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Research article

Statistical modeling with a novel distribution: Inference, information measures, and applications to inflation rates and mechanical failure data

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Abstract: The increasing demand for flexible statistical models defined on the unit interval has led to the development of new distributions capable of capturing various tail behaviors and data shapes. In this study, we introduce an extended form of the power unit inverse Lindley distribution, offering greater modeling flexibility for bounded data. We explore its key statistical properties, including moments, skewness, kurtosis, quantile function, Fisher information matrix, extropy, and negative cumulative extropy. To assess parameter estimation, fifteen classical methods are implemented and evaluated through simulation, demonstrating high efficiency and low bias, even for small samples. The proposed model is then applied to two real datasets: annual inflation rates from 45 Asian countries and failure times of mechanical components. Comparative analysis with several existing unit distributions confirms the superior goodness-of-fit and practical applicability of the proposed model in both economic and engineering contexts.

Keywords: Lindley distribution; information measures; Fisher information matrix; negative cumulative extropy; parameter estimation methods; simulation study; bounded data modeling; real data applications
Mathematics Subject Classification: 62F15, 62G20, 65C60

1. Introduction

A versatile and efficient model for lifespan data, the Lindley distribution (LD) was initially proposed by [1] within the framework of Bayesian statistics. A favorably skewed distribution, the Lindley is a hybrid of the exponential and gamma types; for more details, see [2]. Its simplicity and usefulness in reliability analysis, survival studies, and queueing theory have made it a popular choice. In [3], the mathematical features of the LD with reliability measurements and the application of waiting time data to a bank are examined. A three-parameter generalization that encompasses both the exponential and gamma distributions was introduced by Zakerzadeh and Dolati [4]. Shanker et al. [5] later proposed a two-parameter quasi-LD, while Ghitany et al. [6] developed a novel two-parameter power LD. A more versatile model for lifespan data, an expansion of the LD, and several distributional statistical features are investigated in [7, 8]. For inverse power LD with some applications, see [9]. Reliability engineering is a well-known domain that has made extensive use of the LD and Inverse LD. Mechanical component and system lifespan modeling is well-suited to these distributions because they can account for both growing and decreasing hazard rates (cf. [10, 11]). The versatility of the LD and its extensions, such as the inverse and unit inverse LDs, makes them attractive models across fields requiring handling skewed, heavy-tailed, or bounded data. Their analytical tractability, good empirical fit, and flexibility continue to drive research in distribution theory and its applications. LD and its extensions have been extensively applied in various fields such as, medical application (cf.[12], modeling waiting times (cf. [3]), flood data (cf. [13]), engineering and industrial quality control (cf. [14]), risk life data (cf. [15]), and modeling voltage data, (cf. [16]). Recently, [17] proposed the power unit inverse LD (PUILD), a novel two-parameter generalization of the famous unit inverse LD. Let X be a continuous random variable (RV) that follows the PUILD. Then, the probability density function (PDF) and cumulative distribution function (CDF), as presented in [17], are given as follows:

$$g(x) = \frac{e^{\theta - \frac{\theta}{x^\delta}}\left(\theta + x^\delta\right)}{(\theta + 1)x^\delta} \tag{1.1}$$

and

$$G(x) = \frac{e^{\theta - \frac{\theta}{x^\delta}}\left(\theta + x^\delta\right)}{(\theta + 1)x^\delta}, \tag{1.2}$$

respectively, $0 < x < 1, \theta, \delta > 0$. An essential aspect of non-parametric statistics, and specifically kernel density estimation, is the Epanechnikov family of kernels, which was introduced by [18]. For many tasks, the Epanechnikov kernel is the best option since it minimizes the mean integrated squared error more than any other kernel function in this family. It is computationally convenient and practical to use the conventional Epanechnikov kernel, which is defined as a quadratic function with compact support. Enhancements to smoothness and flexibility in different estimation contexts, as well as adaptations to multivariate settings, are examples of extensions of the Epanechnikov family. The theory of functions that achieve a given differential equation in a fixed domain makes use of kernel functions. The Epanechnikov kernel function is defined as (cf. Alkhazaalh and Al-Zoubi [19]): $K(v) = \frac{3}{4} max\{1 - v^2\}$. It satisfies the symmetry property $(K(v) = K(-v))$, is non-negative, and fulfills the normalization condition $\left(\int_{-\infty}^{\infty} K(v)dv = 1\right)$. The Epanechnikov-Akash distribution was introduced by Alkhazaleh et al. [20]. It was created by combining the Akash distribution with the Epanechnikov function. Using the exponential distribution and the Epanechnikov kernel function, Alkhazaalh and Al-Zoubi [19] suggested a novel continuous distribution. Recently, a bivariate distribution based on the FGM copula, the Epanechnikov exponential-FGM, was introduced by Barakat et al. [21] as a novel statistical model. Theoretically, many academics have generalized distributions using distribution theory. Based on the quadratic transmutation map described by [22], the transmuted Janardan distribution was suggested by [23].

In this paper, we propose a new distribution called the Epanechnikov-PUILD (EPUILD), constructed by integrating the PUILD distribution into the Epanechnikov kernel. In addition to investigating the statistical properties of the EPUILD, employing multiple estimation methods, conducting extensive simulations to demonstrate its efficiency, and applying it to two real data examples, this study also examines FI and various extropy measures for the EPUILD. We will now go over each of these measures, emphasizing their uses and significance. Because it extracts information about unknown parameters from a probability distribution sample, FI is a crucial tool. Sufficiency and efficiency are concepts that are associated with the FI. Suppose the statistics generated from a given sample reveal all there is to know about the unknown parameter inside a particular family of probability distributions. In that case, we can say that the statistics are sufficient and reveal all the information about the parameter, making any additional statistics redundant. The Cramér-Rao lower limit, a lower bound on the variance of unbiased estimators of an unknown parameter, is achieved by the asymptotic variance of an asymptotically efficient estimator. Any unbiased estimator's variance must be at least equal to the inverse of the FI, according to the Cramér-Rao lower bound. To find limits on the variance of an estimate of a parameter and, when the sample size is big enough, the approximate sampling distribution of that estimator, it is helpful to know how much information a sample carries about the unknown parameter. [24] and [25] provide more information regarding the FI and its applications. [26] is another reference. Consider a random variable X with PDF $f(x;\theta)$, where $\theta \in \Theta$ is an unknown parameter from the parameter space Θ. The integral of the real parameter $\theta \in \Theta$ contained in the RV X is defined as

$$\mathcal{I}(X;\theta) = -\mathrm{E}\left(\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2}\right) = \mathrm{E}\left(\frac{\partial \log f(x;\theta)}{\partial\theta}\right)^2. \tag{1.3}$$

Under specific regularity conditions. The FI matrix (FIM) can be defined by expanding this description to include where the parameter θ transforms into a parameter-row-vector Θ. The concept of FI extends naturally to the case where the parameter θ is a vector $\underline{\theta} = (\theta_1, \theta_2)$, leading to the FIM $\boldsymbol{I}$, which is defined as

$$\boldsymbol{I}\left(X;\underline{\theta}\right) = \begin{pmatrix} I_{\theta_1}(X;\underline{\theta}) & -E\left(\frac{\partial^2 \log f(X;\underline{\theta})}{\partial\theta_2\partial\theta_1}\right) \\ -E\left(\frac{\partial^2 \log f(X;\underline{\theta})}{\partial\theta_1\partial\theta_2}\right) & I_{\theta_2}(X;\underline{\theta}) \end{pmatrix}. \tag{1.4}$$

FI sheds light on the accuracy of the distribution's parameter estimations within the framework of our suggested model, which is obtained by combining the Epanechnikov kernel and the PUILD. Parameter identifiability, estimator precision, and the model's statistical behavior can be better understood through the study of FI, especially with the added shape complexity and flexibility introduced by the kernel structure. In order to compare the level of uncertainty between two RVs, Lad et al. [27] proposed the concept of extropy as the complementary dual of entropy. Extropy provides a further perspective on dispersion and randomness, in contrast to entropy's emphasis on the self-information of outcomes. Put simply, when the consequences are uncertain, the entropy is high, but when the outcomes are predictable, the extropy is high. If we consider the discrete probability distribution of an RV X as $P = (p_1, p_2, ..., p_n)$, then $J(X) = -\sum_{i=1}^{n}(1 - p_i)\ln(1 - p_i)$ is the definition of the extropy. This metric provides a two-sided view of entropy by measuring the spread of the complement probabilities $1 - p_i$. Given a continuous non-negative RV X and its corresponding PDF $f_X(.)$, it is possible to express the extropy as

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)dx = -\frac{1}{2}\int_0^1 f_X(F_X^{-1}(u))du \leq 0. \tag{1.5}$$

Tahmasebi and Toomaj [28] proposed a negative cumulative extropy (NC) , defined as

$$C\xi(X) = \frac{1}{2}\int_0^\infty \left[1 - F^2(x)\right]dx = \frac{1}{2}\int_0^1 \frac{1 - u^2}{f(F_X^{-1}(u))}du. \tag{1.6}$$

Our motivation

- The combination of the PUILD with the Epanechnikov kernel is motivated by the need for highly flexible models capable of accurately capturing complex data behaviors, such as bounded support, skewness, and heavy tails, which are commonly observed in fields like reliability engineering, economics, and environmental studies.
- The Epanechnikov kernel enhances the baseline distribution's smoothness and adaptability, allowing for more effective modeling of intricate data patterns.
- By incorporating information-theoretic measures such as FI, extropy, and NC, we can assess the informativeness, uncertainty, and robustness of the model, thus providing a deeper understanding of its theoretical and practical strengths.
- The suggested distribution is used to examine two relevant datasets, inflation rates and mechanical component failure times. The results show that the new model is more practical and fits better than the other unit distribution.

The remainder of this paper is organized as follows: Section 2 presents the formulation of our proposed model. In Section 3, we study the statistical properties of the EPUILD. Section 4 presents the FIM for the EPUILD. Section 5 discusses the extropy and NC. In order to estimate the EPUILD model's parameters, we used fifteen different methods in Section 6. In Section 7, a simulation study is presented. In Section 8, two real-world datasets are used to demonstrate the utility and potential of the EPUILD. Finally, Section 7 provides concluding remarks.

2. Formulation of new model

After some modifications were applied to the CDF of Epanechnikov generalized family of distributions and by the PUILD CDF (1.2) as a baseline distribution, we have a new statistical model called EPUILD. Also, the EPUILD can be obtained by using the transmuted family of generalized distributions. Its CDF and PDF are given by

$$F(x) = \frac{x^{-2\delta} e^{\theta - 2\theta x^{-\delta}} \left(\theta + x^{\delta}\right) \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - e^{\theta}\left(\theta + x^{\delta}\right)\right]}{2(\theta+1)^2}, \quad 0 < x \leq 1,\ \theta, \delta > 0, \tag{2.1}$$

$$f(x) = \frac{\delta \theta^2 x^{-3\delta - 1} e^{\theta - 2\theta x^{-\delta}} \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - 2e^{\theta}\left(\theta + x^{\delta}\right)\right]}{2(\theta+1)^2}. \tag{2.2}$$

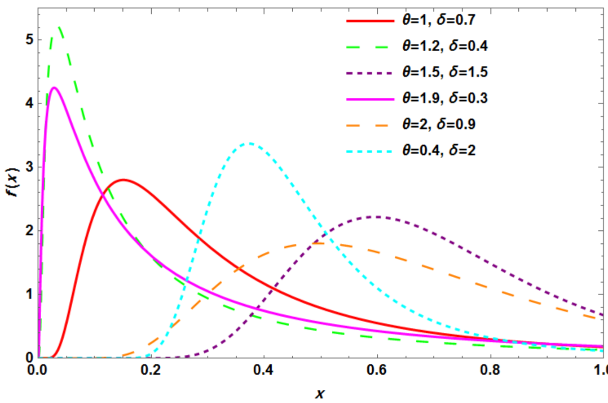
Also, the survival and hazard rate functions are defined as

$$S(x) = 1 - \frac{x^{-2\delta} e^{\theta - 2\theta x^{-\delta}} \left(\theta + x^{\delta}\right) \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - e^{\theta}\left(\theta + x^{\delta}\right)\right]}{2(\theta+1)^2}, \tag{2.3}$$

and

$$h(x) = \frac{\delta \theta^2 x^{-3\delta - 1} e^{\theta - 2\theta x^{-\delta}} \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - 2e^{\theta}\left(\theta + x^{\delta}\right)\right]}{2(\theta+1)^2 - x^{-2\delta} e^{\theta - 2\theta x^{-\delta}} (\theta + x^{\delta}) \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - e^{\theta}(\theta + x^{\delta})\right]}. \tag{2.4}$$

Figure 1 illustrates the shapes of the PDF and hazard function of the EPUILD distribution for various parameter combinations of θ and δ. The PDF demonstrates considerable flexibility, ranging from unimodal to right-skewed shapes, depending on the values of the parameters. For instance, smaller δ values yield heavier tails and sharper peaks, while increasing θ leads to more symmetric density shapes. Similarly, the hazard function exhibits diverse patterns, including increasing, decreasing, and non-monotonic shapes, making the EPUILD suitable for modeling a wide range of real-world lifetime and reliability data. These behaviors confirm the versatility of the proposed model in capturing different hazard structures.



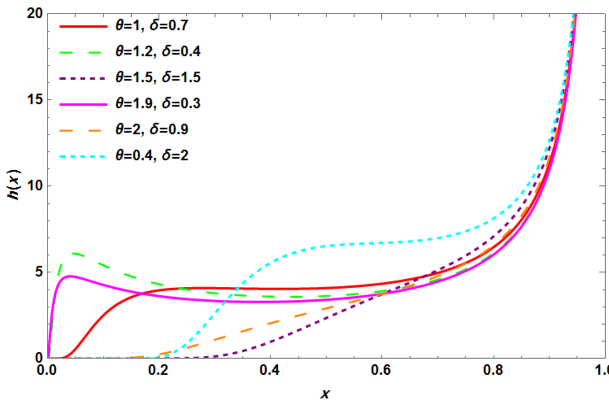


Figure 1. Plots of PDF and hazard function for the EPUILD.

3. Statistical properties

In this section, we derive and explore the key statistical properties of the proposed EPUILD. These properties include moments, skewness, kurtosis, quantile function, and order statistics, which are essential for understanding the model's flexibility and applicability.

3.1. Moments of EPUILD

One way to quantify the features of a statistical distribution is by calculating its moments. The form, central tendency, and variability of a distribution can be summarized and analyzed in an organized way using moments. The kth moment of an RV following EPUILD is given by:

$$\begin{aligned}
\mu_k &= \int_0^1 x^k f(x)dx \\
&= \int_0^1 x^k \frac{\delta\theta^2 x^{-3\delta-1} e^{\theta-2\theta x^{-\delta}} \left[3(\theta+1)x^{\delta} e^{\theta x^{-\delta}} - 2e^{\theta}(\theta + x^{\delta})\right]}{2(\theta+1)^2} dx \\
&= \frac{\delta\theta^2}{2(\theta+1)^2} \int_0^1 x^{k-3\delta-1} e^{\theta-2\theta x^{-\delta}} \left[3(\theta+1)x^{\delta} e^{\theta x^{-\delta}} - 2e^{\theta}(\theta + x^{\delta})\right] dx \\
&= \frac{\delta\theta^2}{2(\theta+1)^2} \left[3(\theta+1) \int_0^1 x^{k-2\delta-1} e^{\theta-\theta x^{-\delta}} dx - 2e^{\theta} \int_0^1 x^{k-3\delta-1}(\theta + x^{\delta}) e^{-2\theta x^{-\delta}} dx\right] \\
&= \frac{\theta^2 \left(-\delta + 3\delta e^{\theta}(\theta+1) E_{\frac{k}{\delta}-1}(\theta) + e^{2\theta}(k - 4\delta) E_{\frac{k}{\delta}-1}(2\theta)\right)}{2\delta(\theta+1)^2},
\end{aligned} \tag{3.1}$$

where, $E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t^n}$ is the exponential integral function. For specific special arguments, $E_n(x)$ automatically evaluates to accurate values, and it is a mathematical function that may be manipulated symbolically and numerically.

3.2. Skewness and kurtosis

Skewness (SK) and kurtosis (KU) are two crucial properties to consider while assessing probability distributions, which can take on many shapes and spreads. A probability distribution's SK and KU can tell us a lot about its structure, help us spot outliers, and provide us with better data to work with. The SK and KU can be obtained as

$$SK(X) = \frac{\mu_3 - \mu_1\mu_2 - 3\mu_1^3}{(\mu_2 - \mu_1^2)^{\frac{3}{2}}} \tag{3.2}$$

and

$$KU(X) = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu 2 - 3\mu_1^4}{(\mu_2 - \mu_1^2)^2}. \tag{3.3}$$

By using (3.2) and (3.3), Figure 2 displays the SK and KU for different values of parameters θ and δ based on EPUILD. Based on Figure 2, we can say that:

- The SK decreases monotonically as θ increases, indicating that the distribution becomes more symmetric with increasing θ.
- Lower values of δ exhibit higher SK, meaning the distribution is more heavily right-skewed.
- The KU also decreases with increasing θ, indicating that the distribution becomes less peaked and has thinner tails.

- Smaller values of δ correspond to extremely high KU, suggesting a more leptokurtic (heavy-tailed) behavior.

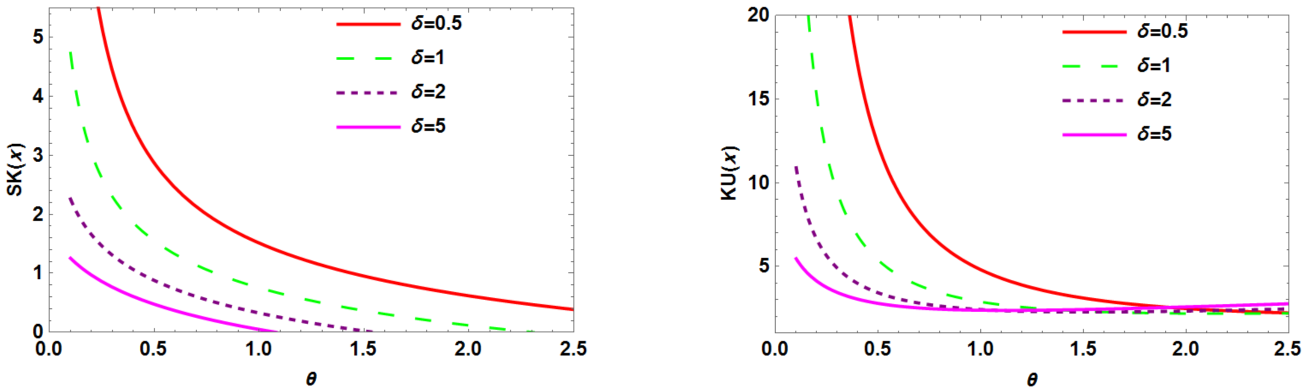


Figure 2. Plots of SK and KU of X based on the EPUILD.

3.3. *Quantile function*

We derive the quantile function $Q(u) = F^{-1}(u)$. Let $A = x^{-\delta}$; then substitute into the CDF (2.1) as:

$$u = \frac{A^2 e^{\theta-2\theta A}(\theta + A^{-1})\left[3(\theta+1)A^{-1}e^{\theta A} - e^{\theta}(\theta + A^{-1})\right]}{2(\theta+1)^2}.$$

Then

$$\begin{aligned} u &= \frac{A^2 e^{\theta-2\theta A}\,\frac{\theta A+1}{A}\left[\frac{3(\theta+1)e^{\theta A} - e^{\theta}(\theta A+1)}{A}\right]}{2(\theta+1)^2} \\ &= \frac{e^{\theta-2\theta A}\cdot(\theta A+1)\left[3(\theta+1)e^{\theta A} - e^{\theta}(\theta A+1)\right]}{2(\theta+1)^2}. \end{aligned}$$

Denote the right-hand side as $G(A)$, and solve

$$u = G(A).$$

This is a transcendental equation in A, which has no closed-form algebraic solution due to the exponential terms involving A. Once A is found numerically or implicitly (denoted A_u), we recover x as

$$x = A_u^{-1/\delta}.$$

The quantile function is given implicitly by

$$Q(u) = [A_u]^{-1/\delta},$$

where A_u is the solution of

$$u = \frac{e^{\theta-2\theta A}(\theta A+1)\left[3(\theta+1)e^{\theta A} - e^{\theta}(\theta A+1)\right]}{2(\theta+1)^2}.$$

Table 1 shows the numeric values of the first, second, and third quartiles based on the EPUILD quantile function.

Table 1. Values of quantile function at $u = 0.25, 0.5, 0.75$.

(θ, δ)	$(1, 0.5)$	$(1, 1)$	$(2, 1)$	$(2, 1.5)$	$(3, 1)$
$Q(0.25)$	0.0792	0.2814	0.4651	0.6003	0.5810
$Q(0.5)$	0.1561	0.3951	0.6001	0.7119	0.7091
$Q(0.75)$	0.3128	0.5593	0.7540	0.8284	0.8347

3.4. Order statistics based on EPUILD

Let $X_{1:n}, ..., X_{n:n}$, denote the order statistics from a random sample from a continuous population with CDF $F(x)$ and PDF $f(x)$. Then the density function of $X_{r:n}$ is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} \left(1 - F(x)\right)^{n-r} f(x).$$

According to EPUILD, we have

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(r-1)!(n-r)!} \left(\frac{x^{-2\delta} e^{\theta - 2\theta x^{-\delta}} \left(\theta + x^{\delta}\right) \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - e^{\theta} \left(\theta + x^{\delta}\right)\right]}{2(\theta+1)^2} \right)^{r-1} \\ &\times \left(1 - \frac{x^{-2\delta} e^{\theta - 2\theta x^{-\delta}} \left(\theta + x^{\delta}\right) \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - e^{\theta} \left(\theta + x^{\delta}\right)\right]}{2(\theta+1)^2} \right)^{n-r} \\ &\times \frac{\delta \theta^2 x^{-3\delta - 1} e^{\theta - 2\theta x^{-\delta}} \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - 2e^{\theta} \left(\theta + x^{\delta}\right)\right]}{2(\theta+1)^2}, \quad 1 \leq r \leq n, 0 < x \leq 1, \ \theta, \delta > 0. \end{aligned}$$

The CDF of $X_{r:n}$ is given by

$$F_{r:n}(X) = \sum_{k=r}^{n} \binom{n}{k} F(x)^k \left(1 - F(x)\right)^{n-k}.$$

For EPUILD, we have

$$\begin{aligned} F_{r:n}(X) &= \sum_{k=r}^{n} \binom{n}{k} \left(\frac{x^{-2\delta} e^{\theta - 2\theta x^{-\delta}} \left(\theta + x^{\delta}\right) \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - e^{\theta} \left(\theta + x^{\delta}\right)\right]}{2(\theta+1)^2} \right)^{k} \\ &\times \left(1 - \left(\frac{x^{-2\delta} e^{\theta - 2\theta x^{-\delta}} \left(\theta + x^{\delta}\right) \left[3(\theta+1) x^{\delta} e^{\theta x^{-\delta}} - e^{\theta} \left(\theta + x^{\delta}\right)\right]}{2(\theta+1)^2} \right) \right)^{n-k}. \end{aligned}$$

We can define the relationship as

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = I_p(r, n-r+1),$$

where

$$I_x(m,n) = \frac{\int_0^x t^{m-1}(1-t)^{n-1}}{\int_0^1 t^{m-1}(1-t)^{n-1}}.$$

The incomplete beta ratio. Then the quantile function of $X_{r:n}$ is

$$Q_r(u_r) = Q_X\left(I_{u_r}^{-1}(r, n-r+1)\right).$$

Therefore, the moments of order statistics can be defined as

$$\mu_{r:n} = \frac{n!}{(r-1)!(n-r)!}\int_0^1 u^{r-1}(1-u)^{n-r}Q(u)du.$$

4. Fisher information matrix in EPUILD

We find the FIM for the EPUILD that has been suggested here. This matrix is the foundation for asymptotic inference and gives crucial information on the efficiency and accuracy of the parameter estimators. The FIM quantifies how much information the data carry about the parameters (θ, δ), enabling theoretical evaluation of estimation accuracy, see [29, 30]. A nonsingular FIM suggests that model parameters are identifiable. This is critical for ensuring the model is well-defined and estimable. Moreover, the FIM is instrumental in assessing parameter identifiability and estimator efficiency, which are key properties in validating the robustness of the proposed model. The log-likelihood for a single observation x based on 2.2 is:

$$\begin{aligned}\ell(\theta,\delta) = \log f(x;\theta,\delta) &= \log\left(\frac{\delta\theta^2}{2(\theta+1)^2}\right) + (-3\delta-1)\log x + (\theta - 2\theta x^{-\delta}) \\ &+ \log\left(3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}(\theta + x^{\delta})\right).\end{aligned}$$

Which simplifies to

$$\begin{aligned}\ell(\theta,\delta) &= \log\delta + 2\log\theta - 2\log(\theta+1) + (-3\delta-1)\log x + \theta - 2\theta x^{-\delta} \\ &+ \log\left(3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}(\theta + x^{\delta})\right).\end{aligned}$$

Now, we can get, $\frac{\partial\ell}{\partial\theta}$ and $\frac{\partial\ell}{\partial\delta}$, as follow

$$\frac{\partial\ell}{\partial\theta} = \frac{2}{\theta} + (1-2x^{-\delta}) - \frac{2}{\theta+1} + \frac{\partial}{\partial\theta}\log\Omega(x;\theta,\delta),$$

and

$$\frac{\partial\ell}{\partial\delta} = \frac{1}{\delta} - 3\log x - 2\theta x^{-\delta}\log x + \frac{\partial}{\partial\delta}\log\Omega(x;\theta,\delta),$$

where $\Omega(x;\theta,\delta) = \left(3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}(\theta + x^{\delta})\right)$. The partial derivative of $\Omega(x;\theta,\delta)$ is

$$\frac{\partial}{\partial\theta}\log\Omega(x;\theta,\delta) = \frac{3x^{\delta}e^{\theta x^{-\delta}} + 3(\theta+1)x^{\delta}x^{-\delta}e^{\theta x^{-\delta}} - 2e^{\theta} - 2e^{\theta}(\theta + x^{\delta})}{3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}(\theta + x^{\delta})},$$

and

$$\frac{\partial}{\partial\delta}\log\Omega(x;\theta,\delta) = \frac{3(\theta+1)(\log x)x^{\delta}e^{\theta x^{-\delta}} + 3(\theta+1)x^{\delta}(\theta x^{-\delta}\log x)e^{\theta x^{-\delta}} - 2e^{\theta}x^{\delta}\log x}{3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}(\theta + x^{\delta})},$$

Then, the FIM $\mathcal{I}(\theta,\delta)$ is defined as

$$\boldsymbol{I}(\theta,\delta) = -\mathbb{E}\begin{pmatrix} \frac{\partial^2\ell}{\partial\theta^2} & \frac{\partial^2\ell}{\partial\theta\partial\delta} \\ \frac{\partial^2\ell}{\partial\theta\partial\delta} & \frac{\partial^2\ell}{\partial\delta^2} \end{pmatrix} = \begin{pmatrix} I_{\theta\theta} & I_{\theta\delta} \\ I_{\theta\delta} & I_{\delta\delta} \end{pmatrix}. \tag{4.1}$$

Alternatively, the FIM can be computed as the expectation of the outer product of the score vector

$$\boldsymbol{I}(\theta,\delta) = \mathbb{E}\left[\begin{pmatrix} \frac{\partial\ln f}{\partial\theta} \\ \frac{\partial\ln f}{\partial\delta} \end{pmatrix}\begin{pmatrix} \frac{\partial\ln f}{\partial\theta} & \frac{\partial\ln f}{\partial\delta} \end{pmatrix}\right].$$

To numerically compute the FIM when closed-form expectations are unavailable, we employ a Monte Carlo approximation. Let $\nabla\ell(\mathbf{x};\boldsymbol{\theta}) = (\partial\ell/\partial\theta_1,\ldots,\partial\ell/\partial\theta_p)^{\top}$ denote the score vector for parameters $\boldsymbol{\theta} = (\theta,\delta)$, where $\ell(\mathbf{x};\boldsymbol{\theta})$ is the log-likelihood function. The FIM is then approximated by

$$\mathbf{I}(\boldsymbol{\theta}) \approx \frac{1}{N}\sum_{i=1}^{N}\nabla\ell(\mathbf{x}_i;\boldsymbol{\theta})\nabla\ell(\mathbf{x}_i;\boldsymbol{\theta})^{\top}, \tag{4.2}$$

where $\{\mathbf{x}_i\}_{i=1}^{N}$ are independent samples drawn from the model distribution $f(\mathbf{x};\boldsymbol{\theta})$. This estimator converges almost surely to the true FIM as $N \to \infty$ by the strong law of large numbers. In practice, we generate samples using rejection sampling with an efficient proposal distribution, ensuring proper mixing and computational feasibility. Table 2 shows the elements of FIM as different values of θ and δ. Based on Table 2, we can say that:

- $I_{\theta\theta}$ decreases with increasing θ and δ, indicating less precision in estimating θ at higher values of the parameters.
- $I_{\delta\delta}$ shows a sharp increase with increasing δ.
- The cross-information $I_{\theta\delta}$ decreases as both parameters increase.

Table 2. Fisher information matrix $\boldsymbol{I}(\theta,\delta)$.

θ	δ	$I_{\theta\theta}$	$I_{\delta\delta}$	$I_{\theta\delta}$
0.5	0.5	8.3446	0.0196	4.10341
0.5	1.0	8.3369	0.0098	1.0257
0.5	1.5	8.3440	0.0068	0.4559
0.5	2.0	8.3418	0.0049	0.2565
1.0	0.5	4.3388	1.5947	3.8432
1.0	1.0	4.5742	0.7998	0.9601
1.0	1.5	4.5786	0.5283	0.4271
1.0	2.0	4.4775	0.3961	0.2402

5. Extropy measures in EPUILD

The extropy and NC of the proposed EPUILD are examined in this section. These measures supplement entropy by reflecting different parts of distributional uncertainty and information dissemination. Based on (1.5) and (2.2), the extropy of EPUILD is given by

$$J(X) = -\frac{1}{2}\int_0^1 f(x)^2\,dx,$$

where

$$f(x)^2 = \left(\frac{\delta\theta^2 x^{-3\delta-1} e^{\theta-2\theta x^{-\delta}}\left[3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}(\theta+x^{\delta})\right]}{2(\theta+1)^2}\right)^2$$

$$= \frac{\delta^2\theta^4 x^{-6\delta-2} e^{2\theta-4\theta x^{-\delta}}\left[3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}(\theta+x^{\delta})\right]^2}{4(\theta+1)^4}.$$

Then,

$$\begin{aligned}
J(X) &= -\frac{1}{2}\int_0^1 \frac{\delta^2\theta^4 x^{-6\delta-2} e^{2\theta-4\theta x^{-\delta}}\left[3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}(\theta+x^{\delta})\right]^2}{4(\theta+1)^4}dx \\
&= -\frac{\delta^2\theta^4 e^{2\theta}}{8(\theta+1)^4}\int_0^1 x^{-6\delta-2}e^{-4\theta x^{-\delta}}\left(3(\theta+1)x^{\delta}e^{\theta x^{-\delta}} - 2e^{\theta}(\theta+x^{\delta})\right)^2 dx \\
&= -\frac{1}{\delta^5(1+\theta)^4}\cdot 2^{-13-\frac{2}{\delta}}\cdot 3^{-4-\frac{1}{\delta}}\cdot \theta^{-\frac{1}{\delta}}\cdot \Bigg[\\
&\times\; 12^{\frac{1}{\delta}}\delta\theta^{\frac{1}{\delta}}(81 - \delta(2233 + 3772\theta) + \delta^2(8069 + 349904\theta + 35664\theta^2) \\
&+\; \delta^3(111529 + 3401556\theta + 308160\theta^2 + 61632\theta^3) \\
&+\; \delta^4(242098 + 7924224\theta + 9147360\theta^2 + 448704\theta^3 + 96768\theta^4) \\
&+\; 24\delta^5(5873 + 234920\theta + 389040\theta^2 + 357120\theta^3 + 195840\theta^4 + 5184\theta^5)) \\
&+\; 6^{6+\frac{1}{\delta}}e^{2\theta}\delta^2(1 + 6\delta + 11\delta^2 + 6\delta^3)(1+\theta)^2\Gamma\left(\frac{1}{\delta}, 2\theta\right) \\
&-\; 4^{6+\frac{1}{\delta}}e^{3\theta}\delta(1 + 13\delta + 53\delta^2 + 83\delta^3 + 42\delta^4)(1+\theta)\Gamma\left(\frac{1}{\delta}, 3\theta\right) \\
&+\; 3^{4+\frac{1}{\delta}}e^{4\theta}(1 + 23\delta + 181\delta^2 + 601\delta^3 + 850\delta^4 + 408\delta^5)\Gamma\left(\frac{1}{\delta}, 4\theta\right)\Bigg].
\end{aligned}$$

Figure 3 displays the 3D plot of extropy for different values of parameters θ and δ based on EPUILD. Based on Figure 3, we can say that:

- The extropy increases as θ and δ increases.
- The extropy reaches a plateau with large values of θ and δ.

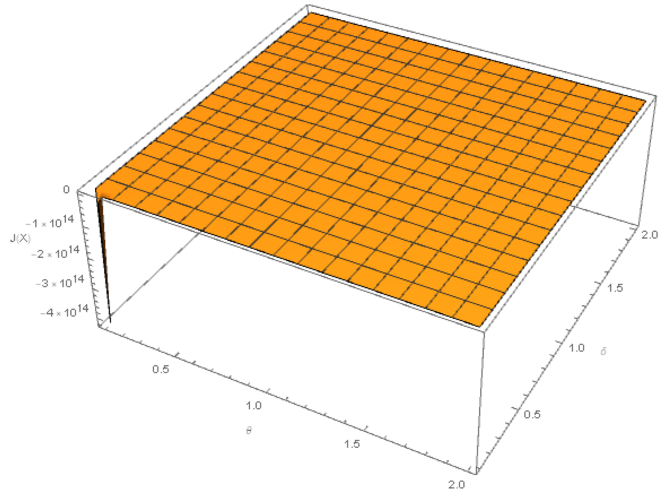


Figure 3. The extropy of X based on EPUILD.

If X is an RV following the EPUILD, then based on Eqs (2.1) and (1.6), the NC in X is defined by:

$$
\begin{aligned}
C\xi(X) \quad &= \frac{1}{2}\int_0^1 \left[1-\left(\frac{x^{-2\delta}e^{\theta-2\theta x^{-\delta}}\left(\theta+x^{\delta}\right)\left[3(\theta+1)x^{\delta}e^{\theta x^{-\delta}}-e^{\theta}\left(\theta+x^{\delta}\right)\right]}{2(\theta+1)^2}\right)^2\right]dx\\
&= \frac{1}{2}\left(1-\frac{9\cdot 4^{\frac{1}{\delta}+4}\delta^4 e^{4\theta}\theta^{1/\delta}\Gamma\left(-\frac{1}{\delta},4\theta\right)+6336\delta^4\theta^3+27888\delta^4\theta^2+30680\delta^4\theta+9974\delta^4}{9216\delta^5(\theta+1)^4}\right.\\
&\quad -\frac{1755\cdot 2^{\frac{\delta+2}{\delta}}\delta^3 e^{4\theta}\theta^{1/\delta}\Gamma\left(-\frac{1}{\delta},4\theta\right)+3792\delta^3\theta^2+3444\delta^3\theta+435\delta^3}{9216\delta^5(\theta+1)^4}\\
&\quad +\frac{81\cdot 2^{\frac{1}{\delta}+6}\delta^2(4\delta^2-5\delta+1)e^{2\theta}(\theta+1)^2\theta^{1/\delta}\Gamma\left(-\frac{1}{\delta},2\theta\right)+1395\cdot 4^{1/\delta}\delta^2 e^{4\theta}\theta^{1/\delta}\Gamma\left(-\frac{1}{\delta},4\theta\right)}{9216\delta^5(\theta+1)^4}\\
&\quad -\frac{476\delta^2\theta+314\delta^2+512\cdot 3^{1/\delta}\delta(27\delta^3-38\delta^2+12\delta-1)e^{3\theta}(\theta+1)\theta^{1/\delta}\Gamma\left(-\frac{1}{\delta},3\theta\right)}{9216\delta^5(\theta+1)^4}\\
&\quad \left.+\frac{9\cdot 4^{1/\delta}e^{4\theta}\theta^{1/\delta}\Gamma\left(-\frac{1}{\delta},4\theta\right)-99\cdot 2^{\frac{\delta+2}{\delta}}\delta e^{4\theta}\theta^{1/\delta}\Gamma\left(-\frac{1}{\delta},4\theta\right)-9\delta}{9216\delta^5(\theta+1)^4}\right).
\end{aligned}
$$

Figure 4 displays the 3D plot of NC for different values of parameters θ and δ based on EPUILD. Based on Figure 4, we can say that:

- The surface increases smoothly with both θ and δ, indicating that NC grows as either parameter increases.

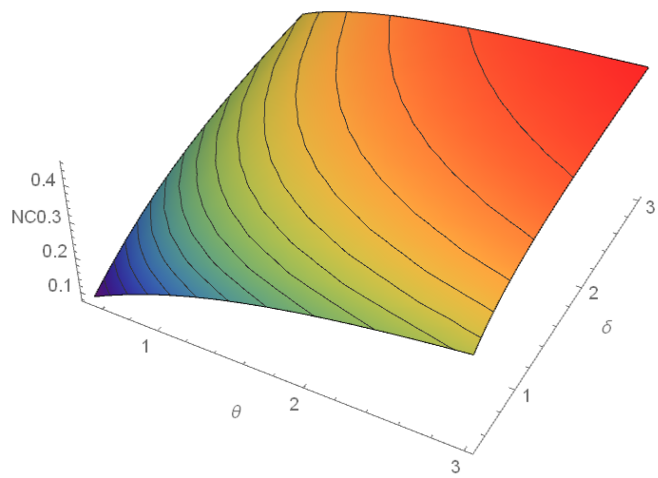


Figure 4. The NC of X based on EPUILD.

6. Estimation methods

This section discusses the fifteen methods utilized to estimate the parameters of the EPUIL model.

6.1. Method of maximum likelihood

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from the EPUILD. The log-likelihood function for the parameter vector (θ, δ) is given by:

$$\ell(\theta,\delta) = n\log\delta + 2n\log\theta - (3\delta+1)\sum_{i=1}^{n}\log x_i + n\theta - 2\theta\sum_{i=1}^{n}x_i^{-\delta} + \sum_{i=1}^{n}\log\left[3(\theta+1)x_i^{\delta}e^{\theta x_i^{-\delta}} - 2e^{\theta}(\theta+x_i^{\delta})\right] - 2\log(2(\theta+1)).$$

The maximum likelihood estimates [31] (MLEs) of θ and δ are obtained by maximizing the log-likelihood function above. This is typically achieved numerically, as closed-form solutions are not available due to the complexity of the log-likelihood. Optimization algorithms such as Newton–Raphson can be employed to compute the estimates.

6.2. Method of Anderson-Darling

The Anderson–Darling (AD) method [32] is a widely used goodness-of-fit-based estimation technique that gives greater weight to discrepancies in the tails between the empirical and theoretical CDFs. This feature makes it particularly useful for applications where tail behavior is critical. For the EPUILD, the AD estimates of the parameters can be obtained by minimizing the following function:

$$A(x_i) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n}) + \log S\left(x_{n-i-1:n}\right)\right]$$

$$= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\Bigg[\log\left(\frac{x_{i:n}^{-2\delta}e^{\theta-2\theta x_{i:n}^{-\delta}}(\theta+x_{i:n}^{\delta})\left[3(\theta+1)x_{i:n}^{\delta}e^{\theta x_{i:n}^{-\delta}}-e^{\theta}(\theta+x_{i:n}^{\delta})\right]}{2(\theta+1)^2}\right)$$
$$+\log\left(1-\frac{x_{n-i-1:n}^{-2\delta}e^{\theta-2\theta x_{n-i-1:n}^{-\delta}}(\theta+x_{n-i-1:n}^{\delta})\left[3(\theta+1)x_{n-i-1:n}^{\delta}e^{\theta x_{n-i-1:n}^{-\delta}}-e^{\theta}(\theta+x_{n-i-1:n}^{\delta})\right]}{2(\theta+1)^2}\right)\Bigg].$$

6.3. *Method of Cramer_von_Mises*

The Cramer von Mises (CvM) estimation method [33] determines the parameters of a theoretical distribution by minimizing the CvM statistic, which measures the discrepancy between the empirical and hypothesized cumulative distribution functions (CDFs). For the EPUIL model, the CvM estimates of the parameters can be obtained by minimizing the following function:

$$\begin{aligned}C(x_i) &= \frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n})-\frac{2i-1}{2n}\right]^2\\ &= \frac{1}{12n}+\sum_{i=1}^{n}\left[\left(\frac{x_{i:n}^{-2\delta}e^{\theta-2\theta x_{i:n}^{-\delta}}\left(\theta+x_{i:n}^{\delta}\right)\left[3(\theta+1)x_{i:n}^{\delta}e^{\theta x_{i:n}^{-\delta}}-e^{\theta}\left(\theta+x_{i:n}^{\delta}\right)\right]}{2(\theta+1)^2}\right)-\frac{2i-1}{2n}\right]^2.\end{aligned}$$

6.4. *Method of maximum product of spacings*

An effective alternative to the maximum likelihood approach [34] is the maximum product of spacings (MPS) method. This estimation technique, known as maximum product of spacings estimation (MPSE), involves maximizing the product of the spacings between ordered data points. For the EPUIL model, the MPS estimates of the parameters are obtained by maximizing the following function:

$$MPS = \frac{1}{n+1}\sum_{i=1}^{n+1}\log I_i(x_i),$$

where

$$\begin{aligned}I_i(x_i) &= F(x_{i:n})-F(x_{i-1:n})\\ &= \left(\frac{x_{i:n}^{-2\delta}e^{\theta-2\theta x_{i:n}^{-\delta}}\left(\theta+x_{i:n}^{\delta}\right)\left[3(\theta+1)x_{i:n}^{\delta}e^{\theta x_{i:n}^{-\delta}}-e^{\theta}\left(\theta+x_{i:n}^{\delta}\right)\right]}{2(\theta+1)^2}\right)\\ &\quad-\left(\frac{x_{i-1:n}^{-2\delta}e^{\theta-2\theta x_{i-1:n}^{-\delta}}\left(\theta+x_{i-1:n}^{\delta}\right)\left[3(\theta+1)x_{i-1:n}^{\delta}e^{\theta x_{i-1:n}^{-\delta}}-e^{\theta}\left(\theta+x_{i-1:n}^{\delta}\right)\right]}{2(\theta+1)^2}\right)\end{aligned} \tag{6.1}$$

$F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$.

6.5. *Method of least squares*

The method of least squares (LSE) [35] is a widely used approach for parameter estimation, which involves minimizing the sum of the squared differences between the observed values and the corresponding values predicted by the model. For the EPUIL model, the least squares estimates of the parameters are obtained by minimizing the following function:

$$V(x_i) = \sum_{i=1}^{n} \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2$$
$$= \sum_{i=1}^{n} \left[\left(\frac{x_{i:n}^{-2\delta} e^{\theta - 2\theta x_{i:n}^{-\delta}} \left(\theta + x_{i:n}^{\delta}\right) \left[3(\theta+1) x_{i:n}^{\delta} e^{\theta x_{i:n}^{-\delta}} - e^{\theta} \left(\theta + x_{i:n}^{\delta}\right) \right]}{2(\theta+1)^2} \right) - \frac{i}{n+1} \right]^2 .$$

6.6. Method of right_tail Anderson_Darling

The right-tail Anderson–Darling (RTAD) [32] method is used to estimate the parameters of a statistical distribution by placing greater emphasis on the right tail of the distribution. This approach tests the goodness-of-fit in the right tail by minimizing a modified version of the Anderson-Darling statistic that focuses on the upper tail behavior of the cumulative distribution function (CDF). For the EPUIL model, the RTAD estimates of the parameters are obtained by minimizing the following objective function:

$$R(x_i) = \frac{n}{2} - 2 \sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log S(x_{i:n})$$
$$= \frac{n}{2} - 2 \sum_{i=1}^{n} \left(\frac{x_{i:n}^{-2\delta} e^{\theta - 2\theta x_{i:n}^{-\delta}} \left(\theta + x_{i:n}^{\delta}\right) \left[3(\theta+1) x_{i:n}^{\delta} e^{\theta x_{i:n}^{-\delta}} - e^{\theta} \left(\theta + x_{i:n}^{\delta}\right) \right]}{2(\theta+1)^2} \right)$$
$$- \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log \left(1 - \frac{x_{i:n}^{-2\delta} e^{\theta - 2\theta x_{i:n}^{-\delta}} \left(\theta + x_{i:n}^{\delta}\right) \left[3(\theta+1) x_{i:n}^{\delta} e^{\theta x_{i:n}^{-\delta}} - e^{\theta} \left(\theta + x_{i:n}^{\delta}\right) \right]}{2(\theta+1)^2} \right).$$

6.7. Method of weighted least squares

Weighted least squares estimation (WLSE) [35] is a statistical method used to estimate the parameters of a probability distribution by assigning different weights to observations. This approach is particularly useful when some data points are considered more reliable or informative than others, leading to more efficient parameter estimates. For the EPUIL model, the WLS estimates of the parameters are obtained by minimizing the following function:

$$W(x_i) = \sum_{i=1}^{n} \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2$$
$$= \sum_{i=1}^{n} \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[\left(\frac{x_{i:n}^{-2\delta} e^{\theta - 2\theta x_{i:n}^{-\delta}} \left(\theta + x_{i:n}^{\delta}\right) \left[3(\theta+1) x_{i:n}^{\delta} e^{\theta x_{i:n}^{-\delta}} - e^{\theta} \left(\theta + x_{i:n}^{\delta}\right) \right]}{2(\theta+1)^2} \right) - \frac{i}{n+1} \right]^2 .$$

6.8. Method of left tail Anderson Darling

The Left Tail Anderson–Darling (LTAD) [36] method is employed due to its simplicity and its effectiveness in measuring the distance between empirical and theoretical distributions, which aids in parameter estimation. Let $X_{1:n}, \ldots, X_{n:n}$ denote an ordered sample from the EPUIL distribution. The LTAD estimates of the parameters are obtained by minimizing the following function:

$$L(x_i) = -\frac{3}{2}n + 2\sum_{i=1}^{n} F(x_{i:n}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F(x_{i:n})$$
$$= -\frac{3}{2}n + 2\sum_{i=1}^{n}\left(\frac{x_{i:n}^{-2\delta} e^{\theta-2\theta x_{i:n}^{-\delta}}\left(\theta + x_{i:n}^{\delta}\right)\left[3(\theta+1)x_{i:n}^{\delta} e^{\theta x_{i:n}^{-\delta}} - e^{\theta}\left(\theta + x_{i:n}^{\delta}\right)\right]}{2(\theta+1)^2}\right)$$
$$-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(\frac{x_{i:n}^{-2\delta} e^{\theta-2\theta x_{i:n}^{-\delta}}\left(\theta + x_{i:n}^{\delta}\right)\left[3(\theta+1)x_{i:n}^{\delta} e^{\theta x_{i:n}^{-\delta}} - e^{\theta}\left(\theta + x_{i:n}^{\delta}\right)\right]}{2(\theta+1)^2}\right).$$

6.9. Method of minimum spacing absolute distance

The method of minimum spacing absolute distance (MSPD) [37] is employed due to its simplicity and its effectiveness in measuring the distance between data points, which aids in parameter estimation. Let $X_{1:n}, \ldots, X_{n:n}$ denote an ordered sample from the EPUIL distribution. The LTAD estimates of the parameters are obtained by minimizing the following function:

$$\zeta(x_i) = \sum_{i=1}^{n+1} |I_i(x_i) - \frac{1}{n+1}|.$$

where $I_i(x_i)$ is given in Eq (6.1).

6.10. Method of minimum spacing absolute-log distance

This method extends the minimum spacing approach [37] by incorporating the logarithm of the spacings, which can offer advantages when analyzing data with varying magnitudes. Using this technique, the estimates of the parameters of the EPUIL model are obtained by minimizing the following expression:

$$\Upsilon(x_i) = \sum_{i=1}^{n+1} |\log I_i(x_i) - \log\frac{1}{n+1}|.$$

where $I_i(x_i)$ is given in Eq (6.1).

6.11. Method of Anderson Darling left tail second order

This method of Anderson–Darling left-tail second-order [38] allows for a more detailed assessment of tail behavior. Using this approach, the estimates of the parameters of the EPUIL model are obtained by minimizing the following expression:

$$LTS = 2\sum_{i=1}^{n}\log F(x_i) + \frac{1}{n}\sum_{i=1}^{n}\frac{(2i-1)}{F(x_i)}.$$

6.12. Kolmogorov method

The Kolmogorov method [38] is widely used due to its simplicity and effectiveness in assessing the maximum deviation between the empirical and theoretical distribution functions. For the EPUIL model, the Kolmogorov estimates of the parameters are obtained by optimizing the following function:

$$\begin{aligned} KM &= \max_{1\le i\le n}\left[\frac{i}{n} - F(x_i),\ F(x_i) - \frac{i-1}{n}\right] \\ &= \max_{1\le i\le n}\left[\frac{i}{n} - \frac{x_{i:n}^{-2\delta} e^{\theta-2\theta x_{i:n}^{-\delta}}(\theta + x_{i:n}^{\delta})\left[3(\theta+1)x_{i:n}^{\delta} e^{\theta x_{i:n}^{-\delta}} - e^{\theta}(\theta + x_{i:n}^{\delta})\right]}{2(\theta+1)^2},\right. \\ &\qquad \left.\frac{x_{i:n}^{-2\delta} e^{\theta-2\theta x_{i:n}^{-\delta}}(\theta + x_{i:n}^{\delta})\left[3(\theta+1)x_{i:n}^{\delta} e^{\theta x_{i:n}^{-\delta}} - e^{\theta}(\theta + x_{i:n}^{\delta})\right]}{2(\theta+1)^2} - \frac{i-1}{n}\right]. \end{aligned}$$

6.13. Method of minimum spacing square distance

The minimum spacing square distance estimation (MSSDE) method [37] is used to obtain parameter estimates for the proposed model by minimizing the following function:

$$\phi_{(x_i)} = \sum_{i=1}^{n+1}(I_i(x_i) - \frac{1}{n+1})^2.$$

where $I_i(x_i)$ is given in Eq (6.1).

6.14. Method of minimum spacing square-log distance

The minimum spacing square-log distance estimation (MSSLDE) method [37] is used to obtain parameter estimates for the proposed model by minimizing the following function:

$$\Psi_{(x_i)} = \sum_{i=1}^{n+1}\left(\log I_i(x_i) - \log\frac{1}{n+1}\right)^2.$$

where $I_i(x_i)$ is given in Eq (6.1).

6.15. Method of minimum spacing Linex distance

The minimum spacing Linex distance estimation (MSLNDE) method [37] is used to obtain parameter estimates for the proposed model by minimizing the following function:

$$\varDelta_{(x_i)} = \sum_{i=1}^{n+1}\left[e^{I_i(x_i)-\frac{1}{n+1}} - \left(I_i(x_i) - \frac{1}{n+1}\right) - 1\right].$$

where $I_i(x_i)$ is given in Eq (6.1).

7. Numerical simulation

A comprehensive evaluation of the efficiency of the estimation techniques described in Section 6.1 is presented in this section. Random datasets were generated in line with the provided model, and these approaches were used to estimate the parameters of the mentioned model. The effectiveness of various estimation methods was assessed by using five distinct metrics, which are outlined as follows

1) Average bias (BIAS), $|Bias(\widehat{\Xi})| = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\Xi} - \Xi|$.

2) Mean squared errors (MSE), $MSE = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\Xi} - \Xi)^2$.

3) Mean relative errors (MRE), $MRE = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\Xi} - \Xi|/\Xi$.

4) Average absolute difference (D_{abs}), $D_{abs} = \frac{1}{nH}\sum_{i=1}^{H}\sum_{j=1}^{n}|F(x_{ij};\Xi) - F(x_{ij};\widehat{\Xi})|$.

5) Maximum absolute difference (D_{max}), $D_{max} = \frac{1}{H}\sum_{i=1}^{H}\max_{j=1,\ldots,n}|F(x_{ij};\Xi) - F(x_{ij};\widehat{\Xi})|, \Xi = (\delta, \theta)$.

A simulation study aims to determine which estimation method is the most efficient for the model provided. The model was used to generate random samples of varying sizes, and then data were gathered about those samples. It was necessary to carry out this technique many times to ensure its robustness. The results of the simulation are shown in Tables 3–7. A heat map for the information that is presented in Table 3 is shown in Figures 5–8. Consider the significance of each result concerning the many different estimation techniques. Table 8 provides a comprehensive presentation of the partial and total ranks for the estimators.

The results of the simulation show that all of the techniques for parameter estimation for the proposed model exhibit a high level of accuracy, closely agreeing with their actual values. When the sample size is increased, the derived metrics decrease. According to the results shown in Table 8, KE has been demonstrated to be the most effective method for parameter estimation, achieving a total score of 32.5.

8. Real life application

In this section, two real-world datasets are used to demonstrate the utility and potential of the EPUIL distribution. The first dataset, obtained from [39], represents the inflation rates of 45 Asian countries as of January 31, 2023. The data are as follows: 0.0077, 0.018, 0.0198, 0.02, 0.0271, 0.0284, 0.031, 0.0315, 0.032, 0.033, 0.036, 0.038, 0.04, 0.0409, 0.042, 0.0435, 0.045, 0.0459, 0.0489, 0.05, 0.053, 0.0551, 0.0572, 0.0589, 0.0593, 0.065, 0.067, 0.0677, 0.0738, 0.081, 0.083, 0.0871, 0.091, 0.098, 0.123, 0.132, 0.139, 0.147, 0.175, 0.1955, 0.203, 0.245, 0.3927, 0.522, 0.542. The second dataset comprises the failure times of 20 mechanical components, as reported by [40]. The values are: 0.067, 0.068, 0.076, 0.081, 0.084, 0.085, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.114, 0.115, 0.121, 0.125, 0.131, 0.149, 0.160, 0.485.

The outcomes of each dataset's descriptive analysis are shown in Table 9. Table 9 shows that both datasets exhibit strong positive skewness (skewness > 1) and are leptokurtic (kurtosis > 3), indicating deviation from normality. These characteristics suggest that the data are heavy-tailed and right-skewed, with sharper peaks than a normal distribution.

Table 3. Numerical values of simulation measures for $\delta = 1.25,\ \theta = 1.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS($\hat{\delta}$)	0.27306[4]	0.33117[10]	0.3857[13]	0.29672[6]	0.3404[11]	0.42775[15]	0.35405[12]	0.31857[7]	0.24686[2]	0.26203[3]	0.38948[14]	0.1426[1]	0.32309[8]	0.29418[5]	0.32524[9]
	BIAS($\hat{\theta}$)	0.42571[3]	0.48879[8]	0.51493[13]	0.50896[11]	0.48815[7]	0.53825[15]	0.50948[12]	0.49613[10]	0.36242[2]	0.43664[4]	0.53469[14]	0.20616[1]	0.49124[9]	0.48034[6]	0.46379[5]
	MSE($\hat{\delta}$)	0.12024[3]	0.16984[10]	0.23307[13]	0.13188[5]	0.18259[11]	0.27[15]	0.19262[12]	0.16792[9]	0.11362[2]	0.12108[4]	0.23631[14]	0.04288[1]	0.16203[7]	0.13575[6]	0.16646[8]
	MSE($\hat{\theta}$)	0.27693[3]	0.36262[9]	0.36813[11]	0.39405[15]	0.35126[8]	0.39007[14]	0.3817[12]	0.36674[10]	0.26358[2]	0.31756[4]	0.38983[13]	0.1011[1]	0.34154[7]	0.34078[6]	0.33753[5]
	MRE($\hat{\delta}$)	0.21845[4]	0.26494[10]	0.30856[13]	0.23738[6]	0.27232[11]	0.3422[15]	0.28324[12]	0.25486[7]	0.19749[2]	0.20963[3]	0.31158[14]	0.11408[1]	0.25847[8]	0.23534[5]	0.26019[9]
	MRE($\hat{\theta}$)	0.2838[3]	0.32586[8]	0.34329[13]	0.33931[11]	0.32543[7]	0.35884[15]	0.33965[12]	0.33075[10]	0.24161[2]	0.29109[4]	0.35646[14]	0.13744[1]	0.32749[9]	0.32023[6]	0.30919[5]
	D_{abs}	0.04377[1]	0.04544[4]	0.04931[10]	0.04386[2]	0.04521[3]	0.04858[9]	0.04937[11]	0.04694[6]	0.05239[13]	0.04659[5]	0.04947[12]	0.0474[7]	0.05995[15]	0.04856[8]	0.05675[14]
	D_{max}	0.07269[2]	0.07527[5]	0.08328[12]	0.07121[1]	0.07545[6]	0.08369[13]	0.0817[9]	0.07725[7]	0.08315[11]	0.07507[4]	0.08194[10]	0.07384[3]	0.09501[15]	0.0778[8]	0.09081[14]
	$\sum Ranks$	23[2]	64[7.5]	98[13]	57[6]	64[7.5]	111[15]	92[12]	66[9]	36[4]	31[3]	105[14]	16[1]	78[11]	50[5]	69[10]
60	BIAS($\hat{\delta}$)	0.17728[2]	0.24361[9]	0.26862[13]	0.2184[5]	0.2988[14]	0.30462[15]	0.26225[11]	0.21423[4]	0.20361[3]	0.24107[8]	0.23974[7]	0.11209[1]	0.24412[10]	0.22928[6]	0.26845[12]
	BIAS($\hat{\theta}$)	0.32959[3]	0.4045[6]	0.42969[13]	0.41775[8]	0.474[15]	0.42444[11]	0.42052[10]	0.35668[4]	0.28226[2]	0.42826[12]	0.40498[7]	0.16256[1]	0.39635[5]	0.41904[9]	0.44289[14]
	MSE($\hat{\delta}$)	0.05087[2]	0.09067[8]	0.11464[12]	0.07205[4]	0.12971[14]	0.14752[15]	0.10206[10]	0.07187[3]	0.07372[5]	0.08858[7]	0.09367[9]	0.02831[1]	0.10484[11]	0.08273[6]	0.11685[13]
	MSE($\hat{\theta}$)	0.20172[3]	0.25602[5]	0.29758[13]	0.28481[10]	0.33009[15]	0.27327[9]	0.26785[8]	0.20789[4]	0.16873[2]	0.29446[12]	0.25935[7]	0.06076[1]	0.25898[6]	0.28784[11]	0.30126[14]
	MRE($\hat{\delta}$)	0.14183[2]	0.19489[9]	0.21489[13]	0.17472[5]	0.23904[14]	0.2437[15]	0.2098[11]	0.17139[4]	0.16289[3]	0.19286[8]	0.19179[7]	0.08967[1]	0.19529[10]	0.18342[6]	0.21476[12]
	MRE($\hat{\theta}$)	0.21973[3]	0.26967[6]	0.28646[13]	0.2785[8]	0.316[15]	0.28296[11]	0.28035[10]	0.23779[4]	0.18818[2]	0.28551[12]	0.26998[7]	0.10838[1]	0.26423[5]	0.27936[9]	0.29526[14]
	D_{abs}	0.02859[1]	0.0313[4]	0.03281[7]	0.03177[5]	0.03037[2]	0.03488[12]	0.03246[6]	0.03122[3]	0.03703[13]	0.03427[9]	0.03411[8]	0.0347[11]	0.03827[14]	0.0345[10]	0.04022[15]
	D_{max}	0.04685[1]	0.05242[4]	0.05528[8]	0.05185[3]	0.05251[5]	0.06018[12]	0.05506[7]	0.0513[2]	0.06049[13]	0.05657[11]	0.0564[10]	0.05423[6]	0.0627[14]	0.05613[9]	0.06536[15]
	$\sum Ranks$	17[1]	51[6]	92[12]	48[5]	94[13]	100[14]	73[9]	28[3]	43[4]	79[11]	62[7]	23[2]	75[10]	66[8]	109[15]
90	BIAS($\hat{\delta}$)	0.15531[2]	0.1873[4]	0.21655[10]	0.19083[6]	0.21676[11]	0.2733[15]	0.20163[8]	0.19058[5]	0.1701[3]	0.21993[12]	0.21217[9]	0.08405[1]	0.22605[13]	0.1936[7]	0.22811[14]
	BIAS($\hat{\theta}$)	0.2743[3]	0.32256[4]	0.33663[6]	0.37638[11]	0.35385[7]	0.40816[15]	0.36454[9]	0.32646[5]	0.26778[2]	0.40704[14]	0.37358[10]	0.1368[1]	0.38956[13]	0.36072[8]	0.38126[12]
	MSE($\hat{\delta}$)	0.04059[2]	0.05505[5]	0.07712[12]	0.05285[3]	0.07557[11]	0.12105[15]	0.0616[8]	0.05667[6]	0.05328[4]	0.07052[9]	0.07331[10]	0.01747[1]	0.07819[13]	0.05707[7]	0.08335[14]
	MSE($\hat{\theta}$)	0.1427[2]	0.16845[4]	0.18589[6]	0.24106[13]	0.20904[7]	0.25399[14]	0.22245[10]	0.17196[5]	0.1483[3]	0.27512[15]	0.21914[9]	0.04636[1]	0.22809[11]	0.21325[8]	0.22951[12]
	MRE($\hat{\delta}$)	0.12424[2]	0.14984[4]	0.17324[10]	0.15266[6]	0.17341[11]	0.21864[15]	0.1613[8]	0.15247[5]	0.13608[3]	0.17595[12]	0.16974[9]	0.06724[1]	0.18084[13]	0.15488[7]	0.18249[14]
	MRE($\hat{\theta}$)	0.18287[3]	0.21504[4]	0.22442[6]	0.25092[11]	0.2359[7]	0.27211[15]	0.24303[9]	0.21764[5]	0.17852[2]	0.27136[14]	0.24905[10]	0.0912[1]	0.25971[13]	0.24048[8]	0.25417[12]
	D_{abs}	0.02463[2]	0.02485[3]	0.0277[8]	0.02638[5]	0.02936[11]	0.02779[9]	0.02567[4]	0.02648[6]	0.0309[13]	0.02985[12]	0.02768[7]	0.02418[1]	0.03206[14]	0.02886[10]	0.03633[15]
	D_{max}	0.04075[2]	0.04137[3]	0.04694[8]	0.04292[4]	0.04915[11]	0.04874[10]	0.04293[5]	0.04398[6]	0.04996[13]	0.04959[12]	0.04579[7]	0.03798[1]	0.05333[14]	0.04708[9]	0.05963[15]
	$\sum Ranks$	18[2]	31[3]	66[9]	59[6]	76[11]	108[14.5]	61[7]	43[4.5]	43[4.5]	100[12]	71[10]	8[1]	104[13]	64[8]	108[14.5]
140	BIAS($\hat{\delta}$)	0.14285[3]	0.16251[8]	0.19878[13]	0.14704[4]	0.18906[11]	0.22194[15]	0.15089[5]	0.16109[7]	0.13696[2]	0.18397[10]	0.17322[9]	0.07798[1]	0.1987[12]	0.15759[6]	0.21737[14]
	BIAS($\hat{\theta}$)	0.25365[3]	0.28169[6]	0.31443[10]	0.2809[5]	0.33319[11]	0.34163[12]	0.254[4]	0.28504[7]	0.21969[2]	0.35907[14]	0.30282[9]	0.12037[1]	0.34279[13]	0.29465[8]	0.37439[15]
	MSE($\hat{\delta}$)	0.03384[2]	0.04337[8]	0.05984[13]	0.034[3]	0.05831[11]	0.07822[15]	0.03727[5]	0.04067[7]	0.03471[4]	0.05324[10]	0.05114[9]	0.01433[1]	0.05917[12]	0.04042[6]	0.0709[14]
	MSE($\hat{\theta}$)	0.10883[3]	0.13599[5]	0.15757[9]	0.14445[7]	0.19237[13]	0.189[12]	0.10953[4]	0.13604[6]	0.103[2]	0.21903[15]	0.16132[10]	0.0347[1]	0.18656[11]	0.14702[8]	0.21067[14]
	MRE($\hat{\delta}$)	0.11428[3]	0.13001[8]	0.15903[13]	0.11763[4]	0.15124[11]	0.17755[15]	0.12071[5]	0.12887[7]	0.10957[2]	0.14718[10]	0.13858[9]	0.06239[1]	0.15896[12]	0.12607[6]	0.17389[14]
	MRE($\hat{\theta}$)	0.1691[3]	0.1878[6]	0.20962[10]	0.18727[5]	0.22212[11]	0.22775[12]	0.16934[4]	0.19003[7]	0.14646[2]	0.23938[14]	0.20188[9]	0.08024[1]	0.22852[13]	0.19643[8]	0.24959[15]
	D_{abs}	0.02052[2]	0.02231[8.5]	0.02231[8.5]	0.02096[4]	0.02313[11]	0.02213[7]	0.02024[1]	0.02117[5]	0.02276[10]	0.02538[13]	0.02171[6]	0.02079[3]	0.028[14]	0.02376[12]	0.02824[15]
	D_{max}	0.03401[3]	0.037[7]	0.0382[9]	0.03426[4]	0.039[12]	0.03842[10]	0.03367[2]	0.03506[5]	0.03703[8]	0.0414[13]	0.03601[6]	0.0328[1]	0.04655[14]	0.03889[11]	0.04703[15]
	$\sum Ranks$	22[2]	56.5[7]	85.5[10]	36[5]	91[11]	98[12]	30[3]	51[6]	32[4]	99[13]	67[9]	10[1]	101[14]	65[8]	116[15]
200	BIAS($\hat{\delta}$)	0.10912[2]	0.14376[9]	0.15594[11]	0.12242[3]	0.17708[14]	0.18438[15]	0.13641[6]	0.14185[7]	0.12408[4]	0.142[8]	0.15132[10]	0.06593[1]	0.16618[12]	0.13099[5]	0.1663[13]
	BIAS($\hat{\theta}$)	0.1993[2]	0.24272[7]	0.25043[9]	0.23665[4]	0.30299[13]	0.29924[12]	0.24536[8]	0.242[6]	0.21232[3]	0.27074[10]	0.28127[11]	0.101[1]	0.30902[14]	0.23725[5]	0.31414[15]
	MSE($\hat{\delta}$)	0.01967[2]	0.03004[7]	0.03771[11]	0.02443[3]	0.04664[14]	0.05444[15]	0.02854[6]	0.03055[8]	0.02633[5]	0.0308[9]	0.03412[10]	0.01086[1]	0.04033[12]	0.02487[4]	0.04279[13]
	MSE($\hat{\theta}$)	0.06752[2]	0.09585[5]	0.09905[8]	0.10609[9]	0.15683[14]	0.14977[12]	0.09748[7]	0.09735[6]	0.09088[3]	0.12782[10]	0.13727[11]	0.027[1]	0.15678[13]	0.09582[4]	0.15807[15]
	MRE($\hat{\delta}$)	0.08729[2]	0.11501[9]	0.12475[11]	0.09793[3]	0.14167[14]	0.1475[15]	0.10913[6]	0.11348[7]	0.09927[4]	0.1136[8]	0.12105[10]	0.05274[1]	0.13294[12]	0.10479[5]	0.13304[13]
	MRE($\hat{\theta}$)	0.13287[2]	0.16181[7]	0.16695[9]	0.15777[4]	0.20199[13]	0.19949[12]	0.16357[8]	0.16133[6]	0.14155[3]	0.18049[10]	0.18751[11]	0.06733[1]	0.20602[14]	0.15817[5]	0.20943[15]
	D_{abs}	0.01845[5]	0.01779[4]	0.01727[2]	0.01719[1]	0.01946[9.5]	0.01955[11]	0.01852[6]	0.01888[8]	0.02168[13]	0.02077[12]	0.01881[7]	0.01763[3]	0.02254[14]	0.01946[9.5]	0.02361[15]
	D_{max}	0.02985[5]	0.0298[4]	0.02971[3]	0.02815[2]	0.03347[10]	0.03362[11]	0.03055[6]	0.03134[8]	0.03516[13]	0.03411[12]	0.03131[7]	0.028[1]	0.03743[14]	0.032[9]	0.03872[15]
	$\sum Ranks$	22[2]	52[6]	64[9]	29[3]	101.5[12]	103[13]	53[7]	56[8]	48[5]	79[11]	77[10]	10[1]	105[14]	46.5[4]	114[15]
250	BIAS($\hat{\delta}$)	0.10478[2]	0.13565[9]	0.1499[14]	0.11858[5]	0.14407[11]	0.17329[15]	0.12856[7]	0.1211[6]	0.11405[3]	0.13226[8]	0.1483[13]	0.0582[1]	0.14824[12]	0.11809[4]	0.13854[10]
	BIAS($\hat{\theta}$)	0.1871[2]	0.231[8]	0.24172[9]	0.21725[7]	0.24355[11]	0.28594[15]	0.21681[6]	0.21105[4]	0.19025[3]	0.24187[10]	0.27638[14]	0.09326[1]	0.26785[13]	0.21583[5]	0.25066[12]
	MSE($\hat{\delta}$)	0.01766[2]	0.02921[9]	0.03414[14]	0.02267[4]	0.03099[11]	0.04673[15]	0.02559[7]	0.02347[6]	0.0222[3]	0.0264[8]	0.03372[12]	0.0087[1]	0.0338[13]	0.02339[5]	0.03074[10]
	MSE($\hat{\theta}$)	0.05853[2]	0.08614[8]	0.0914[9]	0.08393[7]	0.09919[11]	0.13986[15]	0.07293[4]	0.0805[6]	0.07135[3]	0.09877[10]	0.12197[13]	0.02321[1]	0.12937[14]	0.07971[5]	0.1145[12]
	MRE($\hat{\delta}$)	0.08383[2]	0.10852[9]	0.11992[14]	0.09486[5]	0.11526[11]	0.13863[15]	0.10285[7]	0.09688[6]	0.09124[3]	0.1058[8]	0.11864[13]	0.04656[1]	0.11859[12]	0.09447[4]	0.11083[10]
	MRE($\hat{\theta}$)	0.12474[2]	0.154[8]	0.16115[9]	0.14483[7]	0.16236[11]	0.19063[15]	0.14454[6]	0.1407[4]	0.12683[3]	0.16125[10]	0.18425[14]	0.06218[1]	0.17857[13]	0.14389[5]	0.16711[12]
	D_{abs}	0.01587[3]	0.01566[2]	0.0171[8]	0.01552[1]	0.01686[6]	0.0184[10]	0.01669[5]	0.017[7]	0.01884[12]	0.01861[11]	0.01998[13]	0.0162[4]	0.0209[14]	0.01748[9]	0.02193[15]
	D_{max}	0.02585[3]	0.02637[4]	0.02938[9]	0.02581[2]	0.02841[7]	0.03156[12]	0.02798[6]	0.02794[5]	0.03056[10]	0.03074[11]	0.03275[13]	0.02528[1]	0.0345[14]	0.02846[8]	0.0359[15]
	$\sum Ranks$	18[2]	57[8]	86[11]	38[3]	79[10]	112[15]	48[7]	44[5]	40[4]	76[9]	105[13.5]	11[1]	105[13.5]	45[6]	96[12]

Table 4. Numerical values of simulation measures for $\delta = 2.5,\ \theta = 0.6$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS($\hat{\delta}$)	0.31531[2]	0.46291[10]	0.5141[13]	0.42177[5]	0.4291[7]	0.56098[15]	0.44713[9]	0.46408[11]	0.34595[3]	0.39917[4]	0.53131[14]	0.086[1]	0.48241[12]	0.42343[6]	0.43638[8]
	BIAS($\hat{\theta}$)	0.17185[3]	0.18947[5]	0.19167[7]	0.19578[10]	0.19188[8]	0.20642[13]	0.1891[4]	0.19073[6]	0.16362[2]	0.19813[11]	0.20969[14]	0.09054[1]	0.21162[15]	0.19573[9]	0.20265[12]
	MSE($\hat{\delta}$)	0.14506[2]	0.35537[10]	0.44072[13]	0.27248[5]	0.29408[8]	0.51994[15]	0.35737[11]	0.37225[12]	0.24604[4]	0.24011[3]	0.5104[14]	0.0238[1]	0.3508[9]	0.29048[6]	0.29225[7]
	MSE($\hat{\theta}$)	0.05025[3]	0.05324[4]	0.05688[7]	0.05931[10]	0.05787[8]	0.06014[11]	0.05366[5]	0.05486[6]	0.04632[2]	0.06139[12]	0.06352[14]	0.01538[1]	0.06611[15]	0.05879[9]	0.0634[13]
	MRE($\hat{\delta}$)	0.12612[2]	0.18516[10]	0.20564[13]	0.16871[5]	0.17164[7]	0.22439[15]	0.17885[9]	0.18563[11]	0.13838[3]	0.15967[4]	0.21252[14]	0.0344[1]	0.19296[12]	0.16937[6]	0.17455[8]
	MRE($\hat{\theta}$)	0.28642[3]	0.31578[5]	0.31944[7]	0.3263[10]	0.3198[8]	0.34403[13]	0.31517[4]	0.31788[6]	0.27271[2]	0.33022[11]	0.34948[14]	0.1509[1]	0.3527[15]	0.32621[9]	0.33774[12]
	D_{abs}	0.04683[2]	0.0472[3]	0.05206[11]	0.04872[5]	0.04541[1]	0.05509[13]	0.04801[4]	0.05162[10]	0.05275[12]	0.05161[9]	0.05015[6]	0.05084[7]	0.0616[15]	0.05111[8]	0.05958[14]
	D_{max}	0.07627[3]	0.07758[4]	0.08821[12]	0.07885[6]	0.07483[1]	0.09351[13.5]	0.07826[5]	0.0857[11]	0.08301[10]	0.08186[8]	0.08285[9]	0.07516[2]	0.09946[15]	0.0817[7]	0.09351[13.5]
	$\sum Ranks$	20[2]	51[5.5]	83[11]	56[7]	48[4]	108.5[15]	51[5.5]	73[10]	38[3]	62[9]	99[13]	15[1]	108[14]	60[8]	87.5[12]
60	BIAS($\hat{\delta}$)	0.20772[2]	0.31455[6]	0.37263[13]	0.30133[5]	0.31868[7]	0.40021[15]	0.32498[9]	0.31941[8]	0.27165[3]	0.32504[10]	0.38044[14]	0.06887[1]	0.35508[11]	0.2982[4]	0.36659[12]
	BIAS($\hat{\theta}$)	0.11243[2]	0.14766[6]	0.15524[9]	0.15315[8]	0.14394[5]	0.16628[13]	0.15653[10]	0.1385[3]	0.13963[4]	0.16458[12]	0.17195[14]	0.05779[1]	0.16333[11]	0.14795[7]	0.18703[15]
	MSE($\hat{\delta}$)	0.08117[2]	0.15664[6]	0.2181[13]	0.13867[4]	0.1641[10]	0.24956[15]	0.16172[8]	0.16263[9]	0.13263[3]	0.15685[7]	0.23523[14]	0.01757[1]	0.19331[11]	0.14165[5]	0.19493[12]
	MSE($\hat{\theta}$)	0.0246[2]	0.03494[6]	0.0349[5]	0.04201[10]	0.03391[4]	0.04423[11]	0.03862[8]	0.03123[3]	0.03927[9]	0.04531[13]	0.04561[14]	0.00748[1]	0.04452[12]	0.03603[7]	0.05618[15]
	MRE($\hat{\delta}$)	0.08309[2]	0.12582[6]	0.14905[13]	0.12053[5]	0.12747[7]	0.16008[15]	0.12999[9]	0.12776[8]	0.10866[3]	0.13002[10]	0.15217[14]	0.02755[1]	0.14203[11]	0.11928[4]	0.14664[12]
	MRE($\hat{\theta}$)	0.18738[2]	0.2461[6]	0.25873[9]	0.25525[8]	0.2399[5]	0.27713[13]	0.26089[10]	0.23083[3]	0.23271[4]	0.27429[12]	0.28659[14]	0.09632[1]	0.27221[11]	0.24658[7]	0.31172[15]
	D_{abs}	0.03051[1]	0.03109[3]	0.03465[6]	0.03491[7]	0.03425[5]	0.03541[8]	0.03651[11]	0.03132[4]	0.03666[12]	0.03822[13]	0.0362[10]	0.03073[2]	0.04004[15]	0.03572[9]	0.03875[14]
	D_{max}	0.04914[2]	0.05144[3]	0.05916[9]	0.05663[5]	0.05671[6]	0.0607[12]	0.05959[10]	0.05291[4]	0.05908[8]	0.06238[13]	0.06051[11]	0.04633[1]	0.06567[15]	0.05763[7]	0.06419[14]
	$\sum Ranks$	15[2]	42[3.5]	77[10]	52[8]	49[6]	102[13]	75[9]	42[3.5]	46[5]	90[11]	105[14]	9[1]	97[12]	50[7]	109[15]
90	BIAS($\hat{\delta}$)	0.20474[2]	0.24032[4]	0.32283[14]	0.24243[5]	0.29772[10]	0.31972[12]	0.27465[9]	0.26927[8]	0.23008[3]	0.25039[6]	0.33757[15]	0.06146[1]	0.3179[11]	0.25336[7]	0.32188[13]
	BIAS($\hat{\theta}$)	0.10786[2]	0.10907[3]	0.132[10]	0.11634[4]	0.12843[9]	0.13542[12]	0.12096[5]	0.13252[11]	0.12743[7]	0.12277[6]	0.15802[14]	0.04809[1]	0.15398[13]	0.12812[8]	0.16811[15]
	MSE($\hat{\delta}$)	0.06633[2]	0.09453[4]	0.1711[14]	0.09118[3]	0.13676[10]	0.16557[13]	0.12573[9]	0.10384[8]	0.09778[5]	0.10371[7]	0.18386[15]	0.01388[1]	0.15367[11]	0.10175[6]	0.15468[12]
	MSE($\hat{\theta}$)	0.02038[3]	0.01923[2]	0.02582[7]	0.02264[4]	0.02587[8]	0.02858[10]	0.02456[5]	0.02644[9]	0.0306[12]	0.0258[6]	0.03867[13]	0.00492[1]	0.03932[14]	0.0289[11]	0.04623[15]
	MRE($\hat{\delta}$)	0.08189[2]	0.09613[4]	0.12913[14]	0.09697[5]	0.11909[10]	0.12789[12]	0.10986[9]	0.10771[8]	0.09203[3]	0.10016[6]	0.13503[15]	0.02458[1]	0.12716[11]	0.10134[7]	0.12875[13]
	MRE($\hat{\theta}$)	0.17976[2]	0.18179[3]	0.21999[10]	0.1939[4]	0.21405[9]	0.2257[12]	0.2016[5]	0.22087[11]	0.21238[7]	0.20461[6]	0.26336[14]	0.08015[1]	0.25664[13]	0.21353[8]	0.28019[15]
	D_{abs}	0.0276[5]	0.02722[4]	0.0284[8]	0.02498[2]	0.02705[3]	0.02806[7]	0.02772[6]	0.02856[9]	0.03233[12]	0.03132[11]	0.03289[14]	0.02455[1]	0.03302[15]	0.02917[10]	0.03252[13]
	D_{max}	0.04449[3]	0.04493[4]	0.04836[10]	0.04123[2]	0.04571[5]	0.04804[9]	0.04676[6]	0.04738[7]	0.05194[12]	0.05055[11]	0.05516[14]	0.03719[1]	0.05519[15]	0.04751[8]	0.05368[13]
	$\sum Ranks$	21[2]	28[3]	87[11.5]	29[4]	64[8]	87[11.5]	54[5]	71[10]	61[7]	59[6]	114[15]	8[1]	103[13]	65[9]	109[14]
140	BIAS($\hat{\delta}$)	0.15441[2]	0.19636[5]	0.24211[11]	0.18291[3]	0.23404[10]	0.25035[12]	0.20167[7]	0.19159[4]	0.20188[8]	0.20119[6]	0.26515[15]	0.04949[1]	0.26127[14]	0.21734[9]	0.25836[13]
	BIAS($\hat{\theta}$)	0.07876[2]	0.09128[4]	0.11018[11]	0.08826[3]	0.10608[9]	0.10897[10]	0.09331[6]	0.09245[5]	0.1028[7]	0.1029[8]	0.12605[13]	0.04116[1]	0.13242[15]	0.11309[12]	0.12835[14]
	MSE($\hat{\delta}$)	0.03769[2]	0.06109[5]	0.08805[11]	0.05203[3]	0.08592[10]	0.09685[12]	0.06614[7]	0.0561[4]	0.07551[9]	0.06329[6]	0.11666[15]	0.00711[1]	0.10479[14]	0.07147[8]	0.09892[13]
	MSE($\hat{\theta}$)	0.01013[2]	0.01287[3]	0.01919[9]	0.01306[4]	0.01848[8]	0.01938[10]	0.01499[6]	0.01359[5]	0.01959[11]	0.01791[7]	0.02714[13]	0.00318[1]	0.0302[15]	0.02116[12]	0.02716[14]
	MRE($\hat{\delta}$)	0.06176[2]	0.07854[5]	0.09685[11]	0.07317[3]	0.09362[10]	0.10014[12]	0.08067[7]	0.07664[4]	0.08075[8]	0.08047[6]	0.10606[15]	0.0198[1]	0.10451[14]	0.08694[9]	0.10335[13]
	MRE($\hat{\theta}$)	0.13127[2]	0.15213[4]	0.18364[11]	0.14709[3]	0.1768[9]	0.18162[10]	0.15551[6]	0.15409[5]	0.17133[7]	0.17151[8]	0.21009[13]	0.06859[1]	0.22069[15]	0.18849[12]	0.21392[14]
	D_{abs}	0.01998[2]	0.02088[3]	0.02272[7]	0.01967[1]	0.02341[9]	0.02273[8]	0.02112[4]	0.02127[6]	0.02743[13]	0.02351[10]	0.02463[12]	0.02114[5]	0.02767[14]	0.02432[11]	0.03007[15]
	D_{max}	0.03216[2]	0.0342[4]	0.03827[7]	0.03236[3]	0.0393[10]	0.03884[9]	0.03502[6]	0.03481[5]	0.04427[13]	0.03855[8]	0.0414[12]	0.03186[1]	0.0457[14]	0.03963[11]	0.04916[15]
	$\sum Ranks$	16[2]	33[4]	78[10]	23[3]	75[8]	83[11]	49[6]	38[5]	76[9]	59[7]	108[13]	12[1]	115[15]	84[12]	111[14]
200	BIAS($\hat{\delta}$)	0.14492[2]	0.15621[3]	0.17691[7]	0.1617[4]	0.18668[9]	0.19966[12]	0.17264[6]	0.17924[8]	0.16997[5]	0.1898[11]	0.21993[14]	0.04643[1]	0.21624[13]	0.18854[10]	0.22406[15]
	BIAS($\hat{\theta}$)	0.07266[3]	0.07095[2]	0.07961[5]	0.07912[4]	0.08334[7]	0.08795[10]	0.08045[6]	0.08335[8]	0.08576[9]	0.09259[11]	0.11175[15]	0.03172[1]	0.10485[13]	0.09905[12]	0.11128[14]
	MSE($\hat{\delta}$)	0.03201[2]	0.04067[4]	0.05225[7]	0.03915[3]	0.05517[10]	0.06176[12]	0.04607[5]	0.04835[6]	0.05373[8]	0.0554[11]	0.07899[15]	0.00702[1]	0.07214[13]	0.05481[9]	0.07476[14]
	MSE($\hat{\theta}$)	0.0082[2]	0.00856[3]	0.01039[6]	0.00943[4]	0.01134[8]	0.01228[9]	0.01[5]	0.01073[7]	0.01398[11]	0.01387[10]	0.02226[15]	0.00228[1]	0.01813[13]	0.0162[12]	0.02093[14]
	MRE($\hat{\delta}$)	0.05797[2]	0.06249[3]	0.07076[7]	0.06468[4]	0.07467[9]	0.07986[12]	0.06906[6]	0.0717[8]	0.06799[5]	0.07592[11]	0.08797[14]	0.01857[1]	0.0865[13]	0.07541[10]	0.08962[15]
	MRE($\hat{\theta}$)	0.1211[3]	0.11824[2]	0.13268[5]	0.13187[4]	0.1389[7]	0.14658[10]	0.13408[6]	0.13891[8]	0.14294[9]	0.15432[11]	0.18625[15]	0.05287[1]	0.17475[13]	0.16509[12]	0.18546[14]
	D_{abs}	0.01657[2]	0.01808[5]	0.0184[6]	0.01785[3]	0.01955[8]	0.01975[9]	0.01793[4]	0.01901[7]	0.02071[10]	0.02183[11]	0.02221[12]	0.01526[1]	0.02471[14]	0.02233[13]	0.02514[15]
	D_{max}	0.02712[2]	0.02994[5]	0.03068[6]	0.02907[3]	0.03248[8]	0.03312[9]	0.0298[4]	0.03167[7]	0.03375[10]	0.03597[11]	0.03652[13]	0.02345[1]	0.04033[14]	0.03616[12]	0.04148[15]
	$\sum Ranks$	18[2]	27[3]	49[6]	29[4]	66[8]	83[10]	42[5]	59[7]	67[9]	87[11]	113[14]	8[1]	106[13]	90[12]	116[15]
250	BIAS($\hat{\delta}$)	0.12146[2]	0.15136[4]	0.1635[10]	0.14342[3]	0.1585[7]	0.18238[13]	0.1541[6]	0.15879[8]	0.15368[5]	0.16245[9]	0.20901[15]	0.04309[1]	0.19269[14]	0.1711[11]	0.18134[12]
	BIAS($\hat{\theta}$)	0.06083[2]	0.06908[3]	0.07389[7]	0.07346[6]	0.06933[4]	0.08121[11]	0.07067[5]	0.07582[9]	0.0754[8]	0.08047[10]	0.10319[15]	0.0298[1]	0.09285[13]	0.08726[12]	0.09303[14]
	MSE($\hat{\delta}$)	0.02181[2]	0.03452[3]	0.03949[6]	0.03464[4]	0.04087[9]	0.05471[12]	0.03616[5]	0.03982[7]	0.04458[10]	0.04066[8]	0.06974[15]	0.00686[1]	0.05941[14]	0.04866[11]	0.05506[13]
	MSE($\hat{\theta}$)	0.00587[2]	0.00745[3]	0.00801[5]	0.00918[8]	0.00818[6]	0.01082[11]	0.00778[4]	0.00874[7]	0.01066[10]	0.01054[9]	0.01771[15]	0.00203[1]	0.01509[13]	0.01332[12]	0.0156[14]
	MRE($\hat{\delta}$)	0.04858[2]	0.06054[4]	0.0654[10]	0.05737[3]	0.0634[7]	0.07295[13]	0.06164[6]	0.06352[8]	0.06147[5]	0.06498[9]	0.0836[15]	0.01723[1]	0.07708[14]	0.06844[11]	0.07254[12]
	MRE($\hat{\theta}$)	0.10138[2]	0.11513[3]	0.12315[7]	0.12243[6]	0.11555[4]	0.13534[11]	0.11779[5]	0.12636[9]	0.12567[8]	0.13411[10]	0.17199[15]	0.04967[1]	0.15476[13]	0.14544[12]	0.15506[14]
	D_{abs}	0.01653[6]	0.01622[3]	0.01644[5]	0.01689[7]	0.01588[2]	0.01725[8]	0.01641[4]	0.01836[10.5]	0.01842[12]	0.01836[10.5]	0.01974[13]	0.01429[1]	0.02176[14]	0.01822[9]	0.02305[15]
	D_{max}	0.02682[3]	0.02694[4]	0.02743[7]	0.0274[6]	0.02665[2]	0.02925[8]	0.02728[5]	0.03006[10]	0.03014[12]	0.02997[9]	0.03299[13]	0.02186[1]	0.03591[14]	0.03007[11]	0.03687[15]
	$\sum Ranks$	21[2]	27[3]	57[7]	43[6]	41[5]	87[11]	40[4]	68.5[8]	70[9]	74.5[10]	116[15]	8[1]	109[13.5]	89[12]	109[13.5]

Table 5. Numerical values of simulation measures for $\delta = 0.7,\ \theta = 0.2$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS($\hat{\delta}$)	0.07202[2]	0.10347[8]	0.10733[10]	0.0862[4]	0.11324[11]	0.11691[12]	0.0985[7]	0.10438[9]	0.08621[5]	0.08564[3]	0.13883[15]	0.04608[1]	0.1173[13]	0.0919[6]	0.12216[14]
	BIAS($\hat{\theta}$)	0.06924[5]	0.0698[7]	0.07063[9]	0.07087[10]	0.07694[12]	0.07013[8]	0.06822[4]	0.07192[11]	0.06218[2]	0.0651[3]	0.08417[14]	0.04167[1]	0.08174[13]	0.06974[6]	0.08498[15]
	MSE($\hat{\delta}$)	0.00775[2]	0.01968[10]	0.01952[9]	0.01134[3]	0.02118[11]	0.02413[13]	0.01819[7]	0.01896[8]	0.01282[4]	0.01361[6]	0.03703[15]	0.00395[1]	0.02392[12]	0.01332[5]	0.02642[14]
	MSE($\hat{\theta}$)	0.00748[10]	0.00718[7]	0.00717[6]	0.0075[11]	0.0084[12]	0.00712[5]	0.00704[4]	0.0073[9]	0.00648[2]	0.00685[3]	0.00967[14]	0.00343[1]	0.00915[13]	0.00721[8]	0.01007[15]
	MRE($\hat{\delta}$)	0.10289[2]	0.14782[8]	0.15332[10]	0.12314[4]	0.16178[11]	0.16701[12]	0.14071[7]	0.14911[9]	0.12315[5]	0.12235[3]	0.19833[15]	0.06583[1]	0.16757[13]	0.13128[6]	0.17451[14]
	MRE($\hat{\theta}$)	0.34622[5]	0.34899[7]	0.35313[9]	0.35436[10]	0.38472[12]	0.35066[8]	0.34112[4]	0.3596[11]	0.31091[2]	0.3255[3]	0.42086[14]	0.20836[1]	0.40869[13]	0.34868[6]	0.42492[15]
	D_{abs}	0.04429[1]	0.04948[5]	0.04814[4]	0.05317[11]	0.05084[8]	0.05579[12]	0.04769[3]	0.0511[9]	0.05041[7]	0.05212[10]	0.05621[13]	0.04615[2]	0.06227[14]	0.05014[6]	0.06268[15]
	D_{max}	0.0716[2]	0.08074[4]	0.08119[6]	0.08338[10]	0.08435[11]	0.09253[12]	0.07721[3]	0.08304[9]	0.08116[5]	0.08221[8]	0.095[13]	0.07046[1]	0.09997[14]	0.08128[7]	0.10276[15]
	$\sum Ranks$	29[2]	56[7]	63[8.5]	63[8.5]	88[12]	82[11]	39[4.5]	75[10]	32[3]	39[4.5]	113[14]	9[1]	105[13]	50[6]	117[15]
60	BIAS($\hat{\delta}$)	0.05392[2]	0.06597[6]	0.08057[11]	0.06527[5]	0.06766[7]	0.0806[12]	0.07356[10]	0.06992[9]	0.06802[8]	0.06412[3]	0.08375[14]	0.03519[1]	0.08539[15]	0.06506[4]	0.08162[13]
	BIAS($\hat{\theta}$)	0.05202[4]	0.04944[2]	0.05696[9]	0.05586[8]	0.05136[3]	0.05732[11]	0.05709[10]	0.05577[7]	0.05474[6]	0.05378[5]	0.06299[13]	0.03046[1]	0.06687[15]	0.05777[12]	0.06625[14]
	MSE($\hat{\delta}$)	0.00433[2]	0.00705[6]	0.01084[13]	0.00617[4]	0.00754[9]	0.00961[11]	0.00856[10]	0.00739[8]	0.00731[7]	0.00667[5]	0.01211[15]	0.00265[1]	0.01132[14]	0.00616[3]	0.01074[12]
	MSE($\hat{\theta}$)	0.00428[4]	0.0038[2]	0.00478[6.5]	0.00497[9]	0.00423[3]	0.00492[8]	0.00504[11]	0.00449[5]	0.00478[6.5]	0.00499[10]	0.00608[13]	0.0022[1]	0.00669[15]	0.00521[12]	0.00666[14]
	MRE($\hat{\delta}$)	0.07702[2]	0.09425[6]	0.11509[11]	0.09325[5]	0.09666[7]	0.11514[12]	0.10508[10]	0.09989[9]	0.09717[8]	0.0916[3]	0.11964[14]	0.05027[1]	0.12199[15]	0.09294[4]	0.1166[13]
	MRE($\hat{\theta}$)	0.26011[4]	0.24722[2]	0.28481[9]	0.27929[8]	0.25681[3]	0.28659[11]	0.28547[10]	0.27886[7]	0.27372[6]	0.26891[5]	0.31497[13]	0.1523[1]	0.33435[15]	0.28887[12]	0.33123[14]
	D_{abs}	0.02678[1]	0.0313[2]	0.03432[8]	0.03134[4]	0.03154[5]	0.03507[9]	0.03345[7]	0.03225[6]	0.03867[13]	0.03617[11]	0.03767[12]	0.03132[3]	0.04366[15]	0.03529[10]	0.03936[14]
	D_{max}	0.04356[1]	0.05154[4]	0.05766[9]	0.05106[3]	0.05186[5]	0.05977[11]	0.05547[7]	0.0529[6]	0.06268[13]	0.05769[10]	0.06234[12]	0.04912[2]	0.07157[15]	0.05678[8]	0.06417[14]
	$\sum Ranks$	20[2]	30[3]	76.5[11]	46[5]	42[4]	85[12]	75[10]	57[7]	67.5[9]	52[6]	106[13]	11[1]	119[15]	65[8]	108[14]
90	BIAS($\hat{\delta}$)	0.04457[2]	0.05629[4]	0.05991[10]	0.0532[3]	0.05644[6]	0.07063[13]	0.057[7]	0.05633[5]	0.06025[11]	0.05703[8]	0.07342[14]	0.0323[1]	0.07506[15]	0.05806[9]	0.06889[12]
	BIAS($\hat{\theta}$)	0.04114[2]	0.04398[5]	0.04295[4]	0.04547[7]	0.04567[8]	0.05321[12]	0.04463[6]	0.04283[3]	0.04794[10]	0.04754[9]	0.05808[14]	0.0279[1]	0.05988[15]	0.05032[11]	0.05785[13]
	MSE($\hat{\delta}$)	0.003[2]	0.00482[4]	0.00621[11]	0.0044[3]	0.00516[9]	0.00771[13]	0.00507[7]	0.0051[8]	0.00592[10]	0.00498[6]	0.00848[14]	0.00222[1]	0.00905[15]	0.00492[5]	0.00696[12]
	MSE($\hat{\theta}$)	0.0028[2]	0.00309[5]	0.00282[3]	0.00335[8]	0.00318[6]	0.00422[11]	0.00324[7]	0.00292[4]	0.00371[9]	0.00392[10]	0.00504[13]	0.00171[1]	0.00541[15]	0.00423[12]	0.00526[14]
	MRE($\hat{\delta}$)	0.06368[2]	0.08042[4]	0.08558[10]	0.076[3]	0.08063[6]	0.1009[13]	0.08142[7]	0.08047[5]	0.08608[11]	0.08148[8]	0.10488[14]	0.04615[1]	0.10723[15]	0.08295[9]	0.09841[12]
	MRE($\hat{\theta}$)	0.20571[2]	0.21991[5]	0.21474[4]	0.22735[7]	0.22835[8]	0.26606[12]	0.22314[6]	0.21414[3]	0.23969[10]	0.23772[9]	0.29039[14]	0.13948[1]	0.29941[15]	0.25158[11]	0.28923[13]
	D_{abs}	0.02555[2]	0.02588[3]	0.02739[6]	0.02528[1]	0.02813[9]	0.0314[12]	0.02775[7]	0.02705[4]	0.03285[14]	0.02889[10]	0.03002[11]	0.02707[5]	0.03654[15]	0.0279[8]	0.03273[13]
	D_{max}	0.04125[2]	0.04305[4]	0.04618[9]	0.04116[1]	0.04603[8]	0.05261[12]	0.04584[6]	0.0445[5]	0.05392[14]	0.04736[10]	0.04975[11]	0.04239[3]	0.06[15]	0.04591[7]	0.05347[13]
	$\sum Ranks$	16[2]	34[4]	57[7]	33[3]	60[8]	98[12]	53[6]	37[5]	89[11]	70[9]	105[14]	14[1]	120[15]	72[10]	102[13]
140	BIAS($\hat{\delta}$)	0.03391[2]	0.04473[4]	0.05025[11]	0.03951[3]	0.04512[5]	0.05445[12]	0.04917[9]	0.04824[8]	0.05017[10]	0.04715[7]	0.06256[15]	0.02529[1]	0.05756[13]	0.04581[6]	0.05931[14]
	BIAS($\hat{\theta}$)	0.03121[2]	0.03651[5]	0.03939[8]	0.03481[3]	0.03557[4]	0.03945[9]	0.04002[10]	0.03838[6]	0.04347[12]	0.04042[11]	0.05215[15]	0.02199[1]	0.0503[14]	0.03922[7]	0.04989[13]
	MSE($\hat{\delta}$)	0.00184[2]	0.00306[4]	0.004[10]	0.00248[3]	0.00335[6]	0.00457[12]	0.00362[9]	0.00345[8]	0.0041[11]	0.00338[7]	0.00602[15]	0.00151[1]	0.00511[13]	0.00331[5]	0.00513[14]
	MSE($\hat{\theta}$)	0.00169[2]	0.0022[5]	0.00239[7.5]	0.00214[4]	0.00206[3]	0.00239[7.5]	0.00256[9.5]	0.00224[6]	0.00302[12]	0.0028[11]	0.00409[13]	0.00126[1]	0.00425[15]	0.00256[9.5]	0.00414[14]
	MRE($\hat{\delta}$)	0.04844[2]	0.0639[4]	0.07179[11]	0.05644[3]	0.06446[5]	0.07778[12]	0.07025[9]	0.06892[8]	0.07167[10]	0.06735[7]	0.08938[15]	0.03612[1]	0.08223[13]	0.06544[6]	0.08473[14]
	MRE($\hat{\theta}$)	0.15605[2]	0.18256[5]	0.19695[8]	0.17405[3]	0.17784[4]	0.19725[9]	0.20008[10]	0.19191[6]	0.21733[12]	0.2021[11]	0.26074[15]	0.10994[1]	0.25148[14]	0.1961[7]	0.24947[13]
	D_{abs}	0.02127[2]	0.0228[5]	0.02402[10]	0.02146[3]	0.02212[4]	0.02385[8]	0.02379[7]	0.02343[6]	0.02762[13]	0.0241[11]	0.02619[12]	0.02073[1]	0.02886[15]	0.02395[9]	0.02806[14]
	D_{max}	0.03416[2]	0.03765[5]	0.03972[10]	0.03472[3]	0.03637[4]	0.04027[11]	0.0393[8]	0.03846[6]	0.04468[13]	0.03958[9]	0.04337[12]	0.0327[1]	0.04726[15]	0.03874[7]	0.04645[14]
	$\sum Ranks$	16[2]	37[5]	75.5[10]	25[3]	35[4]	80.5[11]	71.5[8]	54[6]	93[12]	74[9]	112[14.5]	8[1]	112[14.5]	56.5[7]	110[13]
200	BIAS($\hat{\delta}$)	0.03124[2]	0.03891[9]	0.03816[5]	0.03494[4]	0.03862[8]	0.0454[12]	0.03471[3]	0.03908[10]	0.03847[6]	0.04312[11]	0.05221[15]	0.02185[1]	0.05095[13]	0.03848[7]	0.05152[14]
	BIAS($\hat{\theta}$)	0.02783[3]	0.03128[7]	0.02849[4]	0.03012[6]	0.03003[5]	0.03522[11]	0.02691[2]	0.03274[8]	0.03375[10]	0.03664[12]	0.04473[15]	0.02085[1]	0.04396[14]	0.03326[9]	0.04189[13]
	MSE($\hat{\delta}$)	0.00157[2]	0.00233[7]	0.0024[9]	0.00184[3]	0.00234[8]	0.00321[12]	0.00201[4]	0.00244[10]	0.00225[5]	0.00282[11]	0.00426[15]	0.00116[1]	0.00399[13]	0.00232[6]	0.00407[14]
	MSE($\hat{\theta}$)	0.00136[4]	0.00149[6]	0.00132[3]	0.00141[5]	0.00152[7]	0.00198[11]	0.00126[2]	0.00175[8.5]	0.00177[10]	0.00208[12]	0.00324[15]	0.00104[1]	0.00308[14]	0.00175[8.5]	0.00278[13]
	MRE($\hat{\delta}$)	0.04462[2]	0.05558[9]	0.05451[5]	0.04992[4]	0.05517[8]	0.06486[12]	0.04958[3]	0.05583[10]	0.05496[6]	0.0616[11]	0.07458[15]	0.03122[1]	0.07279[13]	0.05498[7]	0.07359[14]
	MRE($\hat{\theta}$)	0.13914[3]	0.15638[7]	0.14246[4]	0.1506[6]	0.15015[5]	0.17611[11]	0.13456[2]	0.16369[8]	0.16876[10]	0.18321[12]	0.22366[15]	0.10427[1]	0.21978[14]	0.16631[9]	0.20947[13]
	D_{abs}	0.0169[1]	0.01844[4]	0.01816[2]	0.01865[5]	0.01941[8]	0.01953[9]	0.01899[6]	0.01907[7]	0.02301[13]	0.02201[12]	0.02098[11]	0.0184[3]	0.02517[15]	0.02067[10]	0.02401[14]
	D_{max}	0.02764[1]	0.03041[5]	0.0303[4]	0.02998[3]	0.03203[8]	0.03295[9]	0.03115[6]	0.0312[7]	0.0368[13]	0.03589[12]	0.03473[11]	0.02851[2]	0.04114[15]	0.03332[10]	0.03941[14]
	$\sum Ranks$	18[2]	54[6]	36[4.5]	36[4.5]	57[7]	87[11]	28[3]	68.5[9]	73[10]	93[12]	112[15]	11[1]	111[14]	66.5[8]	109[13]
250	BIAS($\hat{\delta}$)	0.02824[2]	0.0324[6]	0.03478[7]	0.032[4]	0.03541[8]	0.03721[12]	0.03122[3]	0.03238[5]	0.03692[11]	0.03613[9]	0.04702[15]	0.02045[1]	0.04319[14]	0.03642[10]	0.04167[13]
	BIAS($\hat{\theta}$)	0.02421[2]	0.02635[5]	0.02703[6]	0.02713[7]	0.02843[9]	0.02808[8]	0.02599[4]	0.02595[3]	0.03052[11]	0.02938[10]	0.03966[15]	0.0175[1]	0.03703[14]	0.03149[12]	0.03629[13]
	MSE($\hat{\delta}$)	0.0013[2]	0.00174[6]	0.00191[8]	0.00147[3]	0.00188[7]	0.00225[12]	0.00153[4]	0.00165[5]	0.00217[11]	0.00204[10]	0.00354[15]	0.00091[1]	0.00279[14]	0.00202[9]	0.00264[13]
	MSE($\hat{\theta}$)	0.001[2]	0.00119[6.5]	0.00119[6.5]	0.00111[5]	0.00128[8]	0.00129[9]	0.00107[3.5]	0.00107[3.5]	0.00155[12]	0.00147[10]	0.00251[15]	0.00072[1]	0.00217[14]	0.00154[11]	0.00214[13]
	MRE($\hat{\delta}$)	0.04034[2]	0.04628[6]	0.04969[7]	0.04571[4]	0.05059[8]	0.05316[12]	0.04459[3]	0.04626[5]	0.05275[11]	0.05161[9]	0.06717[15]	0.02921[1]	0.0617[14]	0.05202[10]	0.05953[13]
	MRE($\hat{\theta}$)	0.12105[2]	0.13174[5]	0.13517[6]	0.13564[7]	0.14214[9]	0.14041[8]	0.12997[4]	0.12973[3]	0.1526[11]	0.14689[10]	0.19828[15]	0.08752[1]	0.18514[14]	0.15747[12]	0.18146[13]
	D_{abs}	0.01408[1]	0.01575[2]	0.01611[4]	0.01668[6]	0.0167[7]	0.01774[9]	0.01579[3]	0.01651[5]	0.02074[13]	0.018[10]	0.01936[12]	0.01708[8]	0.02156[14]	0.01907[11]	0.02217[15]
	D_{max}	0.0231[1]	0.02582[3]	0.02677[4]	0.02715[7]	0.02772[8]	0.02954[9]	0.02577[2]	0.027[6]	0.03343[13]	0.02957[10]	0.03232[12]	0.02683[5]	0.03537[14]	0.03062[11]	0.03593[15]
	$\sum Ranks$	14[1]	39.5[5]	48.5[7]	43[6]	64[8]	79[10]	26.5[3]	35.5[4]	93[12]	78[9]	114[15]	19[2]	112[14]	86[11]	108[13]

Table 6. Numerical values of simulation measures for $\delta = 0.3,\ \theta = 1.75$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS($\hat{\delta}$)	0.06022[2]	0.08253[9]	0.0974[13]	0.07541[5]	0.08631[11]	0.10372[15]	0.0797[6]	0.08135[8]	0.06233[3]	0.06875[4]	0.09852[14]	0.03435[1]	0.08572[10]	0.07991[7]	0.08807[12]
	BIAS($\hat{\theta}$)	0.48978[4]	0.58471[10]	0.61154[13]	0.56648[7]	0.57228[8]	0.61184[14]	0.57668[9]	0.59331[11]	0.29621[2]	0.47859[3]	0.61991[15]	0.05131[1]	0.52427[6]	0.60459[12]	0.50346[5]
	MSE($\hat{\delta}$)	0.00547[2]	0.01132[9]	0.0144[13]	0.00911[5]	0.01136[10]	0.01644[15]	0.01[6]	0.01106[8]	0.0081[4]	0.00773[3]	0.01463[14]	0.00212[1]	0.01198[11]	0.01014[7]	0.01295[12]
	MSE($\hat{\theta}$)	0.35755[3]	0.49621[10]	0.52943[14]	0.48727[9]	0.47818[7]	0.5246[13]	0.48518[8]	0.49793[11]	0.22268[2]	0.3977[5]	0.51707[12]	0.01627[1]	0.398[6]	0.5369[15]	0.37728[4]
	MRE($\hat{\delta}$)	0.20073[2]	0.27511[9]	0.32466[13]	0.25137[5]	0.28769[11]	0.34575[15]	0.26568[6]	0.27117[8]	0.20777[3]	0.22918[4]	0.32842[14]	0.1145[1]	0.28572[10]	0.26638[7]	0.29357[12]
	MRE($\hat{\theta}$)	0.27987[4]	0.33412[10]	0.34945[13]	0.3237[7]	0.32702[8]	0.34962[14]	0.32953[9]	0.33904[11]	0.16927[2]	0.27348[3]	0.35424[15]	0.02932[1]	0.29958[6]	0.34548[12]	0.28769[5]
	D_{abs}	0.04429[1]	0.04543[4]	0.04685[8]	0.04563[5]	0.04565[6]	0.04959[11]	0.04683[7]	0.04523[3]	0.05051[12]	0.04476[2]	0.0472[10]	0.04717[9]	0.06519[14]	0.05264[13]	0.06872[15]
	D_{max}	0.07186[1]	0.07448[5]	0.07967[9]	0.07268[2]	0.07543[7]	0.08505[13]	0.07606[8]	0.07482[6]	0.08057[11]	0.07366[3]	0.07969[10]	0.07395[4]	0.10267[14]	0.08382[12]	0.10694[15]
	$\sum Ranks$	19[1.5]	66[7.5]	96[13]	45[5]	68[9]	110[15]	59[6]	66[7.5]	39[4]	27[3]	104[14]	19[1.5]	77[10]	85[12]	80[11]
60	BIAS($\hat{\delta}$)	0.04346[3]	0.06263[9]	0.06861[14]	0.05593[5]	0.0683[13]	0.07788[15]	0.06577[11]	0.06521[10]	0.04264[2]	0.05525[4]	0.06652[12]	0.02136[1]	0.05972[7]	0.06076[8]	0.05935[6]
	BIAS($\hat{\theta}$)	0.36106[4]	0.47545[8]	0.4943[10]	0.47515[7]	0.52274[13]	0.52661[14]	0.51597[12]	0.49584[11]	0.23172[2]	0.42235[6]	0.54345[15]	0.04448[1]	0.37636[5]	0.48828[9]	0.34744[3]
	MSE($\hat{\delta}$)	0.00286[2]	0.00588[6]	0.00728[14]	0.00501[4]	0.00699[11]	0.00919[15]	0.00699[11]	0.00647[8]	0.00357[3]	0.00537[5]	0.00695[9]	0.00079[1]	0.00724[13]	0.0059[7]	0.00699[11]
	MSE($\hat{\theta}$)	0.21987[4]	0.34313[7]	0.37006[8]	0.38319[10]	0.42748[13]	0.40502[12]	0.43724[14]	0.38011[9]	0.16096[2]	0.31214[6]	0.46165[15]	0.00997[1]	0.22636[5]	0.38858[11]	0.21103[3]
	MRE($\hat{\delta}$)	0.14487[3]	0.20876[9]	0.2287[14]	0.18644[5]	0.22767[13]	0.25961[15]	0.21922[11]	0.21735[10]	0.14213[2]	0.18418[4]	0.22173[12]	0.07121[1]	0.19908[7]	0.20254[8]	0.19785[6]
	MRE($\hat{\theta}$)	0.20632[4]	0.27168[8]	0.28246[10]	0.27152[7]	0.29871[13]	0.30092[14]	0.29484[12]	0.28334[11]	0.13241[2]	0.24134[6]	0.31054[15]	0.02541[1]	0.21506[5]	0.27902[9]	0.19854[3]
	D_{abs}	0.02837[1]	0.03111[4]	0.03275[10]	0.03124[5]	0.03059[2]	0.03218[8]	0.03291[11]	0.03216[7]	0.03215[6]	0.0333[12]	0.03654[13]	0.03268[9]	0.04203[14]	0.0311[3]	0.04476[15]
	D_{max}	0.04671[1]	0.05232[7]	0.05581[12]	0.05095[3]	0.05154[5]	0.0552[10]	0.05527[11]	0.05368[8]	0.05209[6]	0.05423[9]	0.05999[13]	0.05042[2]	0.06716[14]	0.05116[4]	0.07117[15]
	$\sum Ranks$	22[2]	58[6]	92[12]	46[4]	83[11]	103[14]	93[13]	74[10]	25[3]	52[5]	104[15]	17[1]	70[9]	59[7]	62[8]
90	BIAS($\hat{\delta}$)	0.0407[3]	0.05489[11]	0.05986[13]	0.04757[4]	0.06166[14]	0.0637[15]	0.05575[12]	0.04965[7]	0.03144[2]	0.04784[5]	0.05063[8]	0.01701[1]	0.05265[9]	0.04793[6]	0.05325[10]
	BIAS($\hat{\theta}$)	0.33446[4]	0.42012[8]	0.4542[12]	0.42773[9]	0.46731[15]	0.46079[14]	0.45674[13]	0.39404[6]	0.19622[2]	0.41178[7]	0.42944[10]	0.03732[1]	0.34549[5]	0.43381[11]	0.2912[3]
	MSE($\hat{\delta}$)	0.0027[3]	0.00501[10]	0.00554[12]	0.00339[4]	0.00587[13]	0.00654[15]	0.0049[9]	0.00381[7]	0.00215[2]	0.00356[5]	0.00388[8]	0.00046[1]	0.00511[11]	0.00359[6]	0.00608[14]
	MSE($\hat{\theta}$)	0.18736[4]	0.28865[7]	0.32998[12]	0.3219[11]	0.37272[15]	0.34808[13]	0.35934[14]	0.24912[6]	0.11484[2]	0.30877[10]	0.28902[8]	0.00816[1]	0.19677[5]	0.29759[9]	0.18121[3]
	MRE($\hat{\delta}$)	0.13567[3]	0.18297[11]	0.19955[13]	0.15855[4]	0.20553[14]	0.21234[15]	0.18582[12]	0.16551[7]	0.10479[2]	0.15946[5]	0.16875[8]	0.05669[1]	0.1755[9]	0.15976[6]	0.1775[10]
	MRE($\hat{\theta}$)	0.19112[4]	0.24007[8]	0.25954[12]	0.24441[9]	0.26704[15]	0.26331[14]	0.261[13]	0.22516[6]	0.11212[2]	0.2353[7]	0.24539[10]	0.02133[1]	0.19742[5]	0.24789[11]	0.1664[3]
	D_{abs}	0.02319[1]	0.0256[5]	0.02795[11]	0.02757[10]	0.02635[8]	0.02553[4]	0.02476[3]	0.02475[2]	0.02606[7]	0.02677[9]	0.02837[13]	0.02583[6]	0.03568[14]	0.02825[12]	0.03801[15]
	D_{max}	0.03824[1]	0.04257[6]	0.04681[13]	0.04455[9]	0.0454[10]	0.04436[8]	0.0417[5]	0.04113[3]	0.04136[4]	0.04398[7]	0.04615[12]	0.04008[2]	0.05736[14]	0.0457[11]	0.06103[15]
	$\sum Ranks$	23[2.5]	66[7]	98[13.5]	60[6]	104[15]	98[13.5]	81[12]	44[4]	23[2.5]	55[5]	77[11]	14[1]	72[8.5]	72[8.5]	73[10]
140	BIAS($\hat{\delta}$)	0.03499[4]	0.04333[10]	0.04643[12]	0.04192[7]	0.05252[15]	0.05225[14]	0.04368[11]	0.04266[8]	0.03018[2]	0.04115[6]	0.04865[13]	0.01339[1]	0.0376[5]	0.04302[9]	0.03194[3]
	BIAS($\hat{\theta}$)	0.29337[5]	0.34462[6]	0.35537[9]	0.3622[10]	0.38615[12]	0.40778[14]	0.35102[8]	0.36439[11]	0.17959[3]	0.34774[7]	0.41491[15]	0.02663[1]	0.24215[4]	0.3866[13]	0.17397[2]
	MSE($\hat{\delta}$)	0.00204[3]	0.00299[11]	0.00338[12]	0.00273[7]	0.0041[15]	0.00402[14]	0.00285[9]	0.00275[8]	0.00192[2]	0.00269[6]	0.00358[13]	0.00033[1]	0.00251[5]	0.00286[10]	0.00225[4]
	MSE($\hat{\theta}$)	0.16332[5]	0.19469[6]	0.2135[8]	0.22862[11]	0.23518[12]	0.27347[14]	0.19472[7]	0.22167[10]	0.09683[3]	0.21575[9]	0.27911[15]	0.00278[1]	0.10823[4]	0.25657[13]	0.07233[2]
	MRE($\hat{\delta}$)	0.11664[4]	0.14444[10]	0.15476[12]	0.13972[7]	0.17508[15]	0.17417[14]	0.14561[11]	0.14221[8]	0.1006[2]	0.13718[6]	0.16217[13]	0.04462[1]	0.12532[5]	0.14342[9]	0.10647[3]
	MRE($\hat{\theta}$)	0.16764[5]	0.19693[6]	0.20307[9]	0.20697[10]	0.22066[12]	0.23302[14]	0.20058[8]	0.20823[11]	0.10262[3]	0.19871[7]	0.23709[15]	0.01522[1]	0.13837[4]	0.22091[13]	0.09941[2]
	D_{abs}	0.02045[2]	0.02125[3]	0.0225[9]	0.02146[4]	0.02199[7]	0.02323[12]	0.0224[8]	0.02175[5]	0.02289[10]	0.02196[6]	0.02314[11]	0.01907[1]	0.02809[14]	0.02488[13]	0.02949[15]
	D_{max}	0.03341[2]	0.03516[3]	0.03818[10]	0.03534[4]	0.03767[9]	0.0396[12]	0.03718[8]	0.03592[5]	0.03682[7]	0.03641[6]	0.03854[11]	0.02962[1]	0.04505[14]	0.04041[13]	0.04674[15]
	$\sum Ranks$	30[2]	55[6.5]	81[11]	60[8]	97[13]	108[15]	70[10]	66[9]	32[3]	53[5]	106[14]	8[1]	55[6.5]	93[12]	46[4]
200	BIAS($\hat{\delta}$)	0.03048[4]	0.03641[8]	0.04378[13]	0.03438[5]	0.04476[14]	0.04957[15]	0.03701[10]	0.0378[11]	0.02665[3]	0.03663[9]	0.0399[12]	0.01118[1]	0.03457[6]	0.03491[7]	0.0226[2]
	BIAS($\hat{\theta}$)	0.24274[5]	0.29104[7]	0.32234[12]	0.30558[8]	0.35855[14]	0.37222[15]	0.31026[10]	0.31146[11]	0.17395[3]	0.30698[9]	0.33461[13]	0.0218[1]	0.21721[4]	0.29103[6]	0.14703[2]
	MSE($\hat{\delta}$)	0.00152[4]	0.00207[7]	0.003[14]	0.00187[5]	0.00297[13]	0.00365[15]	0.00219[10]	0.00218[9]	0.00142[3]	0.00212[8]	0.00247[12]	0.00021[1]	0.00232[11]	0.00189[6]	0.00108[2]
	MSE($\hat{\theta}$)	0.09682[4]	0.13791[6]	0.17524[11]	0.16842[9]	0.20375[14]	0.22982[15]	0.17697[12]	0.15772[8]	0.096[3]	0.16986[10]	0.18731[13]	0.00181[1]	0.10244[5]	0.13963[7]	0.03952[2]
	MRE($\hat{\delta}$)	0.10159[4]	0.12137[8]	0.14593[13]	0.1146[5]	0.1492[14]	0.16525[15]	0.12336[10]	0.126[11]	0.08882[3]	0.1221[9]	0.13301[12]	0.03728[1]	0.11523[6]	0.11637[7]	0.07534[2]
	MRE($\hat{\theta}$)	0.13871[5]	0.16631[7]	0.18419[12]	0.17462[8]	0.20488[14]	0.2127[15]	0.17729[10]	0.17798[11]	0.0994[3]	0.17542[9]	0.19121[13]	0.01246[1]	0.12412[4]	0.1663[6]	0.08402[2]
	D_{abs}	0.01712[2]	0.01752[3]	0.01959[11]	0.01801[4]	0.01905[8]	0.01902[7]	0.01828[5]	0.0187[6]	0.01927[9]	0.02039[12]	0.01935[10]	0.01684[1]	0.02532[15]	0.02053[13]	0.02188[14]
	D_{max}	0.02794[2]	0.02915[3]	0.03336[11]	0.02969[4]	0.03232[9]	0.03323[10]	0.03033[5]	0.03084[6]	0.03138[7]	0.0335[12]	0.03207[8]	0.02616[1]	0.04071[15]	0.03352[13]	0.03487[14]
	$\sum Ranks$	30[2]	49[6]	97[13]	48[5]	100[14]	107[15]	72[9]	73[10]	34[3]	78[11]	93[12]	8[1]	66[8]	65[7]	40[4]
250	BIAS($\hat{\delta}$)	0.02831[5]	0.03596[12]	0.03934[13]	0.02968[6]	0.04005[14]	0.04544[15]	0.0344[10]	0.03151[8]	0.02331[3]	0.02998[7]	0.0359[11]	0.00977[1]	0.02563[4]	0.03197[9]	0.02245[2]
	BIAS($\hat{\theta}$)	0.2243[5]	0.27729[11]	0.29605[12]	0.26352[7]	0.32075[14]	0.3247[15]	0.27295[10]	0.26403[8]	0.15831[4]	0.24703[6]	0.30989[13]	0.02025[1]	0.13874[2]	0.27087[9]	0.14551[3]
	MSE($\hat{\delta}$)	0.00133[4]	0.00199[11]	0.0025[14]	0.00141[5]	0.00238[13]	0.00325[15]	0.00179[10]	0.00161[7]	0.00113[3]	0.00149[6]	0.00217[12]	0.00015[1]	0.00164[8]	0.00165[9]	0.00091[2]
	MSE($\hat{\theta}$)	0.08906[5]	0.12336[8]	0.15579[12]	0.13183[10]	0.17788[15]	0.1697[13]	0.12211[7]	0.12506[9]	0.07661[4]	0.11332[6]	0.17697[14]	0.00119[1]	0.05766[3]	0.13383[11]	0.03916[2]
	MRE($\hat{\delta}$)	0.09437[5]	0.11987[12]	0.13113[13]	0.09893[6]	0.13349[14]	0.15146[15]	0.11466[10]	0.10505[8]	0.0771[3]	0.09992[7]	0.11965[11]	0.03257[1]	0.08542[4]	0.10656[9]	0.07485[2]
	MRE($\hat{\theta}$)	0.12817[5]	0.15845[11]	0.16917[12]	0.15058[7]	0.18329[14]	0.18554[15]	0.15597[10]	0.15088[8]	0.09046[4]	0.14116[6]	0.17708[13]	0.01157[1]	0.07928[2]	0.15478[9]	0.08315[3]
	D_{abs}	0.01499[1]	0.01725[9]	0.01701[7]	0.01575[4]	0.01749[11]	0.01754[12]	0.01587[5]	0.0153[3]	0.01632[6]	0.01739[10]	0.01817[14]	0.01511[2]	0.02387[15]	0.01724[8]	0.01788[13]
	D_{max}	0.02469[2]	0.0288[9]	0.02915[11]	0.02599[4]	0.02959[12]	0.03051[14]	0.02692[6]	0.02552[3]	0.02634[5]	0.0286[8]	0.02985[13]	0.0234[1]	0.03758[15]	0.02847[7]	0.02896[10]
	$\sum Ranks$	32[2.5]	83[11]	94[12]	49[5]	107[14]	114[15]	68[9]	54[7]	32[2.5]	56[8]	101[13]	9[1]	53[6]	71[10]	37[4]

Table 7. Numerical values of simulation measures for $\delta = 3.0,\ \theta = 0.9$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
25	BIAS($\hat{\delta}$)	0.48071[3]	0.62338[11]	0.7596[15]	0.55047[4]	0.6177[10]	0.66648[13]	0.66253[12]	0.60443[9]	0.44632[2]	0.58012[6]	0.74443[14]	0.07431[1]	0.59626[8]	0.55971[5]	0.58094[7]
	BIAS($\hat{\theta}$)	0.23295[2]	0.28615[7]	0.32293[15]	0.29478[11]	0.28453[6]	0.28866[8]	0.27817[4]	0.27882[5]	0.24569[3]	0.29968[12]	0.30549[13]	0.10992[1]	0.308[14]	0.2894[9]	0.29282[10]
	MSE($\hat{\delta}$)	0.35962[2]	0.61758[10]	0.92709[14]	0.43539[4]	0.62159[11]	0.73114[12]	0.75566[13]	0.61549[9]	0.3963[3]	0.54538[8]	0.95461[15]	0.02442[1]	0.52406[7]	0.50499[6]	0.47554[5]
	MSE($\hat{\theta}$)	0.08923[2]	0.12571[6]	0.1468[15]	0.13856[13]	0.12599[7]	0.12626[8]	0.11576[4]	0.11846[5]	0.10454[3]	0.13701[11]	0.13724[12]	0.02461[1]	0.14439[14]	0.12839[9]	0.13063[10]
	MRE($\hat{\delta}$)	0.16024[3]	0.20779[11]	0.2532[15]	0.18349[4]	0.2059[10]	0.22216[13]	0.22084[12]	0.20148[9]	0.14877[2]	0.19337[6]	0.24814[14]	0.02477[1]	0.19875[8]	0.18657[5]	0.19365[7]
	MRE($\hat{\theta}$)	0.25883[2]	0.31794[7]	0.35881[15]	0.32754[11]	0.31614[6]	0.32074[8]	0.30908[4]	0.3098[5]	0.27299[3]	0.33298[12]	0.33943[13]	0.12213[1]	0.34222[14]	0.32155[9]	0.32535[10]
	D_{abs}	0.04138[1]	0.04891[7]	0.05027[10]	0.04295[2]	0.04973[9]	0.04803[6]	0.05084[11]	0.0469[5]	0.05739[14]	0.05642[12]	0.04913[8]	0.04561[4]	0.05678[13]	0.04542[3]	0.05777[15]
	D_{max}	0.06909[2]	0.08046[6]	0.08468[11]	0.07009[3]	0.08127[8]	0.08055[7]	0.08402[10]	0.07723[5]	0.08987[13]	0.0886[12]	0.08349[9]	0.06754[1]	0.09219[14]	0.0737[4]	0.09238[15]
	$\sum Ranks$	17[2]	65[7]	110[15]	52[5.5]	67[8]	75[10]	70[9]	52[5.5]	43[3]	79[11.5]	98[14]	11[1]	92[13]	50[4]	79[11.5]
60	BIAS($\hat{\delta}$)	0.31168[2]	0.41933[6]	0.48793[11]	0.415[5]	0.45289[9]	0.51283[14]	0.46904[10]	0.45117[8]	0.36459[3]	0.38771[4]	0.51452[15]	0.06894[1]	0.50149[13]	0.43084[7]	0.49145[12]
	BIAS($\hat{\theta}$)	0.16986[2]	0.21099[3]	0.22641[10]	0.22379[7]	0.22581[9]	0.22479[8]	0.23748[12]	0.22075[6]	0.21145[4]	0.21627[5]	0.25517[14]	0.0782[1]	0.26804[15]	0.23097[11]	0.25344[13]
	MSE($\hat{\delta}$)	0.14892[2]	0.28687[7]	0.39359[13]	0.25565[4]	0.31365[8]	0.44482[15]	0.33199[10]	0.32939[9]	0.26905[5]	0.23572[3]	0.40796[14]	0.01847[1]	0.36297[12]	0.27594[6]	0.34966[11]
	MSE($\hat{\theta}$)	0.05164[2]	0.07189[3]	0.07965[6]	0.08335[10]	0.08235[8]	0.07947[5]	0.08851[12]	0.07822[4]	0.08213[7]	0.08272[9]	0.09873[13]	0.01414[1]	0.1158[15]	0.08647[11]	0.10317[14]
	MRE($\hat{\delta}$)	0.10389[2]	0.13978[6]	0.16264[11]	0.13833[5]	0.15096[9]	0.17094[14]	0.15635[10]	0.15039[8]	0.12153[3]	0.12924[4]	0.17151[15]	0.02298[1]	0.16716[13]	0.14361[7]	0.16382[12]
	MRE($\hat{\theta}$)	0.18873[2]	0.23444[3]	0.25157[10]	0.24866[7]	0.2509[9]	0.24977[8]	0.26386[12]	0.24527[6]	0.23494[4]	0.2403[5]	0.28353[14]	0.08689[1]	0.29782[15]	0.25663[11]	0.2816[13]
	D_{abs}	0.03017[2]	0.03323[5]	0.03497[8]	0.03418[6]	0.03258[4]	0.0371[12]	0.03146[3]	0.03485[7]	0.04133[15]	0.03577[10]	0.03654[11]	0.02982[1]	0.03833[13]	0.03528[9]	0.04119[14]
	D_{max}	0.04933[2]	0.05467[5]	0.05898[10]	0.05531[6]	0.05425[4]	0.06273[12]	0.05298[3]	0.05723[8]	0.06547[14]	0.05704[7]	0.06121[11]	0.04482[1]	0.06324[13]	0.05791[9]	0.06811[15]
	$\sum Ranks$	16[2]	38[3]	79[11]	50[5]	60[8]	88[12]	72[10]	56[7]	55[6]	47[4]	107[14]	8[1]	109[15]	71[9]	104[13]
90	BIAS($\hat{\delta}$)	0.2586[2]	0.34971[5]	0.41533[11]	0.34423[4]	0.39196[9]	0.4362[13]	0.39422[10]	0.36305[6]	0.32324[3]	0.36842[7]	0.4311[12]	0.05619[1]	0.45317[15]	0.38635[8]	0.45084[14]
	BIAS($\hat{\theta}$)	0.14839[2]	0.18948[5]	0.19099[6]	0.19202[7]	0.1964[9]	0.20211[10]	0.2054[11]	0.18765[4]	0.1764[3]	0.19629[8]	0.2344[13]	0.06857[1]	0.24331[15]	0.2173[12]	0.23718[14]
	MSE($\hat{\delta}$)	0.10851[2]	0.19241[4]	0.28225[12]	0.18756[3]	0.24813[10]	0.32839[15]	0.23456[9]	0.20742[6]	0.19557[5]	0.21118[7]	0.26703[11]	0.01398[1]	0.29554[13]	0.22336[8]	0.30698[14]
	MSE($\hat{\theta}$)	0.04148[2]	0.05799[5]	0.05822[6]	0.0627[7]	0.06499[9]	0.07157[11]	0.06709[10]	0.05791[4]	0.05775[3]	0.06369[8]	0.08632[13]	0.01003[1]	0.10221[15]	0.07737[12]	0.08923[14]
	MRE($\hat{\delta}$)	0.0862[2]	0.11657[5]	0.13844[11]	0.11474[4]	0.13065[9]	0.1454[13]	0.13141[10]	0.12102[6]	0.10775[3]	0.12281[7]	0.1437[12]	0.01873[1]	0.15106[15]	0.12878[8]	0.15028[14]
	MRE($\hat{\theta}$)	0.16488[2]	0.21053[5]	0.21221[6]	0.21335[7]	0.21823[9]	0.22457[10]	0.22822[11]	0.2085[4]	0.196[3]	0.2181[8]	0.26045[13]	0.07619[1]	0.27034[15]	0.24144[12]	0.26354[14]
	D_{abs}	0.02451[1]	0.02731[3]	0.02866[7]	0.02842[6]	0.02914[9]	0.02901[8]	0.02797[5]	0.02738[4]	0.02996[10]	0.03052[12]	0.03019[11]	0.02546[2]	0.03352[14]	0.03142[13]	0.03472[15]
	D_{max}	0.03985[2]	0.04486[3]	0.04833[7]	0.046[5]	0.04842[8]	0.04951[10]	0.04612[6]	0.04517[4]	0.04871[9]	0.0497[11]	0.04973[12]	0.03821[1]	0.05633[14]	0.05065[13]	0.05717[15]
	$\sum Ranks$	15[2]	35[3]	66[7]	43[6]	72[9.5]	90[12]	72[9.5]	38[4]	39[5]	68[8]	97[13]	9[1]	116[15]	86[11]	114[14]
140	BIAS($\hat{\delta}$)	0.21756[2]	0.29471[6]	0.34466[11]	0.29024[5]	0.31113[10]	0.38601[15]	0.30865[9]	0.29847[7]	0.27906[3]	0.30217[8]	0.36243[12]	0.05438[1]	0.38015[13]	0.28855[4]	0.38372[14]
	BIAS($\hat{\theta}$)	0.1193[2]	0.14812[3]	0.16994[11]	0.15553[7]	0.15671[8]	0.18608[12]	0.15498[6]	0.15698[9]	0.15155[4]	0.15789[10]	0.20273[13]	0.05608[1]	0.20738[14]	0.15351[5]	0.21413[15]
	MSE($\hat{\delta}$)	0.08238[2]	0.13232[5]	0.1946[11]	0.12705[3]	0.15871[10]	0.24695[15]	0.1516[9]	0.14712[8]	0.14465[6]	0.14516[7]	0.21942[12]	0.01257[1]	0.23113[14]	0.12981[4]	0.21952[13]
	MSE($\hat{\theta}$)	0.02675[2]	0.03806[3]	0.04597[11]	0.0392[4]	0.04145[6]	0.05899[12]	0.04091[5]	0.0443[10]	0.04166[7]	0.04197[8]	0.07449[13]	0.00694[1]	0.07788[14]	0.04232[9]	0.07874[15]
	MRE($\hat{\delta}$)	0.07252[2]	0.09824[6]	0.11489[11]	0.09675[5]	0.10371[10]	0.12867[15]	0.10288[9]	0.09949[7]	0.09302[3]	0.10072[8]	0.12081[12]	0.01813[1]	0.12672[13]	0.09618[4]	0.12791[14]
	MRE($\hat{\theta}$)	0.13255[2]	0.16458[3]	0.18882[11]	0.17281[7]	0.17412[8]	0.20675[12]	0.1722[6]	0.17443[9]	0.16839[4]	0.17543[10]	0.22526[13]	0.06231[1]	0.23042[14]	0.17056[5]	0.23792[15]
	D_{abs}	0.02135[2]	0.02166[4.5]	0.02267[7]	0.02143[3]	0.02166[4.5]	0.02369[9]	0.02219[6]	0.02285[8]	0.02637[13]	0.02409[10]	0.02629[12]	0.02087[1]	0.03015[15]	0.02426[11]	0.02924[14]
	D_{max}	0.03464[2]	0.03616[4]	0.03832[8]	0.03539[3]	0.0363[5]	0.04024[11]	0.03704[6]	0.0376[7]	0.04234[12]	0.03973[10]	0.04313[13]	0.03162[1]	0.04919[15]	0.03942[9]	0.04835[14]
	$\sum Ranks$	16[2]	34.5[3]	81[11]	37[4]	61.5[8]	101[13]	56[7]	65[9]	52[6]	71[10]	100[12]	8[1]	112[14]	51[5]	114[15]
200	BIAS($\hat{\delta}$)	0.19034[2]	0.26558[10]	0.25892[8]	0.22652[3]	0.25837[7]	0.32545[15]	0.24201[4]	0.24431[5]	0.25042[6]	0.28305[11]	0.32262[14]	0.05384[1]	0.30376[12]	0.25967[9]	0.32053[13]
	BIAS($\hat{\theta}$)	0.10192[2]	0.1303[8]	0.12644[7]	0.11671[3]	0.12192[5]	0.16235[12]	0.12004[4]	0.12376[6]	0.132[9]	0.15119[11]	0.17428[15]	0.04749[1]	0.16404[13]	0.13636[10]	0.17173[14]
	MSE($\hat{\delta}$)	0.06144[2]	0.11113[9]	0.10854[8]	0.07628[3]	0.10145[6]	0.16824[15]	0.08632[4]	0.09356[5]	0.11407[10]	0.12264[11]	0.15324[13]	0.01228[1]	0.14553[12]	0.10538[7]	0.15743[14]
	MSE($\hat{\theta}$)	0.01837[2]	0.02652[8]	0.02582[7]	0.02027[3]	0.02431[5]	0.04677[13]	0.02237[4]	0.02512[6]	0.03162[9]	0.03909[11]	0.05[15]	0.00499[1]	0.04469[12]	0.03203[10]	0.04988[14]
	MRE($\hat{\delta}$)	0.06345[2]	0.08853[10]	0.08631[8]	0.07551[3]	0.08612[7]	0.10848[15]	0.08067[4]	0.08144[5]	0.08347[6]	0.09435[11]	0.10754[14]	0.01795[1]	0.10125[12]	0.08656[9]	0.10684[13]
	MRE($\hat{\theta}$)	0.11324[2]	0.14478[8]	0.14049[7]	0.12968[3]	0.13547[5]	0.18039[12]	0.13338[4]	0.13751[6]	0.14667[9]	0.16799[11]	0.19365[15]	0.05276[1]	0.18227[13]	0.15151[10]	0.19082[14]
	D_{abs}	0.01652[1]	0.01933[8]	0.01883[7]	0.01821[6]	0.01817[5]	0.01988[10]	0.01816[4]	0.01798[3]	0.02335[14]	0.02128[12]	0.02085[11]	0.01665[2]	0.02353[15]	0.01978[9]	0.0223[13]
	D_{max}	0.02696[2]	0.03185[8]	0.03138[7]	0.02991[4]	0.03062[6]	0.03374[10]	0.03009[5]	0.02988[3]	0.03788[14]	0.03509[12]	0.03482[11]	0.02514[1]	0.03865[15]	0.03273[9]	0.03741[13]
	$\sum Ranks$	15[2]	69[8]	59[7]	28[3]	46[6]	102[12]	33[4]	39[5]	77[10]	90[11]	108[14.5]	9[1]	104[13]	73[9]	108[14.5]
250	BIAS($\hat{\delta}$)	0.17213[2]	0.22816[7]	0.24222[9]	0.20831[3]	0.2445[10]	0.28516[15]	0.21354[4]	0.21385[5]	0.23047[8]	0.25522[11]	0.28495[14]	0.0499[1]	0.27347[12]	0.22441[6]	0.27481[13]
	BIAS($\hat{\theta}$)	0.09425[2]	0.10879[5]	0.12113[9]	0.11081[6]	0.11644[7]	0.13232[11]	0.10521[3]	0.10839[4]	0.11926[8]	0.13396[12]	0.15787[15]	0.04441[1]	0.14083[13]	0.12192[10]	0.14858[14]
	MSE($\hat{\delta}$)	0.05128[2]	0.0792[6]	0.09519[10]	0.06755[3]	0.09873[11]	0.13132[15]	0.07387[5]	0.07111[4]	0.09179[8]	0.09516[9]	0.12218[14]	0.00983[1]	0.10994[12]	0.08251[7]	0.11778[13]
	MSE($\hat{\theta}$)	0.01495[2]	0.0185[3]	0.02459[8]	0.01921[6]	0.02316[7]	0.03049[12]	0.01861[4]	0.01901[5]	0.02466[9]	0.02752[11]	0.03966[15]	0.00408[1]	0.03265[13]	0.02668[10]	0.03947[14]
	MRE($\hat{\delta}$)	0.05738[2]	0.07605[7]	0.08074[9]	0.06944[3]	0.0815[10]	0.09505[15]	0.07118[4]	0.07128[5]	0.07682[8]	0.08507[11]	0.09498[14]	0.01663[1]	0.09116[12]	0.0748[6]	0.0916[13]
	MRE($\hat{\theta}$)	0.10472[2]	0.12087[5]	0.13459[9]	0.12312[6]	0.12938[7]	0.14702[11]	0.1169[3]	0.12044[4]	0.13251[8]	0.14885[12]	0.17541[15]	0.04935[1]	0.15647[13]	0.13547[10]	0.16509[14]
	D_{abs}	0.01539[3]	0.015[2]	0.01777[8]	0.01605[5]	0.01623[7]	0.01778[9]	0.01579[4]	0.01619[6]	0.01867[11]	0.0188[12]	0.02057[13]	0.01476[1]	0.02161[15]	0.01844[10]	0.02134[14]
	D_{max}	0.02478[2]	0.02532[3]	0.02953[8]	0.02646[5]	0.02732[7]	0.03044[10]	0.02626[4]	0.02683[6]	0.03066[11]	0.03099[12]	0.03363[13]	0.02235[1]	0.03545[15]	0.03006[9]	0.03541[14]
	$\sum Ranks$	17[2]	38[5]	70[9]	37[4]	66[7]	98[12]	31[3]	39[6]	71[10]	90[11]	113[15]	8[1]	105[13]	68[8]	109[14]

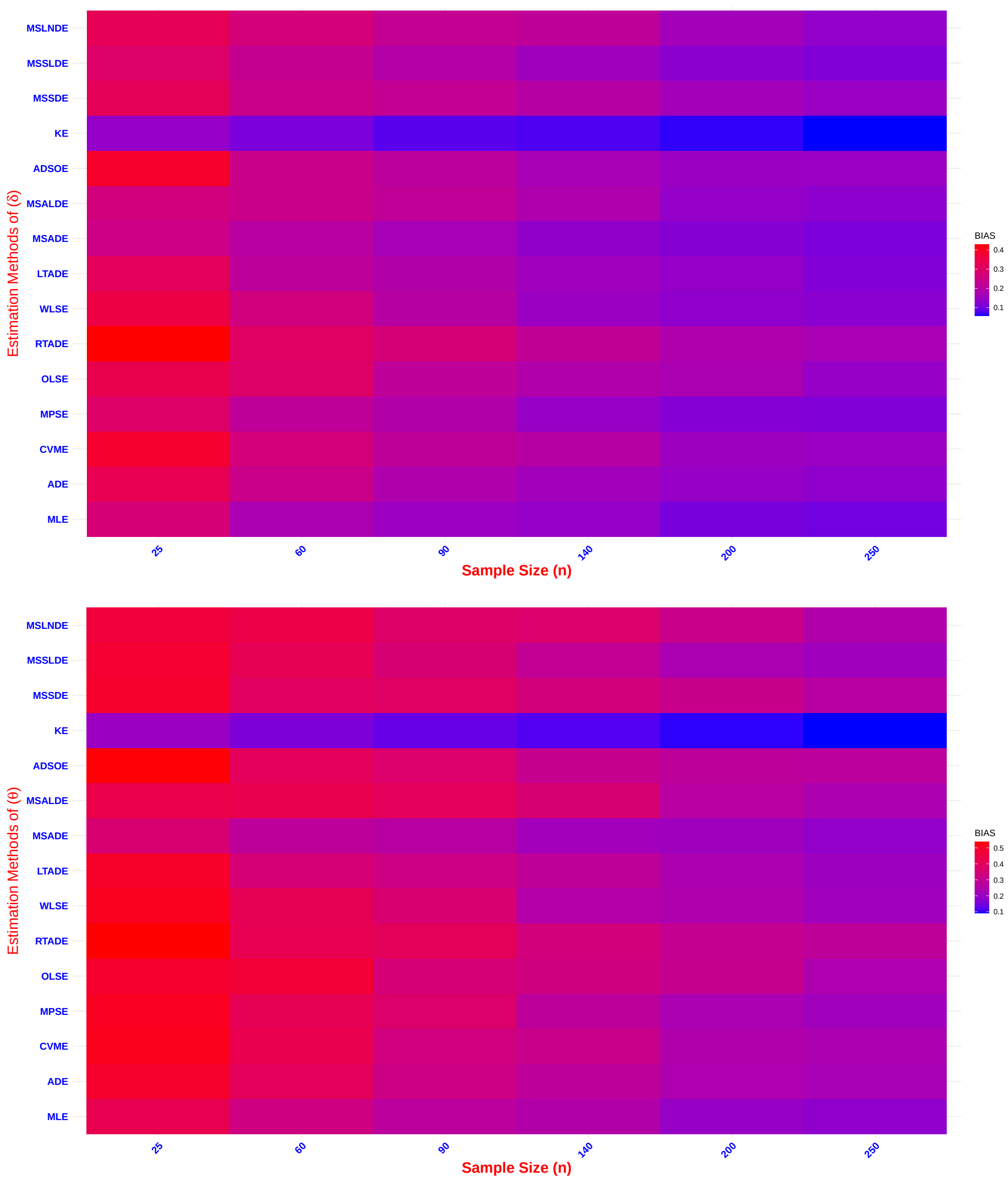


Figure 5. Graphical representation for BIAS values presented in Table 3.

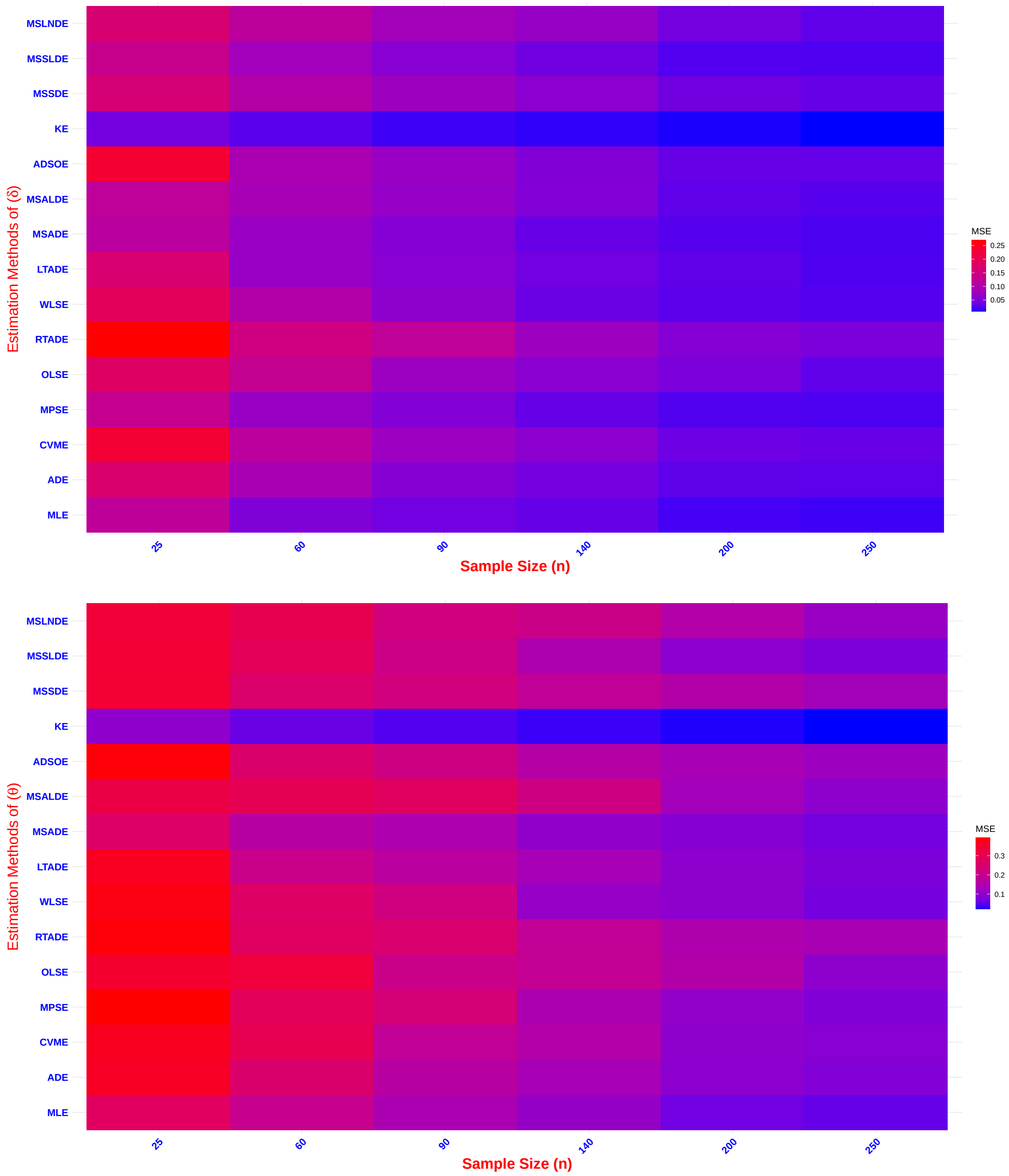


Figure 6. Graphical representation for MSE values presented in Table 3.

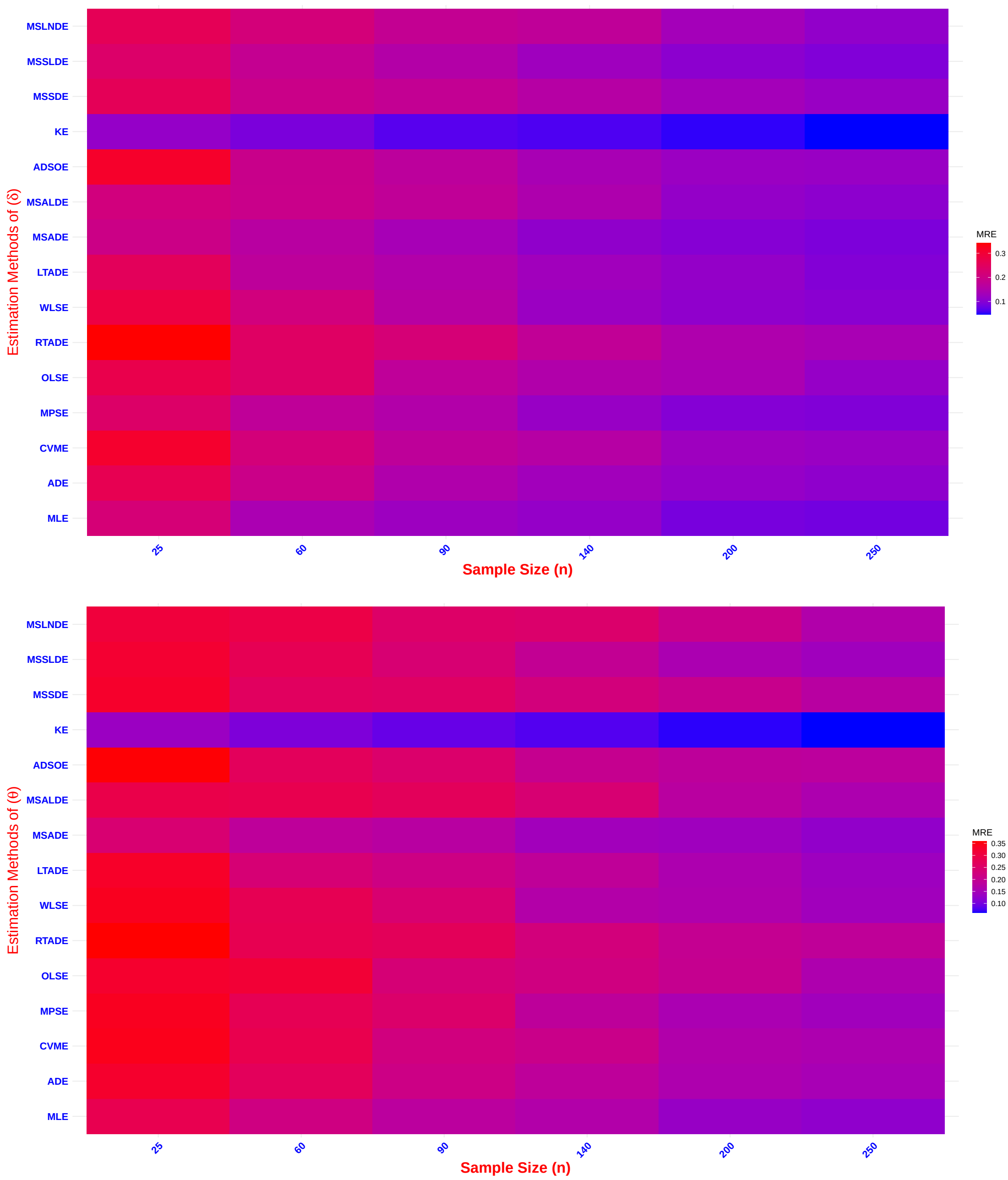


Figure 7. Graphical representation for MRE values presented in Table 3.

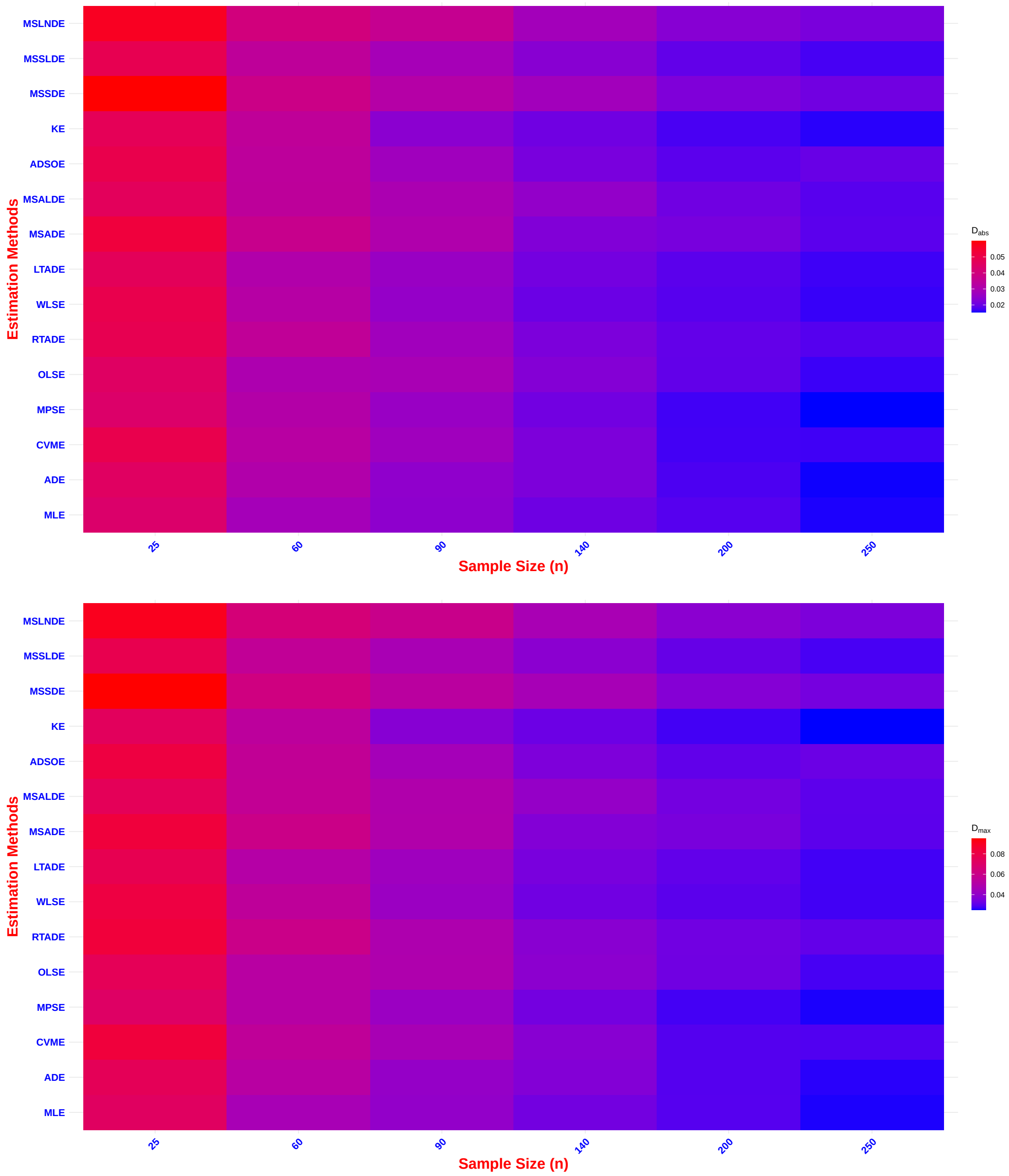


Figure 8. Graphical representation for D_{abs} and D_{max} values presented in Table 3.

Table 8. Partial and overall ranks for all estimation methods of our proposed model.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
$\delta = 1.25,\ \theta = 1.5$	25	2.0	7.5	13.0	6.0	7.5	15.0	12.0	9.0	4.0	3.0	14.0	1.0	11.0	5.0	10.0
	60	1.0	6.0	12.0	5.0	13.0	14.0	9.0	3.0	4.0	11.0	7.0	2.0	10.0	8.0	15.0
	90	2.0	3.0	9.0	6.0	11.0	14.5	7.0	4.5	4.5	12.0	10.0	1.0	13.0	8.0	14.5
	140	2.0	7.0	10.0	5.0	11.0	12.0	3.0	6.0	4.0	13.0	9.0	1.0	14.0	8.0	15.0
	200	2.0	6.0	9.0	3.0	12.0	13.0	7.0	8.0	5.0	11.0	10.0	1.0	14.0	4.0	15.0
	250	2.0	8.0	11.0	3.0	10.0	15.0	7.0	5.0	4.0	9.0	13.5	1.0	13.5	6.0	12.0
$\delta = 2.5,\ \theta = 0.6$	25	2.0	5.5	11.0	7.0	4.0	15.0	5.5	10.0	3.0	9.0	13.0	1.0	14.0	8.0	12.0
	60	2.0	3.5	10.0	8.0	6.0	13.0	9.0	3.5	5.0	11.0	14.0	1.0	12.0	7.0	15.0
	90	2.0	3.0	11.5	4.0	8.0	11.5	5.0	10.0	7.0	6.0	15.0	1.0	13.0	9.0	14.0
	140	2.0	4.0	10.0	3.0	8.0	11.0	6.0	5.0	9.0	7.0	13.0	1.0	15.0	12.0	14.0
	200	2.0	3.0	6.0	4.0	8.0	10.0	5.0	7.0	9.0	11.0	14.0	1.0	13.0	12.0	15.0
	250	2.0	3.0	7.0	6.0	5.0	11.0	4.0	8.0	9.0	10.0	15.0	1.0	13.5	12.0	13.5
$\delta = 0.7,\ \theta = 0.2$	25	2.0	7.0	8.5	8.5	12.0	11.0	4.5	10.0	3.0	4.5	14.0	1.0	13.0	6.0	15.0
	60	2.0	3.0	11.0	5.0	4.0	12.0	10.0	7.0	9.0	6.0	13.0	1.0	15.0	8.0	14.0
	90	2.0	4.0	7.0	3.0	8.0	12.0	6.0	5.0	11.0	9.0	14.0	1.0	15.0	10.0	13.0
	140	2.0	5.0	10.0	3.0	4.0	11.0	8.0	6.0	12.0	9.0	14.5	1.0	14.5	7.0	13.0
	200	2.0	6.0	4.5	4.5	7.0	11.0	3.0	9.0	10.0	12.0	15.0	1.0	14.0	8.0	13.0
	250	1.0	5.0	7.0	6.0	8.0	10.0	3.0	4.0	12.0	9.0	15.0	2.0	14.0	11.0	13.0
$\delta = 0.3,\ \theta = 1.75$	25	1.5	7.5	13.0	5.0	9.0	15.0	6.0	7.5	4.0	3.0	14.0	1.5	10.0	12.0	11.0
	60	2.0	6.0	12.0	4.0	11.0	14.0	13.0	10.0	3.0	5.0	15.0	1.0	9.0	7.0	8.0
	90	2.5	7.0	13.5	6.0	15.0	13.5	12.0	4.0	2.5	5.0	11.0	1.0	8.5	8.5	10.0
	140	2.0	6.5	11.0	8.0	13.0	15.0	10.0	9.0	3.0	5.0	14.0	1.0	6.5	12.0	4.0
	200	2.0	6.0	13.0	5.0	14.0	15.0	9.0	10.0	3.0	11.0	12.0	1.0	8.0	7.0	4.0
	250	2.5	11.0	12.0	5.0	14.0	15.0	9.0	7.0	2.5	8.0	13.0	1.0	6.0	10.0	4.0
$\delta = 3.0,\ \theta = 0.9$	25	2.0	7.0	15.0	5.5	8.0	10.0	9.0	5.5	3.0	11.5	14.0	1.0	13.0	4.0	11.5
	60	2.0	3.0	11.0	5.0	8.0	12.0	10.0	7.0	6.0	4.0	14.0	1.0	15.0	9.0	13.0
	90	2.0	3.0	7.0	6.0	9.5	12.0	9.5	4.0	5.0	8.0	13.0	1.0	15.0	11.0	14.0
	140	2.0	3.0	11.0	4.0	8.0	13.0	7.0	9.0	6.0	10.0	12.0	1.0	14.0	5.0	15.0
	200	2.0	8.0	7.0	3.0	6.0	12.0	4.0	5.0	10.0	11.0	14.5	1.0	13.0	9.0	14.5
	250	2.0	5.0	9.0	4.0	7.0	12.0	3.0	6.0	10.0	11.0	15.0	1.0	13.0	8.0	14.0
$\sum$ Ranks		58.5	162.5	302.0	150.5	269.0	380.5	215.5	204.0	182.5	255.0	394.5	32.5	372.5	251.5	369.0
Overall Rank		2	4	11	3	10	14	7	6	5	9	15	1	13	8	12

Table 9. Some descriptive analysis of all data sets.

	n	Mean	Median	Variance	Skewness	Kurtosis	Range	Min	Max
Dataset I	45	0.0998	0.0572	0.0140	2.5841	9.3441	0.5343	0.0077	0.5420
Dataset II	20	0.1215	0.0980	0.0079	3.5855	15.2033	0.418	0.0670	0.4850

The histograms and kernel density plots (Figures 9 and 10) visually confirm this skewness, with the majority of observations concentrated at lower values and a long right tail. The TTT (total time on test) plots indicate an increasing–decreasing–increasing hazard rate function. This non-monotonic pattern suggests that the instantaneous risk of failure initially rises, then declines, and eventually increases again over time, which is consistent with the hazard rate behavior of the proposed model. Box plots further highlight the presence of outliers in the upper range of the data, consistent with the observed skewness and heavy tails.

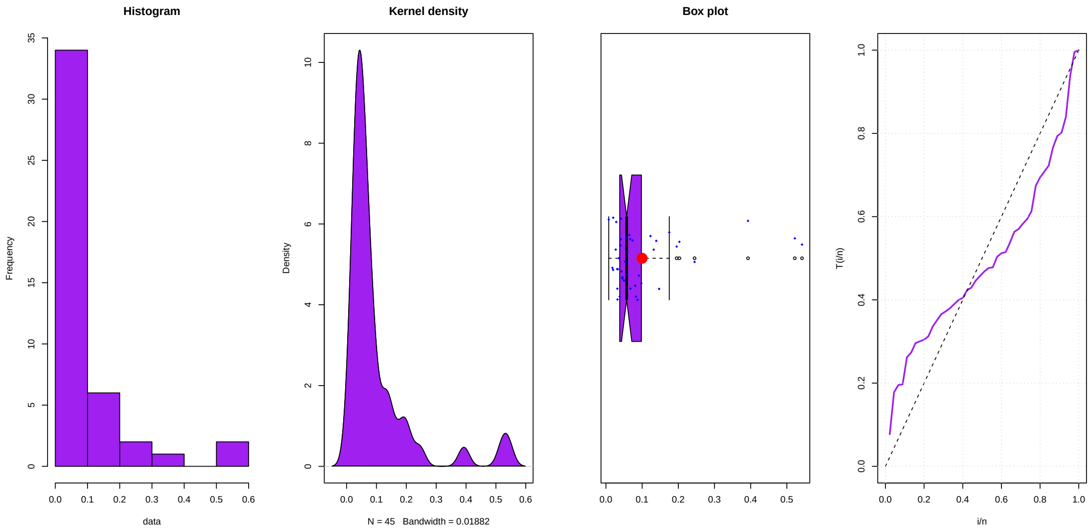


Figure 9. Some basic non-parametric plots for the first dataset.

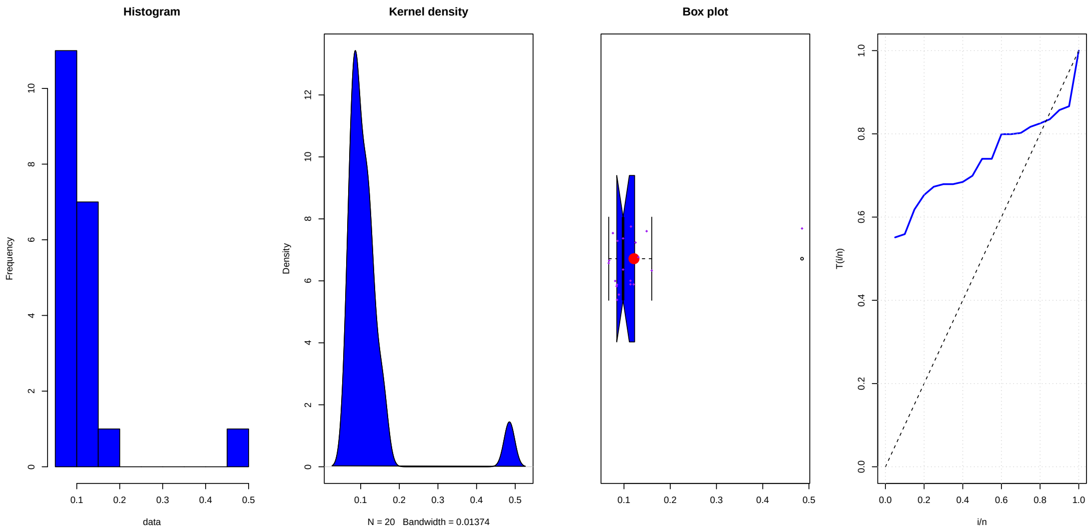


Figure 10. Some basic non-parametric plots for the second dataset.

The goodness-of-fit of the EPUILD is assessed using both datasets. The proposed model is compared with the PUILD, unit-Weibull distribution (UWEBD) [41], unit Mirra distribution (UMIRD) [42], unit-Birnbaum–Saunders distribution (UBSD) [43], Kumaraswamy distribution (KUMD) [44], Beta distribution (BETD) [45], and unit inverse Lindley distribution (UILD) [17]. The PDFs of these competing models are provided in Table 10.

Table 10. PDF of competing models.

Competing Models	PDF
PUILD	$\frac{\delta\beta^2 e^\beta}{1+\beta} x^{-2\delta-1} e^{-\frac{\beta}{x^\delta}}; x \in (0,1), \delta, \beta > 0$
UWEB	$\frac{1}{x}\alpha\beta(-\log x)^{\beta-1}\exp\left[-\alpha(-\log x)^\beta\right]; x \in (0,1), \alpha, \beta > 0$
UMIRD	$\frac{\theta^3\left((\alpha+2)x^2+\alpha-2\alpha x\right)e^{-\frac{\theta(1-x)}{x}}}{2x^4\left(\alpha+\theta^2\right)}; x \in (0,1), \alpha, \theta > 0$
UBSD	$\frac{1}{2x\alpha\beta\sqrt{2\pi}}\left[\left(-\frac{\beta}{\log x}\right)^{1/2}+\left(-\frac{\beta}{\log x}\right)^{3/2}\right]\exp\left[\frac{1}{2\alpha^2}\left(\frac{\log x}{\beta}+\frac{\beta}{\log x}+2\right)\right]; x \in (0,1), \alpha, \beta > 0$
KUMD	$\alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1}; x \in (0,1), \alpha, \beta > 0$
BETD	$\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}; x \in (0,1), \alpha, \beta > 0$
UILD	$\frac{\beta^2 e^\beta}{1+\beta} x^{-3} e^{-\frac{\beta}{x}}; x \in (0,1), \beta > 0$

To evaluate model performance, ten widely used goodness-of-fit (GoF) criteria are considered: the negative log-likelihood (NLL), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Hannan–Quinn information criterion (HQIC), Kolmogorov–Smirnov (KS) statistic and its p-value, Anderson–Darling (AD) statistic and its p-value, and Cramér–von Mises (CVM) statistic and its p-value. The best-fitting model is identified by the highest p-values and the lowest values for NLL, AIC, CAIC, HQIC, KS, AD, and CVM.

The maximum likelihood estimates (MLEs) and standard errors (SEs) for each model applied to the first dataset are presented in Table 11, while the corresponding goodness-of-fit (GoF) measures are reported in Table 12. The results clearly indicate that the EPUIL model outperforms all competing models. Specifically, it achieves the lowest negative log-likelihood (NLL = –66.1691) and the lowest values for AIC, CAIC, HQIC, KS, AD, and CVM, reflecting a superior fit. Moreover, the EPUIL model yields the highest p-values for the KS, AD, and CVM statistics, further confirming its statistical adequacy.

The estimated probability density functions (PDFs) of the fitted models are illustrated in Figure 11, while their cumulative distribution functions (CDFs) are displayed in Figure 12. From these visualizations, it is evident that the EPUIL model closely follows the empirical data distribution, providing the best visual fit among the models. The survival curves shown in Figure 13 further support this conclusion, as the EPUILD curve aligns more closely with the empirical survival pattern.

Table 11. Estimates values and SEs for all models applied to the first dataset

Models	Estimates	SE	Estimates	SE
EPUILD ($\hat{\delta}$, $\hat{\theta}$)	0.7371	0.0955	0.2677	0.0905
PUILD ($\hat{\delta}$, $\hat{\beta}$)	0.8321	0.0993	0.1632	0.0572
UWEB ($\hat{\alpha}$, $\hat{\beta}$)	0.0196	0.0105	3.5448	0.4239
UMIRD ($\hat{\alpha}$, $\hat{\theta}$)	0.0201	0.0202	0.1088	0.0184
UBSD ($\hat{\alpha}$, $\hat{\beta}$)	0.4471	0.0471	2.4815	0.1612
KUMD ($\hat{\alpha}$, $\hat{\beta}$)	0.9771	0.1189	8.2094	2.2620
BETD ($\hat{\alpha}$, $\hat{\beta}$)	1.0848	0.1608	9.2665	1.3954
UILD ($\hat{\beta}$)	0.0923	0.0097		

Table 12. Fitting measures values for the first data set.

Models	NLL	AIC	CAIC	HQIC	KS	KS-Pval	AD	AD-Pval	CVM	CVM-Pval
EPULILD	−66.1691	−128.3385	−128.0527	−126.9914	0.0722	0.9597	0.2844	0.9492	0.0395	0.9372
PULILD	−65.8962	−127.7925	−127.5068	−126.4455	0.0775	0.9303	0.3322	0.9117	0.0474	0.8938
UWEBD	−65.0203	−126.0406	−127.9476	−127.3671	0.1047	0.6675	0.6231	0.6255	0.0943	0.6156
UMIRD	−64.1187	−124.2375	−126.1445	−125.5640	0.0923	0.8039	0.4885	0.7577	0.0645	0.7881
UBSD	−54.4008	−104.8018	−106.7088	−106.1283	0.2023	0.0430	2.8867	0.0314	0.5212	0.0347
KUMD	−56.9868	−109.9738		−111.3003	0.1695	0.1338	2.2085	0.0710	0.3793	0.0816
BETD	−57.0611	−110.1222	−112.0292	−111.4487	0.1833	0.0850	2.3510	0.0596	0.4256	0.0614
UILD	−64.5024	−127.0049	−126.9119	−126.3314	0.0741	0.9499	0.5061	0.7397	0.0497	0.8801

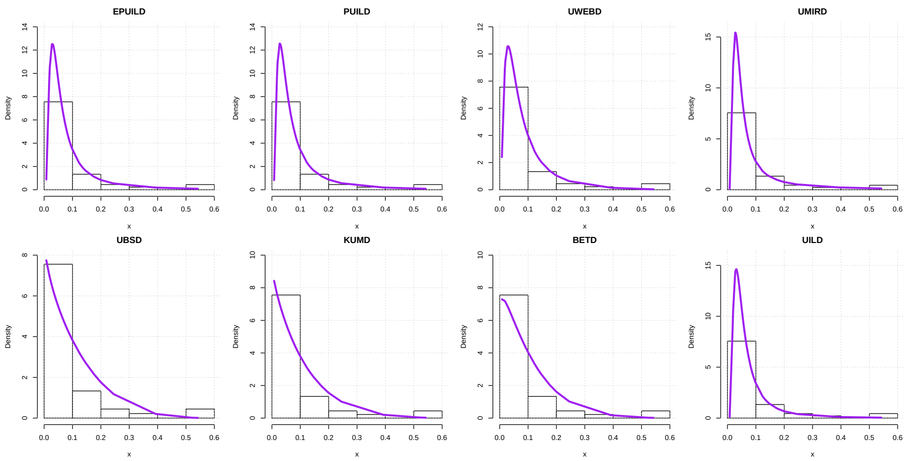


Figure 11. Estimated PDFs for the competing models for the first dataset.

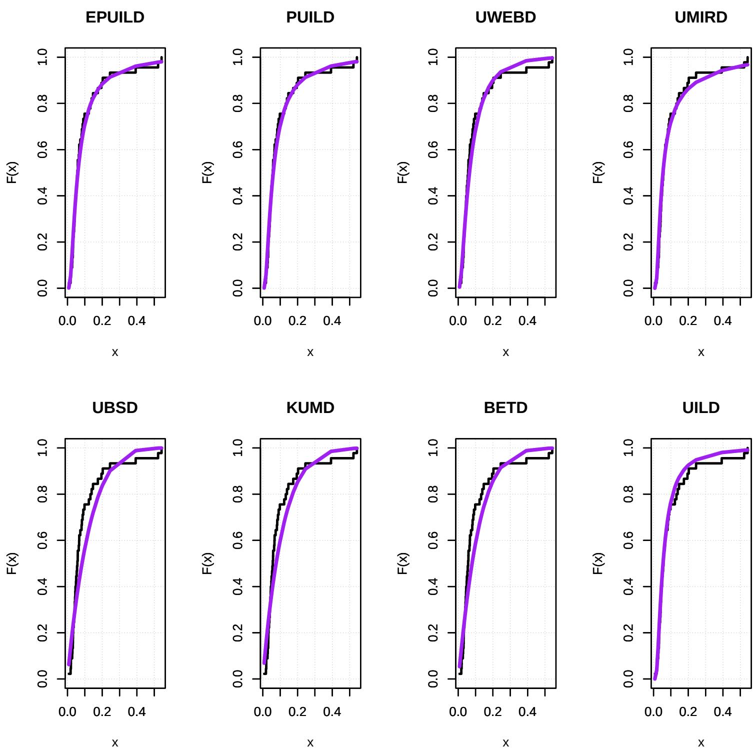


Figure 12. Estimated CDFs for the competing models for the first dataset.

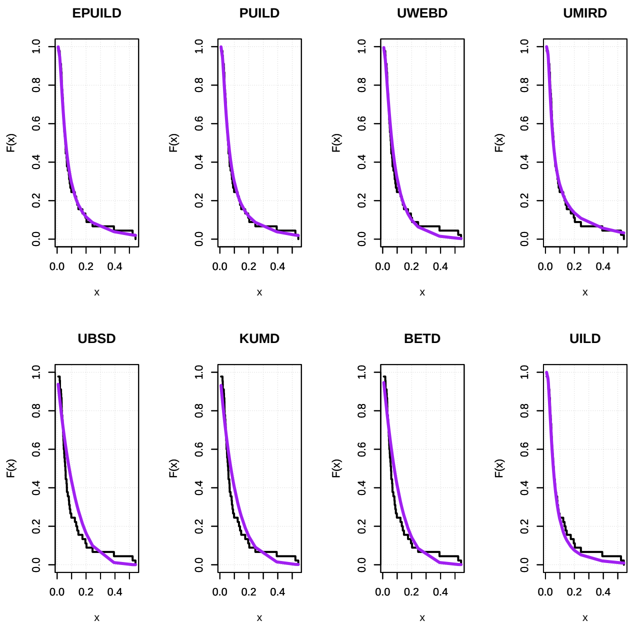


Figure 13. Survival function plots of competing models for the first dataset

Similarly, Tables 13 and 14 report the MLEs, SEs, and GoF measures for the second dataset. Again, the EPUILD exhibits the best performance with the lowest negative log-likelihood (NLL = −38.1449) and the lowest AIC, CAIC, HQIC, and GoF statistics. Its associated p-values are also the highest, underscoring its superior fit relative to the other models.

Table 13. Estimates values and SEs for all models applied to the second dataset.

Models	Estimates	SE	Estimates	SE
EPUILD $(\hat{\delta}, \hat{\theta})$	2.1767	0.3966	0.0143	0.0139
PUILD $(\hat{\delta}, \hat{\beta})$	2.4144	0.4349	0.0067	0.0072
UWEB $(\hat{\alpha}, \hat{\beta})$	0.0012	0.0017	7.7284	1.4898
UMIRD $(\hat{\alpha}, \hat{\theta})$	34.7770	11.8633	0.3344	0.0431
UBSD $(\hat{\alpha}, \hat{\beta})$	0.2840	0.0449	2.1426	0.1347
KUMD $(\hat{\alpha}, \hat{\beta})$	1.5877	0.2359	21.8673	9.7601
BETD $(\hat{\alpha}, \hat{\beta})$	3.1125	0.5472	21.8239	2.6268
UILD $(\hat{\beta})$	0.2046	0.0326		

Table 14. Fitting measures values for the second data set.

Models	NLL	AIC	CAIC	HQIC	KS	KS-Pval	AD	AD-Pval	CVM	CVM-Pval
EPULILD	−38.1449	−72.2899	−71.5840	−71.9011	0.1233	0.9214	0.3718	0.8748	0.0458	0.9061
PULILD		−71.5425	−70.8367	−71.1538	0.1258	0.9093	0.4161	0.8308	0.0499	0.8817
UWEBD	−36.0732	−68.1464	−69.9242	−69.9520	0.1381	0.8399	0.6477	0.6020	0.0752	0.7253
UMIRD	−33.4930	−62.9859	−64.7637	−64.7916	0.2339	0.2236	1.6535	0.1441	0.2971	0.1374
UBSD	−26.1026	−48.2052	−49.9830	−50.0109	0.2763	0.0943	2.6274	0.0430	0.4486	0.0529
KUMD	−25.6484	−47.2968	−49.0746	−49.1024	0.2626	0.1265	2.6890	0.0401	0.4681	0.0469
BETD	−27.8813	−51.7626	−53.5403	−53.5682	0.2537	0.1521	2.2610	0.0669	0.3726	0.0845
UILD	−29.9756	−57.9513	−57.7291	−57.7569	0.3040	0.0496	2.7024	0.0394	0.5323	0.0319

Figures 14 and 15 visually confirm the strong performance of the EPUILD, as its PDF and CDF match the empirical data more closely than the alternatives. The survival function plots in Figure 16 further validate these findings. Overall, both the numerical results and graphical assessments demonstrate that the proposed EPUIL model consistently provides a better fit to the datasets than all competing models.

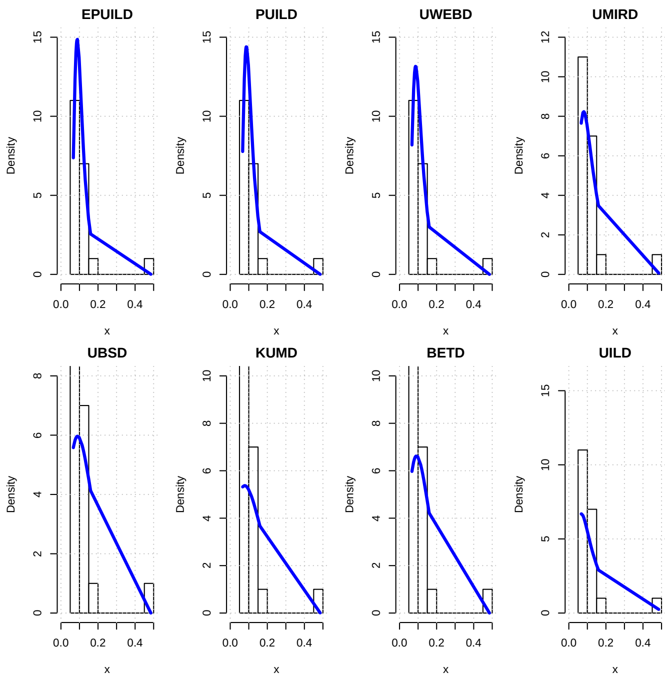


Figure 14. Estimated PDFs for the competing models for the second dataset.

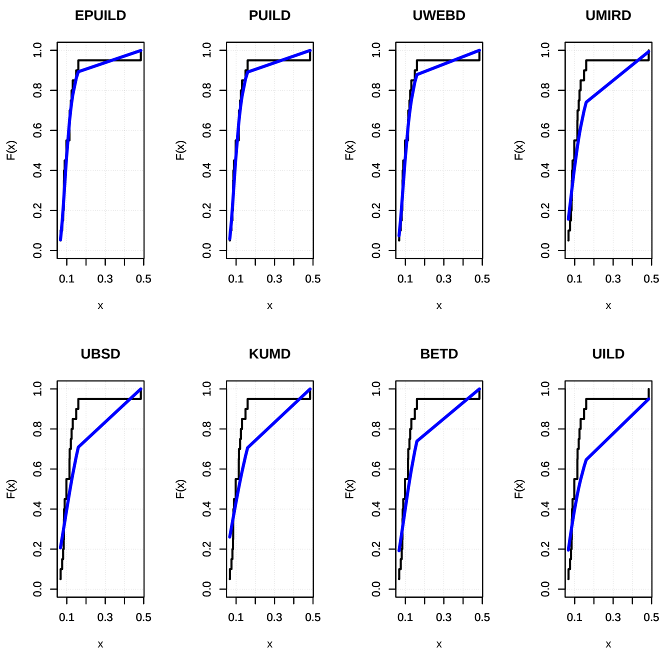


Figure 15. Estimated CDFs for the competing models for the second dataset.

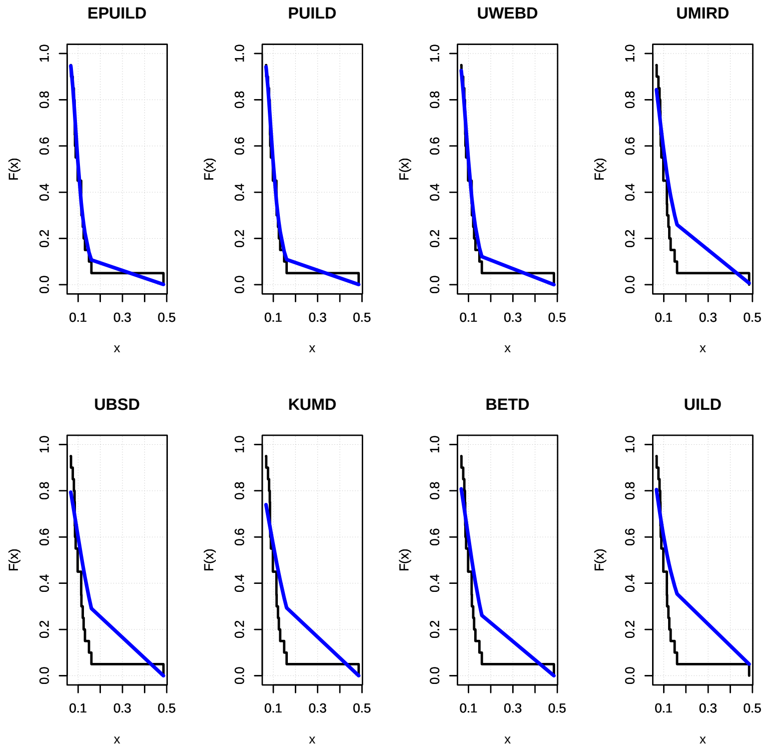


Figure 16. Survival function plots of competing models for the second dataset.

9. Conclusions and future work

We present and thoroughly examine the EPUILD, a novel and adaptable statistical model for unit interval data. Our thorough analytical and empirical investigation proved that the EPUILD has many useful structural features and is quite flexible in terms of modeling. We checked the proposed distribution's density shape, cumulative function, moments, skewness, kurtosis, and quantile function. Our focus was on the information-theoretic aspects of EPUILD, where we examined important metrics like extropy, negative cumulative extropy, and the FIM. These investigations not only revealed the internal coherence of the model but also highlighted its capability to capture a wide spectrum of data behaviors, from symmetry to heavy skewness and peakedness. The comparative analysis of fifteen contemporary estimating methods is a substantial contribution to our work. We validated and assessed the estimation performance using a comprehensive numerical simulation analysis, looking at BIAS, MRE, and MSE. By validating the parameter estimates' correctness and dependability across many scenarios, the simulation results demonstrated that the EPUILD is a practical and viable option for statistical modeling. To further substantiate the proposed model's practical utility, we applied the EPUILD to two real-world datasets, comparing its goodness-of-fit against several well-known unit distributions. In both cases, the EPUILD provided a superior or highly competitive fit, as measured by standard selection criteria (AIC, BIC, etc.), thereby confirming the model's adaptability and efficiency in capturing the underlying structure of real data. Subsequent investigations may advance this study in multiple avenues. The suggested EPUILD distribution can be generalized to encompass multivariate and regression frameworks, facilitating the modeling of interdependence among variables. Secondly, Bayesian estimating methods and their corresponding credible intervals may be investigated to enhance the frequentist techniques utilized in this context. Third, other theoretical advancements, such as stochastic ordering, record statistics, and reliability-based measures, may improve the mathematical foundation of the distribution.

Author contributions

Ahmed M. Gemeay: Writing-original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing-review, Investigation; I. Elbatal: Writing-original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing-review, Investigation; Ehab M. Almetwally: Writing-original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing-review, Investigation; Sule Omeiza Bashiru: Writing-original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing-review, Investigation; I. A. Husseiny: Writing-original draft, Validation, Methodology, Formal analysis, Conceptualization, Writing-review, Investigation.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability

The corresponding author can provide the datasets created and/or studied during the current work upon reasonable request.

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Conflict of interest

The authors declare they have no conflict of interest in this paper.

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