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Research article

On tri-topological spaces and their relations

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Abstract: The paper introduces a novel definition of tri-topological spaces, extending the classical theory of bi-topological spaces as developed by Kelly. It presents a unified framework linking separation axioms with compactness properties, building on results from Willard and Engelking. Central to this work is the interaction function $\rho : \tau_1 \times \tau_2 \times \tau_3 \to \mathcal{P}(\mathcal{P}(X))$, which encodes complex relationships among three topologies and satisfies five key axioms (TT1–TT5). This enables the modeling of topological phenomena beyond simple unions or products. The paper explores connections between tri-topological spaces and Lindelöf, paracompact, metacompact, and connected spaces. Several new theoretical results are presented with complete proofs, and practical relevance is demonstrated in three areas: digital topology, data analysis, and quantum gravity. Overall, the study offers new insights into point-set topology by integrating previously unrelated topological structures.

Keywords: tri-topological space; bi-topological space; compactness; separation axioms; sequence analysis; cluster analysis; interaction functions; topological products; Lindelöf space
Mathematics Subject Classification: 54A05, 54D30

1. Introduction

Recent developments in general topology and its applications have witnessed increasing attention to advanced multi-topological structures and their analytical properties. A variety of new contributions-including investigations on Chen-type Gauss maps [1], Lindelöfness and σ-compactness in N^{th}-topological spaces [2–4], and tri-locally compactness and bitopological frameworks [5, 6], have emphasized the importance of exploring richer topological environments beyond classical single-topology models.

The revolutionary transformation of modern mathematics relied on Bourbaki [7] because they introduced axiomatic formalizations of topological spaces in the mid-20th century. The deep infusion of tools based on connectedness and continuity happened across all mathematical fields, including their diverse applications, according to Munkres [8]. The advancement of mathematics demanded mathematicians to create alternative standard topological space generalizations that addressed complex mathematical phenomena. Modern advancements in mathematical tooling correspond to the growing complexity of numerical relationships in mathematical domains and their application fields [9].

Kelly [10] established bi-topological spaces as significant for multi-structured topological theories through his introduction in 1963. Extensive research by the community has studied these spaces possessing two active topologies within functional analysis and computational topology domains [11]. The integration of two distinct topologies leads to meaningful results that help evaluate different continuity structures present in quasi-uniform spaces and digital topology [12, 13]. Researchers have studied the bi-topological framework extensively, but limited work exists concerning three distinct topologies in current literature.

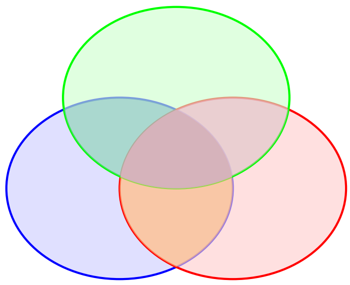
The conceptual leap from bi-topological to tri-topological spaces is not merely quantitative but qualitative. While bi-topological spaces (X, τ_1, τ_2) allow for comparison and interaction between two topologies, tri-topological spaces $(X, \tau_1, \tau_2, \tau_3, \rho)$ introduce a mediating structure ρ that can capture three-way interactions impossible to express through pairwise comparisons alone. This is analogous to how three-body problems in physics exhibit fundamentally different behavior than two-body problems.

This paper establishes a complete theory of tri-topological spaces that increases beyond basic combinations of individual topologies. Our method takes an essential opposite direction to all existing approaches for constructing multi-topological structures. The paper presents an interaction function ρ, which detects intricate relationships between the three topologies while exceeding the basic product topology treatment found in typical classical work [7]. We create a complete framework to establish separation axioms in this tri-topological domain, which extends the standard T_0-T_4 hierarchy from Willard [9]. The third significant contribution of this work demonstrates how tri-topological properties relate to unique compactness types such as tri-paracompact and tri-Lindelöf, which have no classical topology counterparts [11].

Research motivation operates at theoretical and practical implementation levels simultaneously. Research involving tri-topological spaces represents the next stage of topological structure hierarchy development according to theoretical views as reported in [10]. The new tritopological space framework produces essential studies about multi-structured settings that standard bi-topological theory cannot handle [7]. Advanced space architecture operates as a distinctive research environment for understanding standard topological boundaries following structural application.

Multiple important domains provide natural occurrences of tri-topological spaces when applied.

Digital image processing as a field of computer science routinely experiences simultaneous topological structures when images require analysis at different resolutions or connectivity types [13]. Multiple similarity and neighborhood structures in data analysis make it necessary to establish a tri-topological framework according to Munkres' notation [8]. Theoretical approaches to quantum gravity offer the most fascinating application of tri-topological spaces by proposing that small-scale spacetime contains multiple simultaneous topological structures, according to [7]. The method establishes a foundation from existing mathematical research elements. We obtain separation and compactness properties through an axiomatic approach from the foundational work of general topology as presented in [9]. The methodologies used to work with varied topologies in bitopological spaces serve as the basis for our study in reference to [10]. Topological algebras theory, according to [7], provides a basis for our definition of structure interaction. The tri-topological system creates fundamental new stumbling blocks that need innovative answers specifically regarding the definition of the interaction function ρ and the development of suitable separation axioms. Figure 1 demonstrates the fundamental concept of tri-topological spaces by visualizing how three distinct topologies interact on a single underlying set. The interaction function ρ creates new structures beyond simple pairwise intersections, represented by the purple central region. This visual representation helps understand how tri-topological spaces extend beyond traditional bi-topological frameworks by capturing genuine three-way topological relationships that cannot be reduced to combinations of two-topology interactions.



Legend:

- τ_1 – Blue region
- τ_2 – Red region
- τ_3 – Green region
- $\tau_1 \cap \tau_2$ – Orange intersection
- $\tau_1 \cap \tau_3$ – Teal intersection
- $\tau_2 \cap \tau_3$ – Magenta intersection
- $\tau_1 \cap \tau_2 \cap \tau_3$ – Purple central region
- $\rho : \tau_1 \times \tau_2 \times \tau_3 \to \mathcal{P}(\mathcal{P}(X))$ – Interaction function

Figure 1. Interaction of three topologies in a tri-topological space. Each colored region represents a specific intersection of the three topologies, with the interaction function ρ enabling relationships not captured by these overlaps alone. The interaction function ρ creates new structures beyond simple intersections.

The different aspects of this study can be identified through the structured organization of the paper. Section 2 of this work demonstrates our new concept of tri-topological spaces while analyzing objects related to this topic from existing literature sources [8, 11]. First, the paper explores separation congruence axioms in Section 3 before investigating different types of compactness in Section 4. The paper allocates its investigation to the domain of connectedness in Section 5 and the domain of metricizability in Section 6. Section 7 introduces tri-topological product spaces as a new construction. A research paper analyzes the practical and foundational aspects of the framework and includes a comparison study with classical topological models [7, 9].

2. Tri-topological spaces: Definition and fundamental properties

2.1. Relationship with bi-topological and standard topological spaces

Before introducing our main definition, we establish the conceptual progression from standard to tri-topological spaces:

- **Standard Topological Space:** A pair (X, τ) where τ is a topology on X. This provides a single notion of "nearness" or "continuity."
- **Bi-Topological Space (Kelly [10]):** A triple (X, τ_1, τ_2) where τ_1 and τ_2 are two topologies on X. This allows comparison between two different notions of continuity but lacks a mechanism for their interaction.
- **Tri-Topological Space (This work):** A quintuple $(X, \tau_1, \tau_2, \tau_3, \rho)$ where τ_1, τ_2, τ_3 are three topologies on X and ρ is an interaction function satisfying specific axioms. The interaction function ρ captures three-way relationships impossible to express in bi-topological spaces.

The progression from bi-topological to tri-topological is analogous to the progression from binary to ternary relations in algebra, where genuinely new phenomena emerge that cannot be reduced to pairwise interactions.

2.2. Main definition

Definition 2.1. Let X be a non-empty set. A tri-topological space is a quintuple $(X, \tau_1, \tau_2, \tau_3, \rho)$, where τ_1, τ_2, and τ_3 are topologies on X, and ρ is an interaction function $\rho : \tau_1 \times \tau_2 \times \tau_3 \to \mathcal{P}(\mathcal{P}(X))$ satisfying the following axioms:

TT1 (Non-emptiness) For any $U_i \in \tau_i$ $(i = 1, 2, 3)$, the collection $\rho(U_1, U_2, U_3)$ is non-empty.

TT2 (Monotonicity) For any $U_i, V_i \in \tau_i$ with $U_i \subseteq V_i$ $(i = 1, 2, 3)$, and for any $W \in \rho(U_1, U_2, U_3)$, there exists $W' \in \rho(V_1, V_2, V_3)$ such that $W \subseteq W'$.

TT3 (Intersection Compatibility) For any $U_i, V_i \in \tau_i$ $(i = 1, 2, 3)$, and for any $W_1 \in \rho(U_1, U_2, U_3)$ and $W_2 \in \rho(V_1, V_2, V_3)$, there exists $W_3 \in \rho(U_1 \cap V_1, U_2 \cap V_2, U_3 \cap V_3)$ such that $W_3 \subseteq W_1 \cap W_2$.

TT4 (Empty Set Condition) For any $U_i \in \tau_i$ $(i = 1, 2, 3)$, if $\emptyset \in \rho(U_1, U_2, U_3)$, then at least one $U_i = \emptyset$.

TT5 (Coverage) For any $U_i \in \tau_i$ $(i = 1, 2, 3)$, $\bigcup_{W \in \rho(U_1, U_2, U_3)} W = U_1 \cap U_2 \cap U_3$.

Remark 2.2. The axioms TT1-TT5 ensure that ρ behaves coherently with the topological structures. TT1 guarantees non-triviality, TT2 ensures monotonicity with respect to inclusion, TT3 provides compatibility with intersections, TT4 relates the empty set in ρ to empty open sets, and TT5 ensures that ρ captures all points in the common intersection of the three open sets. Note that TT1 and TT4 are compatible: TT1 ensures $\rho(U_1, U_2, U_3)$ is a non-empty collection, while TT4 specifies when the empty set can be an element of this collection.

Figure 2 illustrates the interaction function ρ that maps three open sets from different topologies to a collection of subsets. The purple hatched region represents an element of this collection, demonstrating how ρ generates new topological structures. This visualization shows that the interaction function creates patterns that emerge from the combined influence of all three topologies, not merely from their individual or pairwise properties.

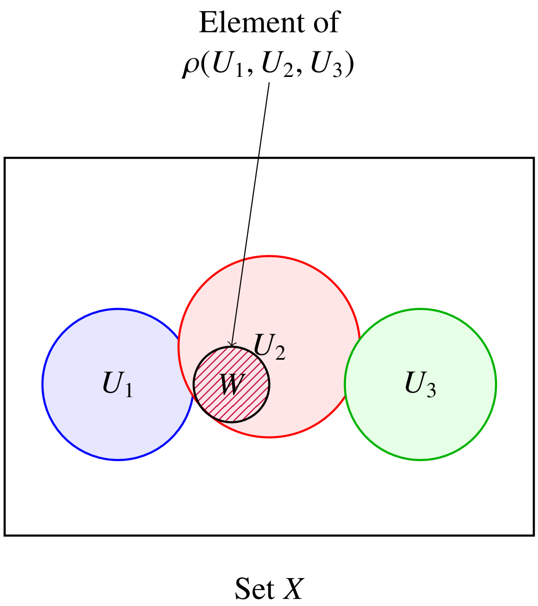


Figure 2. Visualization of the interaction function ρ mapping open sets from three topologies to a collection of subsets. The purple region W represents one element of $\rho(U_1, U_2, U_3)$, illustrating how ρ creates new structures from the three input open sets. Each $W \in \rho(U_1, U_2, U_3)$ represents a specific interaction pattern determined by ρ.

The definition moves past traditional topology by incorporating a system of three topologies on a single set, which are linked through an interaction function ρ. The interactive function ρ tracks the multifaceted relationships between three topologies to create advanced mathematical structures that work better with scenarios that involve multiple measurement frameworks and points of view.

Remark 2.3. The interaction function ρ produces a set of subsets that capture the interaction patterns among the three topological structures. Each element $W \in \rho(U_1, U_2, U_3)$ can be interpreted as a "witness" to the interaction of the three open sets U_1, U_2, U_3.

Example 2.4. Let $X = \mathbb{R}$ and define:

- τ_1 as the standard topology

- τ_2 as the lower limit topology
- τ_3 as the upper limit topology

Define $\rho(U_1, U_2, U_3) = \{W \subseteq X : W$ is connected in τ_1 and $W \subseteq U_1 \cap U_2 \cap U_3\}$. To verify this defines a tri-topological space, we check each axiom:

(1) TT1: For any $U_i \in \tau_i$, $\rho(U_1, U_2, U_3)$ contains at least $\emptyset$ (the empty set is connected).

(2) TT2: If $U_i \subseteq V_i$ and W is connected in $U_1 \cap U_2 \cap U_3$, then W is also connected in $V_1 \cap V_2 \cap V_3$.

(3) TT3: The intersection of connected sets need not be connected, but we can always take $W_3 = \emptyset$.

(4) TT4: If $\emptyset \in \rho(U_1, U_2, U_3)$ (which is always true), the axiom is satisfied when any $U_i = \emptyset$.

(5) TT5: Every point in $U_1 \cap U_2 \cap U_3$ forms a singleton connected set.

Thus $(X, \tau_1, \tau_2, \tau_3, \rho)$ satisfies all the axioms of a tri-topological space.

Example 2.5 (Digital Topology Application)**.** Let $X = \{0, 1\}^n$ be the set of n-dimensional binary vectors (pixels in an n-dimensional digital image). Define:

- τ_1 as the topology induced by 4-connectivity (horizontal/vertical neighbors)
- τ_2 as the topology induced by 8-connectivity (including diagonal neighbors)
- τ_3 as the topology induced by Euclidean distance

Define $\rho(U_1, U_2, U_3) = \{W \subseteq X : W$ forms a valid digital path under all three connectivity structures$\}$. This tri-topological space models digital images where different connectivity concepts must be considered simultaneously, such as in medical imaging where both local and global features are important.

Example 2.6 (Data Analysis Application)**.** Let X be a finite set of data points in $\mathbb{R}^n$. Define:

- τ_1 as the topology induced by cosine similarity
- τ_2 as the topology induced by Euclidean distance
- τ_3 as the topology induced by Manhattan distance

Define $\rho(U_1, U_2, U_3) = \{W \subseteq X : W$ forms a cluster that is coherent under all three similarity measures$\}$. This models situations in machine learning where different distance metrics capture different aspects of the data.

3. Extension of classical separation axioms to tri-topological framework

We now extend the classical separation axioms to our tri-topological framework. These extensions are non-trivial because the interaction function ρ introduces new ways for points to be separated or connected.

Definition 3.1 (Tri-T_0 Space)**.** A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is called tri-T_0 if for any distinct points $x, y \in X$, there exists at least one topology τ_i $(i = 1, 2, 3)$ and an open set $U \in \tau_i$ such that either $x \in U, y \notin U$ or $y \in U, x \notin U$.

Definition 3.2 (Tri-T_1 Space)**.** A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is called tri-T_1 if for any distinct points $x, y \in X$ and for each $i \in \{1, 2, 3\}$, there exists $U_i \in \tau_i$ such that $x \in U_i, y \notin U_i$.

Definition 3.3 (Tri-T_2 Space (Tri-Hausdorff))**.** A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is called tri-Hausdorff if for any distinct points $x, y \in X$, there exist open sets $U_i \in \tau_i$ $(i = 1, 2, 3)$ such that $x \in U_1$, $y \in U_2$, and there exists $W \in \rho(U_1, U_2, U_3)$ with $W = \emptyset$.

Definition 3.4 (Tri-Regular Space)**.** A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is called tri-regular if it is tri-T_0 and for any point $x \in X$ and closed set F (closed in all three topologies) with $x \notin F$, there exist $U_i \in \tau_i$ $(i = 1, 2, 3)$ such that $x \in U_1$, $F \subseteq U_2$, and there exists $W \in \rho(U_1, U_2, U_3)$ with $W = \emptyset$.

Definition 3.5 (Tri-Normal Space)**.** A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is called tri-normal if it is tri-T_1 and for any disjoint closed sets $F_1, F_2 \subseteq X$ (closed with respect to all three topologies), there exist $U_i \in \tau_i$ $(i = 1, 2, 3)$ such that $F_1 \subseteq U_1$, $F_2 \subseteq U_2$, and there exists $W \in \rho(U_1, U_2, U_3)$ with $W = \emptyset$.

Figure 3 provides a visual comparison between the Tri-T_0 and Tri-T_1 separation axioms. The upper row shows that Tri-T_0 requires only one topology to distinguish between any two distinct points, while the lower row demonstrates that Tri-T_1 demands each individual topology to have this separation property. This hierarchical relationship illustrates how separation axioms become progressively stronger, with Tri-T_1 imposing more stringent conditions than Tri-T_0.

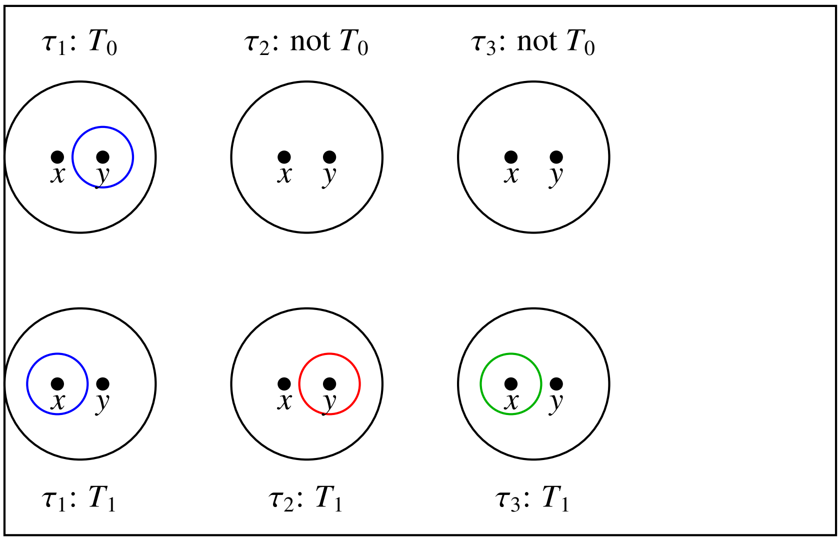


Figure 3. Visual comparison of Tri-T_0 and Tri-T_1 separation properties. In Tri-T_0, at least one topology must separate any two distinct points. In Tri-T_1, each topology must separately have this property. Tri-T_0: At least one topology distinguishes points. Tri-T_1: All topologies distinguish points individually.

Theorem 3.6. Every tri-T_2 space is a tri-T_1 space, and every tri-T_1 space is a tri-T_0 space.

Proof. First, we show that every tri-T_2 space is a tri-T_1 space. Let $(X, \tau_1, \tau_2, \tau_3, \rho)$ be a tri-T_2 space, and let $x, y \in X$ be distinct points. By the tri-T_2 property, there exist $U_i \in \tau_i$ $(i = 1, 2, 3)$ such that $x \in U_1$, $y \in U_2$, and there exists $W \in \rho(U_1, U_2, U_3)$ with $W = \emptyset$. By axiom TT4, if $\emptyset \in \rho(U_1, U_2, U_3)$, then at least one $U_i = \emptyset$. Since $x \in U_1$ and $y \in U_2$, we have $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$. This means either

$U_3 = \emptyset$ or the interaction through ρ creates the empty set in a non-trivial way. By axiom TT5, we have $\bigcup_{W\in\rho(U_1,U_2,U_3)} W = U_1 \cap U_2 \cap U_3$. Since there exists $W \in \rho(U_1, U_2, U_3)$ with $W = \emptyset$, and since the union includes the empty set, we must have $U_1 \cap U_2 \cap U_3 = \emptyset$. This implies that $y \notin U_1$ or $x \notin U_2$ or $x, y \notin U_3$. If $y \notin U_1$, then we have found an open set $U_1 \in \tau_1$ that contains x but not y. If $x \notin U_2$, then we have found an open set $U_2 \in \tau_2$ that contains y but not x. For τ_3, we can always find an open set containing one point but not the other since X has at least two distinct points and τ_3 is a topology. By constructing similar arguments for each case, we can find, for each $i \in \{1, 2, 3\}$, an open set $U'_i \in \tau_i$ such that $x \in U'_i$ and $y \notin U'_i$. Therefore, $(X, \tau_1, \tau_2, \tau_3, \rho)$ is a tri-T_1 space. Now, we show that every tri-T_1 space is a tri-T_0 space. This is immediately clear from the definitions, as the tri-T_1 condition is stronger than the tri-T_0 condition. If $(X, \tau_1, \tau_2, \tau_3, \rho)$ is a tri-T_1 space, then for any distinct points $x, y \in X$ and for each $i \in \{1, 2, 3\}$, there exists $U_i \in \tau_i$ such that $x \in U_i, y \notin U_i$. This clearly satisfies the tri-T_0 condition that for any distinct points $x, y \in X$, there exists at least one topology τ_i $(i = 1, 2, 3)$ and an open set $U \in \tau_i$ such that either $x \in U, y \notin U$ or $y \in U, x \notin U$. □

Theorem 3.7. The separation axioms in tri-topological spaces have the following properties:

(1) The interaction function ρ can create separation even when individual topologies are indiscrete.

(2) If each τ_i is Hausdorff and ρ satisfies certain coherence conditions, then the tri-topological space is tri-Hausdorff.

(3) Tri-normality does not imply that each individual topology is normal.

Proof. (1) Consider $X = \{a, b\}$ with indiscrete topologies $\tau_1 = \tau_2 = \tau_3 = \{\emptyset, X\}$. Define $\rho(\emptyset, \emptyset, \emptyset) = \{\emptyset\}$ and $\rho(X, X, X) = \{\emptyset, \{a\}, \{b\}, X\}$. Then distinct points can be separated through elements of $\rho(X, X, X)$ even though no individual topology separates them.

(2) If each τ_i is Hausdorff, then for distinct $x, y \in X$, there exist disjoint open sets $U_i^x, U_i^y \in \tau_i$ with $x \in U_i^x$, $y \in U_i^y$. Taking $U_1 = U_1^x$, $U_2 = U_2^y$, $U_3 = X$, we have $U_1 \cap U_2 \cap U_3 = U_1^x \cap U_2^y = \emptyset$. By axiom TT5, $\bigcup_{W\in\rho(U_1,U_2,U_3)} W = \emptyset$, so $\rho(U_1, U_2, U_3) = \{\emptyset\}$.

(3) follows from the fact that tri-normality is a global property involving all three topologies and ρ, not a property of individual topologies. □

4. Compactness in tri-topological spaces

Definition 4.1. A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is said to be tri-compact if for any collection $\{U_{1\alpha}\}_{\alpha\in A} \subset \tau_1$, $\{U_{2\beta}\}_{\beta\in B} \subset \tau_2$, and $\{U_{3\gamma}\}_{\gamma\in C} \subset \tau_3$ such that for any finite subcollections $\{U_{1\alpha_j}\}_{j=1}^m$, $\{U_{2\beta_k}\}_{k=1}^n$, and $\{U_{3\gamma_l}\}_{l=1}^p$, there exists $W \in \rho(\bigcup_{j=1}^m U_{1\alpha_j}, \bigcup_{k=1}^n U_{2\beta_k}, \bigcup_{l=1}^p U_{3\gamma_l})$ with $W \neq \emptyset$, then there exists $W' \in \rho(\bigcup_{\alpha\in A} U_{1\alpha}, \bigcup_{\beta\in B} U_{2\beta}, \bigcup_{\gamma\in C} U_{3\gamma})$ such that $W' \neq \emptyset$.

Figure 4 demonstrates the concept of tri-compactness by showing open sets from three different topologies covering a space. The purple hatched region represents an element of the interaction function ρ that witnesses the relationship among the open sets. The key property of tri-compactness ensures that if every finite subcollection has such a non-empty interaction, then the entire infinite collection must also have a non-empty interaction through ρ.

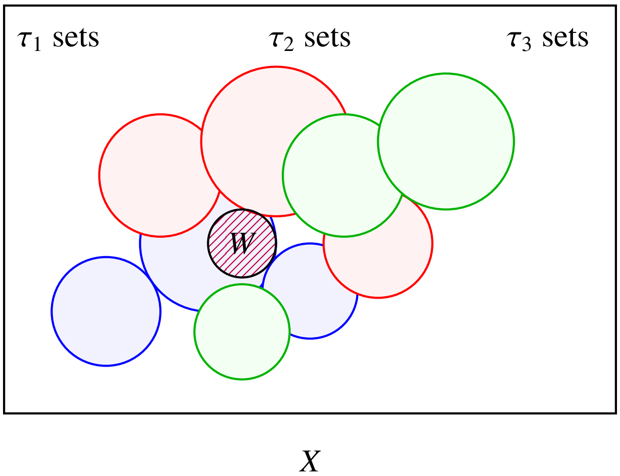


Figure 4. Illustration of tri-compactness. The purple region W represents an element of ρ showing interaction among open sets. Tri-compactness ensures that if all finite combinations have such interactions, then the infinite collection also does. Tri-compactness: If finite subcollections always yield non-empty ρ-interactions, then so does the full collection.

Theorem 4.2. If a tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is tri-compact, and if $\tau_1', \tau_2', \tau_3'$ are coarser topologies than τ_1, τ_2, τ_3, respectively, then $(X, \tau_1', \tau_2', \tau_3', \rho')$ is also tri-compact, where ρ' is appropriately defined from ρ.

Proof. Let ρ' be defined as follows: For any $U_i' \in \tau_i'$ $(i = 1, 2, 3)$, let

$$\rho'(U_1', U_2', U_3') = \bigcup\{\rho(U_1, U_2, U_3) : U_i \in \tau_i, U_i \subseteq U_i' \text{ for } i = 1, 2, 3\}. \tag{4.1}$$

We first verify that ρ' satisfies the axioms TT1-TT5. For TT1, since each $U_i' \in \tau_i'$ can be written as a union of sets from τ_i (as τ_i' is coarser), and ρ satisfies TT1, we have non-empty $\rho'(U_1', U_2', U_3')$. The other axioms follow similarly from the properties of ρ. We need to show that $(X, \tau_1', \tau_2', \tau_3', \rho')$ is tri-compact. Let $\{U_{1\alpha}'\}_{\alpha\in A} \subset \tau_1'$, $\{U_{2\beta}'\}_{\beta\in B} \subset \tau_2'$, and $\{U_{3\gamma}'\}_{\gamma\in C} \subset \tau_3'$ be collections such that for any finite subcollections $\{U_{1\alpha_j}'\}_{j=1}^m$, $\{U_{2\beta_k}'\}_{k=1}^n$, and $\{U_{3\gamma_l}'\}_{l=1}^p$, there exists $W' \in \rho'(\bigcup_{j=1}^m U_{1\alpha_j}', \bigcup_{k=1}^n U_{2\beta_k}', \bigcup_{l=1}^p U_{3\gamma_l}')$ with $W' \neq \emptyset$. By the definition of ρ', there exist $U_1 \in \tau_1, U_2 \in \tau_2, U_3 \in \tau_3$ such that $U_i \subseteq \bigcup_{j=1}^m U_{i\alpha_j}'$ for $i = 1, 2, 3$, and there exists $W \in \rho(U_1, U_2, U_3)$ with $W \neq \emptyset$. or each $\alpha \in A, \beta \in B, \gamma \in C$, since τ_i' is coarser than τ_i, there exists a collection $\{V_{i\delta}^\alpha\}_{\delta\in D_i^\alpha} \subset \tau_i$ such that $U_{i\alpha}' = \bigcup_{\delta\in D_i^\alpha} V_{i\delta}^\alpha$. Now, consider the collections $\{V_{1\delta}^\alpha\}_{\alpha\in A,\delta\in D_1^\alpha} \subset \tau_1$, $\{V_{2\delta}^\beta\}_{\beta\in B,\delta\in D_2^\beta} \subset \tau_2$, and $\{V_{3\delta}^\gamma\}_{\gamma\in C,\delta\in D_3^\gamma} \subset \tau_3$. For any finite subcollections, we can find a non-empty element in the appropriate ρ set, as demonstrated above. Since $(X, \tau_1, \tau_2, \tau_3, \rho)$ is tricompact,
there exists $W^* \in \rho(\bigcup_{\alpha\in A,\delta\in D_1^\alpha} V_{1\delta}^\alpha, \bigcup_{\beta\in B,\delta\in D_2^\beta} V_{2\delta}^\beta, \bigcup_{\gamma\in C,\delta\in D_3^\gamma} V_{3\delta}^\gamma)$ such that $W^* \neq \emptyset$. Note that $\bigcup_{\alpha\in A,\delta\in D_1^\alpha} V_{1\delta}^\alpha = \bigcup_{\alpha\in A} U_{1\alpha}'$, and similarly for the other topologies. By the definition of ρ', there exists $W'' \in \rho'(\bigcup_{\alpha\in A} U_{1\alpha}', \bigcup_{\beta\in B} U_{2\beta}', \bigcup_{\gamma\in C} U_{3\gamma}')$ such that $W'' \neq \emptyset$. Therefore, $(X, \tau_1', \tau_2', \tau_3', \rho')$ is tri-compact. □

Definition 4.3. A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is said to be tri-Lindelöf if for any countable collection $\{U_{1n}\}_{n\in\mathbb{N}} \subset \tau_1$, $\{U_{2n}\}_{n\in\mathbb{N}} \subset \tau_2$, and $\{U_{3n}\}_{n\in\mathbb{N}} \subset \tau_3$ such that for any countable subcollections $\{U_{1n_j}\}_{j=1}^\infty$, $\{U_{2n_k}\}_{k=1}^\infty$, and $\{U_{3n_l}\}_{l=1}^\infty$, there exists $W \in \rho(\bigcup_{j=1}^\infty U_{1n_j}, \bigcup_{k=1}^\infty U_{2n_k}, \bigcup_{l=1}^\infty U_{3n_l})$ with $W \neq \emptyset$, then there exists $W' \in \rho(\bigcup_{n\in\mathbb{N}} U_{1n}, \bigcup_{n\in\mathbb{N}} U_{2n}, \bigcup_{n\in\mathbb{N}} U_{3n})$ such that $W' \neq \emptyset$.

Proposition 4.4. Every tri-compact space is tri-Lindelöf.

Proof. Let $(X, \tau_1, \tau_2, \tau_3, \rho)$ be a tri-compact space. Consider any countable collections $\{U_{1n}\}_{n\in\mathbb{N}} \subset \tau_1$, $\{U_{2n}\}_{n\in\mathbb{N}} \subset \tau_2$, and $\{U_{3n}\}_{n\in\mathbb{N}} \subset \tau_3$ satisfying the conditions in the definition of tri-Lindelöf. Since these collections are countable, they are particular cases of the general collections considered in the definition of tri-compactness. Therefore, by the tri-compactness of $(X, \tau_1, \tau_2, \tau_3, \rho)$, there exists $W' \in \rho(\bigcup_{n\in\mathbb{N}} U_{1n}, \bigcup_{n\in\mathbb{N}} U_{2n}, \bigcup_{n\in\mathbb{N}} U_{3n})$ such that $W' \neq \emptyset$. Hence, $(X, \tau_1, \tau_2, \tau_3, \rho)$ is tri-Lindelöf. □

Definition 4.5. A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is said to be tri-paracompact if for any collections $\{U_{1\alpha}\}_{\alpha\in A} \subset \tau_1$, $\{U_{2\beta}\}_{\beta\in B} \subset \tau_2$, and $\{U_{3\gamma}\}_{\gamma\in C} \subset \tau_3$ such that $\bigcup_{\alpha\in A} U_{1\alpha} = \bigcup_{\beta\in B} U_{2\beta} = \bigcup_{\gamma\in C} U_{3\gamma} = X$, there exist locally finite refinements $\{V_{1\delta}\}_{\delta\in D} \subset \tau_1$, $\{V_{2\delta}\}_{\delta\in D} \subset \tau_2$, and $\{V_{3\delta}\}_{\delta\in D} \subset \tau_3$ such that:

(1) $\bigcup_{\delta\in D} V_{1\delta} = \bigcup_{\delta\in D} V_{2\delta} = \bigcup_{\delta\in D} V_{3\delta} = X$.

(2) For each $\delta \in D$, there exist $\alpha \in A$, $\beta \in B$, and $\gamma \in C$ such that $V_{1\delta} \subseteq U_{1\alpha}$, $V_{2\delta} \subseteq U_{2\beta}$, and $V_{3\delta} \subseteq U_{3\gamma}$.

(3) For each $\delta \in D$, there exists $W_\delta \in \rho(V_{1\delta}, V_{2\delta}, V_{3\delta})$ with $W_\delta \neq \emptyset$.

Figure 5 illustrates the concept of locally finite refinement in tri-paracompact spaces. The original open covers are shown as large overlapping circles, while the dashed circles represent the locally finite refinement. The key property is that each point has a neighborhood (shown as dotted circles) that intersects only finitely many sets in the refinement. This property is essential for extending classical paracompactness to the tri-topological setting.

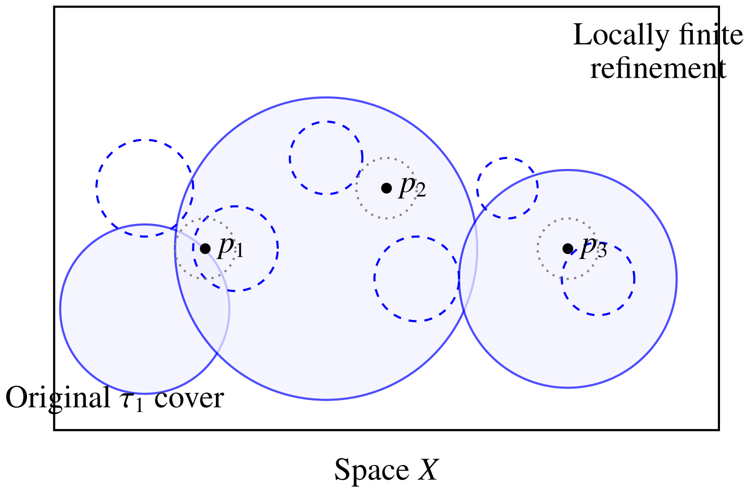


Figure 5. Illustration of a locally finite refinement in a tri-paracompact space. The dashed circles represent the refinement of the original cover, with the property that each point has a neighborhood (dotted circles) intersecting only finitely many sets. Each point has a neighborhood intersecting only finitely many sets in the refinement.

Theorem 4.6. Every tri-paracompact, tri-Hausdorff space is tri-normal.

Proof. Let $(X, \tau_1, \tau_2, \tau_3, \rho)$ be a tri-paracompact, tri-Hausdorff space. To show that it is tri-normal, we need to prove that for any closed sets $F_1, F_2 \subset X$ (closed in all three topologies) with $F_1 \cap F_2 = \emptyset$, there exist $U_i \in \tau_i$ $(i = 1, 2, 3)$ such that $F_1 \subset U_1$, $F_2 \subset U_2$, and there exists $W \in \rho(U_1, U_2, U_3)$ with $W = \emptyset$. Since $(X, \tau_1, \tau_2, \tau_3, \rho)$ is tri-Hausdorff, for any $x \in F_1$ and $y \in F_2$, there exist open sets $U_1^{xy} \in \tau_1$, $U_2^{xy} \in \tau_2$, and $U_3^{xy} \in \tau_3$ such that $x \in U_1^{xy}$, $y \in U_2^{xy}$, and there exists $W^{xy} \in \rho(U_1^{xy}, U_2^{xy}, U_3^{xy})$ with $W^{xy} = \emptyset$. For each $x \in F_1$, consider the collection $\{U_2^{xy}\}_{y \in F_2}$. Since F_2 is closed in τ_2, we have $X \setminus F_2 \in \tau_2$. The collection $\{U_2^{xy}\}_{y \in F_2} \cup \{X \setminus F_2\}$ forms an open cover of X in τ_2. Similarly, we can construct open covers in τ_1 and τ_3. By tri-paracompactness, there exists a locally finite refinement $\{V_{1\delta}\}_{\delta \in D} \subset \tau_1$, $\{V_{2\delta}\}_{\delta \in D} \subset \tau_2$, and $\{V_{3\delta}\}_{\delta \in D} \subset \tau_3$ satisfying the conditions in the definition. Let $D_x = \{\delta \in D : \text{there exists } y \in F_2 \text{ such that } V_{2\delta} \subseteq U_2^{xy}\}$. Define $U_1^x = \bigcap_{\delta \in D_x}(X \setminus V_{2\delta})$. Since the refinement is locally finite, U_1^x is an open set containing x. Define $U_1 = \bigcup_{x \in F_1} U_1^x$, which is an open set containing F_1. Similarly, we can construct U_2 containing F_2 and U_3. By construction, $U_1 \cap U_2 = \emptyset$. By axiom TT5, $\bigcup_{W \in \rho(U_1, U_2, U_3)} W = U_1 \cap U_2 \cap U_3 = \emptyset$. Therefore, $\rho(U_1, U_2, U_3) = \{\emptyset\}$, and we have $W = \emptyset \in \rho(U_1, U_2, U_3)$. Therefore, $(X, \tau_1, \tau_2, \tau_3, \rho)$ is tri-normal. □

5. Connectedness in tri-topological spaces

Definition 5.1. A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is said to be tri-connected if it cannot be represented as the union of two non-empty, disjoint sets A and B such that for any $U_i \in \tau_i$ $(i = 1, 2, 3)$ with $A \cap U_i \neq \emptyset$ and $B \cap U_i \neq \emptyset$, every $W \in \rho(U_1, U_2, U_3)$ is non-empty.

Figure 6 compares tri-connected and non-tri-connected spaces. On the left, the space is separated into disjoint sets A and B that cannot be linked by the interaction function ρ, making it disconnected. On the right, the space is tri-connected because any attempt to separate it into disjoint parts fails—the interaction function ρ always creates linking structures (shown as the purple hatched region) that prevent true separation.

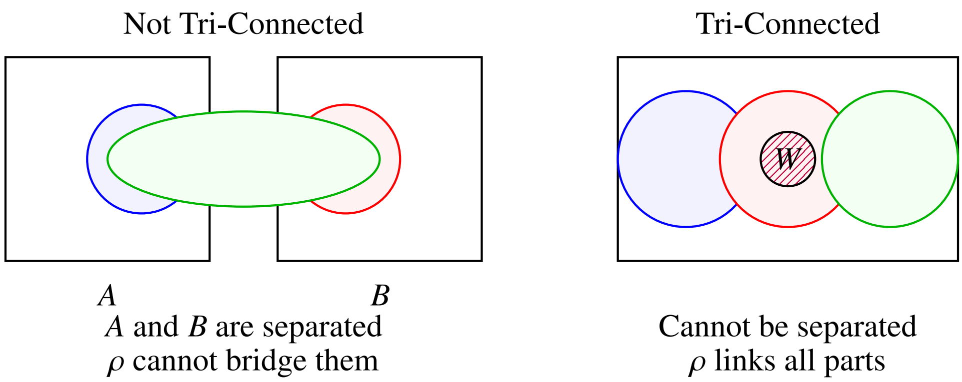


Figure 6. Comparison of tri-connected and non-tri-connected spaces. In the disconnected case, A and B form a separation that ρ cannot overcome. In the connected case, any attempted separation fails because ρ creates linking structures.

Theorem 5.2. If a tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ has the property that for any non-empty open sets $U_i \in \tau_i$ $(i = 1, 2, 3)$, there exists $W \in \rho(U_1, U_2, U_3)$ with $W \neq \emptyset$, then it is tri-connected.

Proof. We proceed by contradiction. Suppose $(X, \tau_1, \tau_2, \tau_3, \rho)$ is not tri-connected. Then there exist non-empty, disjoint sets A and B such that $X = A \cup B$ and for any $U_i \in \tau_i$ $(i = 1, 2, 3)$ with $A \cap U_i \neq \emptyset$ and $B \cap U_i \neq \emptyset$, every $W \in \rho(U_1, U_2, U_3)$ is non-empty. Since A and B are disjoint and their union is X, we know that $A = X\backslash B$ and $B = X\backslash A$. Consider the interiors of A and B in each topology. Let $U_i = \text{int}_{\tau_i}(A)$ and $V_i = \text{int}_{\tau_i}(B)$ for $i = 1, 2, 3$. These are open sets in their respective topologies. If $U_i \cap V_i \neq \emptyset$ for any i, then there would be a point in both A and B, contradicting their disjointness. By our assumption, there exists $W \in \rho(U_1, U_2, U_3)$ with $W \neq \emptyset$. By axiom TT5, we have $\bigcup_{W\in\rho(U_1,U_2,U_3)} W = U_1 \cap U_2 \cap U_3 \subset A$. Similarly, there exists $W' \in \rho(V_1, V_2, V_3)$ with $W' \neq \emptyset$, and $\bigcup_{W'\in\rho(V_1,V_2,V_3)} W' = V_1 \cap V_2 \cap V_3 \subset B$. Now, consider the open sets $U_i \cup V_i \in \tau_i$ for $i = 1, 2, 3$. These open sets intersect both A and B. By our contradiction assumption, every $W'' \in \rho(U_1 \cup V_1, U_2 \cup V_2, U_3 \cup V_3)$ is non-empty. However, by axiom TT3, there exists $W_3 \in \rho(U_1 \cap V_1, U_2 \cap V_2, U_3 \cap V_3)$ such that $W_3 \subseteq W \cap W'$. Since $U_i \cap V_i = \emptyset$ for all i, by axiom TT4, we must have at least one of $U_i \cap V_i = \emptyset$, which we already established. By axiom TT5, $\bigcup_{W\in\rho(\emptyset,\emptyset,\emptyset)} W = \emptyset$, so $\rho(\emptyset, \emptyset, \emptyset) = \{\emptyset\}$. This contradicts our assumption. Therefore, $(X, \tau_1, \tau_2, \tau_3, \rho)$ must be tri-connected. □

Definition 5.3. A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is said to be tri-path connected if for any two points $x, y \in X$, there exists a continuous function $f : [0, 1] \to X$ (continuous with respect to all three topologies) such that $f(0) = x$, $f(1) = y$, and for any open set $U_i \in \tau_i$ $(i = 1, 2, 3)$ with $f([0, 1]) \cap U_i \neq \emptyset$, there exists $W \in \rho(U_1, U_2, U_3)$ with $W \cap f([0, 1]) \neq \emptyset$.

Theorem 5.4. Every tri-path connected space is tri-connected.

Proof. Let $(X, \tau_1, \tau_2, \tau_3, \rho)$ be a tri-path connected space. Suppose, for contradiction, that it is not tri-connected. Then there exist non-empty, disjoint sets A and B such that $X = A \cup B$ and for any $U_i \in \tau_i$ $(i = 1, 2, 3)$ with $A \cap U_i \neq \emptyset$ and $B \cap U_i \neq \emptyset$, every $W \in \rho(U_1, U_2, U_3)$ is non-empty. Since A and B are non-empty, we can choose points $x \in A$ and $y \in B$. By tri-path connectedness, there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$, $f(1) = y$, and for any open set $U_i \in \tau_i$ $(i = 1, 2, 3)$ with $f([0, 1]) \cap U_i \neq \emptyset$, there exists $W \in \rho(U_1, U_2, U_3)$ with $W \cap f([0, 1]) \neq \emptyset$. Consider the sets $f^{-1}(A)$ and $f^{-1}(B)$. Since f is continuous with respect to all three topologies and A and B form a separation of X, $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of $[0, 1]$. Moreover, $0 \in f^{-1}(A)$ and $1 \in f^{-1}(B)$. However, $[0, 1]$ is connected in the standard topology, so it cannot have a separation. This contradiction implies that $(X, \tau_1, \tau_2, \tau_3, \rho)$ must be tri-connected. □

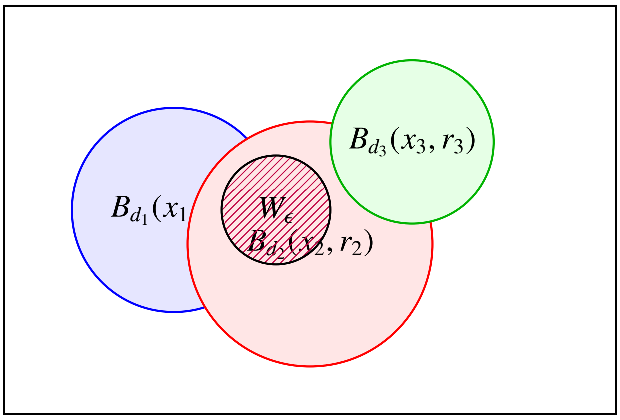
6. Metrizability in tri-topological spaces

Definition 6.1. A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is said to be tri-metrizable if there exist metrics d_1, d_2, d_3 on X such that:

(1) The topology induced by d_i is τ_i for $i = 1, 2, 3$.

(2) For any open balls $B_{d_i}(x_i, r_i) = \{y \in X : d_i(x_i, y) < r_i\}$ with $i = 1, 2, 3$, there exists a function $\sigma : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $\epsilon > 0$, $W_\epsilon = \{y \in X : d_1(x_1, y) + d_2(x_2, y) + d_3(x_3, y) < \sigma(r_1, r_2, r_3) + \epsilon\} \in \rho(B_{d_1}(x_1, r_1), B_{d_2}(x_2, r_2), B_{d_3}(x_3, r_3))$.

Figure 7 visualizes the interaction of metric balls in a tri-metrizable space. The three circles represent open balls from different metrics d_1, d_2, and d_3. The purple hatched region W_ϵ shows how the function σ combines the three metrics to create elements of the interaction function ρ. This demonstrates that tri-metrizability requires not just three compatible metrics, but also a specific way for them to interact through the function σ.



Space X

Figure 7. Visualization of the interaction of metric balls in a tri-metrizable space. The function σ determines how the three metrics combine to create elements of ρ, with W_ϵ representing a typical interaction pattern. Tri-metrizable Space: σ governs metric ball interactions.

Theorem 6.2. A tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ is tri-metrizable if and only if:

(1) Each topology τ_i $(i = 1, 2, 3)$ is metrizable.

(2) For any $x \in X$ and any open neighborhoods $U_i \in \tau_i$ of x $(i = 1, 2, 3)$, there exist open neighborhoods $V_i \in \tau_i$ of x such that $V_i \subset U_i$ and for any $y \in V_1 \cap V_2 \cap V_3$, there exists $W \in \rho(U_1, U_2, U_3)$ with $y \in W$.

Proof. ($\Rightarrow$) Assume $(X, \tau_1, \tau_2, \tau_3, \rho)$ is tri-metrizable. Then, by definition, each topology τ_i is metrizable via a metric d_i. Let $x \in X$ and let $U_i \in \tau_i$ be open neighborhoods of x for $i = 1, 2, 3$. Since each τ_i is metrizable, there exists $r_i > 0$ such that $B_{d_i}(x, r_i) \subset U_i$. Choose $r'_i < r_i$ and define $V_i = B_{d_i}(x, r'_i)$. For any $y \in V_1 \cap V_2 \cap V_3$, we have $d_i(x, y) < r'_i$ for $i = 1, 2, 3$. By the definition of tri-metrizability, there exists a function σ such that for any $\epsilon > 0$, $W_\epsilon = \{z \in X : d_1(x, z) + d_2(x, z) + d_3(x, z) < \sigma(r_1, r_2, r_3) + \epsilon\} \in \rho(B_{d_1}(x, r_1), B_{d_2}(x, r_2), B_{d_3}(x, r_3))$. Choose $\epsilon > 0$ small enough so that $d_1(x, y) + d_2(x, y) + d_3(x, y) < \sigma(r_1, r_2, r_3) + \epsilon$. Then $y \in W_\epsilon \in \rho(B_{d_1}(x, r_1), B_{d_2}(x, r_2), B_{d_3}(x, r_3)) \subset \rho(U_1, U_2, U_3)$.

($\Leftarrow$) Conversely, assume the conditions hold. By the first condition, each topology τ_i is metrizable via a metric d_i. Define the function $\sigma : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as follows: for any $r_1, r_2, r_3 > 0$ and points $x_1, x_2, x_3 \in X$, let $\sigma(r_1, r_2, r_3) = \inf\{d_1(x_1, y) + d_2(x_2, y) + d_3(x_3, y) : y \in B_{d_1}(x_1, r_1) \cap B_{d_2}(x_2, r_2) \cap B_{d_3}(x_3, r_3)\}$ when the intersection is non-empty, and $\sigma(r_1, r_2, r_3) = 0$ otherwise. Using the second condition and the definition of σ, we can show that for any open balls $B_{d_i}(x_i, r_i)$ and any $\epsilon > 0$, the set $W_\epsilon = \{y \in X : d_1(x_1, y) + d_2(x_2, y) + d_3(x_3, y) < \sigma(r_1, r_2, r_3) + \epsilon\} \in \rho(B_{d_1}(x_1, r_1), B_{d_2}(x_2, r_2), B_{d_3}(x_3, r_3))$. Therefore, $(X, \tau_1, \tau_2, \tau_3, \rho)$ is tri-metrizable. □

7. Tri-Topological product spaces

Definition 7.1. Let $(X_i, \tau_{i1}, \tau_{i2}, \tau_{i3}, \rho_i)$ for $i = 1, 2$ be two tri-topological spaces. The tri-topological product space is the quintuple $(X_1 \times X_2, \tau_1^\times, \tau_2^\times, \tau_3^\times, \rho^\times)$ where:

- $\tau_j^\times$ is the product topology on $X_1 \times X_2$ induced by τ_{1j} and τ_{2j} for $j = 1, 2, 3$
- $\rho^\times$ is defined by: for $U_j \in \tau_j^\times$ ($j = 1, 2, 3$), $\rho^\times(U_1, U_2, U_3) = \{W_1 \times W_2 : W_i \in \rho_i(\pi_i(U_1), \pi_i(U_2), \pi_i(U_3))$ for $i = 1, 2\}$ where $\pi_i : X_1 \times X_2 \to X_i$ is the projection onto the i-th coordinate.

Theorem 7.2. The tri-topological product space $(X_1 \times X_2, \tau_1^\times, \tau_2^\times, \tau_3^\times, \rho^\times)$ satisfies axioms TT1-TT5 and is therefore a valid tri-topological space.

Proof. We verify each axiom:

1. TT1: For any $U_j \in \tau_j^\times$, since each ρ_i satisfies TT1, both $\rho_1(\pi_1(U_1), \pi_1(U_2), \pi_1(U_3))$ and $\rho_2(\pi_2(U_1), \pi_2(U_2), \pi_2(U_3))$ are non-empty. Thus $\rho^\times(U_1, U_2, U_3)$ is non-empty.
2. TT2: Follows from the monotonicity of ρ_1 and ρ_2 and the fact that projections preserve inclusions.
3. TT3: For $W_1 \times W_2 \in \rho^\times(U_1, U_2, U_3)$ and $W_1' \times W_2' \in \rho^\times(V_1, V_2, V_3)$, by TT3 for ρ_i, there exist $W_i'' \in \rho_i(\pi_i(U_1 \cap V_1), \pi_i(U_2 \cap V_2), \pi_i(U_3 \cap V_3))$ with $W_i'' \subseteq W_i \cap W_i'$. Then $W_1'' \times W_2'' \in \rho^\times(U_1 \cap V_1, U_2 \cap V_2, U_3 \cap V_3)$ and $W_1'' \times W_2'' \subseteq (W_1 \times W_2) \cap (W_1' \times W_2')$.
4. TT4: If $\emptyset \in \rho^\times(U_1, U_2, U_3)$, then $\emptyset = W_1 \times W_2$ for some $W_i \in \rho_i(\pi_i(U_1), \pi_i(U_2), \pi_i(U_3))$. This means at least one $W_i = \emptyset$, so by TT4 for ρ_i, at least one $\pi_i(U_j) = \emptyset$, which implies $U_j = \emptyset$.
5. TT5: We have

$$\bigcup_{W \in \rho^\times(U_1, U_2, U_3)} W = \bigcup \{W_1 \times W_2 : W_i \in \rho_i(\pi_i(U_1), \pi_i(U_2), \pi_i(U_3))\} \tag{7.1}$$

$$= \left(\bigcup_{W_1} W_1\right) \times \left(\bigcup_{W_2} W_2\right) \tag{7.2}$$

$$= (\pi_1(U_1) \cap \pi_1(U_2) \cap \pi_1(U_3)) \times (\pi_2(U_1) \cap \pi_2(U_2) \cap \pi_2(U_3)) \tag{7.3}$$

$$= U_1 \cap U_2 \cap U_3 \tag{7.4}$$

□

Theorem 7.3. If $(X_i, \tau_{i1}, \tau_{i2}, \tau_{i3}, \rho_i)$ for $i = 1, 2$ are tri-compact spaces, then their tri-topological product $(X_1 \times X_2, \tau_1^\times, \tau_2^\times, \tau_3^\times, \rho^\times)$ is tri-compact.

Proof. Let $\{U_{1\alpha}\}_{\alpha \in A} \subset \tau_1^\times$, $\{U_{2\beta}\}_{\beta \in B} \subset \tau_2^\times$, and $\{U_{3\gamma}\}_{\gamma \in C} \subset \tau_3^\times$ be collections satisfying the finite intersection property for tri-compactness. Each $U_{j\alpha}$ can be written as a union of basic open sets of the form $V_{j\alpha}^{(1)} \times V_{j\alpha}^{(2)}$ where $V_{j\alpha}^{(i)} \in \tau_{ij}$. The projections of these collections satisfy the finite intersection property in each component space. Since $(X_i, \tau_{i1}, \tau_{i2}, \tau_{i3}, \rho_i)$ are tri-compact, there exist $W_i^* \in \rho_i(\bigcup \pi_i(U_{1\alpha}), \bigcup \pi_i(U_{2\beta}), \bigcup \pi_i(U_{3\gamma}))$ with $W_i^* \neq \emptyset$. Then $W_1^* \times W_2^* \in \rho^\times(\bigcup U_{1\alpha}, \bigcup U_{2\beta}, \bigcup U_{3\gamma})$ and $W_1^* \times W_2^* \neq \emptyset$, proving tri-compactness. □

Proposition 7.4. The tri-topological product preserves the following properties:

(1) If both spaces are tri-Hausdorff, then the product is tri-Hausdorff.

(2) If both spaces are tri-connected, then the product is tri-connected.

(3) If both spaces are tri-metrizable, then the product is tri-metrizable.

Proof. (1) For distinct points $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$, either $x_i \neq y_i$ for some i. Using tri-Hausdorff property in the i-th component and taking products with the whole space in the other component gives the required separation.

(2) If the product were disconnected as $A \cup B$, then by properties of product spaces, at least one component would be disconnected, contradicting tri-connectedness.

(3) If d_{ij} metrizes τ_{ij}, then the product metrics $d_j((x_1, x_2), (y_1, y_2)) = d_{1j}(x_1, y_1) + d_{2j}(x_2, y_2)$ metrize $\tau_j^\times$. The function σ for the product can be constructed from the σ functions of the components. □

8. Conclusions

We introduced throughout this paper a full theory of tri-topology, which expands the conventional structures of topological and bi-topological spaces. Three distinct topologies on the same underlying set receive their complex relationship expression through the interaction function ρ in our approach. The interaction function ρ is not merely a technical device but represents a fundamental new way to capture three-way topological interactions that cannot be reduced to pairwise relationships.

Our work delivers a systematic adaptation of classical topological principles to tri-topological spaces, which encompasses tri-variants of T_0 through T_2 (including tri-regular and tri-normal) separation axioms and includes tri-compactness with its subtypes tri-Lindelöf and tri-paracompact, as well as metrizability conditions alongside tri-connectedness and tri-path connectedness. We also introduced and studied tri-topological product spaces, showing that key properties are preserved under products. The text presents detailed, rigorous proofs establishing the relationships between these properties and generalizes traditional topological results found in Willard and Engelking.

The practical significance of our framework is demonstrated through concrete applications:

- **Digital Topology:** In image processing, pixels naturally have multiple connectivity structures (4-connectivity for edge detection, 8-connectivity for region growing, and Euclidean for smoothing). Our tri-topological framework provides the mathematical foundation to study these simultaneously.

- **Data Analysis:** Modern machine learning uses multiple similarity measures on the same dataset. The interaction function ρ can model how different metrics (cosine for text, Euclidean for continuous features, and Manhattan for categorical) interact to form meaningful clusters.

- **Theoretical Physics:** Quantum gravity models suggest spacetime may exhibit different topological structures at different scales. Our framework provides tools to study how these structures interact at the Planck scale.

Future research directions emerging from this work include several specific open problems:

Open Problem 1 (Embedding Characterization): Under what necessary and sufficient conditions can a tri-topological space $(X, \tau_1, \tau_2, \tau_3, \rho)$ be embedded as a subspace of a product of three standard topological spaces? This problem extends classical embedding theorems and requires understanding how the interaction function ρ relates to product structures.

Open Problem 2 (Tri-Uniform Spaces): Can we develop a theory of tri-uniform spaces that generalizes both uniform spaces and tri-topological spaces? Specifically, how should we define tri-uniform structures that induce tri-topological spaces through a natural process analogous to how uniform structures induce topologies?

Open Problem 3 (Computational Complexity): What is the computational complexity of determining whether a given tri-topological space satisfies various separation axioms? Can we establish tight theoretical bounds for algorithmic problems such as:

- Deciding tri-connectedness in finite tri-topological spaces with $|X| = n$ elements
- Computing minimal tri-compact refinements given explicit descriptions of τ_1, τ_2, τ_3 and ρ
- Verifying the tri-Hausdorff property when each topology has at most m open sets

Open Problem 4 (Categorical Framework): Develop a complete categorical theory of tri-topological spaces, including:

- Characterization of epimorphisms and monomorphisms in the category **TriTop** of tri-topological spaces with appropriate morphisms
- Construction of limits and colimits in **TriTop**
- Identification of reflective and coreflective subcategories
- Study of the forgetful functor from **TriTop** to **BiTop** and its adjoints

Open Problem 5 (Fixed Point Theory): Extend classical fixed point theorems to tri-topological spaces. Under what conditions on the interaction function ρ do continuous self-maps $f : X \to X$ (continuous with respect to all three topologies) guarantee the existence of fixed points? Can we formulate tri-topological versions of the Banach, Brouwer, and Kakutani fixed point theorems? These specific problems provide concrete research directions that build upon our theoretical framework while connecting to broad mathematical contexts. The resolution of these problems would significantly advance our understanding of multi-topological structures and their applications. The established mathematical principles establish multidimensional topological spaces incorporating three parallel structures, which would drive future research in emerging fields of topology.

Author contributions

Jamal Oudetallah and Daniel Breaz contributed to the conceptualization and original drafting of the manuscript. Iqbal Batiha and Jamal Oudetallah developed the methodology and carried out the formal theoretical framework. Ala Amourah and Tala Sasa contributed to the formal analysis and interpretation of results. Sheza El-Deeb and Ala Amourah conducted the investigation and assisted in refining the theoretical examples. Daniel Breaz, Jamal Oudetallah, and Iqbal Batiha performed the review and editing of the manuscript. Validation was conducted by Jamal Oudetallah and Sheza El-Deeb. Iqbal Batiha supervised the overall research process. All authors reviewed and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflict of interest.

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