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Research article

Fuzziness and randomness in fixed point theory: Measurable selections and applications

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Abstract: This paper presents a novel fixed point theorem for measurable selections arising from pairs of fuzzy set-valued operators defined on Polish spaces. By establishing conditions under which such selections exist, we provided a rigorous framework for analyzing the solvability of random multivalued operator equations in fuzzy environments. Our approach seamlessly integrated fuzziness and randomness, extending classical fixed point theory into a more realistic setting where uncertainty is both probabilistic and vague. To demonstrate the utility and applicability of our results, we constructed well-structured and insightful examples rooted in engineering-inspired scenarios. These examples not only validate the theoretical framework but also highlight its effectiveness in modeling complex systems affected by dual sources of uncertainty.

Keywords: fixed point; coincidence point; random operators; contractive-type mapping; fuzzy mapping
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1. Introduction

Fixed point theorems in fuzzy set theory and random operator theory have been extensively studied over the past few decades due to their significant applications in various branches of mathematics and applied sciences. The concept of fuzzy sets, introduced by Zadeh [1], laid the groundwork for the development of fuzzy analysis.

Fuzzy game models also have gained prominence due to their ability to represent strategic interactions under uncertainty using fuzzy constraint mappings and generalized equilibria [2]. Recent developments utilize fuzzy versions of classical fixed point theorems, such as the Kakutani-Fan-Glicksberg theorem, to establish the existence of Nash equilibria in fuzzy environments [3]. Moreover, the well-posedness and structural stability of the α-core in generalized fuzzy games have been rigorously addressed, enhancing solution robustness in multi-agent settings [4].

Building upon this, Heilpern [5] initiated the study of fixed point theorems for fuzzy mappings, while Bharucha-Reid [6] pioneered fixed point results in probabilistic frameworks. Himmelberg [7] and Kuratowski et al. [8] provided essential tools for the measurability of multifunctions and selections, which later became instrumental in establishing random fixed point theorems [9].

In the realm of random operator theory, notable contributions include those by Beg et al. [10, 11], who investigated random fixed points and iterative procedures in Banach spaces, and Shahzad et al. [12, 13], who generalized these results to Fréchet spaces and f-nonexpansive mappings. More recently, Banikirane et al. [14] studied random fixed point results with updated perspectives and improved contractive conditions. These developments motivate the need for further exploration of measurable selections for fuzzy and random multivalued operators, which is the focus of the present work.

In recent developments, the study of boundary value problems involving fuzzy random differential equations has gained notable attention, particularly for systems governed by integral-type random operators in fuzzy metric spaces. Srivastava et al. [15] presented a comprehensive investigation into fuzzy random boundary value problems by establishing random fixed point theorems in fuzzy normed spaces and applying them to a class of nonlinear differential systems with boundary conditions. Their approach blends fuzzy set theory with random operator methods and demonstrates the practical relevance of fuzzy random modeling in applied sciences such as biology, physics, and engineering.

The advances in the study of fuzzy random differential equations have highlighted the significance of integrating fuzzy set theory with random operator methods. For instance, Saadati et al. [16] examined uncertain control problems within symmetric F-n-NLS spaces, focusing on the existence and continuity of solutions to random multivalued equations. Similarly, Abdulsahib [17] employed the Laplace variational iteration method to approximate solutions of linear fuzzy random ordinary differential equations, demonstrating the method's efficiency and applicability. Khan [18] studied fuzzy delay impulsive fractional differential equations using fixed point approaches in credibility spaces. Additionally, Shagari [19] applied fuzzy fixed point techniques to analyze differential inclusions, with applications to real-world dynamics such as epidemiological models.

Moreover, recent developments in fuzzy random analysis have emphasized the interplay between randomness and fuzziness in modeling real-world phenomena. In particular, Srivastava [15] presented a comprehensive study of fuzzy random boundary value problems by developing random fixed point theorems in fuzzy normed spaces and applying them to nonlinear differential systems with boundary conditions. Their approach effectively integrated fuzzy set theory with random operator methods, highlighting their applicability in fields such as biology, physics, and engineering. In a related direction, Tassaddiq et al. [20] investigated fuzzy fixed point results for fuzzy mappings under specific fuzzy contraction conditions within complete fuzzy metric spaces. These fuzzy fixed point techniques are instrumental in solving mathematical models involving imprecise or uncertain data-scenarios where traditional deterministic methods are often insufficient. Furthermore, Saadati et al. [16] studied an uncertain control problem governed by a second-order nonlinear Schrodinger-type system within the framework of the symmetric F-n-NLS space. This setting, induced by a dynamic norm inspired by random norms, distribution functions, and fuzzy sets, allowed them to analyze a class of random multivalued equations with parameters and explore the existence and unbounded continuity of their solution sets. Motivated by these foundational works, our paper establishes a new fixed point theorem for measurable selections generated by a pair of fuzzy set-valued operators. This result facilitates the existence of measurable solutions to a class of random multivalued operator equations defined on Polish spaces. Our goal is to advance the theoretical landscape by developing a selection-based measurable fixed point theorem for fuzzy random multivalued mappings and demonstrating its utility through explicit examples. This contribution not only builds upon the theoretical advances of the aforementioned studies but also provides concrete tools for applications in fuzzy random analysis.

Beyond probabilistic and fuzzy uncertainties, recent advances in fractal-based uncertainty quantification offer powerful tools for modeling hierarchical irregularities in complex systems. Higher-order fractal belief Rényi divergence [21] enables robust pattern classification under multi-scale uncertainties by measuring distributional discrepancies in fractal domains. Similarly, fractal belief Rényi divergence [22] captures self-similar structures in uncertain data, enhancing feature extraction in high-noise environments. While our work focuses on integrating probabilistic randomness and fuzzy vagueness in operator equations, these fractal divergences present promising future extensions for handling multi-layered uncertainties with scale-invariant properties.

Let (X, d) be a Polish space, that is, a separable complete metric space, and let (Ω, Σ, m) be a measure space consisting of a nonempty set Ω called the sample space, a σ-algebra Σ of subsets of Ω, and a nonnegative countably additive measure $m : \Sigma \to [0, \infty]$ such that $m(\emptyset) = 0$.

Let $C(X)$ be the family of nonempty compact subsets of X. Then the Hausdorff metric H on $C(X)$ induced by d is defined as:

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\}.$$

A fuzzy set in X is a function with domain X and values in $[0, 1]$, and I^X is the collection of all fuzzy sets in X. If A is a fuzzy set and $x \in X$, then the function value $A(x)$ are called the grade of membership of x in A. The α-level set of a fuzzy set A is denoted by $[A]_\alpha$ and is defined as

$$[A]_\alpha = \begin{cases} \{x \in X : A(x) \geq \alpha\}, & \text{if } \alpha \in (0, 1], \\ \text{closure of } \{x \in X : A(x) > 0\}, & \text{if } \alpha = 0. \end{cases}$$

For two sets X and Y, a fuzzy set-valued mapping $F : Y \to I^X$ called a fuzzy mapping on Y into X.

The deterministic version of the following lemma was proved by Abu-Donia [9].

Lemma 1.1. *[9] Let $\varphi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable and non-decreasing function in t for each $\omega \in \Omega$. Assume that φ is right-continuous in t and satisfies*

$$\sum_{i=0}^{\infty} \varphi^i(\omega, t) < \infty, \quad \forall t > 0, \quad \forall\, \omega \in \Omega,$$

where φ^i denotes the i-th iteration of φ. Then,

$$\varphi(\omega, t) < t, \quad \forall t > 0, \quad \forall \omega \in \Omega.$$

2. Fixed point results for fuzzy mappings

In what follows, we assume that the function $\varphi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the conditions of Lemma 1.1. We are now in a position to derive an existence result. This theorem asserts the existence of a measurable function that selects a common value from the (α)-cut sets of two fuzzy mappings.

Theorem 2.1. *Let (X, d) be a Polish space and $S, T : \Omega \times X \to I^X$ be two fuzzy mappings where (Ω, Σ, m) is a complete σ-finite measure space. Suppose the following conditions hold:*

(i) *For each $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that $[S(\omega, x)]_{\alpha_S(x)}$, $[T(\omega, x)]_{\alpha_T(x)} \in C(X)$.*

(ii) *The set-valued mappings*

$$S_{\alpha_S(x)}(\omega, x) := [S(\omega, x)]_{\alpha_S(x)}, \quad T_{\alpha_T(x)}(\omega, x) := [T(\omega, x)]_{\alpha_T(x)},$$

satisfy the Kuratowski measurability condition with respect to the sigma-algebra Σ. That is, for every open subset $C \subset X$, the set

$$\{\omega \in \Omega : S_{\alpha_S(x)}(\omega, x) \cap C \neq \emptyset\} \in \Sigma \quad \textit{and} \quad \{\omega \in \Omega : T_{\alpha_T(x)}(\omega, x) \cap C \neq \emptyset\} \in \Sigma.$$

(iii) *The mappings $S(\omega, \cdot)$ and $T(\omega, \cdot)$ are continuous in the sense that for every sequence $x_n \to x$,*

$$\lim_{n\to\infty} H([S(\omega, x_n)]_{\alpha_S(x)}, [S(\omega, x)]_{\alpha_S(x)}) = 0, \quad \forall\, \alpha \in [0, 1].$$

The same holds for T.

(iv) *There exists a measurable function $\varphi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in X$ and $\omega \in \Omega$,*

$$H([S(\omega, x)]_{\alpha_S(x)}, [T(\omega, y)]_{\alpha_T(y)}) \leq \varphi(\omega, M(\omega, x, y, \alpha_S(x), \alpha_T(y))),$$

where

$$M(\omega, x, y, \alpha, \beta) = \max \left\{ \begin{array}{c} d(x, y), d(x, [S(\omega, x)]_\alpha), d(y, [T(\omega, y)]_\beta), \\ \frac{1}{2}(d(x, [T(\omega, y)]_\beta) + d(y, [S(\omega, x)]_\alpha)) \end{array} \right\}.$$

Then, there exists a measurable mapping $u : \Omega \to X$ such that

$$u(\omega) \in [S(\omega, u(\omega))]_{\alpha_S(u(\omega))} \cap [T(\omega, u(\omega))]_{\alpha_T(u(\omega))}, \quad \forall\, \omega \in \Omega.$$

Proof. Let $x_0 : \Omega \to X$ be an arbitrary measurable mapping. By hypothesis (i), there exists $\alpha_S(x_0(\omega)) \in (0, 1]$ such that the α-cut $[S(\omega, x_0(\omega))]_{\alpha_S(x_0(\omega))}$ is a nonempty compact subset X, for each $\omega \in \Omega$. Denote $\alpha_S(x_0(\omega))$ by α_1. By the compactness of $[S(\omega, x_0(\omega))]_{\alpha_1}$ and assumption (ii), we can find a measurable mapping $x_1 : \Omega \to X$ such that for each $\omega \in \Omega$, $x_1(\omega) \in [S(\omega, x_0(\omega))]_{\alpha_1}$,

$$d(x_0(\omega), x_1(\omega)) = d(x_0(\omega), [S(\omega, x_0(\omega))]_{\alpha_1}).$$

Similarly, there exists $\alpha_T(x_1(\omega)) \in (0,1]$ such that the α-cut $[T(\omega, x_1(\omega))]_{\alpha_T(x_1(\omega))}$ is a nonempty subset of X. Denote $\alpha_T(x_1(\omega))$ by α_2, and by the compactness of $[T(\omega, x_1(\omega)))]_{\alpha_2}$ and assumption (ii), choose a measurable mapping $x_2 : \Omega \to X$ such that for each $\omega \in \Omega$, $x_2(\omega) \in [T(\omega, x_1(\omega))]_{\alpha_2}$,

$$d(x_1(\omega), x_2(\omega)) = d(x_1(\omega), [T(\omega, x_1(\omega))]_{\alpha_1}).$$

By induction, we produce a sequence of measurable mappings $x_n : \Omega \to X$ with

$$\begin{aligned} x_{2k+1}(\omega) &\in [S(\omega, x_{2k}(\omega))]_{\alpha_{2k+1}}, \\ x_{2k+2}(\omega) &\in [T(\omega, x_{2k+1}(\omega))]_{\alpha_{2k+2}}, \quad k = 0, 1, 2, \ldots, \end{aligned}$$

and

$$\begin{aligned} d(x_{2k}(\omega), x_{2k+1}(\omega)) &= d(x_{2k}(\omega), [S(\omega, x_{2k}(\omega))]_{\alpha_{2k+1}}), \\ d(x_{2k+1}(\omega), x_{2k+2}(\omega)) &= d(x_{2k+1}(\omega), [T(\omega, x_{2k+1}(\omega))]_{\alpha_{2k+2}}). \end{aligned}$$

By the definition of H, we have

$$d(x_{2k}(\omega), x_{2k+1}(\omega)) \leq H([T(\omega, x_{2k-1}(\omega))]_{\alpha_{2k}}, [S(\omega, x_{2k}(\omega))]_{\alpha_{2k+1}}),$$

$$d(x_{2k+1}(\omega), x_{2k+2}(\omega)) \leq H([S(\omega, x_{2k}(\omega))]_{\alpha_{2k+1}}, [T(\omega, x_{2k+1}(\omega))]_{\alpha_{2k+2}}).$$

Now, assume that $x_{2k}(\omega) = x_{2k+1}(\omega)$ for some $k \geq 0$. Then,

$$\begin{aligned} & M(\omega, x_{2k}(\omega), x_{2k+1}(\omega), \alpha_{2k+1}, \alpha_{2k+2}) \\ = \ & \max \left\{ \begin{array}{c} d(x_{2k+1}(\omega), x_{2k+2}(\omega)), d(x_{2k}(\omega), [S(\omega, x_{2k}(\omega))]_{\alpha_{2k+1}}), \\ d(x_{2k+1}(\omega), [T(\omega, x_{2k+1}(\omega))]_{\alpha_{2k+2}}), \\ \frac{1}{2}\left[d(x_{2k}(\omega), [T(\omega, x_{2k+1}(\omega))]_{\alpha_{2k+2}}) + d(x_{2k+1}(\omega), [S(\omega, x_{2k}(\omega))]_{\alpha_{2k+1}})\right] \end{array} \right\} \\ = \ & \max \left\{ \begin{array}{c} d(x_{2k+1}(\omega), x_{2k+2}(\omega)), d(x_{2k}(\omega), x_{2k+1}(\omega)), d(x_{2k+1}(\omega), x_{2k+2}(\omega)), \\ \frac{1}{2}\left[d(x_{2k}(\omega), x_{2k+2}(\omega)) + d(x_{2k+1}(\omega), x_{2k+1}(\omega)\right] \end{array} \right\} \\ = \ & \max \left\{ d(x_{2k+1}(\omega), x_{2k+2}(\omega)), 0, \frac{1}{2}\left[d(x_{2k}(\omega), x_{2k+2}(\omega)) + 0\right] \right\} \\ = \ & \max \left\{ d(x_{2k+1}(\omega), x_{2k+2}(\omega)), \frac{1}{2}\left[d(x_{2k}(\omega), x_{2k+1}(\omega)) + d(x_{2k+1}(\omega), x_{2k+2}(\omega))\right] \right\}. \end{aligned}$$

The assumption $x_{2k}(\omega) = x_{2k+1}(\omega)$ implies that

$$\begin{aligned} & M(\omega, x_{2k}(\omega), x_{2k+1}(\omega), \alpha_{2k+1}, \alpha_{2k+2}) \\ = \ & \max\{d(x_{2k+1}(\omega), x_{2k+2}(\omega)), 0, \frac{1}{2}[d(x_{2k}(\omega), x_{2k+2}(\omega)) + 0]\} \\ = \ & \max\{d(x_{2k+1}(\omega), x_{2k+2}(\omega)), \frac{1}{2}[d(x_{2k}(\omega), x_{2k+1}(\omega)) + d(x_{2k+1}(\omega), x_{2k+2}(\omega))]\} \\ = \ & d(x_{2k+1}(\omega), x_{2k+2}(\omega)). \end{aligned}$$

Thus, using (iv), we have

$$d(x_{2k+1}(\omega), x_{2k+2}(\omega)) \quad \leq \quad H([S(\omega, x_{2k}(\omega))]_{\alpha_{2k+1}}, [T(\omega, x_{2k+1}(\omega))]_{\alpha_{2k+2}})$$

$$\begin{aligned} &\leq \varphi(\omega, M(\omega, x_{2k(\omega)}, x_{2k+1}(\omega), \alpha_{2k+1}, \alpha_{2k+2})) \\ &\leq \varphi(\omega, d(x_{2k+1}(\omega), x_{2k+2}(\omega))). \end{aligned}$$

Since $\varphi(\omega, t) < t$ for all $t > 0$, it follows that $d(x_{2k+1}(\omega), x_{2k+2}(\omega)) = 0$. This implies that

$$x_{2k}(\omega) = x_{2k+1}(\omega) \in [S(\omega, x_{2k}(\omega))]_{\alpha_{2k+1}},$$

$$x_{2k}(\omega) = x_{2k+1}(\omega) = x_{2k+2}(\omega) \in [T(\omega, x_{2k+1}(\omega))]_{\alpha_{2k+2}} = [T(\omega, x_{2k}(\omega))]_{\alpha_{2k+2}}.$$

Thus,

$$x_{2k}(\omega) \in [S(\omega, x_{2k}(\omega))]_{\alpha_{2k+1}} \cap [T(\omega, x_{2k}(\omega))]_{\alpha_{2k+2}}.$$

For the remaining case where $x_{n+1}(\omega) \neq x_n(\omega)$ for all n, using the contractive condition, we obtain

$$d(x_{2k+1}(\omega), x_{2k+2}(\omega)) \leq \varphi(\omega, M(\omega, x_{2k}(\omega), x_{2k+1}(\omega), \alpha_{2k+1}, \alpha_{2k+2})).$$

Since

$$M(\omega, x_{2k}(\omega), x_{2k+1}(\omega), \alpha_{2k+1}, \alpha_{2k+2}) = \max\{d(x_{2k}(\omega), x_{2k+1}(\omega)), d(x_{2k+1}(\omega), x_{2k+2}(\omega))\},$$

if

$$M(\omega, x_{2k}(\omega), x_{2k+1}(\omega), \alpha_{2k+1}, \alpha_{2k+2}) = d(x_{2k+1}(\omega), x_{2k+2}(\omega)),$$

we get

$$d(x_{2k+1}(\omega), x_{2k+2}(\omega)) \leq \varphi(\omega, d(x_{2k+1}(\omega), x_{2k+2}(\omega))) < d(x_{2k+1}(\omega), x_{2k+2}(\omega)),$$

which is a contradiction. Hence,

$$M(\omega, x_{2k}(\omega), x_{2k+1}(\omega), \alpha_{2k+1}, \alpha_{2k+2}) = d(x_{2k}(\omega), x_{2k+1}(\omega)).$$

Thus,

$$d(x_{2k+1}(\omega), x_{2k+2}(\omega)) \leq \varphi(\omega, d(x_{2k}(\omega), x_{2k+1}(\omega))).$$

In a similar way, we prove that

$$d(x_{2k+1}(\omega), x_{2k}(\omega)) \leq \varphi(\omega, d(x_{2k}(\omega), x_{2k-1}(\omega))).$$

Thus,

$$d(x_{n+1}(\omega), x_n(\omega)) \leq \varphi(\omega, d(x_n(\omega), x_{n-1}(\omega)))$$

and

$$d(x_n(\omega), x_{n-1}(\omega)) \leq \varphi(\omega, d(x_{n-1}(\omega), x_{n-2}(\omega))).$$

Since $\varphi(\omega, \cdot)$ is non-decreasing, applying φ to both sides gives:

$$\varphi(\omega, d(x_n(\omega), x_{n-1}(\omega))) \leq \varphi(\omega, \varphi(\omega, d(x_{n-1}(\omega), x_{n-2}(\omega)))).$$

By the definition of φ^2, we conclude:

$$\varphi(\omega, d(x_n(\omega), x_{n-1}(\omega))) \leq \varphi^2(\omega, d(x_{n-1}(\omega), x_{n-2}(\omega))).$$

Thus, by induction, we obtain:

$$d(x_{n+1}(\omega), x_n(\omega)) \leq \varphi^n(\omega, d(x_1(\omega), x_0(\omega))).$$

It implies that, for each positive integer m, n with $n > m$, we have:

$$\begin{aligned}
&(x_m(\omega), x_n(\omega)) \leq d(x_m(\omega), x_{m+1}(\omega)) + d(x_{m+1}(\omega), x_{m+2}(\omega)) + \cdots + d(x_{n-1}(\omega), x_n(\omega)) \\
&\leq \varphi^m(\omega, d(x_1(\omega), x_0(\omega))) + \varphi^{m+1}(\omega, d(x_1(\omega), x_0(\omega))) + \cdots + \varphi^{n-1}(\omega, d(x_1(\omega), x_0(\omega))) \\
&\leq \sum_{i=m}^{n-1} \varphi^i(\omega, d(x_1(\omega), x_0(\omega))) \leq \sum_{i=m}^{\infty} \varphi^i(\omega, d(x_1(\omega), x_0(\omega))).
\end{aligned}$$

Since $\sum_{i=1}^{\infty} \varphi^i(\omega, t) < \infty$ for all $t > 0$, the sequence $\{x_n(\omega)\}$ is Cauchy. Since X is complete, there exists a measurable mapping $u : \Omega \to X$ such that $x_n(\omega) \to u(\omega)$. This, with (iv), implies that

$$\begin{aligned}
d(u(\omega), [S(\omega, u(\omega))]_{\alpha_S(u(\omega))}) &\leq d(u(\omega), x_{2n}(\omega)) + d(x_{2n}(\omega), [S(\omega, u(\omega))]_{\alpha_S(u(\omega))}) \\
&\leq d(u(\omega), x_{2n}(\omega)) + H([T(\omega, x_{2n-1})]_{\alpha_{2n}(\omega)}, [S(\omega, u(\omega))]_{\alpha_S(u(\omega))}) \\
&\leq d(u(\omega), x_{2n}(\omega)) + \varphi(\omega, M(u(\omega), x_{2n-1}(\omega), \alpha_S(u(\omega)), \alpha_{2n}(\omega))),
\end{aligned}$$

where

$$\begin{aligned}
&M(u(\omega), x_{2n-1}(\omega), \alpha_S(u(\omega)), \alpha_{2n}(\omega)) \\
&= \max\Big\{d(u(\omega), x_{2n-1}(\omega)), d(u(\omega), [S(\omega, u(\omega))]_{\alpha_S(u(\omega))}), d(x_{2n-1}(\omega), x_{2n}(\omega)), \\
&\quad \frac{1}{2}[d(u(\omega), x_{2n}(\omega)) + d(x_{2n-1}(\omega), [S(\omega, u(\omega))]_{\alpha_S(u(\omega))})]\Big\}.
\end{aligned}$$

Thus by using the assumption (iii), we get

$$\lim_{n\to\infty} M(u(\omega), x_{2n-1}(\omega), \alpha_S(u(\omega)), \alpha_{2n}(\omega)) = d(u(\omega), [S(\omega, u(\omega))]_{\alpha_S(u(\omega))}).$$

Since $\varphi(\omega, t)$ is continuous from the right, as $n \to \infty$, we obtain

$$d(u(\omega), [S(\omega, u(\omega))]_{\alpha_S(u(\omega))}) \leq \varphi(\omega, d(u(\omega), [S(\omega, u(\omega))]_{\alpha_S(u(\omega))}).$$

Then Lemma 1.1 implies $u(\omega) \in [S(\omega, u(\omega))]_{\alpha_S(u(\omega))}$. Similarly, by using

$$d(u(\omega), [T(\omega, u(\omega))]_{\alpha_T(u(\omega))}) \leq d(u(\omega), x_{2n+1}(\omega)) + d(x_{2n+1}(\omega), [T(\omega, u(\omega))]_{\alpha_T(u(\omega))}),$$

we conclude $u(\omega) \in [T(\omega, u(\omega))]_{\alpha_T(u(\omega))}$. Hence,

$$u(\omega) \in [S(\omega, u(\omega))]_{\alpha_S(u(\omega))} \cap [T(\omega, u(\omega))]_{\alpha_T(u(\omega))}, \quad \forall\, \omega \in \Omega.$$

□

Now to demonstrate the applicability of Theorem 2.1 and validate its hypotheses, we present the following example built in a well-structured and intuitive way.

Example 2.1. *Let us consider the metric space $X = [0, 1]$, endowed with the usual Euclidean metric $d(x, y) = |x - y|$, which makes X a compact, complete, and separable space, and hence a Polish space. Let (Ω, Σ, μ) be a complete probability space.*

Suppose that $a(\omega)$, $b(\omega) : \Omega \to [0, 1]$ are measurable functions such that, for every $\omega \in \Omega$, we have

$$|a(\omega) - b(\omega)| \leq \frac{1}{4}.$$

Now, consider a pair of fuzzy mappings $S, T : \Omega \times X \to I^X$ in terms of triangular membership functions centered at $a(\omega)$ and $b(\omega)$, respectively. Fix a constant $\delta > 0$ such that $\delta \leq \frac{1}{2}$, and define

$$S(\omega, x)(t) = \begin{cases} 1 - \dfrac{2|t - a(\omega)|}{\delta}, & \text{if } |t - a(\omega)| \leq \dfrac{\delta}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

$$T(\omega, x)(t) = \begin{cases} 1 - \dfrac{2|t - b(\omega)|}{\delta}, & \text{if } |t - b(\omega)| \leq \dfrac{\delta}{2}, \\ 0, & \text{otherwise}. \end{cases}$$

For each $x \in X$, let us define $\alpha_S(x) = \alpha_T(x) = \frac{1}{2}$. Then the α-level sets of the fuzzy mappings become

$$[S(\omega, x)]_{1/2} = \left[a(\omega) - \frac{\delta}{4},\ a(\omega) + \frac{\delta}{4}\right], \quad [T(\omega, x)]_{1/2} = \left[b(\omega) - \frac{\delta}{4},\ b(\omega) + \frac{\delta}{4}\right].$$

Clearly, these sets are compact intervals in X, and their dependence on ω through measurable functions $a(\omega)$ and $b(\omega)$ ensures that the mappings

$$S_{\alpha_S(x)}(\omega, x) := [S(\omega, x)]_{\alpha_S(x)}, \quad T_{\alpha_T(x)}(\omega, x) := [T(\omega, x)]_{\alpha_T(x)}$$

are Kuratowski measurable, satisfying the second hypothesis.

Moreover, the fuzzy membership only depends on ω so continuity in the variable x is immediate in the Hausdorff metric. Thus, the third condition is also fulfilled. Now define a function $\varphi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\varphi(\omega, t) = \frac{t}{2}.$$

This function is measurable in ω, non-decreasing, right-continuous in t, and satisfies

$$\sum_{n=0}^{\infty} \varphi^n(\omega, t) = \sum_{n=0}^{\infty} \frac{t}{2^n} = 2t < \infty.$$

To verify the contraction-type condition, consider:

$$M(\omega, x, y, \tfrac{1}{2}, \tfrac{1}{2}) = \max \left\{ \begin{array}{c} |x - y|,\ d(x, [S(\omega, x)]_{1/2}),\ d(y, [T(\omega, y)]_{1/2}), \\ \dfrac{1}{2} \left(d(x, [T(\omega, y)]_{1/2}) + d(y, [S(\omega, x)]_{1/2})\right) \end{array} \right\}.$$

Given the symmetry and compact support of the α-level sets, we estimate each term using the triangle inequality:

$$d(x, [S(\omega, x)]_{1/2}) \leq |x - a(\omega)| + \frac{\delta}{4}, \quad d(y, [T(\omega, y)]_{1/2}) \leq |y - b(\omega)| + \frac{\delta}{4}.$$

So the maximum term in M will always be greater than or equal to $|a(\omega) - b(\omega)|$. Since, $M(\omega, x, y, \frac{1}{2}, \frac{1}{2}) \geq |a(\omega) - b(\omega)|$ and $\varphi(\omega, t) = \frac{t}{2}$, then

$$H([S(\omega, x)]_{1/2}, [T(\omega, y)]_{1/2}) = |a(\omega) - b(\omega)| \leq \varphi(\omega, M(\omega, x, y, \tfrac{1}{2}, \tfrac{1}{2})),$$

which shows that the fourth condition of Theorem 2.1 is also satisfied.

Therefore, by Theorem 2.1, there exists a measurable selection $u : \Omega \to X$ such that for every $\omega \in \Omega$,

$$u(\omega) \in [S(\omega, u(\omega))]_{1/2} \cap [T(\omega, u(\omega))]_{1/2}.$$

3. Applications in engineering and robotics

The fixed point results discussed in this work are not only of theoretical interest, but also have practical relevance in fields such as engineering systems, robotics, and control theory. The following example provides concrete applications of our main theorem under uncertain and fuzzy environments. It models a real-world scenario in robotic control where the actual position of a robot is subject to environmental noise and fuzzy uncertainty. The fuzzy random operators represent imprecise actuation or sensor estimates depending on both randomness (e.g., ambient noise) and fuzziness (e.g., vague boundaries of motion tolerance). The constructed mappings simulate how a robot might interpret its current position, and the fixed point represents a self-consistent state accepted by both models. This is particularly relevant in adaptive control systems, autonomous mobile robotics, and feedback-based correction mechanisms under uncertainty.

Example 3.1. *Let $X = [0, 3]$ be the working range (in meters) of a robotic arm segment. Consider the probability space $(\Omega, \mathcal{F}, P) = ([0, 1], \mathcal{B}, \lambda)$, where each $\omega \in \Omega$ represents a realization of random environmental conditions such as vibration, latency, or temperature variation. The uncertainty in measurement is modeled using fuzzy set-valued mappings, while the randomness accounts for environmental noise.*

Define a fuzzy random operator $T : \Omega \times X \to I^X$ by specifying its α-cuts as follows:

$$[T(\omega, x)]_\alpha = \left[\omega x - \ln\left(\tfrac{1}{\alpha}\right),\ \omega x + \ln\left(\tfrac{1}{\alpha}\right)\right] \cap [0, 3],$$

for all $\alpha \in (0, 1]$, $x \in X$, and $\omega \in \Omega$. This defines a fuzzy number whose membership function is given by

$$T(\omega, x)(t) = \begin{cases} e^{-|\omega x - t|}, & \text{if } t \in [0, 3], \\ 0, & \text{otherwise.} \end{cases}$$

Here, $T(\omega, x)$ represents the uncertainty in the sensed position of the robotic arm when the true position is x, distorted by the environmental noise factor ω. The membership function is maximized at $t = \omega x$, representing the most likely measurement under distortion, and decays exponentially away from this center.

We aim to find a measurable function $u : \Omega \to X$ such that

$$\{u(\omega)\} \subseteq [T(\omega, u(\omega))]_\alpha \quad \text{for all } \alpha \in (0, 1],\ \text{a.e. } \omega \in \Omega,$$

i.e., the robotic arm stabilizes at a position that lies within every α-cut of the fuzzy random measurement meaning that $u(\omega)$ is a fuzzy fixed point of T in the sense of α-cuts.

To verify the conditions of Theorem 2.1, consider the Hausdorff distance between the α-cuts:

$$H([T(\omega,x)]_\alpha,[T(\omega,y)]_\alpha) = |\omega x-\omega y| = \omega|x-y|.$$

Define the function $\varphi:\Omega\times\mathbb{R}^+\to\mathbb{R}^+$ by

$$\varphi(\omega,t)=\omega t,$$

which is measurable in ω, continuous, and non-decreasing in t, and satisfies the integrability condition assumed in Theorem 2.1 (for $S=T$). Then,

$$H([T(\omega,x)]_\alpha,[T(\omega,y)]_\alpha)\le\varphi(\omega,|x-y|),\quad \text{for all } x,y\in X,\ \alpha\in(0,1].$$

Hence, all the conditions of Theorem 2.1 are satisfied, and it follows that there exists a measurable selection $u:\Omega\to X$ such that

$$\{u(\omega)\}\subseteq[T(\omega,u(\omega))]_\alpha,\quad \text{for all } \alpha\in(0,1],\ \text{a.e. } \omega\in\Omega.$$

Interpretation: This result implies that the robotic system can adapt its position control strategy to ensure that the actual position coincides with a value contained in the uncertain sensed data, even under random environmental disturbances. The model integrates fuzziness (sensor inaccuracy) and randomness (environmental variation), showcasing the utility of fuzzy random fixed point theorems in intelligent control and feedback systems.

In the following, Figure 1 illustrates a 2-D visualization of the fuzzy random mapping

$$T(\omega,x)(t)=e^{-|t-\omega x|},$$

where $\omega\in[0,1]$ is a random parameter and $x\in\mathbb{R}$ is fixed. The curve represents the membership function of the fuzzy set for a fixed value of $\omega=0.6$ and $x=2$, so that the center of the fuzzy set is at $t=1.2$. The vertical axis corresponds to the membership degree and the horizontal axis is the universe $t\in\mathbb{R}$. The peak of the curve shows maximum membership at $t=\omega x$, and decays exponentially as $|t-\omega x|$ increases. This representation captures the effect of randomness in shifting the fuzzy set.

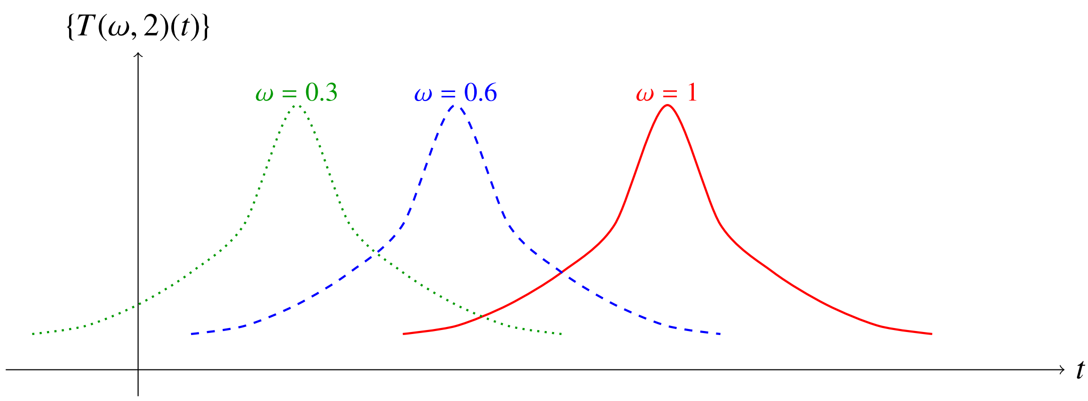

Figure 1. 2-D visualization of the fuzzy random mapping.

Figure 2 shows a 3-D plot of the fuzzy random mapping

$$T(\omega,x)(t)=e^{-|t-\omega x|},$$

where $x = 2$ is fixed and $\omega \in [0,1]$ varies continuously. The surface illustrates how the fuzzy membership function changes with respect to both the random parameter ω and the universe $t \in \mathbb{R}$. For each ω, the fuzzy set is centered at $t = \omega x$, and the membership function decays exponentially from this center. This 3-D plot provides an intuitive view of how randomness affects the position of the fuzzy set, and how the level of uncertainty varies across the domain. It effectively visualizes the coupling between fuzziness (via the exponential decay) and randomness (via ω) in the mapping. Additional perspectives of Figure 2 (contour visualization) have been incorporated as other figures (Figures 5 and 6) in the Appendix.

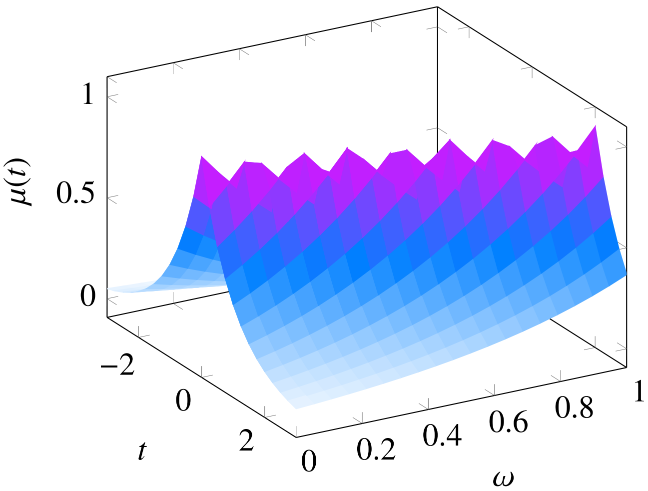

Figure 2. 3-D plot of the fuzzy random mapping.

4. Applications to random operator equations

In this section, using more general kinds of contractive conditions, we extend well-known deterministic results for contractive-type multivalued mappings to the random case by utilizing the findings from the preceding section on fuzzy mappings and measurable selections.

Theorem 4.1. *Let (X, d) be a Polish space and $A, B : \Omega \times X \to C(X)$ be continuous random multivalued operators. Suppose that there exists a measurable function $\varphi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in X$ and $\omega \in \Omega$,*

$$H(A(\omega, x), B(\omega, y)) \leq \varphi\left(\omega, \max\left\{\begin{array}{c} d(x,y), d(x, A(\omega,x)), d(y, B(\omega,y)), \\ \frac{1}{2}\left(d(x, B(\omega,y)) + d(y, A(\omega,x))\right) \end{array}\right\}\right).$$

Then, there exists a measurable mapping $u : \Omega \to X$ such that

$$u(\omega) \in A(\omega, u(\omega)) \cap B(\omega, u(\omega)), \quad \forall\, \omega \in \Omega.$$

Proof. Define two fuzzy mappings $S, T : \Omega \times X \to I^X$ by

$$S(\omega, x)(\ell) = \chi_{A(\omega,x)}(\ell) = \begin{cases} 1, & \ell \in A(\omega, x), \\ 0, & \text{otherwise}, \end{cases}$$

and similarly

$$T(\omega, x)(\ell) = \chi_{B(\omega,x)}(\ell) = \begin{cases} 1, & \ell \in B(\omega, x), \\ 0, & \text{otherwise}. \end{cases}$$

We now verify that the conditions of Theorem 2.1 are satisfied for these fuzzy mappings.

For each $x \in X$, define $\alpha_S(x) = \alpha_T(x) = 1 \in (0, 1]$. Then

$$\begin{aligned} [S(\omega, x)]_{\alpha_S(x)} &= \{\ell \in X : S(\omega, x)(\ell) = 1\}, \\ [S(\omega, x)]_{\alpha_S(x)} &= A(\omega, x) \in C(X), \\ [T(\omega, x)]_{\alpha_T(x)} &= B(\omega, x) \in C(X), \end{aligned}$$

so condition (i) is satisfied.

Let us see about measurability. Since $A(\omega, x)$ and $B(\omega, x)$ are continuous random multivalued operators with compact values, the mappings $(\omega, x) \mapsto A(\omega, x)$ and $(\omega, x) \mapsto B(\omega, x)$ are measurable. Hence, the mappings

$$S^{\alpha_S(x)}(\omega, x) = A(\omega, x), \quad T^{\alpha_T(x)}(\omega, x) = B(\omega, x)$$

are measurable. Thus, condition (ii) is satisfied.

Let us check the condition of continuity. Since $A(\omega, \cdot)$ and $B(\omega, \cdot)$ are continuous for each fixed $\omega \in \Omega$, it follows that for any sequence $x_n \to x$ in X,

$$\lim_{n\to\infty} H([S(\omega, x_n)]_1, [S(\omega, x)]_1) = H(A(\omega, x_n), A(\omega, x)) \to 0,$$

and similarly for T. Therefore, condition (iii) is satisfied.

Now we see the fulfillment of the contractive-type condition. By assumption, there exists a measurable function $\varphi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in X$ and $\omega \in \Omega$,

$$H(A(\omega, x), B(\omega, y)) \leq \varphi\left(\omega, \max\left\{ \begin{array}{c} d(x, y), d(x, A(\omega, x)), d(y, B(\omega, y)), \\ \frac{1}{2}(d(x, B(\omega, y)) + d(y, A(\omega, x))) \end{array} \right\}\right).$$

Rewriting in terms of S and T, we obtain

$$H([S(\omega, x)]_1, [T(\omega, y)]_1) \leq \varphi(\omega, M(\omega, x, y, 1, 1)),$$

where M is defined in Theorem 2.1. Thus, condition (iv) is satisfied. Having verified all conditions of Theorem 2.1, we conclude that there exists a measurable mapping $u : \Omega \to X$ such that

$$u(\omega) \in [S(\omega, u(\omega))]_1 \cap [T(\omega, u(\omega))]_1 = A(\omega, u(\omega)) \cap B(\omega, u(\omega)), \quad \forall\, \omega \in \Omega.$$

□

This theorem ensures the existence of a random fixed point common to two random operators, under a general contraction condition expressed via a measurable control function φ. The flexibility of φ allows for various types of nonlinear contractive conditions, making the result applicable in a wide range of probabilistic and fuzzy modeling scenarios.

We now present a concrete example to demonstrate the application of this result. In this example, we explicitly compute a measurable selection $u(\omega)$ satisfying the conclusion of Theorem 4.1, by carefully choosing suitable random operators and a control function φ that meets the required criteria. As a practical situation, this example is constructed about a situation in robotics or engineering systems where two distinct subsystems measure or estimate a physical quantity (such as temperature, position, or velocity). The outputs of these subsystems are imprecise due to random environmental influences (such as noise, calibration error, and fluctuating humidity), but they can be expressed by random intervals rather than exact values.

Example 4.1. *Let $(X, d) = (\mathbb{R}, |\cdot|)$ be the real line with the usual metric. Define continuous random multivalued operators $A, B : \Omega \times \mathbb{R} \to C(\mathbb{R})$ by*

$$A(\omega, x) = [\omega x, \omega x + 1], \quad B(\omega, x) = [\omega x + \sin(\omega), \omega x + \sin(\omega) + 1],$$

where $\omega \in \Omega \subseteq \mathbb{R}$. The Hausdorff distance between the compact intervals $A(\omega, x)$ and $B(\omega, y)$ satisfies

$$H(A(\omega, x), B(\omega, y)) = \max\{|\omega x - \omega y - \sin(\omega)|, |\omega x - \omega y - \sin(\omega)|\} = |\omega(x - y) - \sin(\omega)|.$$

Define the control function

$$\varphi(\omega, t) = |\omega|t + |\sin(\omega)|, \quad \text{for } t \in \mathbb{R}^+, \omega \in \Omega.$$

Then, for all $x, y \in \mathbb{R}$,

$$H(A(\omega, x), B(\omega, y)) = |\omega(x - y) - \sin(\omega)| \leq |\omega||x - y| + |\sin(\omega)| = \varphi(\omega, |x - y|).$$

Since the control function is measurable in ω, increasing in t, and satisfies $\varphi(\omega, t) < t$ for all ω and sufficiently small t, the hypotheses of the main theorem are satisfied. Hence, there exists a measurable mapping $u : \Omega \to \mathbb{R}$ such that

$$u(\omega) \in A(\omega, u(\omega)) \cap B(\omega, u(\omega)), \quad \forall\, \omega \in \Omega.$$

We now explicitly compute such a selection $u(\omega)$. For a fixed $\omega \in (0, 1)$, let $u = u(\omega)$. Then the intervals become

$$A(\omega, u) = [\omega u, \omega u + 1], \quad B(\omega, u) = [\omega u + \sin(\omega), \omega u + \sin(\omega) + 1].$$

To find $u \in A(\omega, u) \cap B(\omega, u)$, we require

$$\omega u \leq u \leq \omega u + 1, \quad \text{and} \quad \omega u + \sin(\omega) \leq u \leq \omega u + \sin(\omega) + 1.$$

The first inequality gives $0 \leq (1 - \omega)u \leq 1$, so $u \in \left[0, \frac{1}{1-\omega}\right]$. The second gives $\sin(\omega) \leq (1 - \omega)u \leq \sin(\omega) + 1$, so $u \in \left[\frac{\sin(\omega)}{1-\omega}, \frac{\sin(\omega)+1}{1-\omega}\right]$. Therefore, the intersection of both conditions yields

$$u(\omega) \in \left[\max\left(0, \frac{\sin(\omega)}{1 - \omega}\right), \min\left(\frac{1}{1 - \omega}, \frac{\sin(\omega) + 1}{1 - \omega}\right)\right].$$

Thus, an explicit measurable selection is given by

$$u(\omega) := \max\left(0, \frac{\sin(\omega)}{1 - \omega}\right), \quad \text{for } \omega \in (0, 1).$$

This choice is measurable and satisfies $u(\omega) \in A(\omega, u(\omega)) \cap B(\omega, u(\omega))$ for all $\omega \in (0, 1)$. To see it practically, consider a scenario in engineering systems or robotics, where a physical quantity (like temperature, position, or velocity) is measured or estimated by two different subsystems. Due to random environmental influence (e.g., noise, calibration error, varying humidity), the outputs of these subsystems are uncertain but can be described by random intervals instead of precise values.

Let $X = \mathbb{R}$, representing the range of possible values (e.g., position on a line), $\omega \in \Omega \subset \mathbb{R}$ represent a random environmental parameter (e.g., a noise level or time-varying coefficient), and $A(\omega, x)$ and $B(\omega, x)$ represent two interval-valued estimates for the system state from two different sensors or models, both depending on ω and current estimate x.

Define the two random interval-valued outputs $A(\omega, x), B(\omega, x)$ as above: This could model sensor A giving a baseline interval with uncertainty of ± 0.5 and sensor B being influenced by a periodic disturbance $\sin(\omega)$ (e.g., due to oscillating background noise).

We want to find a self-consistent estimate $u(\omega)$ of the system state that is accepted by both sensors, i.e., it lies in both intervals. To ensure this is possible, consider the control function:

$$\varphi(\omega, t) = |\omega| t + |\sin(\omega)|,$$

which bounds the discrepancy between sensor intervals. This function is measurable in ω, increasing in t, and satisfies $\varphi(\omega, t) < t$ for all ω and sufficiently small t.

The Hausdorff distance between $A(\omega, x)$ and $B(\omega, y)$ is:

$$H(A(\omega, x), B(\omega, y)) = |\omega(x - y) - \sin(\omega)| \leq \varphi(\omega, |x - y|).$$

By the above theorem, a measurable selection $u : \Omega \to \mathbb{R}$ exists such that:

$$u(\omega) \in A(\omega, u(\omega)) \cap B(\omega, u(\omega)), \quad \forall\ \omega \in \Omega.$$

To compute $u(\omega)$, fix $\omega \in (0, 1)$ and write $u = u(\omega)$. Then:

$$A(\omega, u) = [\omega u, \omega u + 1], \quad B(\omega, u) = [\omega u + \sin(\omega), \omega u + \sin(\omega) + 1].$$

We seek u such that

$$\omega u \leq u \leq \omega u + 1, \quad \omega u + \sin(\omega) \leq u \leq \omega u + \sin(\omega) + 1.$$

From the first inequality:

$$0 \leq (1 - \omega) u \leq 1 \quad \Rightarrow \quad u \in \left[0, \frac{1}{1 - \omega}\right].$$

From the second:

$$\sin(\omega) \leq (1 - \omega) u \leq \sin(\omega) + 1 \quad \Rightarrow \quad u \in \left[\frac{\sin(\omega)}{1 - \omega}, \frac{\sin(\omega) + 1}{1 - \omega}\right].$$

The intersection yields:

$$u(\omega) \in \left[\max\left(0, \frac{\sin(\omega)}{1 - \omega}\right), \min\left(\frac{1}{1 - \omega}, \frac{\sin(\omega) + 1}{1 - \omega}\right)\right].$$

Thus, an explicit measurable selection is:

$$u(\omega) := \max\left(0, \frac{\sin(\omega)}{1-\omega}\right), \quad \textit{for } \omega \in (0,1).$$

This choice is measurable and satisfies $u(\omega) \in A(\omega, u(\omega)) \cap B(\omega, u(\omega))$ *for all* $\omega \in (0,1)$.

Figure 3 illustrates a 2-D view of $A(\omega, u(\omega))$ and $B(\omega, u(\omega))$ at a fixed value $\omega = 0.6$. The blue and green horizontal bars represent the interval outputs of two sensing subsystems, while the red dashed line marks the measurable selection $u(\omega)$ that lies in their intersection.

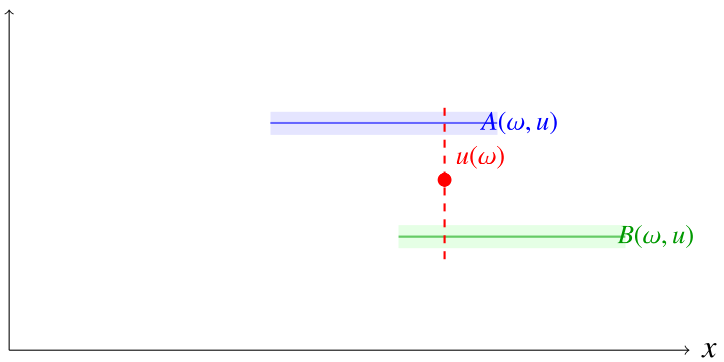

Figure 3. 2-D plot of $A(\omega, u(\omega))$ and $B(\omega, u(\omega))$ for $\omega = 0.6$, showing overlap and the selection $u(\omega) \approx 1.92$.

Figure 4 presents the blue curves to show the lower and upper bounds of the interval

$$A(\omega, u(\omega)) = [\omega u, \omega u + 1]$$

plotted at layer $z = 1$.

The green curves show the interval

$$B(\omega, u(\omega)) = [\omega u + \sin(\omega), \omega u + \sin(\omega) + 1]$$

plotted at layer $z = 2$.

The layer $z = 1.5$ represents the computed measurable selection

$$u(\omega) = \max\left(0, \frac{\sin(\omega)}{1-\omega}\right),$$

which lies within both intervals for each $\omega \in (0,1)$. This demonstrates how the selection $u(\omega)$ varies with randomness ω and remains consistent with both interval-valued sensors in the presence of fuzzy uncertainty. Additional perspectives of Figure 4 have been incorporated as other figures (Figures 7 and 8) in the Appendix.

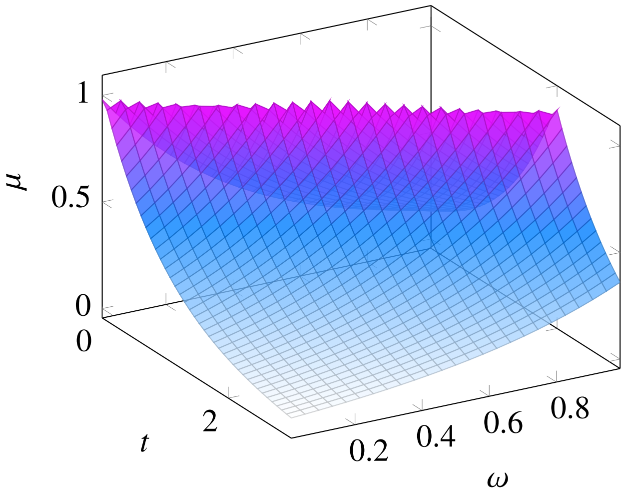

Figure 4. 3-D plot of the fuzzy random mapping $T(\omega, x)(t)$ for fixed $x = 2$ showing variation with ω and fuzzy membership decay.

Theorem 4.2. *Let (X, d) be a Polish space and $A, B : \Omega \times X \to C(X)$ be continuous random multivalued operators. Suppose there exists a measurable function $\lambda : \Omega \to (0, 1)$ such that for all $x, y \in X$ and $\omega \in \Omega$,*

$$H(A(\omega, x), B(\omega, y)) \leq \lambda(\omega) \max \left\{ \begin{array}{c} d(x, y), d(x, A(\omega, x)), d(y, B(\omega, y)), \\ \frac{1}{2}(d(x, B(\omega, y)) + d(y, A(\omega, x))) \end{array} \right\}.$$

Then, there exists a measurable mapping $u : \Omega \to X$ such that

$$u(\omega) \in A(\omega, u(\omega)) \cap B(\omega, u(\omega)), \quad \forall\, \omega \in \Omega.$$

Define the function $\varphi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\varphi(\omega, t) := \lambda(\omega)t.$$

Then for all $x, y \in X$ and $\omega \in \Omega$, we have:

$$\begin{aligned} H(A(\omega, x), B(\omega, y)) &\leq \lambda(\omega) \max \left\{ \begin{array}{c} d(x, y), d(x, A(\omega, x)), d(y, B(\omega, y)), \\ \frac{1}{2}(d(x, B(\omega, y)) + d(y, A(\omega, x))) \end{array} \right\} \\ &= \varphi\left(\omega, \max \left\{ \begin{array}{c} d(x, y), d(x, A(\omega, x)), d(y, B(\omega, y)), \\ \frac{1}{2}(d(x, B(\omega, y)) + d(y, A(\omega, x))) \end{array} \right\}\right). \end{aligned}$$

So, the contractive condition in Theorem 4.1 is satisfied with this choice of φ. We now verify the properties of φ: $\varphi(\omega, \cdot)$ are non-decreasing for each fixed $\omega \in \Omega$. Since $\lambda(\omega) > 0$, $\varphi(\omega, \cdot)$ is right-continuous in t and $\varphi(\omega, t) < t$ for all $t > 0$, since $\lambda(\omega) \in (0, 1)$.

The series of iterates of φ converges:

$$\sum_{i=0}^{\infty} \varphi^i(\omega, t) = \sum_{i=0}^{\infty} \lambda(\omega)^i t = t \sum_{i=0}^{\infty} \lambda(\omega)^i = \frac{t}{1 - \lambda(\omega)} < \infty,$$

for all $t > 0$ and all $\omega \in \Omega$.

Hence, all conditions of Theorem 4.1 are satisfied. By applying Theorem 4.1, we conclude that there exists a measurable mapping $u : \Omega \to X$ such that

$$u(\omega) \in A(\omega, u(\omega)) \cap B(\omega, u(\omega)), \quad \forall\ \omega \in \Omega.$$

Corollary 4.1. *Let (X, d) be a Polish space, and let $C(X)$ be the family of nonempty compact subsets of X. Let $A, B : \Omega \times X \to C(X)$ be continuous random multivalued operators. Suppose that there exist measurable functions $\lambda_i : \Omega \to [0, 1]$ for $i = 1, \ldots, 5$, with $\lambda_4(\omega) = \lambda_5(\omega)$ and*

$$\sum_{i=1}^{5} \lambda_i(\omega) < 1, \quad \forall\ \omega \in \Omega,$$

such that for all $x, y \in X$ and $\omega \in \Omega$,

$$\begin{aligned} H(A(\omega, x), B(\omega, y)) \leq\ & \lambda_1(\omega)d(x, y) + \lambda_2(\omega)d(x, A(\omega, x)) + \lambda_3(\omega)d(y, B(\omega, y)) \\ & + \lambda_4(\omega)d(x, B(\omega, y)) + \lambda_5(\omega)d(y, A(\omega, x)). \end{aligned}$$

Then, there exists a measurable mapping $u : \Omega \to X$ such that

$$u(\omega) \in A(\omega, u(\omega)) \cap B(\omega, u(\omega)), \quad \forall \omega \in \Omega.$$

Proof. Let us denote

$$m := \max\left\{ d(x, y),\ d(x, A(\omega, x)),\ d(y, B(\omega, y)),\ \frac{1}{2}\left[d(x, B(\omega, y)) + d(y, A(\omega, x))\right] \right\}.$$

Then each term on the right-hand side of the assumed inequality is less than or equal to m, except the last two, which involve asymmetric terms. However, since $\lambda_4(\omega) = \lambda_5(\omega)$, we can write

$$\lambda_4(\omega)d(x, B(\omega, y)) + \lambda_5(\omega)d(y, A(\omega, x)) = \lambda_4(\omega)\left[d(x, B(\omega, y)) + d(y, A(\omega, x))\right] \leq 2\lambda_4(\omega)m.$$

Therefore, we obtain

$$\begin{aligned} H(A(\omega, x), B(\omega, y)) \leq\ & \lambda_1(\omega)d(x, y) + \lambda_2(\omega)d(x, A(\omega, x)) + \lambda_3(\omega)d(y, B(\omega, y)) \\ & + \lambda_4(\omega)\left[d(x, B(\omega, y)) + d(y, A(\omega, x))\right] \\ \leq\ & \left[\lambda_1(\omega) + \lambda_2(\omega) + \lambda_3(\omega) + 2\lambda_4(\omega)\right] m. \end{aligned}$$

Define the function

$$\lambda(\omega) := \lambda_1(\omega) + \lambda_2(\omega) + \lambda_3(\omega) + 2\lambda_4(\omega).$$

Since each λ_i is measurable and the class of measurable functions is closed under addition and scalar multiplication, it follows that $\lambda : \Omega \to [0, 1]$ is also a measurable function. Furthermore, $\lambda(\omega) < 1$.

Thus, we get

$$H(A(\omega, x), B(\omega, y)) \leq \lambda(\omega)m,$$

which is the contraction condition of Theorem 4.2. Therefore, by applying Theorem 4.2, there exists a measurable mapping $u : \Omega \to X$ such that

$$u(\omega) \in A(\omega, u(\omega)) \cap B(\omega, u(\omega)), \quad \forall\ \omega \in \Omega.$$

□

5. Conclusions

In this work, we developed a fixed point framework for a class of fuzzy operators and demonstrated its applicability to random multivalued operator equations. By utilizing properties of fuzzy mappings and measurable selections, we extended deterministic results to the random setting under generalized contractive conditions. These findings provide a flexible approach for analyzing stochastic systems involving fuzzy and random operators. Future research may explore broader classes of control functions and further applications in stochastic differential equations and decision-making models.

Author contributions

Akbar Azam: Conceptualization, methodology, investigation, writing-original draft preparation, supervision, project administration; Faryad Ali: Conceptualization, methodology, investigation, writing-review and editing, funding acquisition; Sehar Afsheen: Formal analysis, writing-original draft preparation, visualization; Mohammed Shehu Shagari: Formal analysis, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that there are no conflicts of interest.

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A. Appendix

To illuminate the geometric and functional nuances of our core visualizations, this appendix presents alternative viewpoints of Figures 2 and 4 from the main text. These supplementary perspectives, carefully curated in response to reviewer feedback, reveal topological details and contextual relationships that further validate our observations while preserving the manuscript's narrative flow.

A.1. Alternative perspective of Figure 2

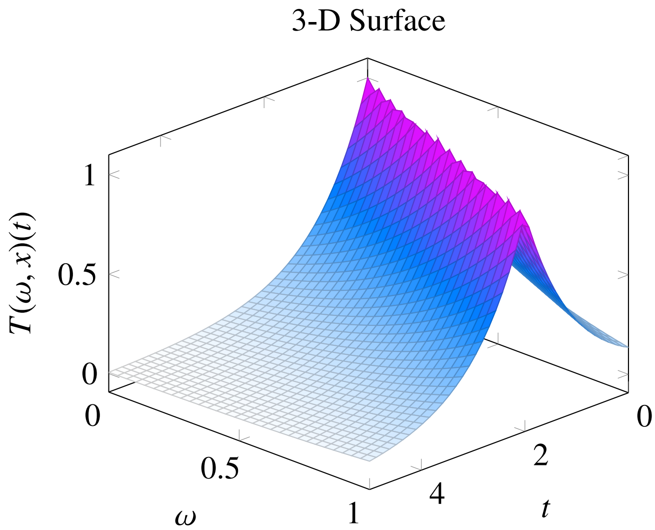

Figure 5. 3-D view of fuzzy mapping $T(\omega, x)(t) = e^{-|t-\omega x|}$ ($x = 2$ fixed).

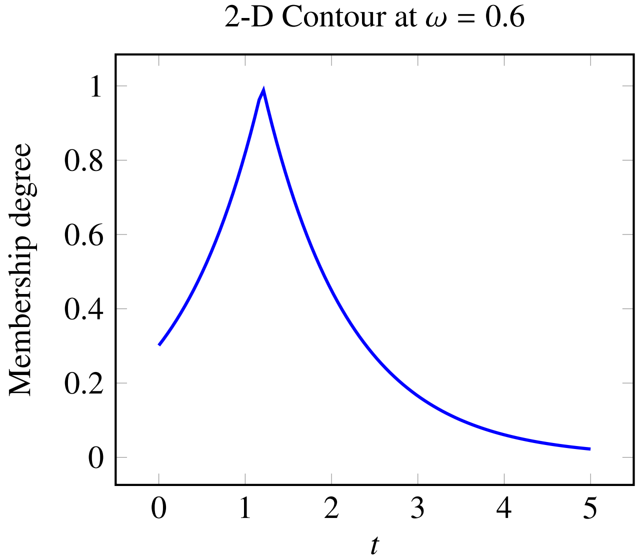

Figure 6. Contour view for $\omega = 0.6$ showing exponential membership decay.

A.2. Alternative perspective of Figure 4

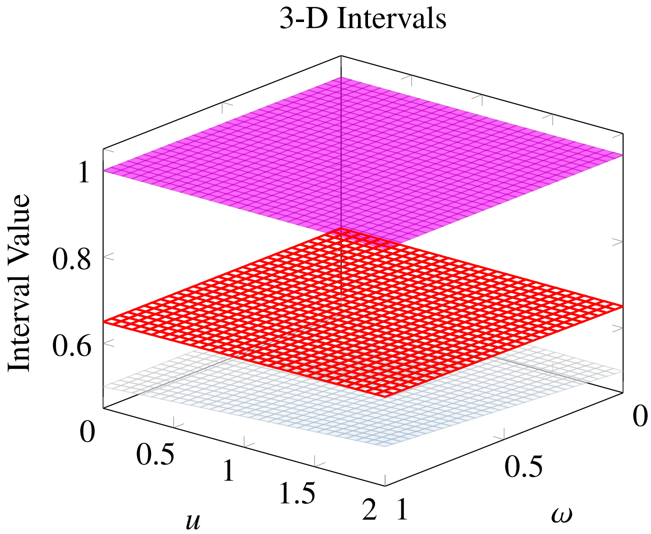

Figure 7. 3-D view of sensor intervals $A(\omega, u)$, $B(\omega, u)$ and selection $u(\omega)$.

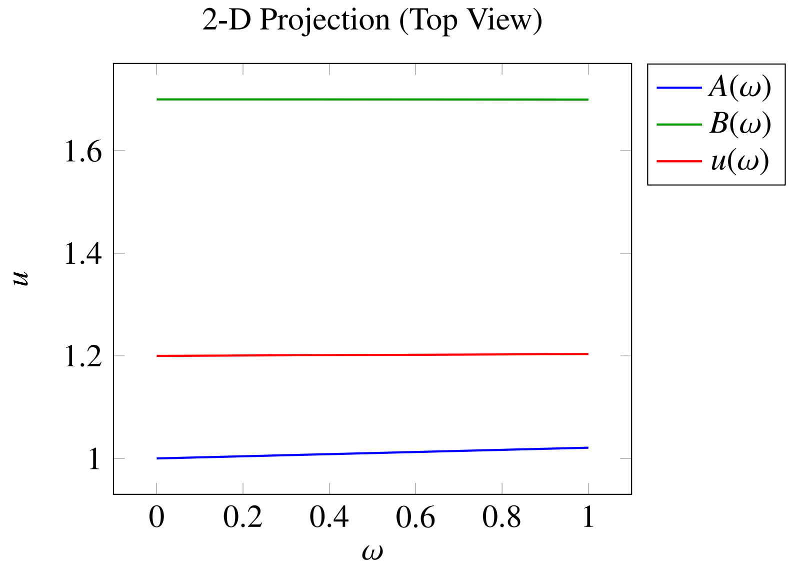

Figure 8. $u(\omega)$ (red) lies within A (blue) and B (green) for all ω.