right) \\ \vdots \quad \ddots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ X_n\left(\frac{\partial u}{\partial\varphi_1}\right) \cdots X_n\left(\frac{\partial u}{\partial\varphi_n}\right) X_n\left(\frac{\partial u}{\partial\psi_1}\right) \cdots X_n\left(\frac{\partial u}{\partial\psi_n}\right) X_n\left(\frac{\partial u}{\partial\tau}\right) \\ Y_1\left(\frac{\partial u}{\partial\varphi_1}\right) \cdots Y_1\left(\frac{\partial u}{\partial\varphi_n}\right) Y_1\left(\frac{\partial u}{\partial\psi_1}\right) \cdots Y_1\left(\frac{\partial u}{\partial\psi_n}\right) Y_1\left(\frac{\partial u}{\partial\tau}\right) \\ \vdots \quad \ddots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ Y_n\left(\frac{\partial u}{\partial\varphi_1}\right) \cdots Y_n\left(\frac{\partial u}{\partial\varphi_n}\right) Y_n\left(\frac{\partial u}{\partial\psi_1}\right) \cdots Y_n\left(\frac{\partial u}{\partial\psi_n}\right) Y_n\left(\frac{\partial u}{\partial\tau}\right) \end{pmatrix} \\ &= \Upsilon_1 + \Upsilon_2, \end{aligned}$$

where

$$\begin{aligned} \Upsilon_1 &= \sum_{i=1}^{n}\left(\frac{\partial u}{\partial\varphi_i}\cdot\Delta_{\mathbb{H}^n}\varphi_i + \frac{\partial u}{\partial\psi_i}\cdot\Delta_{\mathbb{H}^n}\psi_i\right) + \frac{\partial u}{\partial\tau}\cdot\Delta_{\mathbb{H}^n}\tau \\ &= \sum_{i=1}^{n}\left(-nT\psi_i\cdot\frac{\partial u}{\partial\varphi_i} + nT\varphi_i\cdot\frac{\partial u}{\partial\psi_i}\right) + \frac{8nt|z|^2}{\left(|z|^4+16t^2\right)^2}\cdot\frac{\partial u}{\partial\tau}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\Upsilon_2 &= \sum_{i,j=1}^{n}\left[X_i\varphi_j\cdot X_i\left(\frac{\partial u}{\partial\varphi_j}\right)+X_i\psi_j\cdot X_i\left(\frac{\partial u}{\partial\psi_j}\right)+Y_i\varphi_j\cdot Y_i\left(\frac{\partial u}{\partial\varphi_j}\right)+Y_i\psi_j\cdot Y_i\left(\frac{\partial u}{\partial\psi_j}\right)\right]\\ &\quad+\sum_{i=1}^{n}\left[X_i\tau\cdot X_i\left(\frac{\partial u}{\partial\tau}\right)+Y_i\tau\cdot Y_i\left(\frac{\partial u}{\partial\tau}\right)\right].\end{aligned}$$

To deal with Υ_2, by the derivative rule of composite functions and the definition of horizontal gradients, we note that

$$\begin{aligned}X_i\left(\frac{\partial u}{\partial\varphi_j}\right) &= \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial\varphi_j\partial\varphi_k}\cdot X_i\varphi_k+\frac{\partial^2 u}{\partial\varphi_j\partial\psi_k}\cdot X_i\psi_k\right)+\frac{\partial^2 u}{\partial\varphi_j\partial\tau}\cdot X_i\tau,\\ X_i\left(\frac{\partial u}{\partial\psi_j}\right) &= \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial\psi_j\partial\varphi_k}\cdot X_i\varphi_k+\frac{\partial^2 u}{\partial\psi_j\partial\psi_k}\cdot X_i\psi_k\right)+\frac{\partial^2 u}{\partial\psi_j\partial\tau}\cdot X_i\tau,\\ X_i\left(\frac{\partial u}{\partial\tau}\right) &= \sum_{k=1}^{n}\left(\frac{\partial^2 u}{\partial\tau\partial\varphi_k}\cdot X_i\varphi_k+\frac{\partial^2 u}{\partial\tau\partial\psi_k}\cdot X_i\psi_k\right)+\frac{\partial^2 u}{\partial\tau^2}\cdot X_i\tau.\end{aligned}$$

There hold similar equalities involving Y_i. It follows from Proposition 3.1 that

$$\begin{aligned}\Upsilon_2 &= \sum_{j,k=1}^{n}\left[\frac{\partial^2 u}{\partial\varphi_j\partial\varphi_k}\left(\nabla_{\mathbb{H}^n}\varphi_j\cdot\nabla_{\mathbb{H}^n}\varphi_k\right)+\frac{\partial^2 u}{\partial\psi_j\partial\psi_k}\left(\nabla_{\mathbb{H}^n}\psi_j\cdot\nabla_{\mathbb{H}^n}\psi_k\right)+2\frac{\partial^2 u}{\partial\varphi_j\partial\psi_k}\left(\nabla_{\mathbb{H}^n}\varphi_j\cdot\nabla_{\mathbb{H}^n}\psi_k\right)\right]\\ &\quad+2\sum_{j=1}^{n}\left[\frac{\partial^2 u}{\partial\tau\partial\varphi_j}\left(\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\varphi_j\right)+\frac{\partial^2 u}{\partial\tau\partial\psi_j}\left(\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\psi_j\right)\right]+\frac{\partial^2 u}{\partial\tau^2}\left(\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\tau\right)\\ &= \frac{1}{|z|^4+16t^2}\sum_{j=1}^{n}\left(\frac{\partial^2 u}{\partial\varphi_j^2}+\frac{\partial^2 u}{\partial\psi_j^2}+\psi_j\frac{\partial^2 u}{\partial\tau\partial\varphi_j}-\varphi_j\frac{\partial^2 u}{\partial\tau\partial\psi_j}\right)+\frac{|z|^2}{4(|z|^4+16t^2)^2}\frac{\partial^2 u}{\partial\tau^2}.\end{aligned}$$

By (3.5), one gets

$$\begin{aligned}\Upsilon_2 &= \frac{1}{|z|^4+16t^2}\sum_{j=1}^{n}\left(\frac{\partial^2 u}{\partial\varphi_j^2}+\frac{\partial^2 u}{\partial\psi_j^2}+\psi_j\frac{\partial^2 u}{\partial\tau\partial\varphi_j}-\varphi_j\frac{\partial^2 u}{\partial\tau\partial\psi_j}+\frac{\varphi_i^2+\psi_i^2}{4}\frac{\partial^2 u}{\partial\tau^2}\right)\\ &= \frac{1}{|z|^4+16t^2}\left(\Delta_{\mathbb{H}^n}u\right)\left(\sigma(z,t)\right).\end{aligned}\tag{3.10}$$

From (3.9) and (3.10), we deduce that

$$\begin{aligned}\Delta_{\mathbb{H}^n}\left(u\circ\sigma\right) &= \sum_{i=1}^{n}\left(-nT\psi_i\cdot\frac{\partial u}{\partial\varphi_i}+nT\varphi_i\cdot\frac{\partial u}{\partial\psi_i}\right)+\frac{8nt|z|^2}{\left(|z|^4+16t^2\right)^2}\cdot\frac{\partial u}{\partial\tau}\\ &\quad+\frac{1}{|z|^4+16t^2}\left(\Delta_{\mathbb{H}^n}u\right)\left(\sigma(z,t)\right).\end{aligned}\tag{3.11}$$

(2) We calculate the second term $\nabla_{\mathbb{H}^n}\varpi\cdot\nabla_{\mathbb{H}^n}\left(u\circ\sigma\right)$. For $\varpi=\left(|z|^4+16t^2\right)^{-\frac{n}{2}}$, it is clear that

$$X_j\varpi = \frac{\partial\varpi}{\partial x_i}+\frac{y_i}{2}\frac{\partial\varpi}{\partial t}=-2n\left(|z|^4+16t^2\right)^{-\frac{n+2}{2}}\left[x_j|z|^2+4ty_j\right],$$

$$Y_j\varpi \;=\; \frac{\partial\varpi}{\partial y_i} - \frac{x_i}{2}\frac{\partial\varpi}{\partial t} = -2n\left(|z|^4 + 16t^2\right)^{-\frac{n+2}{2}}\left[y_j|z|^2 - 4tx_j\right].$$

For the sake of convenience, we set

$$\nabla_{\mathbb{H}^n}\varpi = -2n\left(|z|^4 + 16t^2\right)^{-\frac{n+2}{2}}\mathbf{F}\,\mathbf{z},$$

where the matrix $\mathbf{F} = \begin{pmatrix} |z|^2\,\mathbf{I}_n & 4t\,\mathbf{I}_n \\ -4t\,\mathbf{I}_n & |z|^2\,\mathbf{I}_n \end{pmatrix}_{2n\times 2n}$. Applying the chain rule for $\nabla_{\mathbb{H}^n}(u\circ\sigma)$, we have

$$\begin{aligned} X_j(u\circ\sigma) &= \sum_{i=1}^{n}\left(\frac{\partial u}{\partial\varphi_i}\cdot X_j\varphi_i + \frac{\partial u}{\partial\psi_i}\cdot X_j\psi_i\right) + \frac{\partial u}{\partial\tau}\cdot X_j\tau, \\ Y_j(u\circ\sigma) &= \sum_{i=1}^{n}\left(\frac{\partial u}{\partial\varphi_i}\cdot Y_j\varphi_i + \frac{\partial u}{\partial\psi_i}\cdot Y_j\psi_i\right) + \frac{\partial u}{\partial\tau}\cdot Y_j\tau. \end{aligned}$$

Thus, we have the expressions by

$$\begin{aligned} &\nabla_{\mathbb{H}^n}(u\circ\sigma)\cdot\nabla_{\mathbb{H}^n}\varpi \qquad (3.12)\\ =\;& \sum_{j=1}^{n}\left[X_j(u\circ\sigma)\cdot X_j\varpi + Y_j(u\circ\sigma)\cdot Y_j\varpi\right] \\ =\;& \sum_{j=1}^{n}\Bigg[\sum_{i=1}^{n}\left(\frac{\partial u}{\partial\varphi_i}\cdot X_j\varphi_i\cdot X_j\varpi + \frac{\partial u}{\partial\psi_i}\cdot X_j\psi_i\cdot X_j\varpi + \frac{\partial u}{\partial\varphi_i}\cdot Y_j\varphi_i\cdot Y_j\varpi + \frac{\partial u}{\partial\psi_i}\cdot Y_j\psi_i\cdot Y_j\varpi\right) \\ &\quad + \frac{\partial u}{\partial\tau}\cdot X_j\tau\cdot X_j\varpi + \frac{\partial u}{\partial\tau}\cdot Y_j\tau\cdot Y_j\varpi\Bigg] \\ =\;& \sum_{i=1}^{n}\left[\frac{\partial u}{\partial\varphi_i}\left(\nabla_{\mathbb{H}^n}\varphi_i\cdot\nabla_{\mathbb{H}^n}\varpi\right) + \frac{\partial u}{\partial\psi_i}\left(\nabla_{\mathbb{H}^n}\psi_i\cdot\nabla_{\mathbb{H}^n}\varpi\right)\right] + \frac{\partial u}{\partial\tau}\left(\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\varpi\right). \end{aligned}$$

Using the matrices and vectors introduced in Lemma 2.2, we note that

$$\nabla_{\mathbb{H}^n}\varphi_i\cdot\nabla_{\mathbb{H}^n}\varpi = -2n\left(|z|^4 + 16t^2\right)^{-\frac{n+2}{2}}\left[\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{M}_i^{\mathrm{T}}\mathbf{F}\,\mathbf{z} + \mathbf{z}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{U}_i\right],$$

where

$$\mathbf{M}_i^{\mathrm{T}}\mathbf{F} = \begin{pmatrix} \left(-|z|^2T\psi_i + 4tT\varphi_i\right)\mathbf{I}_n & \left(-|z|^2T\varphi_i - 4tT\psi_i\right)\mathbf{I}_n \\ \left(|z|^2T\varphi_i + 4tT\psi_i\right)\mathbf{I}_n & \left(-|z|^2T\psi_i + 4tT\varphi_i\right)\mathbf{I}_n \end{pmatrix} = \begin{pmatrix} -4\varphi_i\mathbf{I}_n & 4\psi_i\mathbf{I}_n \\ -4\psi_i\mathbf{I}_n & -4\varphi_i\mathbf{I}_n \end{pmatrix}.$$

Hence, one obtains $\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{M}_i^{\mathrm{T}}\mathbf{F}\,\mathbf{z} = -2|z|^2\varphi_i$. We calculate

$$\mathbf{F}^{\mathrm{T}}\mathbf{U}_i = \left(\frac{-|z|^4 - 16t^2}{|z|^4 + 16t^2}\,\mathbf{e_i},\quad \frac{-4t|z|^2 + 4t|z|^2}{|z|^4 + 16t^2}\,\mathbf{e_i}\right)^{\mathrm{T}} = \left(-\mathbf{e_i},\quad \mathbf{0}\right)^{\mathrm{T}},$$

and so

$$\frac{1}{2}\,\mathbf{z}^{\mathrm{T}}\,\mathbf{M}_i^{\mathrm{T}}\,\mathbf{F}\,\mathbf{z}+\mathbf{z}^{\mathrm{T}}\,\mathbf{F}^{\mathrm{T}}\,\mathbf{U}_i \;=\; \frac{-2|z|^2\left(4ty_i-x_i|z|^2\right)-x_i\left(|z|^4+16t^2\right)}{|z|^4+16t^2} \;=\; -\frac{1}{4}\left(|z|^4+16t^2\right)T\psi_i.$$

Thus, we can draw a conclusion

$$\nabla_{\mathbb{H}^n}\varphi_i\cdot\nabla_{\mathbb{H}^n}\varpi=\frac{n}{2}\left(|z|^4+16t^2\right)^{-\frac{n}{2}}T\psi_i, \tag{3.13}$$

and similarly

$$\nabla_{\mathbb{H}^n}\psi_i\cdot\nabla_{\mathbb{H}^n}\varpi=-\frac{n}{2}\left(|z|^4+16t^2\right)^{-\frac{n}{2}}T\varphi_i. \tag{3.14}$$

For $\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\varpi$, we know

$$\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\varpi=-n\left(|z|^4+16t^2\right)^{-\frac{n+6}{2}}\,\mathbf{z}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\,\mathbf{F}\,\mathbf{z},$$

where

$$\mathbf{P}^{\mathrm{T}}\,\mathbf{F}=\left(|z|^4+16t^2\right)\begin{pmatrix}4t\,\mathbf{I}_n & |z|^2\,\mathbf{I}_n\\ -|z|^2\,\mathbf{I}_n & 4t\,\mathbf{I}_n\end{pmatrix}.$$

It follows that $\mathbf{z}^{\mathrm{T}}\,\mathbf{P}^{\mathrm{T}}\,\mathbf{F}\,\mathbf{z}=4t|z|^2\left(|z|^4+16t^2\right)$, which means

$$\nabla_{\mathbb{H}^n}\tau\cdot\nabla_{\mathbb{H}^n}\varpi=-4n\left(|z|^4+16t^2\right)^{-\frac{n+4}{2}}t|z|^2. \tag{3.15}$$

Accordingly, we apply (3.12)–(3.15) to conclude that

$$2\,\nabla_{\mathbb{H}^n}u\cdot\nabla_{\mathbb{H}^n}\varpi=-n\left(|z|^4+16t^2\right)^{-\frac{n}{2}}\left[\sum_{i=1}^{n}\left(-T\psi_i\frac{\partial u}{\partial\varphi_i}+T\varphi_i\frac{\partial u}{\partial\psi_i}\right)+\frac{8t|z|^2}{(|z|^4+16t^2)^2}\frac{\partial u}{\partial\tau}\right]. \tag{3.16}$$

Plugging (3.11) and (3.16) in (3.6), we have

$$\varpi\,\Delta_{\mathbb{H}^n}\left(u\circ\sigma\right)+2\,\nabla_{\mathbb{H}^n}\varpi\cdot\nabla_{\mathbb{H}^n}u=\left(|z|^4+16t^2\right)^{-\frac{n+2}{2}}\left(\Delta_{\mathbb{H}^n}u\right)\left(\sigma(z,t)\right),$$

which completes the proof of (3.6) and so (1.6). □

4. Proof of Theorem 1.2

Building upon the Kelvin transform and sub-Laplacian results, we investigate their implications for Hardy-Sobolev type inequality on the Heisenberg group.

We state the following inequality.

Lemma 4.1 (Hardy-Sobolev type inequality). *Let $0\le s<2$ and $Q=2n+2$. Denote by $D^{1,2}\left(\mathbb{H}^n\right)$ the closure of $C_0^\infty\left(\mathbb{H}^n\right)$ with respect to the norm $\|u\|=\left(\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^2\,\mathrm{d}z\,\mathrm{d}t\right)^{\frac{1}{2}}$. For any $u\in D^{1,2}\left(\mathbb{H}^n\backslash\{0\}\right)$, there holds*

$$\int_{\mathbb{H}^n}\frac{|z|^2}{\rho^2}\frac{|u|^{2^*(s)}}{\rho^s}\,\mathrm{d}z\,\mathrm{d}t\;\le\;C\left(\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^2\,\mathrm{d}z\,\mathrm{d}t\right)^{\frac{Q-s}{Q-2}},$$

where $2^*(s)=\dfrac{2(Q-s)}{Q-2}$ and $C=C\left(s,Q\right)$ is a positive constant independent of u.

In [20], the Hardy-Sobolev type inequality

$$\int_{\mathbb{H}^n} \frac{|z|^s}{\rho^s} \frac{|u|^{2^*(s)}}{\rho^s} \,\mathrm{d}z\,\mathrm{d}t \le \left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{G}} u|^2 \,\mathrm{d}z\,\mathrm{d}t \right)^{\frac{Q-s}{Q-2}}$$

is established, where $|z| < \rho$ and $s < 2$. Thus, one deduces the generalized Hardy-Sobolev type inequality (1.7).

Remark 4.2. The weight $\frac{|z|^2}{\rho^2}$ reflects the non-isotropic geometry of $\mathbb{H}^n$, contrasting with the Euclidean case (1.2) with $\frac{|x|^2}{|x|^2} \equiv 1$. The exponent $\frac{2(Q-s)}{Q-2}$ interpolates between two endpoints, i.e., the Sobolev case ($s = 0$) and the Hardy case ($s = 2$) .

Lemma 4.3 (Invariance under the Kelvin transform)**.** *If u is a solution to* (1.8)*, then its Kelvin transform u^* satisfies Eq* (1.8)*.*

Proof. We consider $-\Delta_{\mathbb{H}^n} u^*(z,t)$. From the definition of inversion, it is clear that

$$-\Delta_{\mathbb{H}^n} u\,(\sigma(z,t)) = (Q-s)(Q-2)\frac{|z|^2}{\rho^{2-s}} \cdot \left[\rho^{Q-2} u^*\right]^{\frac{Q-2s+2}{Q-2}} = (Q-s)(Q-2)\frac{|z|^2}{\rho^{s-Q}} \cdot (u^*)^{\frac{Q-2s+2}{Q-2}} .$$

By Theorem (1.1), one has

$$-\Delta_{\mathbb{H}^n} u^*(z,t) = \frac{1}{\rho^{Q+2}} \cdot (-\Delta_{\mathbb{H}^n} u\,(\sigma(z,t))) = (Q-s)(Q-2)\frac{|z|^2}{\rho^2} \frac{(u^*)^{\frac{Q-2s+2}{Q-2}}}{\rho^s}.$$

This gives us that, if a cylindrical function u satisfies Eq (1.8) in $\mathbb{H}^n$, then the Kelvin transform

$$u^*\,(z,t) = \frac{1}{\rho^{Q-2}} u\,(\sigma\,(z,t))$$

of u satisfies the same Eq (1.8). This fact reveals the invariance of the equation in the Heisenberg group. □

We establish a conformally invariant Kelvin transform on $\mathbb{H}^n$ in Theorem 1.1 and shall apply it to derive the best constant of the Hardy-Sobolev type inequality.

Proof of Theorem 1.2. We divide the proof into two steps.

Step 1: Find the cylindrically symmetric solution.

We introduce polar coordinates of z and set $r = |z|$. From (2.9), we express $\Delta_{\mathbb{H}^n} u$ by

$$\Delta_{\mathbb{H}^n} u = \frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r}\frac{\partial u}{\partial r} + \frac{r^2}{4}\frac{\partial^2 u}{\partial t^2}. \tag{4.1}$$

Substituting this into (1.8), it yields the following PDE in (r,t)

$$\frac{\partial^2 u}{\partial r^2} + \frac{2n-1}{r}\frac{\partial u}{\partial r} + \frac{r^2}{4}\frac{\partial^2 u}{\partial t^2} = -(Q-s)(Q-2)\frac{r^2}{(r^4+16t^2)^{\frac{2+s}{4}}} \cdot u(r,t)^{\frac{Q-2s+2}{Q-2}}. \tag{4.2}$$

Since $u = u(g)$, then by performing the substitution $g = \left(r^4 + 16t^2\right)^{\frac{1}{4}}$, we reduce (4.2) to the following ODE:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}g^2} + \frac{2n+1}{g}\frac{\mathrm{d}u}{\mathrm{d}g} = -(Q-s)(Q-2)\frac{u(g)^{\frac{Q-2s+2}{Q-2}}}{g^s}. \tag{4.3}$$

Since both u and its inversion u^* satisfy Eq (4.3), by we have $u(g) = g^{2-Q}u(1/g)$. It follows that $u(g) = 1/\left(1 + g^{2-s}\right)^{\frac{Q-2}{2-s}}$. That is, we find the cylindrically symmetric solution (1.9) to Eq (1.8). It is easy to check that

$$u^*(z,t) = \frac{1}{\rho^{Q-2}\left[1 + \rho^{-(2-s)}\right]^{\frac{Q-2}{2-s}}} = \frac{1}{\left(1 + \rho^{2-s}\right)^{\frac{Q-2}{2-s}}} = u(z,t),$$

and the solution is invariant under the Kelvin transform.

Step 2: Calculating the best constant of the Hardy-Sobolev type inequality.

We consider the extremal problem

$$I = \inf\left\{ \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 \,\mathrm{d}z\,\mathrm{d}t \,\middle|\, u \in D^{1,2}(\mathbb{H}^n), \int_{\mathbb{H}^n} \frac{|x|^2}{\rho^2}\frac{|u|^{2^*(s)}}{\rho^s}\,\mathrm{d}z\,\mathrm{d}t = 1 \right\}.$$

By Remark 3.4 in [20], the extremal function of (1.7) is essentially the solution (1.9), which gives us the way to calculate the best constant.

Differentiating the solution (1.9), one has

$$\frac{\partial u}{\partial x_i} = \frac{-(Q-2)\,x_i|z|^2\rho^{-2-s}}{\left(1+\rho^{2-s}\right)^{\frac{Q-s}{2-s}}}, \quad \frac{\partial u}{\partial t} = \frac{-8\,(Q-2)\,t\rho^{-2-s}}{\left(1+\rho^{2-s}\right)^{\frac{Q-s}{2-s}}}, \tag{4.4}$$

and it follows that

$$\begin{aligned} X_i u &= -(Q-2)\frac{\left[x_i|z|^2 + 4ty_i\right]\rho^{-2-s}}{\left(1+\rho^{2-s}\right)^{\frac{Q-s}{2-s}}}, \\ Y_i u &= -(Q-2)\frac{\left[y_i|z|^2 - 4tx_i\right]\rho^{-2-s}}{\left(1+\rho^{2-s}\right)^{\frac{Q-s}{2-s}}}. \end{aligned}$$

Therefore,

$$|\nabla_{\mathbb{H}^n} u|^2 = \sum_{i=1}^n \left[(X_i u)^2 + (Y_i u)^2\right] = (Q-2)^2 \frac{|z|^2}{\rho^{2s}\left(1+\rho^{2-s}\right)^{\frac{2(Q-s)}{2-s}}}. \tag{4.5}$$

In order to find the best constant for inequality (1.7), we introduce

$$\mathrm{I} = \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 \,\mathrm{d}z\,\mathrm{d}t, \tag{4.6}$$

$$\mathrm{II} = \int_{\mathbb{H}^n} \frac{|z|^2\,|u|^{2^*(s)}}{\rho^{2+s}}\,\mathrm{d}z\,\mathrm{d}t. \tag{4.7}$$

Then the desired inequality $\mathrm{II} \le C \times \mathrm{I}^{\frac{Q-s}{Q-2}}$ is transformed into

$$C \ge \frac{\mathrm{II}}{\mathrm{I}^{\frac{Q-s}{Q-2}}}. \tag{4.8}$$

The calculation of the constant C requires evaluation of two key integrals.

The first integral is expressed by

$$\mathrm{I} = (Q-2)^2 \int_{\mathbb{H}^n} \frac{|z|^2}{(|z|^4+16t^2)^{\frac{s}{2}} \left[1+(|z|^4+16t^2)^{\frac{2-s}{4}}\right]^{\frac{2(Q-s)}{2-s}}} \, \mathrm{d}z \, \mathrm{d}t. \tag{4.9}$$

For $z \in \mathbb{R}^{2n}$, we let $r = |z|$ with the spherical measure $\mathrm{d}z = \dfrac{2\pi^n}{\Gamma(n)} r^{2n-1} \, \mathrm{d}r$. Here, Γ denotes the classical Gamma function

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} \, \mathrm{d}t.$$

Hence

$$\mathrm{I} = \frac{2\,(Q-2)^2\,\pi^n}{\Gamma(n)} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{r^{2n+1}}{(r^4+16t^2)^{\frac{s}{2}} \left[1+(r^4+16t^2)^{\frac{2-s}{4}}\right]^{\frac{2(Q-s)}{2-s}}} \, \mathrm{d}r \, \mathrm{d}t. \tag{4.10}$$

Perform variable substitution; let $\varrho = \left(r^4+16t^2\right)^{\frac{1}{4}}$ and $w = \dfrac{t}{r^2}$ with the Jacobian determinant $J = \left|\dfrac{\partial(r,t)}{\partial(\varrho,w)}\right| = \varrho^2 \left(1+16w^2\right)^{-\frac{3}{4}}$. We deduce that

$$\begin{aligned}
\mathrm{I} &= \frac{2\,(Q-2)^2\,\pi^n}{\Gamma(n)} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{\varrho^{2n+1}(1+16w^2)^{-\frac{2n+1}{4}}}{\varrho^{2s}(1+\varrho^{2-s})^{\frac{2(Q-s)}{2-s}}} \varrho^2 \left(1+16w^2\right)^{-\frac{3}{4}} \mathrm{d}\varrho \, \mathrm{d}w \\
&= \frac{2\,(Q-2)^2\,\pi^n}{\Gamma(n)} \int_{-\infty}^{+\infty} \left(1+16w^2\right)^{-\frac{n+2}{2}} \mathrm{d}w \int_0^{+\infty} \frac{\varrho^{2n+3-2s}}{(1+\varrho^{2-s})^{\frac{2(Q-s)}{2-s}}} \, \mathrm{d}\varrho \\
&= \frac{2\,(Q-2)^2\,\pi^n}{\Gamma(n)} \mathrm{I}_1 \times \mathrm{I}_2.
\end{aligned}$$

In order to calculate $\mathrm{I}_1 = \displaystyle\int_{-\infty}^{+\infty} \left(1+16w^2\right)^{-\frac{n+2}{2}} \mathrm{d}w$, we perform the substitution $u = 4w$ to get that

$$\mathrm{I}_1 = \frac{1}{2} \int_0^{+\infty} \left(1+u^2\right)^{-\frac{n+2}{2}} \mathrm{d}u = \frac{1}{4} B\left(\frac{n+1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{4\,\Gamma\left(\frac{n}{2}+1\right)},$$

where $B(x,y)$ is the Beta function presented by

$$B\,(x,y) = \int_0^{+\infty} \frac{t^{x-1}}{(1+t)^{x+y}} \, \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Meanwhile, we let $v = \varrho^{2-s}$ for the estimate of $\mathrm{I}_2 = \displaystyle\int_0^{+\infty} \frac{\varrho^{2n+3-2s}}{(1+\varrho^{2-s})^{\frac{2(Q-s)}{2-s}}} \, \mathrm{d}\varrho$. It follows that

$$\begin{aligned}
\mathrm{I}_2 &= \frac{1}{2-s}\int_0^{+\infty}\frac{v^{\frac{2(Q-s)}{2-s}-1}}{(1+v)^{\frac{2(Q-s)}{2-s}}}\,\mathrm{d}v\\
&= \frac{1}{2-s}B\left(\frac{Q-2s+2}{2-s},\frac{Q-2}{2-s}\right)=\frac{\Gamma\left(\frac{Q-2s+2}{2-s}\right)\Gamma\left(\frac{Q-2}{2-s}\right)}{(2-s)\,\Gamma\left(\frac{2(Q-s)}{2-s}\right)}.
\end{aligned}$$

In conclusion, we derive the value of (4.6) by

$$\begin{aligned}
\mathrm{I} &= \frac{2\,(Q-2)^2\,\pi^n}{\Gamma(n)}\,\mathrm{I}_1\times\mathrm{I}_2\\
&= \frac{(Q-2)^2\,\pi^{n+\frac{1}{2}}}{2(2-s)}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{Q-2s+2}{2-s}\right)\Gamma\left(\frac{Q-2}{2-s}\right)}{\Gamma(n)\,\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{2(Q-s)}{2-s}\right)}. \qquad (4.11)
\end{aligned}$$

In order to compute (4.7), i.e.,

$$\mathrm{II}=\int_{\mathbb{H}^n}\frac{|z|^2}{(|z|^4+16t^2)^{\frac{s+2}{4}}\left(1+(|z|^4+16t^2)^{\frac{2-s}{4}}\right)^{\frac{2(Q-s)}{2-s}}}\,\mathrm{d}z\,\mathrm{d}t, \qquad (4.12)$$

we adopt a similar argument to deduce that

$$\begin{aligned}
\mathrm{II} &= \frac{2\pi^n}{\Gamma(n)}\int_{-\infty}^{+\infty}\left(1+16w^2\right)^{-\frac{n+2}{2}}\,\mathrm{d}w\int_0^{+\infty}\frac{\varrho^{2n-s+1}}{(1+\varrho^{2-s})^{\frac{2(Q-s)}{2-s}}}\,\mathrm{d}\varrho\\
&= \frac{2\,(Q-2)^2\,\pi^n}{\Gamma(n)}\,\mathrm{II}_1\times\mathrm{II}_2, \qquad (4.13)
\end{aligned}$$

where $\mathrm{II}_1 = \mathrm{I}_1 = \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{4\,\Gamma\left(\frac{n}{2}+1\right)}$, and

$$\begin{aligned}
\mathrm{II}_2 &= \int_0^{+\infty}\frac{\varrho^{2n-s+1}}{(1+\varrho^{2-s})^{\frac{2(Q-s)}{2-s}}}\,\mathrm{d}\varrho = \frac{1}{2-s}\int_0^{+\infty}\frac{v^{\frac{2n-s+2}{2-s}-1}}{(1+v)^{\frac{2(Q-s)}{2-s}}}\,\mathrm{d}v\\
&= \frac{1}{2-s}B\left(\frac{Q-s}{2-s},\frac{Q-s}{2-s}\right)=\frac{\Gamma\left(\frac{Q-s}{2-s}\right)^2}{(2-s)\,\Gamma\left(\frac{2(Q-s)}{2-s}\right)}.
\end{aligned}$$

Therefore,

$$\mathrm{II} = \frac{2\pi^n}{\Gamma(n)}\,\mathrm{II}_1\times\mathrm{II}_2 = \frac{\pi^{n+\frac{1}{2}}}{2(2-s)}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{Q-s}{2-s}\right)^2}{\Gamma(n)\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{2(Q-s)}{2-s}\right)}. \qquad (4.14)$$

Combining (4.8), (4.11), and (4.14), we finally conclude that

$$
\begin{aligned}
C &\geq \frac{\mathrm{II}}{\mathrm{I}^{\frac{Q-s}{Q-2}}} \\
&= \frac{\dfrac{\pi^{n+\frac{1}{2}}}{2(2-s)} \cdot \dfrac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{Q-s}{2-s}\right)^2}{\Gamma(n)\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{2(Q-s)}{2-s}\right)}}{\left[\dfrac{(Q-2)^2\,\pi^{n+\frac{1}{2}}}{2(2-s)} \cdot \dfrac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{Q-2s+2}{2-s}\right)\Gamma\left(\frac{Q-2}{2-s}\right)}{\Gamma(n)\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{2(Q-s)}{2-s}\right)}\right]^{\frac{Q-s}{Q-2}}} \\
&= \left[\frac{\pi^{n+\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{2(2-s)\Gamma(n)\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{2(Q-s)}{2-s}\right)}\right]^{\frac{s-2}{Q-2}} \cdot \frac{\Gamma\left(\frac{Q-s}{2-s}\right)^2}{\left[(Q-2)^2\,\Gamma\left(\frac{Q-2s+2}{2-s}\right)\Gamma\left(\frac{Q-2}{2-s}\right)\right]^{\frac{Q-s}{Q-2}}},
\end{aligned}
$$

which completes the proof of Theorem 1.2. □

5. Conclusions

In this paper, we revisit the Kelvin transform on the Heisenberg group and present an algebraic proof of the formula involving the sub-Laplacian. A key contribution of this paper is the computation of the best constant for the generalized Hardy-Sobolev type inequality on $\mathbb{H}^n$, achieved by deriving cylindrically symmetric solutions to a related sub-elliptic equation. This systematic approach extends the results of analysis and PDEs on the Heisenberg group.

Author contributions

Zimiao Mu: Conceptualization, Methodology, Writing (Original Draft Preparation); Feng Zhou: Supervision, Funding Acquisition, Project Administration, Writing (Review and Editing). All authors have read and approved the final version of the manuscript for publication.

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The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest regarding the publication of this article.

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